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A variational principle for weighted topological pressure under $\mathbb{Z}^d$-actions

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Abstract. Let $k \geq 2$ and $(X_i, \mathcal{T}_i), i = 1, \ldots, k$, be $\mathbb{Z}^d$-actions topological dynamical systems with $\mathcal{T}_i := \{T_i^{\mathbf{g}} : X_i \to X_i\}_{\mathbf{g}\in\mathbb{Z}^d}$, where $d \in \mathbb{N}$ and $f \in C(X_1)$. Assume that for each $1 \leq i \leq k-1$, $(X_{i+1}, \mathcal{T}_{i+1})$ is a factor of $(X_i, \mathcal{T}_i)$. In this paper, we introduce the weighted topological pressure $P^{\mathbf{a}}(\mathcal{T}_1, f)$ and weighted measure-theoretic entropy $h_\mu^{\mathbf{a}}(\mathcal{T}_1)$ for $\mathbb{Z}^d$-actions, and establish a weighted variational principle as

$$P^{\mathbf{a}}(\mathcal{T}_1, f) = \sup \left\{ h_\mu^{\mathbf{a}}(\mathcal{T}_1) + \int_{X_1} f \, d\mu : \mu \in \mathcal{M}(X_1, \mathcal{T}_1) \right\}.$$

This result not only generalizes some well-known variational principles about topological pressure for compact or non-compact sets, but also improves the variational principle for weighted topological pressure in [16] from $\mathbb{Z}_+$-action topological dynamical systems to $\mathbb{Z}^d$-actions topological dynamical systems.

Key words: weighted topological pressure, $\mathbb{Z}^d$-actions, variational principle, factor maps, geometric measure theory
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1. *Introduction*

Topological pressure, a term motivated by statistical mechanics, was introduced by Ruelle [36] for expansive dynamical system and later by Walters [42] in the general case. Considering continuous potential, topological pressure generalizes the definition of topological entropy by Bowen [4]. Moreover, topological pressure plays an important role in dimensional theory. For example, for repellers of $C^{1+\gamma}$ conformal expanding maps, Bowen [5] and Ruelle [37] discovered that their Hausdorff dimension is a solution of Bowen's equation involving topological pressure. In the non-conformal setting, Cao, Pesin and Zhao [7] establish continuity of the sub-additive topological pressure with singular valued potential and obtain a sharp lower bound of the Hasudorff dimension of the repeller. Inspired by the entropy variational principle [11, 19, 20] which reveals the basic relationship between topological entropy and measure-theoretic entropy (see [24, 38] by Kolmogorov and Sinai), Walters in [42] developed a variational principle for topological pressure. Precisely, let (X, T) be a topological dynamical system (TDS) with a compact metric space X and a continuous map $T : X \to X$, and f be an arbitrary continuous real-valued function on X. Then,

$$P(T, f) = \sup\left\{h_\mu(T) + \int_X f\, d\mu : \mu \in \mathcal{M}(X, T)\right\},$$

where $\mathcal{M}(X, T)$ denotes all the T-invariant Borel probability measures on X and $h_\mu(T)$ denotes the measure-theoretic entropy of T with respect to μ.

In [31], Misiurewicz gave a short and elegant proof of the variational principle of pressure for an action of the group $\mathbb{Z}_+^N$. Soon afterwards, increasingly more attention has been drawn to extend the classical variational principle of topological pressure to any countable amenable group actions instead of $\mathbb{Z}_+^N$, including [26, 32, 33, 39, 41]. It is worth mentioning that Bowen [3] defined sofic entropy for measure-preserving actions of countable sofic groups on standard probability measure spaces admitting a generating partition with finite entropy. Later, Kerr and Li [22, 23] extended Bowen's sofic measure-theoretic entropy to all measure-preserving actions of countable sofic groups on standard probability measure spaces. They also defined sofic topological entropy for continuous actions of countable sofic groups on compact metric spaces and established the variational principle between sofic measure-theoretic entropy and sofic topological entropy. Inspired by their work, Chung [10] introduced the topological pressure of a continuous function for continuous actions of countable sofic groups on compact metric spaces, and established the variational principle for it in the sofic context.

However, Carvalho, Rodrigues and Varandas [9] point out the fact that some non-trivial challenges appear when considering the variational principle for free group actions. For example, differing from amenable group actions, Borel probability measures which are invariant by all the generators of a free group action may fail to exist. Due to this obstacle, [2, 8, 27] only obtained a partial variational principle for free semigroup actions. To overcome this difficulty, Carvalho, Rodrigues and Varandas [9] defined the metric-theoretic entropy of a Borel probability measure via the topological pressure for continuous free semigroup actions inspired by the fact that pressure determines both its Borel invariant probability measures and the entropy function, cf. Theorems 9.11 and 9.12 of [43]. They also obtained the variational principle of pressure for continuous free semigroup actions.

Next we elaborate our motivations and main results. Let (X, T) and (Y, S) be two TDSs. Suppose that (Y, S) is a factor of (X, T), that is to say, there exists a continuous surjective map $\pi : X \to Y$ such that $\pi \circ T = S \circ \pi$. The map π is called the factor map from X to Y. Let f be a continuous real-valued function on X and $\mathbf{a} = (a_1, a_2)$ with $a_1 > 0, a_2 \geq 0$. Following Pesin–Pistskel's [35] definition of topological pressure of non-compact subsets, which resembles the Hausdorff dimension, Feng and Huang [16] defined the $\mathbf{a}$-weighted topological pressure of *f*, denoted by $P^{\mathbf{a}}(T, f)$, by **a**-weighted Bowen balls instead of Bowen balls. They also obtained the following variational principle:

$$P^{\mathbf{a}}(T, f) = \sup \left\{ a_1 h_\mu(T) + a_2 h_{\mu\circ\pi^{-1}}(S) + \int_X f \, d\mu : \mu \in \mathcal{M}(X, T) \right\}. \tag{1.1}$$

Equation (1.1) is also a version of the Ledrappier–Young dimension formula [**25**]. In the end of [**16**], the authors asked whether the variational principle for weighted topological pressure remains valid for $\mathbb{Z}^d$-actions or not. In this paper, we give an affirmative answer to this question and prove the variational principle for weighted topological pressure under $\mathbb{Z}^d$-actions. Additionally, before Feng and Huang, Barral and Feng [**1**, **14**] defined $P^{\mathbf{a}}(X, f)$ (and called it weighted topological pressure) by relative thermodynamic formalism and subadditive thermodynamic formalism, in particular when the underlying dynamical systems X and Y are shifts over finite alphabets. However, their way to define $P^{\mathbf{a}}(X, f)$ relies on certain properties of subshifts and therefore does not extend to a general TDS. For this reason, in this paper, we extend Feng and Huang's [**16**] approach of defining weighted topological pressure in the setting of $\mathbb{Z}^d$-actions.

Now we introduce the definitions of weighted topological pressure for continuous potential for $\mathbb{Z}^d$-actions topological dynamical systems. Let $(X, \mathcal{T})$ be a $\mathbb{Z}^d$-actions TDS, where X is a compact metric space with a family of continuous transformations $\mathcal{T} := \{T^{\mathbf{g}} : X \to X\}_{\mathbf{g}\in\mathbb{Z}^d}$ satisfying that $T^{\mathbf{0}}$ is the identity map and $T^{\mathbf{g}+\mathbf{h}} = T^{\mathbf{g}} \circ T^{\mathbf{h}}$ for all $\mathbf{g}, \mathbf{h} \in \mathbb{Z}^d$. For $n, m \in \mathbb{N}$ with $n < m$, let

$$\Lambda_n := \{\mathbf{g} = (g_1, \ldots, g_d) \in \mathbb{Z}^d : |g_i| < n, 1 \le i \le d\}, \quad \Lambda_n^m = \Lambda_m \setminus \Lambda_n,$$

and $\lambda_n := \operatorname{Card} \Lambda_n = (2n-1)^d$. For a compact metric space X, let $\mathcal{M}(X)$ be the set of all Borel probability measures on X with the weak*-topology. A measure $\mu \in \mathcal{M}(X)$ is invariant under $\mathbb{Z}^d$-actions if $\mu(T^{-\mathbf{g}}B \triangle B) = 0$ for all $\mathbf{g} \in \mathbb{Z}^d$ and $B \subset \mathcal{B}(X)$, where $\triangle$ denotes the symmetric difference and $\mathcal{B}(X)$ is the σ-algebra of subsets of X. In addition, the $\mathbb{Z}^d$-action is called ergodic if any set $B \subset \mathcal{B}(X)$ with $\mu(T^{-\mathbf{g}}B \triangle B) = 0$ for all $\mathbf{g} \in \mathbb{Z}^d$ has $\mu(B) = 0$ or $\mu(B) = 1$. Denote by $\mathcal{M}(X, \mathcal{T})$ and $E(X, \mathcal{T})$ the sets of all $\mathcal{T}$-invariant Borel probability measures and ergodic measures on X, respectively. Then $\mathcal{M}(X, \mathcal{T}) \neq \emptyset$. Denote the set of finite Borel-measurable partitions of X by $\mathcal{P}_X$. Given $\alpha \in \mathcal{P}_X$ and $\mu \in \mathcal{M}(X)$, define

$$H_\mu(\alpha) := -\sum_{A\in\alpha} \mu(A) \log \mu(A).$$

When $\mu \in \mathcal{M}(X, \mathcal{T})$, the function $n \in \mathbb{N} \mapsto H_\mu(\bigvee_{\mathbf{g}\in\Lambda_n} T^{-\mathbf{g}}\alpha)$ is non-negative sub-additive for a given $\alpha \in \mathcal{P}_X$. We can define the measure-theoretic entropy of $\mathcal{T}$ with respect to α as

$$h_\mu(\mathcal{T}, \alpha) := \lim_{n\to\infty} \frac{1}{\lambda_n} H_\mu\bigg(\bigvee_{\mathbf{g}\in\Lambda_n} T^{-\mathbf{g}}\alpha\bigg) = \inf_{n\in\mathbb{N}} \frac{1}{\lambda_n} H_\mu\bigg(\bigvee_{\mathbf{g}\in\Lambda_n} T^{-\mathbf{g}}\alpha\bigg).$$

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It is easy to show that the limit exists, cf. [44, Lemma 2.3]. Additionally, the measure-theoretic entropy of $\mathcal{T}$ with respect to μ is defined by

$$h_\mu(\mathcal{T}) := \sup_{\alpha\in\mathcal{P}_X} h_\mu(\mathcal{T},\alpha).$$

Let $k \geq 2$, (X_i, d_i), $i = 1, \ldots, k$, be compact metric spaces and $(X_i, \mathcal{T}_i)$ be $\mathbb{Z}^d$-actions topological dynamical systems with $\mathcal{T}_i := \{T_i^{\mathbf{g}} : X_i \to X_i\}_{\mathbf{g}\in\mathbb{Z}^d}$. Assume that for each $1 \leq i \leq k-1$, $(X_{i+1}, \mathcal{T}_{i+1})$ is a factor of $(X_i, \mathcal{T}_i)$ with a factor map $\pi : X_i \to X_{i+1}$; in other words, there exist continuous surjective maps $\pi_i : X_i \to X_{i+1}$ such that $\pi_i \circ T_i^{\mathbf{g}} = T_{i+1}^{\mathbf{g}} \circ \pi_i$ holds for all $1 \leq i \leq k-1$ and $\mathbf{g} \in \mathbb{Z}^d$. Let $\pi_0 :=$ id on X_1 and define $\tau_i : X_1 \to X_{i+1}$ by $\tau_i = \pi_i \circ \pi_{i-1} \circ \cdots \circ \pi_0$ for $i = 0, 1, \ldots, k-1$.

Let $\mathcal{M}(X_i)$ be the set of all Borel probability measures on X_i with the weak*-topology. Denote by $\mathcal{M}(X_i, \mathcal{T}_i)$ the sets of all $\mathcal{T}_i$-invariant (that is, $T_i^{\mathbf{g}}$-invariant for each $\mathbf{g} \in \mathbb{Z}^d$) Borel probability measures on X_i. Fix $\mathbf{a} = (a_1, a_2, \ldots, a_k) \in \mathbb{R}^k$ with $a_1 > 0$ and $a_i \geq 0$ for $i \geq 2$. Let $a_0 = 0$. Write for brevity that $c_i = (a_0 + \cdots + a_i)^d - (a_0 + \cdots + a_{i-1})^d$ for $i = 1, \ldots, k$. For $\mu \in \mathcal{M}(X_1, \mathcal{T}_1)$, denote by

$$h_\mu^{\mathbf{a}}(\mathcal{T}_1) := \sum_{i=1}^{k} c_i h_{\mu\circ\tau_{i-1}^{-1}}(\mathcal{T}_i)$$

the weighted measure-theoretic entropy of $\mathcal{T}_1$ with respect to μ.

Remark 1.1. If $d = 1$, then $c_i = a_i$ for all $i = 1, \ldots, k$. In this case, the above definition coincides with Feng and Huang's weighted measure-theoretic entropy in [16]. So we extend their work.

