Some generalisations of Weitzenböck's inequality

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Throughout this Article, we use the following notations for the triangle ABC

- a, b and c are the lengths of the sides BC, CA and AB, respectively,
- Δ denotes the area of triangle ABC,
- h_a, h_b and h_c are the lengths of the altitudes through the vertices A, B and C, respectively,
- m_a, m_b and m_c are the lengths of the medians through the vertices A, B and C, respectively.

1. *Introduction*

Roland Weitzenböck, an Austrian mathematician, first proposed the following geometric inequality (see [1, 2, 3, 4, 5, 6]):

$$a^2 + b^2 + c^2 \geqslant 4\sqrt{3}\Delta. \tag{1}$$

Weitzenböck's inequality is an important inequality in the system of geometrical inequalities.

The following inequality of Finsler and Hadwiger (see [1, 2, 3, 4, 5, 6])

$$a^2 + b^2 + c^2 \geqslant 4\sqrt{3}\Delta + (b - c)^2 + (c - a)^2 + (a - b)^2$$

seems to be a refinement of Weitzenböck's inequality, but it is actually equivalent to it [6]. A direct refinement of both inequalities is

$$a^2 + b^2 + c^2 \geqslant 4\Delta\sqrt{3 + \frac{R - 2r}{R}} + (b - c)^2 + (c - a)^2 + (a - b)^2,$$

which is equivalent to Kooi's inequality [7, 8].

It is well known that $\frac{1}{3}\left(a^2 + b^2 + c^2\right) = GA^2 + GB^2 + GC^2$, where G is the centroid of triangle ABC, so we can rewrite Weitzenböck's inequality (1) as

$$GA^2 + GB^2 + GC^2 \geqslant \frac{4\Delta}{\sqrt{3}}.$$

Actually for any interior point P we have

$$PA^2 + PB^2 + PC^2 \geqslant GA^2 + GB^2 + GC^2,$$

so the following inequality holds

$$PA^2 + PB^2 + PC^2 \geqslant \frac{4\Delta}{\sqrt{3}}$$

and we shall improve this inequality in the following theorem.

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Theorem (main theorem): Let ABC be a triangle. Let P be the interior point of triangle ABC. Let A', B' and C' be the midpoints of sides BC, CA and AB, respectively. Let O and R be the circumcentre and circumradius of triangle ABC, respectively. Then

$$PA^2 + PB^2 + PC^2 \geqslant \frac{4\Delta}{\sqrt{3}}\left(1 + \frac{OP^2}{3R^2}\right). \tag{2}$$

$$PA^2 + PB^2 + PC^2 \geqslant \frac{4\Delta}{\sqrt{3}} \cdot \max\left\{\frac{PA + PA'}{h_a}, \frac{PA + PB'}{h_b}, \frac{PC + PC'}{h_c}\right\}. \tag{3}$$

Remark: When P coincides with G, the centroid of triangle ABC, the inequality (2) becomes

$$GA^2 + GB^2 + GC^2 \geqslant \frac{4\Delta}{\sqrt{3}}\left(1 + \frac{OP^2}{3R^2}\right) \geqslant \frac{4\Delta}{\sqrt{3}},$$

and then, using $GA^2 + GB^2 + GC^2 = \frac{1}{3}(a^2 + b^2 + c^2)$, we get

$$a^2 + b^2 + c^2 \geqslant 4\sqrt{3}\Delta,$$

which is Weitzenböck's inequality.

The inequality (3) becomes

$$GA^2 + GB^2 + GC^2 \geqslant \frac{4\Delta}{\sqrt{3}} \cdot \max\left\{\frac{GA + GA'}{h_a}, \frac{GA + GB'}{h_b}, \frac{GC + GC'}{h_c}\right\}$$

$$= \frac{4\Delta}{\sqrt{3}} \cdot \max\left\{\frac{m_a}{h_a}, \frac{m_b}{h_b}, \frac{m_c}{h_c}\right\} \geqslant \frac{4\Delta}{\sqrt{3}}$$

or

$$a^2 + b^2 + c^2 \geqslant 4\sqrt{3}\Delta,$$

which is Weitzenböck's inequality.

Thus we see that (2) and (3) are generalisations of Weitzenböck's inequality.

2. *Proof of the main theorem*

In this section, we give solutions to the inequalities (2) and (3) in turn. First of all, we would like to introduce the following lemmas:

Lemma 1: Let R be the circumradius of triangle ABC. Then,

$$3\sqrt{3}R^2 \geqslant 4\Delta.$$

Proof: Multiplying the Euler's inequality $R \geqslant 2r$ and the well-known $3\sqrt{3}R \geqslant 2s$, where $s = \frac{1}{2}(a + b + c)$ is the semiperimeter of triangle ABC, we have

$$3\sqrt{3}R^2 \geqslant (2a)(2s) = 4\Delta.$$

This completes proof of Lemma 1.

Lemma 2 (Area of pedal triangle; see [9, 10]): Let ABC be a triangle. Let P be the interior point of triangle ABC. Let O and R be the circumcentre and circumradius of triangle ABC, respectively. Let DEF be pedal triangle of P with respect to triangle ABC; and Δ_{DEF} be the area of triangle DEF. Then

$$\Delta_{DEF} = \frac{R^2 - OP^2}{4R^2} \cdot \Delta,$$

(see proof at [9, 10]).

Lemma 3: Given triangle ABC. Then

$$a^2 + b^2 + c^2 \geqslant 2\sqrt{3}a \cdot m_a, \tag{4}$$

$$3\left(b^2 + c^2\right) - a^2 \geqslant 2\sqrt{3}a \cdot m_a. \tag{5}$$

Equality in (4) holds if, and only if, ABC is an equilateral triangle. Equality in (5) holds if, and only if, $AB = AC$ and $\angle A = \frac{2}{3}\pi$.

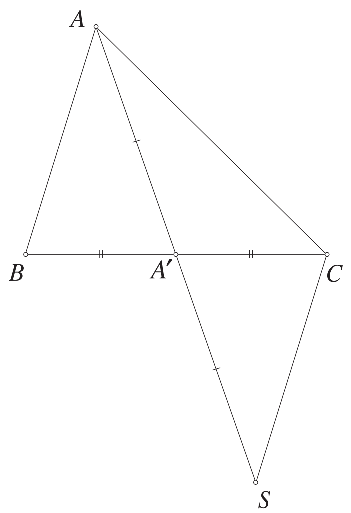

FIGURE 1: Proof of Lemma 3

Proof: Using the AM-GM inequality and the formula for medians, we have

$$2a \cdot \frac{2}{\sqrt{3}}m_a \leqslant a^2 + \frac{4}{3}m_a^2 = a^2 + \frac{2\left(b^2 + c^2\right) - a^2}{3} = \frac{2}{3}\left(a^2 + b^2 + c^2\right).$$

Therefore

$$a^2 + b^2 + c^2 \geqslant 2\sqrt{3}a \cdot m_a.$$

Equality holds if, and only if, ABC is an equilateral triangle.

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Let S be the reflection of A in the midpoint A' of BC. Applying inequality (4) to triangle SAC with median $CA' = \frac{1}{2}a$, we obtain

$$AS^2 + SC^2 + CA^2 \geqslant 2\sqrt{3}AS \cdot \frac{a}{2}$$

or

$$4m_a^2 + b^2 + c^2 \geqslant 2\sqrt{3}a \cdot m_a$$

or

$$3\left(b^2 + c^2\right) - a^2 \geqslant 2\sqrt{3}a \cdot m_a.$$

Equality holds if, and only if, SAC is an equilateral triangle or $AB = AC$ and $A = \frac{2}{3}\pi$. This finishes the proof.

We return to the main theorem. Let DEF be the pedal triangle of P with respect to triangle ABC; and Δ_{DEF} be the area of triangle DEF. We see that triangle AEF is inscribed in circle diameter PA, and denote by Δ_a the area of triangle AEF. Applying Lemma 1 to triangle AEF, we get

$$3\sqrt{3}\left(\frac{PA}{2}\right)^2 \geqslant 4\Delta_a \tag{6}$$

Similarly for triangles BFD and CDE

$$3\sqrt{3}\left(\frac{PB}{2}\right)^2 \geqslant 4\Delta_b \tag{7}$$

and

$$3\sqrt{3}\left(\frac{PC}{2}\right)^2 \geqslant 4\Delta_c. \tag{8}$$

Summing the inequalities (6), (7) and (8), we deduce that

$$\frac{3\sqrt{3}}{4}\left(PA^2 + PB^2 + PC^2\right) \geqslant 4\left(\Delta_a + \Delta_b + \Delta_c\right) = 4\left(\Delta - \Delta_{DEF}\right). \tag{9}$$

Applying Lemma 2 to (9), we see that

$$\frac{3\sqrt{3}}{4}\left(PA^2 + PB^2 + PC^2\right) \geqslant 4\left(\Delta - \frac{R^2 - OP^2}{4R^2} \cdot \Delta\right) = \frac{3R^2 + OP^2}{R^2} \cdot \Delta,$$

which is equivalent to

$$PA^2 + PB^2 + PC^2 \geqslant \frac{4\Delta}{\sqrt{3}}\left(1 + \frac{OP^2}{3R^2}\right).$$

The equality holds if, and only if, triangle ABC is equilateral and P coincides with its centre. To continue, we apply Lemma 3 (inequality (5)) to triangle PBC with median PA'; then

$$3\left(PC^2 + PB^2\right) - a^2 \geqslant 2\sqrt{3}a \cdot PA',$$

which is equivalent to

$$3\left(PC^2 + PB^2\right) \geqslant 2\sqrt{3}a \cdot PA' + a^2. \tag{10}$$

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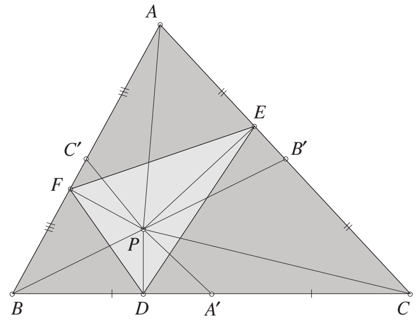