Definition 1.1. ($\mathbf{a}$-weighted Bowen ball) For $x \in X_1$, $n \in \mathbb{N}$, $\epsilon > 0$, denote

$$\begin{aligned} B_n^{\mathbf{a}}(x,\epsilon) :&= \{y \in X_1 : d_i(T_i^{\mathbf{g}}\tau_{i-1}x, T_i^{\mathbf{g}}\tau_{i-1}y) < \epsilon \text{ for } \mathbf{g} \in \Lambda_{\lceil (a_1+\cdots+a_i)n\rceil}, i = 1, \ldots, k\} \\ &= \{y \in X_1 : d_i(\tau_{i-1}T_1^{\mathbf{g}}x, \tau_{i-1}T_1^{\mathbf{g}}y) < \epsilon \text{ for } \mathbf{g} \in \Lambda_{\lceil (a_1+\cdots+a_i)n\rceil}, i = 1, \ldots, k\}, \end{aligned}$$

where $\lceil u\rceil$ denotes the least integer $\geq u$. For $n \in \mathbb{N}$, define a metric $d_n^{\mathbf{a}}$ on X_1 by

$$d_n^{\mathbf{a}}(x, y) = \sup\{d_i(T_i^{\mathbf{g}}\tau_{i-1}x, T_i^{\mathbf{g}}\tau_{i-1}y) \quad \text{for } i = 1, \ldots, k, \mathbf{g} \in \Lambda_{\lceil (a_1+\cdots+a_i)n\rceil}\}.$$

Then

$$B_n^{\mathbf{a}}(x,\epsilon) = \{y \in X_1 : d_n^{\mathbf{a}}(x, y) < \epsilon\}.$$

We call $B_n^{\mathbf{a}}(x,\epsilon)$ the nth $\mathbf{a}$-weighted Bowen ball of radius ϵ centred at x.

Let $C(X_1)$ be the space of all continuous real-valued functions on X_1 with norm $\|f\| := \sup_{x\in X_1} |f(x)|$. Let $Z \subseteq X_1$, $s \geq 0$, $\epsilon > 0$, $N \in \mathbb{N}$, $f \in C(X_1)$, and define

$$\Lambda_{f,N,\epsilon}^{\mathbf{a},s}(Z) = \inf \sum_j \exp\bigg(-s\lambda_{n_j} + \frac{1}{a_1^d}\sup_{x\in A_j}\sum_{\mathbf{g}\in\Lambda_{\lceil a_1 n_j\rceil}} f(T_1^{\mathbf{g}}x)\bigg),$$

where the infimum is taken over all countable collections $\Gamma = \{(n_j, A_j)\}_j$ satisfying $n_j \geq N$, A_j is Borel subset of $B_{n_j}^{\mathbf{a}}(x,\epsilon)$ for some $x \in X_1$ and $Z \subseteq \bigcup_j A_j$. The quantity

$\Lambda^{\mathbf{a},s}_{f,N,\epsilon}(Z)$ does not decrease as N increases and ϵ decreases, and hence the following limits exist:

$$\Lambda^{\mathbf{a},s}_{f,\epsilon}(Z) = \lim_{N\to\infty} \Lambda^{\mathbf{a},s}_{f,N,\epsilon}(Z), \quad \Lambda^{\mathbf{a},s}_{f}(Z) = \lim_{\epsilon\to 0} \Lambda^{\mathbf{a},s}_{f,\epsilon}(Z).$$

There exists a critical value of the parameter s, which we will denote by $P^{\mathbf{a}}(\mathcal{T}_1, f, Z)$, where $\Lambda^{\mathbf{a},s}_{f}(Z)$ jumps from ∞ to 0, that is,

$$\Lambda^{\mathbf{a},s}_{f}(Z) = \begin{cases} 0, & s > P^{\mathbf{a}}(\mathcal{T}_1, f, Z), \\ \infty, & s < P^{\mathbf{a}}(\mathcal{T}_1, f, Z). \end{cases}$$

In other words, $P^{\mathbf{a}}(\mathcal{T}_1, f, Z) = \inf\{s : \Lambda^{\mathbf{a},s}_{f}(Z) = 0\} = \sup\{s : \Lambda^{\mathbf{a},s}_{f}(Z) = \infty\}$.

Definition 1.2. We call $P^{\mathbf{a}}(\mathcal{T}_1, f) := P^{\mathbf{a}}(\mathcal{T}_1, f, X_1)$ the **a**-weighted topological pressure of f with respect to $\mathcal{T}_1$. Denote by $h^{\mathbf{a}}_{\mathrm{top}}(\mathcal{T}_1) := P^{\mathbf{a}}(\mathcal{T}_1, 0)$ the **a**-weighted topological entropy of $\mathcal{T}_1$.

Now we can establish our main result about the variational principle as follows.

THEOREM 1.1. *Let $f \in C(X_1)$. Then*

$$P^{\boldsymbol{a}}(\mathcal{T}_1, f) = \sup\left\{h^{\boldsymbol{a}}_{\mu}(\mathcal{T}_1) + \int_{X_1} f\,d\mu : \mu \in \mathcal{M}(X_1, \mathcal{T}_1)\right\}.$$

If we take $f = 0$ in Theorem 1.1, we can directly obtain the following corollary, which reveals the relationship between **a**-weighted topological entropy and weighted measure-theoretic entropy.

COROLLARY 1.2. $h^{\boldsymbol{a}}_{top}(\mathcal{T}_1) = \sup\{h^{\boldsymbol{a}}_{\mu}(\mathcal{T}_1) : \mu \in \mathcal{M}(X_1, \mathcal{T}_1)\}$.

The proof of Theorem 1.1 (see §3.3 for details) consists of two parts. In part (i), we prove the lower of weighted topological pressure $P^{\mathbf{a}}(\mathcal{T}_1, f)$, which means $P^{\mathbf{a}}(\mathcal{T}_1, f) \geq h^{\mathbf{a}}_{\mu}(\mathcal{T}_1) + \int_{X_1} f\,d\mu$ for all $\mu \in \mathcal{M}(X_1, \mathcal{T}_1)$. In part (ii), we give the upper bound estimate of the weighted topological pressure $P^{\mathbf{a}}(\mathcal{T}_1, f)$. That is to say, for any $\delta > 0$, there exists $\mu \in \mathcal{M}(X_1, \mathcal{T}_1)$ such that $P^{\mathbf{a}}(\mathcal{T}_1, f) \leq h^{\mathbf{a}}_{\mu}(\mathcal{T}_1) + \int_{X_1} f\,d\mu + \delta$.

Feng and Huang's techniques in [16] provide the motivation for our paper. While considering a $\mathbb{Z}^d$-actions TDS rather than (X, T), there are still some problems that need attention. One should be more careful when dealing with $\{T^{\mathbf{g}} : X \to X\}_{\mathbf{g}\in\mathbb{Z}^d}$, a family of transformations on compact metric space X, than with single T on X. First, in the study of ergodic theory, the invariant measure is necessary. For $\mathbb{Z}$-action (X, T), the T-invariant Borel probability measure always exists, cf. [43, Corollary 6.9.1]. As for actions of some groups G, a well-known result says that when G is an Abelian group, there exists a G-invariant measure, cf. [13, Theorem 8.11]. Obviously, $\mathbb{Z}^d$ is an Abelian group. Also, the ergodic decomposition for a continuous measure-preserving action of $\mathbb{Z}^d$ (see [13, Theorem 8.20]) may be deduced by Choquet's theorem, just as for single transformation. In addition, we need to use Birkhoff's ergodic theorem in part (i). Given an arbitrary invertible measure-preserving transformation T on the probability space (X, T, μ), Birkhoff's

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pointwise ergodic theorem asserts that for any $f \in L^1(X)$, the averages of f along an orbit of T, namely the expressions $(f(T^{-n}(x)) + \cdots + f(T^n(x)))/(2n+1)$ converge to $f^*(x)$ for μ-almost every (a.e) $x \in X$, where f^* is the conditional expectation of f with respect to the σ-algebra of T-invariant sets. In particular, if T is ergodic, we have

$$\lim_{n\to\infty} \frac{f(T^{-n}(x)) + \cdots + f(T^n(x))}{2n+1} = \int_X f\, d\mu$$

for μ-a.e. $x \in X$. Then it is natural to ask whether, given a family of measure-preserving transformations $\{T^{\mathbf{g}} : X \to X\}_{\mathbf{g}\in\mathbb{Z}^d}$, there is a natural way to average a function f along the orbits of the group generated by $\{T^{\mathbf{g}} : X \to X\}_{\mathbf{g}\in\mathbb{Z}^d}$. Luckily, since $\mathbb{Z}^d$ is an Abelian group, $T^{\mathbf{g}_1}$ and $T^{\mathbf{g}_2}$ commute for all $\mathbf{g}_1, \mathbf{g}_2 \in \mathbb{Z}^d$. Ornstein and Weiss [34] proved that the pointwise ergodic theorem still holds with finite measure-preserving actions of an Abelian group. Lindenstrauss [28] obtained pointwise ergodic theorem for amenable groups with respect to tempered Følner sequences. Then we have

$$\lim_{n\to\infty} \frac{\sum_{\mathbf{g}\in\Lambda_n} f(T^{\mathbf{g}}x)}{\lambda_n} = \int_X f\, d\mu$$

for μ-a.e. $x \in X$, which will be used in equation (3.10) later. Furthermore, a weighted version of the Brin–Katok theorem on local entropy is needed. We postpone the proof of it in Appendix A, based on the Shannon–McMillan–Breiman theorem (see [17] or [34]) for a family of transformations under $\mathbb{Z}^d$-actions. Owing to the work of Lindenstrauss [28], general covering lemmas were developed to generalize classical pointwise convergence results to general discrete amenable groups, which are powerful to obtain Shannon–McMillan–Breiman theorem for discrete amenable groups. The above facts together ensure that we can answer Feng and Huang's question [16] of extending the weighted variational principle from $\mathbb{Z}_+$-action to $\mathbb{Z}^d$-actions, see Theorem 1.1. Theorem 1.1 also generalizes some well-known variational principles about topological pressure for compact or non-compact sets in the literature.

Finally, we give the organization of this paper. In §2, we investigate some properties of certain entropy functions. Section 3 is divided into three subsections. In §3.1, we list four lemmas which are crucial to prove the main result, including a weighted version of the Brin–Katok formula, Yan's lemma [44, Lemma 4.4], a combinatoric lemma and a dynamical Frostman lemma. In §3.2, we introduce the definition of average **a**-weighted topological pressure $P_W^{\mathbf{a}}(\mathcal{T}_1, f)$ to prove the dynamical Frostman lemma. In §3.3, we prove our main result, a variational principle for weighted topological pressure in the $\mathbb{Z}^d$-actions setting. In §4, we investigate how the pressure $P^{\mathbf{a}}(\mathcal{T}_1, f)$ determines the weighted measure-theoretic entropy $h_\mu^{\mathbf{a}}(\mathcal{T}_1)$. In §5, we give some remarks. In Appendix A, we prove the weighted version of the Brin–Katok formula.

2. *Properties of certain entropy functions*

In this section, we first investigate the upper semi-continuity of certain entropy functions, which are crucial to the upper bound estimate of topological pressure in Theorem 1.1. First, we give the definition of upper semi-continuity for convenience.

Definition 2.1. Let X be a compact metric space. A function $f : X \to [-\infty, \infty)$ is called upper semi-continuous if one of the following equivalent conditions holds:

(C1) $\limsup_{x_n \to x} f(x_n) \leq f(x)$ for each $x \in X$;

(C2) for each $r \in \mathbb{R}$, the set $\{x \in X : f(x) \geq r\}$ is closed.

Remark 2.1. Theorems 6.4 and 6.5 in [43] together show that if X is a compact metrizable space, then $\mathcal{M}(X)$ is compact and metrizable in the weak*-topology. More precisely, let $\{f_n\}_{n=1}^{\infty}$ be a dense subset of $C(X)$ with $\|f_n\| \neq 0$, then

$$D(m, \mu) = \sum_{n=1}^{\infty} \frac{|\int f_n\, dm - \int f_n\, d\mu|}{2^n \|f_n\|}$$

is a metric on $\mathcal{M}(X)$ giving the weak*-topology. Additionally, in the weak*-topology, $\mu_n \to \mu$ in $\mathcal{M}(X)$ if and only if $\int f\, d\mu_n \to \int f\, d\mu$ for all $f \in C(X)$. Due to these facts, we can still use Definition 2.1 to define the upper semi-continuity of some entropy functions.

Let $(X, \mathcal{T})$ be a $\mathbb{Z}^d$-actions TDS with metric ρ. For $\epsilon > 0$ and $M \in \mathbb{N}$, we define

$$\mathcal{P}_X(\epsilon, M) = \{\alpha \in \mathcal{P}_X : \mathrm{diam}(\alpha) < \epsilon \text{ and } \mathrm{Card}(\alpha) \leq M\}, \tag{2.1}$$

and $\mathcal{P}_X(\epsilon) = \bigcup_{M \in \mathbb{N}, \mathcal{P}_X(\epsilon, M) \neq \emptyset} \mathcal{P}_X(\epsilon, M)$, where $\mathrm{diam}(\alpha) := \max\{\mathrm{diam}(A) : A \in \alpha\}$.

The following lemma is a slight variant of [16, Lemma 2.3], we omit the proof.

LEMMA 2.1. *Let $(X, \mathcal{T})$ be a $\mathbb{Z}^d$-actions TDS and $\epsilon > 0$. Then the following hold.*

(1) *If $M \in \mathbb{N}$ satisfies $\mathcal{P}_X(\epsilon, M) \neq \emptyset$, then the map*

$$\theta \in \mathcal{M}(X) \mapsto H_\theta(\epsilon, M; l) := \inf_{\alpha \in \mathcal{P}_X(\epsilon, M)} \frac{1}{\lambda_l} H_\theta\bigg(\bigvee_{\mathbf{g} \in \Lambda_l} T^{-\mathbf{g}}\alpha\bigg) \tag{2.2}$$

is upper semi-continuous from $\mathcal{M}(X)$ to $[0, \log M]$ for each $l \in \mathbb{N}$.

(2) *The map*

$$\theta \in \mathcal{M}(X) \mapsto H_\theta(\epsilon; l) := \inf_{\alpha \in \mathcal{P}_X(\epsilon)} \frac{1}{\lambda_l} H_\theta\bigg(\bigvee_{\mathbf{g} \in \Lambda_l} T^{-\mathbf{g}}\alpha\bigg)$$

is a bounded upper semi-continuous non-negative function for each $l \in \mathbb{N}$.

(3) *The map*

$$\mu \in \mathcal{M}(X, \mathcal{T}) \mapsto h_\mu(\mathcal{T}, \epsilon) := \inf_{\alpha \in \mathcal{P}_X(\epsilon)} h_\mu(\mathcal{T}, \alpha)$$

is a bounded upper semi-continuous non-negative function.

Remark 2.2. Since $\mathcal{P}_X(\epsilon) = \bigcup_{M \in \mathbb{N}, \mathcal{P}_X(\epsilon, M) \neq \emptyset} \mathcal{P}_X(\epsilon, M)$, we have

$$H_\theta(\epsilon; l) = \inf_{M \in \mathbb{N}, \mathcal{P}_X(\epsilon, M) \neq \emptyset} H_\theta(\epsilon, M; l)$$

for $\theta \in \mathcal{M}(X)$ and

$$h_\mu(\mathcal{T},\epsilon) = \inf_{\alpha\in\mathcal{P}_X(\epsilon)} h_\mu(\mathcal{T},\alpha) = \inf_{\alpha\in\mathcal{P}_X(\epsilon)} \inf_{l\geq 1} \frac{1}{\lambda_l} H_\mu\bigg(\bigvee_{\mathbf{g}\in\Lambda_l} T^{-\mathbf{g}}\alpha\bigg)$$
$$= \inf_{l\geq 1} \inf_{\alpha\in\mathcal{P}_X(\epsilon)} \frac{1}{\lambda_l} H_\mu\bigg(\bigvee_{\mathbf{g}\in\Lambda_l} T^{-\mathbf{g}}\alpha\bigg) = \inf_{l\geq 1} H_\theta(\epsilon; l)$$

for $\theta \in \mathcal{M}(X,\mathcal{T})$.

LEMMA 2.2. *Let $(X,\mathcal{T})$ be a $\mathbb{Z}^d$-actions TDS and $\mu\in\mathcal{M}(X)$. Let $\alpha\in\mathcal{P}_X$ with* $\mathrm{Card}(\alpha)=M$. *For $n,m\in\mathbb{N}$ with $n<m$, denote*

$$h(n) := H_{1/\lambda_n \sum_{g\in\Lambda_n} \mu\circ T^{-g}}(\alpha) \quad \text{and} \quad h(n,m) := H_{1/(\lambda_m-\lambda_n)\sum_{g\in\Lambda_n^m}\mu\circ T^{-g}}(\alpha),$$

then:

(i) $h(n)\leq \log M$ *and* $h(n,m)\leq \log M$*;*
(ii) $|h(n+1)-h(n)| \leq -(\lambda_n/\lambda_{n+1})\log(\lambda_n/\lambda_{n+1}) - ((\lambda_{n+1}-\lambda_n)/\lambda_{n+1})$ $\log((\lambda_{n+1}-\lambda_n)/\lambda_{n+1}) + 2((\lambda_{n+1}-\lambda_n)/\lambda_{n+1})\log M$*;*
(iii) $|h(m)-(\lambda_n/\lambda_m)h(n)-((\lambda_m-\lambda_n)/\lambda_m)h(n,m)|\leq \log 2$.

Proof. (i) is obtained directly from [43, Corollary 4.2.1].

(ii) Given $\mu_1,\mu_2\in\mathcal{M}(X)$ and $p\in[0,1]$, since the function $\phi(x)=x\log(x)$ is convex, if $A\in\alpha$, then

$$\begin{aligned}
0 &\geq \phi(p\mu_1(A)+(1-p)\mu_2(A)) - p\phi(\mu_1(A)) - (1-p)\phi(\mu_2(A))\\
&= (p\mu_1(A)+(1-p)\mu_2(A))\log(p\mu_1(A)+(1-p)\mu_2(A)) - p\mu_1(A)\log(\mu_1(A))\\
&\quad - (1-p)\mu_2(A)\log(\mu_2(A))\\
&= p\mu_1(A)[\log(p\mu_1(A)+(1-p)\mu_2(A)) - \log(p\mu_1(A))]\\
&\quad + (1-p)\mu_2(A)[\log(p\mu_1(A)+(1-p)\mu_2(A)) - \log((1-p)\mu_2(A))]\\
&\quad + p\mu_1(A)[\log(p\mu_1(A)) - \log(\mu_1(A))]\\
&\quad + (1-p)\mu_2(A)[\log((1-p)\mu_2(A)) - \log(\mu_2(A))]\\
&\geq 0+0+\mu_1(A)p\log p + \mu_2(A)(1-p)\log(1-p) \quad \text{because log is increasing.}
\end{aligned}$$

In addition,

$$\begin{aligned}
0 &\leq H_{p\mu_1+(1-p)\mu_2}(\alpha) - pH_{\mu_1}(\alpha) - (1-p)H_{\mu_2}(\alpha)\\
&\leq -p\log p - (1-p)\log(1-p) \leq \log 2. \qquad (2.3)
\end{aligned}$$

For $n\in\mathbb{N}$, by (i) and equation (2.3), we have

$$\begin{aligned}
|h(n+1)-h(n)| &= \bigg|h(n+1) - \frac{\lambda_n}{\lambda_{n+1}}h(n) - \frac{\lambda_{n+1}-\lambda_n}{\lambda_{n+1}}h(n,n+1) - \frac{\lambda_{n+1}-\lambda_n}{\lambda_{n+1}}h(n)\\
&\quad + \frac{\lambda_{n+1}-\lambda_n}{\lambda_{n+1}}h(n,n+1)\bigg|\\
&\leq -\frac{\lambda_n}{\lambda_{n+1}}\log\frac{\lambda_n}{\lambda_{n+1}} - \frac{\lambda_{n+1}-\lambda_n}{\lambda_{n+1}}\log\frac{\lambda_{n+1}-\lambda_n}{\lambda_{n+1}} + 2\frac{\lambda_{n+1}-\lambda_n}{\lambda_{n+1}}\log M.
\end{aligned}$$

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(iii) Since $\Lambda_m = \Lambda_n \bigcup \Lambda_n^m$, we have

$$\frac{1}{\lambda_m}\sum_{\mathbf{g}\in\Lambda_m}\mu\circ T^{-\mathbf{g}} = \frac{\lambda_n}{\lambda_m}\bigg(\frac{1}{\lambda_n}\sum_{\mathbf{g}\in\Lambda_n}\mu\circ T^{-\mathbf{g}}\bigg) + \frac{\lambda_m-\lambda_n}{\lambda_m}\bigg(\frac{1}{\lambda_m-\lambda_n}\sum_{\mathbf{g}\in\Lambda_n^m}\mu\circ T^{-\mathbf{g}}\bigg)$$

for $m, n \in \mathbb{N}$ with $n < m$. Taking $p = \lambda_n/\lambda_m$, $\mu_1 = (1/\lambda_n)\sum_{\mathbf{g}\in\Lambda_n}\mu\circ T^{-\mathbf{g}}$ and $\mu_2 = (1/(\lambda_m-\lambda_n))\sum_{\mathbf{g}\in\Lambda_n^m}\mu\circ T^{-\mathbf{g}}$, then equation (2.3) implies (iii). □

Remark 2.3. Combining (ii) with the fact $\lambda_n/(\lambda_{n+1}) = (2n-1)^d/(2n+1)^d \to 1$ as $n\to\infty$ and $0\cdot\log 0 = 0$, we gain $\limsup_{n\to\infty}|h(n+1)-h(n)| = 0$.

LEMMA 2.3. *Let $(X, \mathcal{T})$ be a $\mathbb{Z}^d$-actions TDS and $\mu \in \mathcal{M}(X)$. For $\epsilon > 0$ and $l, M \in \mathbb{N}$, let $H_\bullet(\epsilon, M; l)$ be defined as equation (2.2). Then the following statements hold.*

(1) *For all $n \in \mathbb{N}$,*

$$\bigg|H_{1/\lambda_n\sum_{g\in\Lambda_n}\mu\circ T^{-g}}(\epsilon, M; l) - H_{1/\lambda_{n+1}\sum_{g\in\Lambda_{n+1}}\mu\circ T^{-g}}(\epsilon, M; l)\bigg|$$
$$\leq -\frac{\lambda_n}{\lambda_l\lambda_{n+1}}\log\frac{\lambda_n}{\lambda_{n+1}} - \frac{\lambda_{n+1}-\lambda_n}{\lambda_l\lambda_{n+1}}\log\frac{\lambda_{n+1}-\lambda_n}{\lambda_{n+1}} + 2\frac{\lambda_{n+1}-\lambda_n}{\lambda_{n+1}}\log M.$$

(2) *For all $n, m \in \mathbb{N}$ with $n < m$,*

$$\frac{\lambda_n}{\lambda_m}H_{1/\lambda_n\sum_{g\in\Lambda_n}\mu\circ T^{-g}}(\epsilon, M; l) + \frac{\lambda_m-\lambda_n}{\lambda_m}H_{1/(\lambda_m-\lambda_n)\sum_{g\in\Lambda_n^m}\mu\circ T^{-g}}(\epsilon, M; l)$$
$$\leq H_{1/\lambda_m\sum_{g\in\Lambda_m}\mu\circ T^{-g}}(\epsilon, M; l) + \frac{\log 2}{\lambda_l}. \tag{2.4}$$

Proof. The statements follow from the definition of $H_\bullet(\epsilon, M; l)$ as well as Lemma 2.2. □

3. *Variational principle for weighted topological pressure*

3.1. *Some lemmas.* In this section, we give some lemmas which play a significant role in the proof of weighted variational principle. Recall that for $x \in X_1$, $n \in \mathbb{N}$, $\epsilon > 0$, the nth $\mathbf{a}$-weighted Bowen ball of radius ϵ centred at x is defined by

$$\begin{aligned}B_n^{\mathbf{a}}(x,\epsilon) &:= \{y \in X_1 : d_i(T_i^{\mathbf{g}}\tau_{i-1}x, T_i^{\mathbf{g}}\tau_{i-1}y) < \epsilon \text{ for } \mathbf{g} \in \Lambda_{\lceil(a_1+\cdots a_i)n\rceil}, i = 1, \ldots, k\}\\ &= \{y \in X_1 : d_i(\tau_{i-1}T_1^{\mathbf{g}}x, \tau_{i-1}T_1^{\mathbf{g}}y) < \epsilon \text{ for } \mathbf{g} \in \Lambda_{\lceil(a_1+\cdots a_i)n\rceil}, i = 1, \ldots, k\}.\end{aligned}$$

Let $\mathbf{e}_1, \ldots, \mathbf{e}_d$ be the canonical basis for $\mathbb{Z}^d$, then

$$T_1^{\mathbf{g}} = T_{1,1}^{g_1}\circ\cdots\circ T_{1,d}^{g_d} \quad \text{for all } \mathbf{g} = (g_1, \ldots, g_d) \in \mathbb{Z}^d,$$

where $T_{1,j} = T_1^{\mathbf{e}_j}$ and $T_{1,j}^{g_j}$ denotes the g_j-fold iteration of $T_{1,j}$. Hence, if $d = 1$, then $T_1^{\mathbf{g}}$ is the g_1-fold iteration of the map T_1. We have extended Feng and Huang's definition of a weighted Bowen ball in the $\mathbb{Z}_+$-action setting (see [16, Definition 1.2]). For the $\mathbf{a}$-weighted Bowen ball and weighted measure-theoretic entropy of $\mathcal{T}_1$, we can establish the following theorem similar to the Brin–Katok theorem (see [6, 29]), which contributes to the lower bound estimate of the weighted topological pressure $P^{\mathbf{a}}(\mathcal{T}_1, f)$.

THEOREM 3.1. *For each $\mu \in E(X_1, \mathcal{T}_1)$, we have*

$$\lim_{\epsilon\to 0}\liminf_{n\to\infty} \frac{-\log \mu(B_n^{\boldsymbol{a}}(x,\epsilon))}{\lambda_n} = \lim_{\epsilon\to 0}\limsup_{n\to\infty} \frac{-\log \mu(B_n^{\boldsymbol{a}}(x,\epsilon))}{\lambda_n} = h_\mu^{\boldsymbol{a}}(\mathcal{T}_1)$$

for μ-a.e. $x \in X_1$.