FIGURE 2: Proof of main theorem

Adding $3PA^2$ to each side of (10) and now applying the AM-GM inequality to give $a^2 + 3PA^2 \geqslant 2\sqrt{3}a \cdot PA$, we have

$$3\left(PA^2 + PB^2 + PC^2\right) \geqslant 2\sqrt{3}a \cdot PA' + a^2 + 3PA^2 \geqslant 2\sqrt{3}a \cdot PA' + 2\sqrt{3}a \cdot PA. \quad (11)$$

Also, we have

$$\begin{aligned} 2\sqrt{3}a \cdot PA' + 2\sqrt{3}a \cdot PA &= 2\sqrt{3}a\left(PA + PA'\right) \\ &= 2\sqrt{3}a \cdot h_a \cdot \frac{PA + PA'}{h_a} \quad (12) \\ &= 4\sqrt{3}\Delta \cdot \frac{PA + PA'}{h_a}. \end{aligned}$$

From (11) and (12), it follows that

$$PA^2 + PB^2 + PC^2 \geqslant \frac{4\Delta}{\sqrt{3}} \cdot \frac{PA + PA'}{h_a}.$$

Similarly with vertices B and C, we get

$$PA^2 + PB^2 + PC^2 \geqslant \frac{4\Delta}{\sqrt{3}} \cdot \max\left\{\frac{PA + PA'}{h_a}, \frac{PB + PB'}{h_b}, \frac{PC + PC'}{h_c}\right\}.$$

The equality holds if, and only if, triangle ABC is equilateral and P coincides with its centre. That completes the proof of the main theorem.

Remark: Actually, we do not need P to be an interior point of triangle ABC for the inequality (3) and the same goes for the proof.

3. *Weitzenböck's inequality via triangle identity*

In this section, we give a further extension for the Weitzenböck's inequality using the identity of the triangle. First, we recall the definition of two Fermat points or isogonic centres in a triangle.

Definition (See [10, 11, 12]): Given triangle ABC, construct the equilateral triangles BCA', CAB' and ABC' having bases BC, CA and AB, respectively, outwardly (inwardly) from triangle ABC. The lines AA', BB' and CC' concur in the first (the second) Fermat point F_1 (F_2) of triangle ABC.

The author has also proposed an extension of Weitzenböck's inequality under the identity as follows (see [14]):

$$a^2 + b^2 + c^2 = 4\Delta\sqrt{3 + \frac{4}{3}\cdot\frac{OH^2}{F_1F_2^2}} \geqslant 4\sqrt{3}\Delta, \tag{13}$$

where O, H, F_1 and F_2 are the circumcentre, the orthocentre, the first Fermat point and the second Fermat point, respectively.

Here we shall suggest a method to prove (13) using *Maple* with the available barycentric coordinates formulae for the two points F_1 and F_2. The first and second Fermat points F_1 and F_2 are Kimberling centre $X(13)$ and $X(14)$, respectively (see [12, 13, 14, 15]), with barycentric coordinates

$$F_1\left(a^4 - 2\left(b^2 - c^2\right)^2 + a^2\left(b^2 + c^2 + 4\sqrt{3}\Delta\right), \dots, \dots\right) \tag{14}$$

and

$$F_2\left(a^4 - 2\left(b^2 - c^2\right)^2 + a^2\left(b^2 + c^2 - 4\sqrt{3}\Delta\right), \dots, \dots\right). \tag{15}$$

Using (14), (15) and the formula for distance in areal coordinates (barycentric coordinates) [16] with *Maple's* expression reduction support, we obtain

$$F_1F_2^2 = \frac{4}{3}\cdot\frac{9a^2b^2c^2 - 16\Delta\left(a^2 + b^2 + c^2\right)}{\left(a^2 + b^2 + c^2\right)^2 - 48\Delta^2}$$

or

$$F_1F_2^2 = \frac{4}{3}\cdot\frac{16\Delta^2\left(9R^2 - \left(a^2 + b^2 + c^2\right)\right)}{\left(a^2 + b^2 + c^2\right)^2 - 48\Delta^2} = \frac{4}{3}\cdot\frac{OH^2}{\left(\frac{a^2 + b^2 + c^2}{4\Delta}\right)^2 - 3}$$

where R is the circumradius of triangle ABC. Hence, we easily obtain equation (13).

4. *Conclusion*

We have replaced the sum of squares $\frac{1}{3}\left(a^2 + b^2 + c^2\right)$ by $\left(GA^2 + GB^2 + GC^2\right)$, from which we come up with the idea of expanding instead because G becomes any point P. Since the Finsler-Hadwiger inequality follows from similar inequality,

$$PA^2 + PB^2 + PC^2 \geqslant \frac{4\Delta}{\sqrt{3}} + \frac{(a - b)^2 + (b - c)^2 + (c - a)^2}{3},$$

the open question is can we work similarly for this form of the Finsler-Hadwiger inequality?

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