Remark 3.1. The above theorem extends some well-known results. When $\mathbf{a} = (1, 0, \ldots, 0)$ and $d = 1$, Theorem 3.1 reduces to the classical Brin–Katok theorem on local entropy. When $d = 1$, Feng and Huang [**16**] established a weighted version of the Brin–Katok theorem based on the weighted version of the classical Schannon–McMilian–Breiman theorem, which is a special case of Theorem 3.1. For the reader's convenience, we give the proof of Theorem 3.1 in detail and postpone it to Appendix A.

Inspired by Misiurewicz's [**31**] elegant proof of the entropy variational principle, Yan [**44**, Lemma 4.4] proved the following.

LEMMA 3.2. *Let $\nu \in \mathcal{M}(X)$ and $\alpha = \{A_1, \ldots, A_M\} \in \mathcal{P}_X$. Then for any $n, l \in \mathbb{N}$ with $n \geq 2l$, we have*

$$\frac{1}{\lambda_n} H_\nu\bigg(\bigvee_{g\in\Lambda_n} T^{-g}\alpha\bigg) \leq \frac{1}{\lambda_l} H_{\nu_n}\bigg(\bigvee_{g\in\Lambda_l} T^{-g}\alpha\bigg) + \frac{\gamma_{l,n}}{\lambda_n}\log M,$$

where $\nu_n = (1/\lambda_n)\sum_{g\in\Lambda_n} \nu\circ T^{-g}$ and $\gamma_{l,n} := \lambda_n - \lambda_{n-2l}$.

The following combinatoric lemma was obtained by Feng and Huang [**16**, Lemma 5.4], as a slight variant of [**21**, Lemma 4.1] by Kenyon and Peres.

LEMMA 3.3. *Let $p \in \mathbb{N}$ and $u_j : \mathbb{N}\to\mathbb{R}$ $(j = 1, \ldots, p)$ be bounded functions with*

$$\lim_{n\to\infty} |u_j(n+1) - u_j(n)| = 0.$$

Then for any positive numbers $c_1, \ldots, c_p$ and $r_1, \ldots, r_p$,

$$\limsup_{n\to\infty} \sum_{j=1}^{p} (u_j(\lceil c_i n\rceil) - u_j(\lceil r_j n\rceil)) \geq 0.$$

To give the upper bound estimate in Theorem 1.1, see equation (3.15) later, we show the following lemma similar to a result due to Frostman.

LEMMA 3.4. *Let $f \in C(X_1)$. Suppose that $P^{\boldsymbol{a}}(\mathcal{T}_1, f) > 0$. Then for all $0 < s < P^{\boldsymbol{a}}(\mathcal{T}_1, f)$, there exist $\nu \in \mathcal{M}(X_1)$ and $\epsilon > 0, N \in \mathbb{N}$ such that for any $x \in X_1$ and $n \geq N$, we have*

$$\nu(B_n^{\boldsymbol{a}}(x,\epsilon)) \leq \sup_{y\in B_n^{\boldsymbol{a}}(x,\epsilon)} \exp\bigg(-s\lambda_n + \frac{1}{a_1^d}\sum_{g\in\Lambda_{\lceil a_1 n\rceil}} f(T_1^{g}y)\bigg).$$

Remark 3.2. The classical Frostman's lemma [**30**] says that for any compact set $E \subset X$ with Hausdorff dimension greater than t, there exists $\mu \in \mathcal{M}(X)$ with $\mu(E) = 1$ so

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that $\mu(B(x,r)) < cr^t$ for some constant $c > 0$ and any $r > 0, x \in X$. Adapted from Howroyd's elegant argument, Feng and Huang obtained the corresponding non-weighted version and weighted version of the dynamical Frostman lemma in [16], combining some ideas in geometric measure theory. The main tool of the proof is the notion of an averaged **a**-weighted topological pressure, which is similar to the weighted Hausdorff measure in geometric measure theory. In our setting of a $\mathbb{Z}^d$-actions topological dynamical system, we give the definition of averaged **a**-weighted topological pressure and the complete proof of Lemma 3.4 in the next subsection.

Remark 3.3. We can see from Theorem 3.1 and Lemma 3.4 that the **a**-weighted Bowen ball $B_n^{\mathbf{a}}(x,\epsilon)$ constructs the bridge to relate weighted measure-theoretic entropy $h_\mu^{\mathbf{a}}(\mathcal{T}_1)$ to weighted topological pressure $P^{\mathbf{a}}(\mathcal{T}_1, f)$.

3.2. *Averaged **a**-weighted topological pressure and proof of Lemma 3.4.* Let g be an arbitrary real-valued function on X_1, $f \in C(X_1)$, $s \geq 0$, $\epsilon > 0$, $N \in \mathbb{N}$, and define

$$\mathcal{W}_{f,N,\epsilon}^{\mathbf{a},s}(g) = \inf \sum_j b_j \exp\bigg(-s\lambda_{n_j} + \frac{1}{a_1^d} \sup_{x \in A_j} \sum_{\mathbf{g} \in \Lambda_{\lceil a_1 n_j \rceil}} f(T_1^{\mathbf{g}} x)\bigg). \tag{3.1}$$

Here the infimum is taken over all countable collections $\Gamma = \{(n_j, A_j, b_j)\}_j$ satisfying $n_j \geq N$, A_j is Borel subset of $B_{n_j}^{\mathbf{a}}(x,\epsilon)$ for some $x \in X_1$, $0 < b_j < \infty$ and $\sum_j b_j \chi_{A_j} \geq g$, where χ_A denotes the characteristic function of A, that is, $\chi_A(x) = 1$ if $x \in A$ and $\chi_A(x) = 0$ if $x \in X_1 \setminus A$. For $Z \subset X_1$, set $\mathcal{W}_{f,N,\epsilon}^{\mathbf{a},s}(Z) := \mathcal{W}_{f,N,\epsilon}^{\mathbf{a},s}(\chi_Z)$. The quantity $\mathcal{W}_{f,N,\epsilon}^{\mathbf{a},s}(Z)$ does not decrease as N increases and ϵ decreases, and hence the following limits exist:

$$\mathcal{W}_{f,\epsilon}^{\mathbf{a},s}(Z) = \lim_{N\to\infty} \mathcal{W}_{f,N,\epsilon}^{\mathbf{a},s}(Z), \quad \mathcal{W}_{f}^{\mathbf{a},s}(Z) = \lim_{\epsilon\to 0} \mathcal{W}_{f,\epsilon}^{\mathbf{a},s}(Z).$$

There exists a critical value of the parameter s, which we will denote by $P_W^{\mathbf{a}}(\mathcal{T}_1, f, Z)$, where $\mathcal{W}_f^{\mathbf{a},s}(Z)$ jumps from ∞ to 0, that is,

$$\mathcal{W}_f^{\mathbf{a},s}(Z) = \begin{cases} 0, & s > P_W^{\mathbf{a}}(\mathcal{T}_1, f, Z), \\ \infty, & s < P_W^{\mathbf{a}}(\mathcal{T}_1, f, Z). \end{cases}$$

In other words, $P_W^{\mathbf{a}}(\mathcal{T}_1, f, Z) = \inf\{s : \mathcal{W}_f^{\mathbf{a},s}(Z) = 0\} = \sup\{s : \mathcal{W}_f^{\mathbf{a},s}(Z) = \infty\}$.

Definition 3.1. We call $P_W^{\mathbf{a}}(\mathcal{T}_1, f) := P_W^{\mathbf{a}}(\mathcal{T}_1, f, X_1)$ the average **a**-weighted topological pressure of f with respect to $\mathcal{T}_1$.

Essentially, for any $s \geq 0$, $N \in \mathbb{N}$, $\epsilon > 0$, $f \in C(X_1)$, both $\Lambda_{f,N,\epsilon}^{\mathbf{a},s}$ and $\mathcal{W}_{f,N,\epsilon}^{\mathbf{a},s}$ are outer measures on X_1, as a direct consequence of their definitions. The next proposition reveals that they are equivalent to a certain extent.

PROPOSITION 3.5. *Let $Z \subset X_1$. Then for any $s \geq 0$ and $\epsilon, \delta > 0$, we have*

$$\Lambda_{f,N,6\epsilon}^{\boldsymbol{a},s+\delta}(Z) \leq \mathcal{W}_{f,\epsilon}^{\boldsymbol{a},s}(Z) \leq \Lambda_{f,N,\epsilon}^{\boldsymbol{a},s}(Z)$$

when $N \in \mathbb{N}$ is large enough. Moreover, $P^{\boldsymbol{a}}(\mathcal{T}_1, f) = P_W^{\boldsymbol{a}}(\mathcal{T}_1, f)$.

To prove Proposition 3.5, we need the following lemma, obtained in [16, Lemma 3.7].

LEMMA 3.6. *Let (X, d) be a compact metric space and $\epsilon > 0$. Let $(E_i)_{i\in\mathcal{I}}$ be a finite or countable family of subsets of X with $\mathrm{diam}(E_i) \le \epsilon$, and $(c_i)_{i\in\mathcal{I}}$ a family of positive numbers. Let $t > 0$. Assume that $F \subset X$ is such that*

$$F \subset \bigg\{x \in X : \sum_{i\in\mathcal{I}} c_i \chi_{E_i}(x) > t\bigg\}.$$

Then F can be covered by no more than $(1/t)\sum_{i\in\mathcal{I}} c_i$ balls with centres in $\bigcup_{i\in\mathcal{I}} E_i$ and radius 6ϵ.

LEMMA 3.7. *Let $\epsilon > 0$. Then there exists $\gamma > 0$ so that for any $n \in \mathbb{N}$, X_1 can be covered by no more than $\exp(\gamma\lambda_n)$ balls of radius ϵ in metric $d_n^{\mathbf{a}}$.*

Proof. For $i = 1, \ldots, k$, since X_i is compact, there exists a finite open cover α_i of X_i with $\mathrm{diam}(\alpha_i) < \epsilon$ (in metric $d_n^{\mathbf{a}}$). Let $n \in \mathbb{N}$. Denote

$$\beta = \bigvee_{i=1}^{k}\bigg(\bigvee_{\mathbf{g}\in\Lambda_{\lceil(a_1+\cdots+a_i)n\rceil}} T_1^{-\mathbf{g}}\tau_{i-1}^{-1}\alpha_i\bigg).$$

Then β is an open cover of X_1 with $\mathrm{diam}(\beta) < \epsilon$ (in metric $d_n^{\mathbf{a}}$). Hence, X_1 can be covered by at most Card β many balls of radius ϵ in metric $d_n^{\mathbf{a}}$. Choose $\gamma > 0$ so that $\exp\gamma = \prod_{i=1}^{k}(\mathrm{Card}\,\alpha_i)^{(a_1+\cdots+a_i+1)^d}$. Then

$$\mathrm{Card}\,\beta \le \prod_{i=1}^{k}(\mathrm{Card}\,\alpha_i)^{\lambda_{\lceil(a_1+\cdots+a_i)n\rceil}} \le \exp(\gamma\lambda_n),$$

which completes the proof. □

Proof of Proposition 3.5. Let $Z \subset X_1$, $s \ge 0$, $\epsilon, \delta > 0$. If we take $g = \chi_Z$ and $b_j \equiv 1$ in the definition of equation (3.1), then $\mathcal{W}_{f,\epsilon}^{\mathbf{a},s}(Z) \le \Lambda_{f,N,\epsilon}^{\mathbf{a},s}(Z)$ for each $N \in \mathbb{N}$. Next, we show that $\Lambda_{f,N,6\epsilon}^{\mathbf{a},s+\delta}(Z) \le \mathcal{W}_{f,\epsilon}^{\mathbf{a},s}(Z)$ when $N \in \mathbb{N}$ is large enough. Given $\gamma > 0$ as in Lemma 3.7, assume $N \ge 2$ so that

$$n^2(\lambda_n + 1)\exp(\gamma - \lambda_n\delta) \le 1 \quad \text{when } n \ge N. \tag{3.2}$$

Let $\{(n_i, A_i, b_i)\}_{i\in\mathcal{I}}$ be a family so that $\mathcal{I} \subset \mathbb{N}$, $A_i \subset B_{n_i}^{\mathbf{a}}(x, \epsilon)$ for some $x \in X_1$, $0 < b_i < \infty$, $n_i \ge N$ and

$$\sum_{i\in\mathcal{I}} b_i\chi_{A_i} \ge \chi_Z. \tag{3.3}$$

So we only need to prove that

$$\Lambda_{f,N,6\epsilon}^{\mathbf{a},s+\delta}(Z) \le \sum_{i\in\mathcal{I}} b_i \exp\bigg(-s\lambda_{n_i} + \frac{1}{a_1^d}\sup_{x\in A_i}\sum_{\mathbf{g}\in\Lambda_{\lceil a_1 n_i\rceil}} f(T_1^{\mathbf{g}}x)\bigg), \tag{3.4}$$

which implies $\Lambda_{f,N,6\epsilon}^{\mathbf{a},s+\delta}(Z) \le \mathcal{W}_{f,\epsilon}^{\mathbf{a},s}(Z)$. Denote $\mathcal{I}_n := \{i \in \mathcal{I} : n_i = n\}$,

$$g_n(x) := \frac{1}{a_1^d}\sum_{\mathbf{g}\in\Lambda_{\lceil a_1 n\rceil}} f(T_1^{\mathbf{g}}x), \;\; g_n(E) := \sup_{x\in E} g_n(x)$$

for $N \in \mathbb{N}$, $x \in X_1$, $E \subset X_1$ and

$$Z_{n,t} := \left\{ x \in Z : \sum_{i \in \mathcal{I}_n} b_i \chi_{A_i}(x) > t \right\}.$$

Now we claim that

$$\Lambda^{\mathbf{a},s+\delta}_{f,N,6\epsilon}(Z_{n,t}) \leq \frac{1}{tn^2} \sum_{i \in \mathcal{I}_n} b_i \exp(-s\lambda_n + g_n(A_i)) \tag{3.5}$$

for all $n \geq N$ and $0 < t < 1$.

To prove the claim, assume that $n \geq N$ and $0 < t < 1$. Set $D = (1/\lambda_n) g_n(Z_{n,t})$. For $\ell = 1, 2, \ldots, \lambda_n$ and $i \in \mathcal{I}_n$, let

$$Z^{\ell}_{n,t} := \left\{ x \in Z_{n,t} : \frac{1}{\lambda_n} g_n(x) \in \left(D - \frac{\gamma \ell}{\lambda_n}, D - \frac{\gamma(\ell-1)}{\lambda_n} \right] \right\}, \quad A_{i,\ell} := A_i \cap Z^{\ell}_{n,t}$$

and

$$Z^0_{n,t} := \left\{ x \in Z_{n,t} : \frac{1}{\lambda_n} g_n(x) \leq D - \gamma \right\}, \quad A_{0,\ell} := A_0 \cap Z^{\ell}_{n,t}.$$

For $\ell = 0, 1, 2, \ldots, \lambda_n$, denote $\mathcal{I}_{n,\ell} := \{ i \in \mathcal{I}_n : A_{i,\ell} \neq \emptyset \}$, then

$$Z^{\ell}_{n,t} = \left\{ x \in X_1 : \sum_{i \in \mathcal{I}_{n,\ell}} b_i \chi_{A_{i,\ell}}(x) > t \right\}.$$

By Lemma 3.6, $Z^{\ell}_{n,t}$ can be covered by at most $(1/t) \sum_{i \in \mathcal{I}_{n,\ell}} b_i$ balls with centre in $\bigcup_{i \in \mathcal{I}_{n,\ell}} A_{i,\ell}$ and radius 6ϵ in metric $d^{\mathbf{a}}_n$. Then for $\ell = 1, 2, \ldots, \lambda_n$,

$$\begin{aligned}
\Lambda^{\mathbf{a},s+\delta}_{f,N,6\epsilon}(Z^{\ell}_{n,t}) &\leq \frac{1}{t} \sum_{i \in \mathcal{I}_{n,\ell}} b_i \exp(-(s+\delta)\lambda_n + g_n(Z^{\ell}_{n,t})) \\
&\leq \frac{1}{t} \exp(-(s+\delta)\lambda_n)) e^{\gamma} \sum_{i \in \mathcal{I}_{n,\ell}} b_i \exp(g_n(A_{i,\ell})) \\
&\leq e^{\gamma - \lambda_n \delta} \frac{1}{t} \sum_{i \in \mathcal{I}_n} b_i \exp(-\lambda_n s + g_n(A_i)).
\end{aligned} \tag{3.6}$$

In addition, by Lemma 3.7, $Z^0_{n,t}$ can be covered by at most $\exp(\lambda_n \delta)$ balls of radius 6ϵ in metric $d^{\mathbf{a}}_n$. Note that $g_n(Z_{n,t}) = \lambda_n D$. For any $u < \lambda_n D$, there exists $x \in Z_{n,t}$ so that $u \leq g_n(x)$. For another thing, since $x \in Z_{n,t}$, we have $\sum_{i \in \mathcal{I}_n} b_i \chi_{A_i}(x) \geq t$, and therefore

$$\frac{1}{t} \sum_{i \in \mathcal{I}_n} b_i g_n(A_i) \geq \frac{1}{t} \sum_{i \in \mathcal{I}_n, x \in A_i} b_i g_n(A_i) \geq \frac{1}{t} \sum_{i \in \mathcal{I}_n, x \in A_i} b_i u \geq u. \tag{3.7}$$

Thus, by equation (3.7),

$$\begin{aligned}
\Lambda^{\mathbf{a},s+\delta}_{f,N,6\epsilon}(Z^0_{n,t}) &\leq \exp(\lambda_n \gamma) \exp(-\lambda_n(s+\delta) + g_n(Z^0_{n,t})) \\
&\leq \exp[\lambda_n \gamma - \lambda_n(s+\delta) + \lambda_n(D-\gamma)] \\
&\leq e^{-\lambda_n \delta} \frac{1}{t} \sum_{i \in \mathcal{I}_n} b_i \exp(-\lambda_n s + g_n(A_i)).
\end{aligned} \tag{3.8}$$

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Combining equations (3.2), (3.6) and (3.8), we have

$$\Lambda^{\mathbf{a},s+\delta}_{f,N,6\epsilon}(Z_{n,t}) \leq \sum_{\ell=0}^{\lambda_n} \Lambda^{\mathbf{a},s+\delta}_{f,N,6\epsilon}(Z^{\ell}_{n,t}) \leq (\lambda_n+1)e^{\gamma-\lambda_n\delta}\frac{1}{t}\sum_{i\in\mathcal{I}_n} b_i\exp(-\lambda_n s+g_n(A_i))$$
$$\leq \frac{1}{n^2 t}\sum_{i\in\mathcal{I}_n} b_i\exp(-\lambda_n s+g_n(A_i)), \tag{3.9}$$

which finishes the proof of the claim in advance. It is clear that $\sum_{n=N}^{\infty}(1/n^2) \leq \sum_{n=2}^{\infty}(1/n^2) \leq 1$. Hence, if $x \notin \bigcup_{n\geq N} Z_{n,t/n^2}$, then

$$\sum_{i\in\mathcal{I}} b_i\chi_{A_i}(x) = \sum_{i\in\bigcup_{n=N}^{\infty}\mathcal{I}_n} b_i\chi_{A_i}(x) \leq \sum_{n=N}^{\infty}\sum_{i\in\mathcal{I}_n} b_i\chi_{A_i}(x) \leq \sum_{n=N}^{\infty}\frac{t}{n^2} \leq t < 1,$$

thus $x \notin Z$ by equation (3.3). We can infer that $Z \subset \bigcup_{n\geq N} Z_{n,t/n^2}$. By equation (3.9),

$$\Lambda^{\mathbf{a},s+\delta}_{f,N,6\epsilon}(Z) \leq \sum_{n=N}^{\infty}\Lambda^{\mathbf{a},s+\delta}_{f,N,6\epsilon}(Z_{n,t/n^2}) \leq \frac{1}{t}\sum_{n=N}^{\infty}\sum_{i\in\mathcal{I}_n} b_i\exp(-\lambda_n s+g_n(A_i))$$
$$\leq \frac{1}{t}\sum_{i\in\mathcal{I}} b_i\exp(-\lambda_n s+g_n(A_i)).$$

Letting $t \nearrow 1$, we have $\Lambda^{\mathbf{a},s+\delta}_{f,N,6\epsilon}(Z) \leq \sum_{i\in\mathcal{I}} b_i\exp(-\lambda_n s+g_n(A_i))$, which implies that equation (3.4) holds. $\square$

LEMMA 3.8. *Let $s \geq 0$, $N \in \mathbb{N}$, $\epsilon > 0$. Assume that $c := \mathcal{W}^{\boldsymbol{a},s}_{f,N,\epsilon}(X_1) > 0$. Then there exists $\mu \in \mathcal{M}(X_1)$ so that*

$$\mu(B^{\boldsymbol{a}}_n(x,\epsilon)) \leq \frac{1}{c}\exp(-s\lambda_n+g_n(B^{\boldsymbol{a}}_n(x,\epsilon))),$$

where

$$g_n(z) := \frac{1}{a_1^d}\sum_{\boldsymbol{g}\in\Lambda_{\lceil a_1 n\rceil}} f(T_1^{\boldsymbol{g}}z),\quad g_n(E) := \sup_{z\in E} g_n(z)$$

for $z \in X_1$, $E \subset X_1$.

Proof. Obviously, $c < \infty$. Define a functional $p : C(X_1)\to\mathbb{R}$ by

$$p(g) := \frac{1}{c}\mathcal{W}^{\mathbf{a},s}_{f,N,\epsilon}(g),\quad g \in C(X_1).$$

Let $\mathbf{1} \in C(X_1)$ denote the constant function $\mathbf{1}(x) \equiv 1$. One can verify that:

(1) $p(g+h) \leq p(g)+p(h)$ for all $g, h \in C(X_1)$;
(2) $p(tg) = tp(g)$ for all $t \geq 0$ and $g \in C(X_1)$;
(3) $p(\mathbf{1}) = 1$, $0 \leq p(g) \leq \|g\|$ for all $g \in C(X_1)$ and $p(g) = 0$ if $g \in C(X_1)$ with $g \leq 0$.

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By the Hahn–Banach theorem, we can extend the linear functional $t \mapsto tp(\mathbf{1})$, $t \in \mathbb{R}$, from the subspace of the constant functions to a linear functional $\mathcal{L} : C(X_1) \to \mathbb{R}$ satisfying

$$\mathcal{L}(\mathbf{1}) = p(\mathbf{1}) = 1 \quad \text{and} \quad -p(-g) \le \mathcal{L}(g) \le p(g) \text{ for all } g \in C(X_1).$$

If $g \in C(X_1)$ with $g \ge 0$, then $p(-g) = 0$ and therefore $\mathcal{L}(g) \ge 0$. Furthermore, $\mathcal{L}(\mathbf{1}) = 1$. By the Riesz representation theorem [**43**, Theorem 6.3], there exists $\mu \in \mathcal{M}(X_1)$ so that $\mathcal{L}(g) = \int_{X_1} g d\mu$ for all $g \in C(X_1)$. Let $x \in X_1, n \ge N$ and $K \subset B_n^{\mathbf{a}}(x, \epsilon)$ be compact. Then there exists an open set V with $K \subset V \subset B_n^{\mathbf{a}}(x, \epsilon)$ so that $g_n(V) \le g_n(K) + \delta$. By the Uryson lemma, there exists $g \in C(X_1)$ such that $0 \le g \le 1$, $g(z) = 1$ for $z \in K$ and $g(z) = 0$ for $z \in X_1 \setminus V$. Then $\mu(K) \le \mathcal{L}(g) \le p(g)$. Since $g \le \chi_V, n \ge N$, by the definition of $\mathcal{W}^{\mathbf{a},s}_{f,N,\epsilon}(g)$ in equation (3.1), we have $\mathcal{W}^{\mathbf{a},s}_{f,N,\epsilon}(g) \le \exp(-s\lambda_n + g_n(V))$. Therefore, $p(g) \le (1/c)\exp(-s\lambda_n + g_n(V))$ and

$$\mu(K) \le \frac{1}{c}\exp(-s\lambda_n + g_n(V)) \le \frac{1}{c}\exp(-s\lambda_n + g_n(K) + \delta).$$

Letting $\delta \to 0$, we conclude that $\mu(K) \le (1/c)\exp(-s\lambda_n + g_n(K))$. Finally, since μ is regular, for the arbitrariness of $K \subset B_n^{\mathbf{a}}(x, \epsilon)$, we have

$$\mu(B_n^{\mathbf{a}}(x, \epsilon)) \le \frac{1}{c}\exp(-s\lambda_n + g_n(B_n^{\mathbf{a}}(x, \epsilon))). \qquad \square$$

Proposition 3.5 and Lemma 3.8 together imply Lemma 3.4.

3.3. *Proof of Theorem 1.1*

Part (i): lower bound. First, we prove that

$$P^{\mathbf{a}}(\mathcal{T}_1, f) \ge \sup\left\{h^{\mathbf{a}}_\mu(\mathcal{T}_1) + \int_{X_1} f\, d\mu : \mu \in \mathcal{M}(X_1, \mathcal{T}_1)\right\}.$$

Recall that for each $\mu \in \mathcal{M}(X_1, \mathcal{T}_1)$, there is a unique measure τ on Borel subsets of $\mathcal{M}(X_1, \mathcal{T}_1)$ such that $\tau(E(X_1, \mathcal{T}_1)) = 1$ and for all $f \in C(X_1)$,

$$\int_{X_1} f(x)\, d\mu(x) = \int_{E(X_1,\mathcal{T}_1)} \left(\int_{X_1} f(x)\, dm(x)\right) d\tau(m).$$

We write $\mu = \int_{E(X_1,\mathcal{T}_1)} m d\tau(m)$ and call this the ergodic decomposition (see [**13**]) of μ. For $\mu \in E(X_1, \mathcal{T}_1)$, via Birkhoff's ergodic theorem improved by Ornstein and Weiss [**34**], we have

$$\lim_{n\to\infty} \frac{\sum_{\mathbf{g}\in\Lambda_n} f(T_1^{\mathbf{g}}x)}{\lambda_n} = \int_{X_1} f\, d\mu \tag{3.10}$$

for μ-a.e. $x \in X_1$. By Jacob's theorem [**43**, Theorem 8.4], if $\mu = \int_{E(X_1,\mathcal{T}_1)} m d\tau(m)$ is the ergodic decomposition for $\mu \in \mathcal{M}(X_1, \mathcal{T}_1)$, we have

$$h^{\mathbf{a}}_\mu(\mathcal{T}_1) = \int_{E(X_1,\mathcal{T}_1)} h^{\mathbf{a}}_m(\mathcal{T}_1)\, d\tau(m). \tag{3.11}$$

So we only need to prove that

$$P^{\mathbf{a}}(\mathcal{T}_1, f) \geq \int_{X_1} f\, d\mu + \min\{\delta^{-1}, h_\mu^{\mathbf{a}}(\mathcal{T}_1) - \delta\} - \delta \tag{3.12}$$

for each $\delta > 0$ and $\mu \in E(X_1, \mathcal{T}_1)$. Denote $H := \min\{\delta^{-1}, h_\mu^{\mathbf{a}}(\mathcal{T}_1) - \delta\}$. By Theorem 3.1, choose $\epsilon > 0$ such that

$$\liminf_{n\to\infty} \frac{-\log \mu(B_n^{\mathbf{a}}(x, \epsilon))}{\lambda_n} > H \quad \text{for } \mu - \text{a.e. } x \in X_1. \tag{3.13}$$

Considering equations (3.10), (3.13) and using the Egorov theorem, there exist $N \in \mathbb{N}$ and a Borel set $E_N \subset X_1$ with $\mu(E_N) \geq \frac{1}{2}$ such that for any $x \in E_N$ and $n \geq N$,

$$\mu(B_n^{\mathbf{a}}(x, \epsilon)) < \exp(-\lambda_n H), \sum_{\mathbf{g}\in\Lambda_{\lceil a_1 n\rceil}} f(T_1^{\mathbf{g}} x) \geq \lambda_{\lceil a_1 n\rceil}\bigg(\int_{X_1} f\, d\mu - \delta\bigg). \tag{3.14}$$

Choose a countable set $\Gamma = \{(n_j, A_j)\}_j$ such that $n_j \geq N$, $\bigcup_j A_j = X_1$ and there exists $x_j \in X_1$ satisfying $A_j \subset B_{n_j}^{\mathbf{a}}(x_j, \epsilon/2)$ for each j. Denote by $\mathcal{I} := \{j : A_j \cap E_N \neq \emptyset\}$. For each j, taking $y_j \in A_j \cap E_N$, then $A_j \subset B_{n_j}^{\mathbf{a}}(x_j, \frac{\epsilon}{2}) \subset B_{n_j}^{\mathbf{a}}(y_j, \epsilon)$. Therefore, by equation (3.14), we have

$$\mu(A_j) \leq \mu(B_{n_j}^{\mathbf{a}}(y_j, \epsilon)) < \exp(-\lambda_{n_j} H)$$

and

$$\begin{aligned} \frac{1}{a_1^d} \sup_{x\in A_j} \sum_{\mathbf{g}\in\Lambda_{\lceil a_1 n_j\rceil}} f(T_1^{\mathbf{g}} x) &\geq \frac{1}{a_1^d} \sum_{\mathbf{g}\in\Lambda_{\lceil a_1 n_j\rceil}} f(T_1^{\mathbf{g}} y_j) \\ &\geq \frac{\lambda_{\lceil a_1 n_j\rceil}}{a_1^d}\bigg(\int_{X_1} f\, d\mu - \delta\bigg) \geq \lambda_{n_j}\bigg(\int_{X_1} f\, d\mu - \delta\bigg). \end{aligned}$$

If we choose $s = \int_{X_1} f\, d\mu + H - \delta$, then for all $j \in \mathcal{I}$,

$$\begin{aligned} &\exp\bigg(-s\lambda_{n_j} + \frac{1}{a_1^d} \sup_{x\in A_j} \sum_{\mathbf{g}\in\Lambda_{\lceil a_1 n_j\rceil}} f(T_1^{\mathbf{g}} x)\bigg) \\ &\quad\geq \exp\bigg(-\lambda_{n_j}\bigg(\int_{X_1} f\, d\mu + H - \delta\bigg) + \lambda_{n_j}\bigg(\int_{X_1} f\, d\mu - \delta\bigg)\bigg) \\ &\quad= \exp(-\lambda_{n_j} H) \geq \mu(A_j). \end{aligned}$$

Summing over $j \in \mathcal{I}$, we obtain that

$$\sum_{j\in\mathcal{I}} \exp\bigg(-s\lambda_{n_j} + \frac{1}{a_1^d} \sup_{x\in A_j} \sum_{\mathbf{g}\in\Lambda_{\lceil a_1 n_j\rceil}} f(T_1^{\mathbf{g}} x)\bigg) \geq \sum_{j\in\mathcal{I}} \mu(A_j) \geq \mu\bigg(\bigcup_{j\in\mathcal{I}} A_j\bigg) \geq \mu(E_N) \geq \frac{1}{2}.$$

Then $\Lambda_f^{\mathbf{a},s}(X_1) \geq \Lambda_{f,\epsilon}^{\mathbf{a},s}(X_1) \geq \Lambda_{f,N,\epsilon}^{\mathbf{a},s}(X_1) \geq \frac{1}{2} > 0$ and therefore

$$P^{\mathbf{a}}(\mathcal{T}_1, f) \geq s = \int_{X_1} f\, d\mu + \min\{\delta^{-1}, h_\mu^{\mathbf{a}}(\mathcal{T}_1) - \delta\} - \delta.$$

Thus, equation (3.12) holds as desired.

Part (ii): upper bound. In this section, we will prove that for any $f \in C(X_1)$ and $\delta > 0$, there exists $\mu \in \mathcal{M}(X_1, \mathcal{T}_1)$ such that

$$P^{\mathbf{a}}(\mathcal{T}_1, f) \leq h_\mu^{\mathbf{a}}(\mathcal{T}_1) + \int_{X_1} f \, d\mu + \delta.$$

Suppose that $P^{\mathbf{a}}(\mathcal{T}_1, f) > 0$. Take $0 < s < s' < P^{\mathbf{a}}(\mathcal{T}_1, f)$. Denote by $S_n f(x) := \sum_{\mathbf{g} \in \Lambda_n} f(T_1^{\mathbf{g}} y)$. Take $\epsilon_0 > 0$ such that for $x, y \in X_1$: if $d_1(x, y) \leq \epsilon_0$, then $|f(x) - f(y)| < (s' - s) a_1^d / (\lceil a_1 \rceil + 1)^d$. By Lemma 3.4, there exist $\nu \in \mathcal{M}(X_1, \mathcal{T}_1)$, $\epsilon \in (0, \epsilon_0)$ and $N \in \mathbb{N}$ such that

$$\begin{aligned} \nu(B_n^{\mathbf{a}}(x, \epsilon)) &\leq \sup_{y \in B_n^{\mathbf{a}}(x, \epsilon)} \exp\left(- s' \lambda_n + \frac{1}{a_1^d} S_{\lceil a_1 n \rceil} f(y) \right) \\ &\leq \exp\left(- s \lambda_n + \frac{1}{a_1^d} S_{\lceil a_1 n \rceil} f(x) \right) \end{aligned} \tag{3.15}$$

for any $n \geq N$ and $x \in X_1$. Additionally, there exists $\tau \in (0, \epsilon)$ such that for any $1 \leq i \leq j \leq k$: if $x_i, y_i \in X_i$ with $d_i(x_i, y_i) < \tau$, then

$$d_j(\pi_{j-1} \circ \cdots \circ \pi_i(x_i), \pi_{j-1} \circ \cdots \circ \pi_i(y_i)) < \epsilon.$$

Take $M_0 \in \mathbb{N}$ with $\mathcal{P}_{X_i}(\tau, M_0) \neq \emptyset$ for $i = 1, \ldots, k$. Let $M \in \mathbb{N}$ with $M \geq M_0$ and $\alpha_i \in \mathcal{P}_{X_i}(\tau, M)$ for $i = 1, \ldots, k$. Denote $\beta_i = \tau_{i-1}^{-1} \alpha_i$ and

$$\Pi_1(n) = \Lambda_{\lceil a_1 n \rceil}, \quad \Pi_i(n) = \Lambda_{\lceil (a_1 + \cdots + a_{i-1}) n \rceil}^{\lceil (a_1 + \cdots + a_i) n \rceil}$$

for $n \in \mathbb{N}$ and $i = 2, \ldots, k$. Then for any $n \geq N$ and $x \in X_1$, we have

$$\bigvee_{i=1}^{k} \bigvee_{\mathbf{g} \in \Pi_i(n)} T_1^{-\mathbf{g}} \beta_i(x) \subseteq B_n^{\mathbf{a}}(x, \epsilon). \tag{3.16}$$

Here $\beta_i(x)$ represents the element in β containing x. Combining equations (3.15) and (3.16), we conclude that for any $x \in X_1$,

$$\nu\left(\bigvee_{i=1}^{k} \bigvee_{\mathbf{g} \in \Pi_i(n)} T_1^{-\mathbf{g}} \beta_i(x) \right) \leq \exp\left(- s \lambda_n + \frac{1}{a_1^d} S_{\lceil a_1 n \rceil} f(x) \right), \tag{3.17}$$

which implies that

$$\begin{aligned} H_\nu\left(\bigvee_{i=1}^{k} \bigvee_{\mathbf{g} \in \Pi_i(n)} T_1^{-\mathbf{g}} \beta_i \right) &= - \int \log \nu\left(\bigvee_{i=1}^{k} \bigvee_{\mathbf{g} \in \Pi_i(n)} T_1^{-\mathbf{g}} \beta_i(x) \right) d\nu(x) \\ &\geq s \lambda_n - \frac{1}{a_1^d} \int S_{\lceil a_1 n \rceil} f(x) \, d\nu(x). \end{aligned}$$

Thus,

$$\sum_{i=1}^{k} H_\nu\left(\bigvee_{\mathbf{g} \in \Pi_i(n)} T_1^{-\mathbf{g}} \beta_i \right) \geq s \lambda_n - \frac{1}{a_1^d} \int S_{\lceil a_1 n \rceil} f(x) \, d\nu(x). \tag{3.18}$$

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Denote by $t_0(n)=0$, $t_i(n)=\lceil(a_1+\cdots+a_i)n\rceil$ for $n\in\mathbb{N}$ and $i=1,\ldots,k$. Fix $l\in\mathbb{N}$. By Lemma 3.2, for sufficiently large n, the left-hand side of equation (3.18) is bounded from above by

$$\sum_{i=1}^{k}\frac{\lambda_{t_i(n)}-\lambda_{t_{i-1}(n)}}{\lambda_l}H_{w_{i,n}}\bigg(\bigvee_{\mathbf{g}\in\Lambda_l}T_1^{-\mathbf{g}}\beta_i\bigg)+(\lambda_{t_k(n)}-\lambda_{t_k(n)-2l})\log M,$$

where

$$w_{i,n}:=\frac{\sum_{\mathbf{g}\in\Pi_i(n)}\nu\circ T_1^{-\mathbf{g}}}{\lambda_{t_i(n)}-\lambda_{t_{i-1}(n)}}.$$

Notice that

$$\int S_{\lceil a_1n\rceil}f(x)\,d\nu(x)=\int\sum_{\mathbf{g}\in\Lambda_{\lceil a_1n\rceil}}f(T_1^{\mathbf{g}}x)\,d\nu(x)=\int\sum_{\mathbf{g}\in\Lambda_{\lceil a_1n\rceil}}f\,d\nu\circ T_1^{-\mathbf{g}}=\lambda_{t_1(n)}\int f\,dw_{1,n},$$

then by equation (3.18) and the definition of $H_\bullet(\tau,M;l)$, we have

$$\begin{aligned}&\sum_{i=1}^{k}(\lambda_{t_i(n)}-\lambda_{t_{i-1}(n)})H_{w_{i,n}\circ\tau_{i-1}^{-1}}(\tau,M;l)\\&\quad\geq s\lambda_n-\frac{\lambda_{\lceil a_1n\rceil}}{a_1^d}\int f\,dw_{1,n}-(\lambda_{t_k(n)}-\lambda_{t_k(n)-2l})\log M.\end{aligned}\tag{3.19}$$

Define $\nu_m=(\sum_{\mathbf{g}\in\Lambda_m}\nu\circ T_1^{-\mathbf{g}})/\lambda_m$ for $m\in\mathbb{N}$. Since $\pi_i\circ T_i^{\mathbf{g}}=T_{i+1}^{\mathbf{g}}\circ\pi_i$ holds for all $1\leq i\leq k-1$ and $\mathbf{g}\in\mathbb{Z}^d$, we have $\tau_{i-1}\circ T_1^{\mathbf{g}}=T_i^{\mathbf{g}}\circ\tau_{i-1}$. Thus, for $i=1,\ldots,k$, we obtain that

$$\nu_m\circ\tau_{i-1}^{-1}=\frac{\sum_{\mathbf{g}\in\Lambda_m}\nu\circ\tau_{i-1}^{-1}\circ T_i^{-\mathbf{g}}}{\lambda_m},\quad w_{i,n}\circ\tau_{i-1}^{-1}=\frac{\sum_{\mathbf{g}\in\Pi_i(n)}\nu\circ\tau_{i-1}^{-1}\circ T_i^{-\mathbf{g}}}{\lambda_{t_i(n)}-\lambda_{t_{i-1}(n)}}$$

and therefore

$$\nu_{t_i(n)}\circ\tau_{i-1}^{-1}=\frac{\lambda_{t_{i-1}(n)}}{\lambda_{t_i(n)}}\nu_{t_{i-1}(n)}\circ\tau_{i-1}^{-1}+\frac{\lambda_{t_i(n)}-\lambda_{t_{i-1}(n)}}{\lambda_{t_i(n)}}w_{i,n}\circ\tau_{i-1}^{-1}.\tag{3.20}$$

Here we recall that $t_0(n)=0$ and $t_i(n)=\lceil(a_1+\cdots+a_i)n\rceil$ for $n\in\mathbb{N}$.

To apply Lemma 2.3, we replace the terms $\mathcal{T},\mu,n,m$ by $\mathcal{T}_i,\nu\circ\tau_{i-1}^{-1},t_{i-1}(n),t_i(n)$, respectively, and obtain

$$\begin{aligned}&\frac{\lambda_{t_{i-1}(n)}}{\lambda_{t_i(n)}}H_{\nu_{t_{i-1}(n)}\circ\tau_{i-1}^{-1}}(\tau,M;l)+\frac{\lambda_{t_i(n)}-\lambda_{t_{i-1}(n)}}{\lambda_{t_i(n)}}H_{w_{i,n}\circ\tau_{i-1}^{-1}}(\tau,M;l)\\&\quad\leq H_{\nu_{t_i(n)}\circ\tau_{i-1}^{-1}}(\tau,M;l)+\frac{\log 2}{\lambda_l},\end{aligned}$$

and

$$\begin{aligned}&\lambda_{t_i(n)}H_{\nu_{t_i(n)}\circ\tau_{i-1}^{-1}}(\tau,M;l)-\lambda_{t_{i-1}(n)}H_{\nu_{t_{i-1}(n)}\circ\tau_{i-1}^{-1}}(\tau,M;l)\\&\quad\geq(\lambda_{t_i(n)}-\lambda_{t_{i-1}(n)})H_{w_{i,n}\circ\tau_{i-1}^{-1}}(\tau,M;l)-\frac{\lambda_{t_i(n)}\log 2}{\lambda_l}.\end{aligned}$$

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Summing this over i from 1 to k and considering equation (3.19), we conclude that

$$\begin{aligned}\Theta_n :&= \sum_{i=1}^{k}(\lambda_{t_i(n)}H_{\nu_{t_i(n)}\circ\tau_{i-1}^{-1}}(\tau, M; l) - \lambda_{t_{i-1}(n)}H_{\nu_{t_{i-1}(n)}\circ\tau_{i-1}^{-1}}(\tau, M; l))\\ &\geq s\lambda_n - \frac{\lambda_{\lceil a_1 n\rceil}}{a_1^d}\int f\, dw_{1,n} - (\lambda_{t_k(n)} - \lambda_{t_k(n)-2l})\log M - \frac{k\lambda_{t_k(n)}\log 2}{\lambda_l}.\end{aligned} \tag{3.21}$$

Let $\Upsilon_i(n) := H_{\nu_n\circ\tau_{i-1}^{-1}}(\tau, M; l)$. By Lemma 2.3(1),

$$|\Upsilon_i(n)-\Upsilon_i(n+1)|\leq -\frac{\lambda_n}{\lambda_l\lambda_{n+1}}\log\frac{\lambda_n}{\lambda_{n+1}} - \frac{\lambda_{n+1}-\lambda_n}{\lambda_l\lambda_{n+1}}\log\frac{\lambda_{n+1}-\lambda_n}{\lambda_{n+1}} + 2\frac{\lambda_{n+1}-\lambda_n}{\lambda_{n+1}}\log M. \tag{3.22}$$

Let

$$\begin{aligned}\Xi(n) :&= \sum_{i=2}^{k}\lambda_{t_i(n)}(\Upsilon_i(t_i(n)) - \Upsilon_i(t_1(n))) - \sum_{i=2}^{k}\lambda_{t_{i-1}(n)}(\Upsilon_i(t_{i-1}(n)) - \Upsilon_i(t_1(n)))\\ &= \Theta_n - \sum_{i=1}^{k}(\lambda_{t_i(n)} - \lambda_{t_{i-1}(n)})\Upsilon_i(t_1(n)).\end{aligned}$$

By equation (3.21), we have

$$\begin{aligned}&\sum_{i=1}^{k}\frac{\lambda_{t_i(n)} - \lambda_{t_{i-1}(n)}}{\lambda_n}\Upsilon_i(t_1(n)) + \frac{\lambda_{\lceil a_1 n\rceil}}{a_1^d\lambda_n}\int f\, dw_{1,n}\\ &\quad\geq -\frac{\Xi(n)}{\lambda_n} + s - \frac{\lambda_{t_k(n)} - \lambda_{t_k(n)-2l}}{\lambda_n}\log M + \frac{k\log 2}{\lambda_l}\cdot\frac{\lambda_{t_k(n)}}{\lambda_n}.\end{aligned} \tag{3.23}$$

Next we claim that $\limsup_{n\to\infty}(-\Xi(n)/\lambda_n) \geq 0$. Define

$$\begin{aligned}\Delta(n) :&= \sum_{i=2}^{k}(a_1 + \cdots + a_{i-1})^d(\Upsilon_i(t_{i-1}(n)) - \Upsilon_i(t_1(n)))\\ &\quad - \sum_{i=2}^{k}(a_1 + \cdots + a_i)^d(\Upsilon_i(t_i(n)) - \Upsilon_i(t_1(n))).\end{aligned}$$

Then $\limsup_{n\to\infty}(-\Xi(n)/\lambda_n) = \limsup_{n\to\infty}\Delta(n)$. To apply Lemma 3.3, in which we take $p = 2k-2$, let

$$u_j(n) = \begin{cases}(a_1 + \cdots + a_j)^d\Upsilon_{j+1}(n) & \text{if } 1 \leq j \leq k-1,\\ -(a_1 + \cdots + a_{j-k+2})^d\Upsilon_{j-k+2}(n) & \text{if } k \leq j \leq 2k-2,\end{cases}$$

and

$$c_j(n) = \begin{cases}a_1 + \cdots + a_j & \text{if } 1 \leq j \leq k-1,\\ a_1 + \cdots + a_{j-k+2} & \text{if } k \leq j \leq 2k-2,\end{cases}$$

and $r_j = a_1$ for all $1 \leq j \leq 2k-2$. Hence, by equation (3.22), we have $\lim_{n\to\infty}|u_j(n+1) - u_j(n)| = 0$. Thus, $\limsup_{n\to\infty}(-\Xi(n)/\lambda_n) = \limsup_{n\to\infty}\Delta(n) \geq 0$.

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Letting $n \to \infty$ and taking the upper limit in equation (3.23), we have

$$\limsup_{n\to\infty}\bigg(\sum_{i=1}^{k}[(a_1+\cdots+a_i)^d-(a_1+\cdots+a_{i-1})^d]\Upsilon_i(t_1(n))+\int f\,d\nu_{t_1(n)}\bigg)$$
$$\geq s-\frac{k(a_1+\cdots+a_k)^d\log 2}{\lambda_l}. \tag{3.24}$$

Write for brevity that $\kappa=(k(a_1+\cdots+a_k)^d\log 2)/\lambda_l$. Since

$$c_i=(a_1+\cdots+a_i)^d-(a_1+\cdots+a_{i-1})^d$$

for $i=1,\ldots,k$, then equation (3.24) can be rewritten as

$$\limsup_{n\to\infty}\bigg(\sum_{i=1}^{k}c_i\Upsilon_i(t_1(n))+\int f\,d\nu_{t_1(n)}\bigg)\geq s-\kappa. \tag{3.25}$$

Since $\mathcal{M}(X_1,\mathcal{T}_1)$ is compact, we can choose a subsequence $\{n_j\}$ such that the left-hand side of equation (3.25) equals

$$\lim_{j\to\infty}\bigg(\sum_{i=1}^{k}c_iH_{\nu_{t_1(n_j)}\circ\tau_{i-1}^{-1}}(\tau,M;l)+\int f\,d\nu_{t_1(n_j)}\bigg)\geq s-\kappa$$

and $\{\nu_{t_1(n_j)}\}$ converges in $\mathcal{M}(X_1,\mathcal{T}_1)$ for some $\vartheta\in\mathcal{M}(X_1,\mathcal{T}_1)$. Since $H_\bullet(\tau,M;l)$ is upper semi-continuous by Lemma 2.1, we conclude that

$$\lim_{j\to\infty}\sum_{i=1}^{k}c_iH_{\vartheta\circ\tau_{i-1}^{-1}}(\tau,M;l)+\int f\,d\vartheta\geq s-\kappa. \tag{3.26}$$

Define

$$\Phi:=\bigg\{(M,l,\delta):M,l\in\mathbb{N},\delta>0\text{ with }M\geq M_0,\,\lambda_l\geq\frac{k(a_1+\cdots+a_k)^d\log 2}{\delta}\bigg\}$$

and

$$\Omega_{M,l,\delta}:=\bigg\{\eta\in\mathcal{M}(X_1,\mathcal{T}_1):H_\eta^{\mathbf{a}}(\tau,M;l)+\int f\,d\eta\geq s-\delta\bigg\},$$

where $H_\eta^{\mathbf{a}}(\tau,M;l):=\sum_{i=1}^{k}c_iH_{\eta\circ\tau_{i-1}^{-1}}(\tau,M;l)$. Then $\Omega_{M,l,\delta}\neq\emptyset$ since equation (3.26) holds whenever $(M,l,\delta)\in\Phi$. Moreover, the mapping $\eta\in\mathcal{M}(X_1,\mathcal{T}_1)\mapsto H_\eta^{\mathbf{a}}(\tau,M;l)+\int f\,d\eta$ is upper semi-continuous since the sum of finitely many upper semi-continuous functions is still upper semi-continuous. By Definition 2.1(C2), $\Omega_{M,l,\delta}$ is a non-empty closed subset of $\mathcal{M}(X_1,\mathcal{T}_1)$. Additionally,

$$\Omega_{M_1,l_1,\delta_1}\cap\Omega_{M_2,l_2,\delta_2}\supseteq\Omega_{M_1+M_2,l_1l_2,\min\{\delta_1,\delta_2\}}$$

for any $(M_1,l_1,\delta_1),(M_2,l_2,\delta_2)\in\Phi$. Hence, $\bigcap_{(M,l,\delta)\in\Phi}\Omega_{M,l,\delta}\neq\emptyset$ for the finite intersection property characterization of compactness, that is, there exists a $\mu_s\in\bigcap_{(M,l,\delta)\in\Phi}\Omega_{M,l,\delta}$. That is to say,

$$H_{\mu_s}^{\mathbf{a}}(\tau,M;l)+\int f\,d\mu_s\geq s-\delta.$$

Therefore,

$$\sum_{i=1}^{k} c_i H_{\mu_s\circ\tau_{i-1}^{-1}}(\tau; l) + \int f\, d\mu_s = \inf_{M\in\mathbb{N}, M\geq M_0} H^{\mathbf{a}}_{\mu_s}(\tau, M; l) + \int f\, d\mu_s \geq s - \delta.$$

Fix $\delta > 0$, since $\lambda_l \geq (k(a_1 + \cdots + a_k)^d \log 2)/\delta$ when $l \in \mathbb{N}$ is large enough, we have

$$\begin{aligned}\sum_{i=1}^{k} c_i h_{\mu_s\circ\tau_{i-1}^{-1}}(\mathcal{T}_1, \tau) + \int f\, d\mu_s &= \inf_{l\in\mathbb{N}} \sum_{i=1}^{k} c_i H_{\mu_s\circ\tau_{i-1}^{-1}}(\tau; l) + \int f\, d\mu_s \\ &= \lim_{l\to\infty} \sum_{i=1}^{k} c_i H_{\mu_s\circ\tau_{i-1}^{-1}}(\tau; l) + \int f\, d\mu_s \geq s - \delta.\end{aligned}$$

Notice that the mapping $\theta \in \mathcal{M}(X_1, \mathcal{T}_1) \mapsto \sum_{i=1}^{k} c_i h_\theta(\mathcal{T}_1, \tau)$ is upper semi-continuous, there exists $\mu \in \mathcal{M}(X_1, \mathcal{T}_1)$ satisfying $\sum_{i=1}^{k} c_i h_\mu(\mathcal{T}_1, \tau) + \int f\, d\mu \geq s - \delta$. Furthermore, $h_\mu(\mathcal{T}_1) \geq h_\mu(\mathcal{T}_1, \tau)$. Then $h^{\mathbf{a}}_\mu(\mathcal{T}_1) + \int f\, d\mu \geq s - \delta$. Letting $s \nearrow P^{\mathbf{a}}(\mathcal{T}_1, f)$, for the arbitrariness of $\delta > 0$, we conclude that $P^{\mathbf{a}}(\mathcal{T}_1, f) \leq h^{\mathbf{a}}_\mu(\mathcal{T}_1) + \int_{X_1} f\, d\mu$.

4. *Pressure determines measure-theoretic entropy*

In this section, based on the weighted variation principle in Theorem 1.1, we investigate how the pressure $P^{\mathbf{a}}(\mathcal{T}_1, f)$ determines the weighted measure-theoretic entropy $h^{\mathbf{a}}_\mu(\mathcal{T}_1)$. We need the following lemma in [12].

LEMMA 4.1. *If K_1, K_2 are disjoint closed convex subsets of a locally convex linear topological space V and if K_1 is compact, then there exists a continuous real-valued linear functional F on V such that $F(x) < F(y)$ for all $x \in K_1$, $y \in K_2$.*

THEOREM 4.2. *Let $\mu_0 \in \mathcal{M}(X_1, \mathcal{T}_1)$. Assume that $h^{\boldsymbol{a}}_{top}(\mathcal{T}_1) < \infty$ and the entropy map $\theta \in \mathcal{M}(X_i, \mathcal{T}_i) \mapsto h_\theta(\mathcal{T}_i)$, $i = 1, \ldots, k$ are upper semi-continuous at μ_0. Then*

$$h^{\boldsymbol{a}}_{\mu_0}(\mathcal{T}_1) = \inf\left\{P^{\boldsymbol{a}}(\mathcal{T}_1, f) - \int_{X_1} f\, d\mu_0 | f \in C(X_1)\right\}.$$

Proof. By the variational principle in Theorem 1.1, we have

$$h^{\mathbf{a}}_{\mu_0}(\mathcal{T}_1) \leq \inf\left\{P^{\mathbf{a}}(\mathcal{T}_1, f) - \int_{X_1} f\, d\mu_0 | f \in C(X_1)\right\}.$$

To prove the opposite inequality, fix $b > h^{\mathbf{a}}_{\mu_0}(\mathcal{T}_1)$ and let

$$C := \{(\mu, t) \in \mathcal{M}(X_1, \mathcal{T}_1) \times \mathbb{R} | 0 \leq t \leq h^{\mathbf{a}}_\mu(\mathcal{T}_1)\}.$$

Now we prove that C is a convex set. Given $(\mu_1, t_1), (\mu_2, t_2) \in C$, that is to say $0 \leq t_1 \leq h^{\mathbf{a}}_{\mu_1}(\mathcal{T}_1)$ and $0 \leq t_2 \leq h^{\mathbf{a}}_{\mu_2}(\mathcal{T}_1)$, for $p \in [0, 1]$, since the entropy function $\mu \in \mathcal{M}(X_1, \mathcal{T}_1) \mapsto h^{\mathbf{a}}_\mu(\mathcal{T}_1)$ is affine, we have $h^{\mathbf{a}}_{p\mu_1+(1-p)\mu_2}(\mathcal{T}_1) = ph^{\mathbf{a}}_{\mu_1}(\mathcal{T}_1) + (1-p)h^{\mathbf{a}}_{\mu_2}(\mathcal{T}_1) \geq pt_1 + (1-p)t_2 \geq 0$. Then $p(\mu_1, t_1) + (1-p)(\mu_2, t_2) \in C$. Thus C is a convex set. Additionally, let $C(X_1)^*$ be the dual space of $C(X_1)$ endowed with the weak* topology and consider C as a subset of $C(X_1)^* \times \mathbb{R}$. Under the assumption of the

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lemma, the mapping $\mu \in \mathcal{M}(X_1, \mathcal{T}_1) \mapsto h^{\mathbf{a}}_{\mu}(\mathcal{T}_1)$ is upper semi-continuous at μ_0, then $(\mu_0, b) \notin \overline{C}$. To apply Lemma 4.1, let $V = C(X_1)^* \times \mathbb{R}$, $K_1 = \overline{C}$, $K_2 = (\mu_0, b)$, then there exists a continuous linear functional $F : C(X_1)^* \times \mathbb{R} \to \mathbb{R}$ such that

$$F((\mu, t)) \leq F((\mu_0, b)) \quad \text{for all } (\mu, t) \in \overline{C}.$$

Since we are using the weak* topology on $C(X_1)^*$, F must have the form $F((\mu, t)) = \int_{X_1} f\, d\mu + dt$ for some $f \in C(X_1)$ and $d \in \mathbb{R}$. It follows that $\int_{X_1} f\, d\mu + dt \leq \int_{X_1} f\, d\mu_0 + db$ for all $(\mu, t) \in \overline{C}$. In particular, $\int_{X_1} f\, d\mu + dh^{\mathbf{a}}_{\mu}(\mathcal{T}_1) \leq \int_{X_1} f\, d\mu_0 + db$ for all $\mu \in \mathcal{M}(X_1, \mathcal{T}_1)$. Taking $\mu = \mu_0$, since $b > h^{\mathbf{a}}_{\mu_0}(\mathcal{T}_1)$, we have $d > 0$. Hence,

$$h^{\mathbf{a}}_{\mu}(\mathcal{T}_1) + \int_{X_1} \frac{f}{d}\, d\mu < b + \int_{X_1} \frac{f}{d}\, d\mu_0, \quad \text{for all } \mu \in \mathcal{M}(X_1, \mathcal{T}_1).$$

By Theorem 1.1, we have $P^{\mathbf{a}}(\mathcal{T}_1, f/d) \leq b + \int_{X_1} f/d\, d\mu_0$. Then

$$b \geq P^{\mathbf{a}}\bigg(\mathcal{T}_1, \frac{f}{d}\bigg) - \int_{X_1} \frac{f}{d}\, d\mu_0 \geq \inf\bigg\{P^{\mathbf{a}}(\mathcal{T}_1, g) - \int_{X_1} g\, d\mu_0 | g \in C(X_1)\bigg\}.$$

Letting $b \searrow h^{\mathbf{a}}_{\mu_0}(\mathcal{T}_1)$, we conclude that $h^{\mathbf{a}}_{\mu_0}(\mathcal{T}_1) \geq \inf\{P^{\mathbf{a}}(\mathcal{T}_1, f) - \int_{X_1} f\, d\mu_0 | f \in C(X_1)\}$. $\square$

5. *Final remarks*

As emphasized in the introduction, owing to the research of Ornstein and Weiss [34] and Lindenstrauss [28], we can extend Feng and Huang's weighted variational principle for topological pressure (see [16, Theorem 1.4]) from TDS (X, T) to $\mathbb{Z}^d$-actions TDS. In other words, [28, 34] generalized classical pointwise convergence results to general amenable discrete groups, and therefore contributed to obtain Birkhoff's ergodic theorem and the Schannon–McMillan–Breiman theorem for discrete amenable groups. In this paper, we only consider the $\mathbb{Z}^d$-action, which is a special case of amenable group actions. Additionally, we believe that the weighted variational principle obtained (see Theorem 1.1) is valid for pressure under general amenable group actions.

However, while considering a finitely generated free group or semigroup G on a compact metric space X, [18] or [27, Example 5.3] shows that $\mathcal{M}(X, G)$, the invariant measure space, can be empty. Consequently, the conclusion in Theorem 1.1 may fail in the free group setting. Alternatively, we can only obtain a partial variational principle like [2, 8, 27]. We propose that this difficulty can be overcome by two different approaches. First, Theorem 4.2 in this paper shows that the pressure $P^{\mathbf{a}}(\mathcal{T}_1, f)$ determines the weighted measure-theoretic entropy $h^{\mathbf{a}}_{\mu}(\mathcal{T}_1)$. Combined with the explanation in [9], it might be reasonable to define the weighted measure-theoretic entropy by weighted topological pressure, rather than the traditional Kolmogrov–Sinai entropy. Second, Feng and Huang [15] investigate whether there is certain variational relation between Bowen topological entropy and measure-theoretic entropy for arbitrary non-invariant compact set or Borel set in general. In this case, one does not expect to have such variational principle on the invariant measure space. Following the Brin–Katok formula (see [6, 29]), they defined the measure-theoretic lower entropy and upper entropy and obtained the desired variational principle. Later, [40, 45] extend Feng and Huang's work to topological pressure. Inspired by [15, 40, 45], to establish the variational principle for weighted topological pressure in the free group setting, one can similarly define a weighted version of measure-theoretic lower entropy and upper entropy by weighted Bowen balls. This will avoid the difficulty that the invariant measure under free group actions may fail to exist. Since new ideas and techniques must be considered, we leave the above meaningful work for further research.

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A. *Appendix. A weighted version of the Brin–Katok theorem*

In this section, we give the proof of a weighted version of the Brin–Katok theorem. First, we recall some notation. Let $(X, \mathcal{T})$ be a $\mathbb{Z}^d$-actions TDS. Set $\mathbf{a} = (a_1, \ldots, a_k) \in \mathbb{R}^k$ satisfying $a_1 > 0$ and $a_i \geq 0$ for $i \geq 2$. Make the convention $a_0 = 0$. Write for brevity that $c_i = (a_0 + \cdots + a_i)^d - (a_0 + \cdots + a_{i-1})^d$ for $i = 1, \ldots, k$. Denote $\Pi_i(n) = \Lambda^{\lceil (a_0+\cdots+a_i)n\rceil}_{\lceil (a_0+\cdots+a_{i-1})n\rceil}$, $t_0(n) = 0$, $t_i(n) = \lceil (a_1 + \cdots + a_i)n\rceil$ for $n \in \mathbb{N}$ and $i = 1, \ldots, k$.

LEMMA A.1. ([17] Shannon–McMillan–Breiman) *Let $(X, \mathcal{B}(X), \mu, \mathcal{T})$ be an ergodic measure preserving dynamical system and $\alpha \in \mathcal{P}_X$ with $H_\mu(\alpha) < \infty$. Then*

$$\lim_{n\to\infty} \frac{1}{\lambda_n} I_\mu\bigg(\bigvee_{\boldsymbol{g}\in\Lambda_n} T^{-\boldsymbol{g}}\alpha\bigg)(x) = h_\mu(\mathcal{T}, \alpha)$$

for μ-a.e. $x \in X$, where $I_\mu(\alpha)(x) := -\sum_{A\in\alpha} \chi_A(x) \log \mu(A)$ for $\alpha \in \mathcal{P}_X$ denotes the information function.

As a consequence of Lemma A.1, we have the following lemma.

LEMMA A.2. *Let $\mu \in E(X, \mathcal{T})$. Let $k \geq 1$ and $\alpha_1, \ldots, \alpha_k \in \mathcal{P}_X$ be k finite partitions with $H_\mu(\alpha_i) < \infty$ for each i. Then*

$$\lim_{N\to\infty} \frac{I_\mu(\bigvee_{i=1}^k \bigvee_{\boldsymbol{g}\in\Lambda_{t_i(N)}} T^{-\boldsymbol{g}}\alpha_i)(x)}{\lambda_N} = \sum_{i=1}^k c_i h_\mu\bigg(\mathcal{T}, \bigvee_{j=i}^k \alpha_j\bigg) \tag{A.1}$$

for μ-a.e. $x \in X$. In particular, if $\alpha_1 \succeq \alpha_2 \succeq \cdots \succeq \alpha_k$, then

$$\lim_{N\to\infty} \frac{I_\mu(\bigvee_{i=1}^k \bigvee_{\boldsymbol{g}\in\Pi_i(N)} T^{-\boldsymbol{g}}\alpha_i)(x)}{\lambda_N} = \sum_{i=1}^k c_i h_\mu(\mathcal{T}, \alpha_i) \tag{A.2}$$

for μ-a.e. $x \in X$.

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Proof. Fix $N \in \mathbb{N}$. Note that $\Lambda_{t_i(N)} = \Lambda_{t_{i-1}(N)}^{t_i(N)} \cup \Lambda_{t_{i-1}(N)}$ for $i = 1, \ldots, k$ and $\bigvee_{i=1}^k \bigvee_{\mathbf{g}\in\Lambda_{t_i(N)}} T^{-\mathbf{g}}\alpha_i = \bigvee_{i=1}^k \bigvee_{\mathbf{g}\in\Lambda_{t_{i-1}(N)}^{t_i(N)}} T^{-\mathbf{g}}(\bigvee_{k=i}^k \alpha_j)$. In addition, $I_\mu(\alpha \vee \beta) = I_\mu(\alpha) + I_\mu(\beta)$ for all $\alpha, \beta \in \mathcal{P}_X$. Thus

$$\frac{I_\mu(\bigvee_{i=1}^k \bigvee_{\mathbf{g}\in\Lambda_{t_i(N)}} T^{-\mathbf{g}}\alpha_i)(x)}{\lambda_N} = \frac{I_\mu(\bigvee_{i=1}^k \bigvee_{\mathbf{g}\in\Lambda_{t_{i-1}(N)}^{t_i(N)}} T^{-\mathbf{g}}(\bigvee_{j=i}^k \alpha_j))(x)}{\lambda_N}$$
$$= \frac{\sum_{i=1}^k I_\mu(\bigvee_{\mathbf{g}\in\Lambda_{t_{i-1}(N)}^{t_i(N)}} T^{-\mathbf{g}}(\bigvee_{j=i}^k \alpha_j))(x)}{\lambda_N}$$
$$= \frac{\sum_{i=1}^k I_\mu(\bigvee_{\mathbf{g}\in\Lambda_{t_i(N)}} T^{-\mathbf{g}}(\bigvee_{j=i}^k \alpha_j))(x) - \sum_{i=1}^k I_\mu(\bigvee_{\mathbf{g}\in\Lambda_{t_{i-1}(N)}} T^{-\mathbf{g}}(\bigvee_{j=i}^k \alpha_j))(x)}{\lambda_N},$$

and therefore by Lemma A.2,

$$\lim_{N\to\infty} \frac{I_\mu(\bigvee_{i=1}^k \bigvee_{\mathbf{g}\in\Lambda_{t_i(N)}} T^{-\mathbf{g}}\alpha_i)(x)}{\lambda_N}$$
$$= \sum_{i=1}^k (a_0 + \cdots + a_i)^d \lim_{N\to\infty} \frac{I_\mu(\bigvee_{\mathbf{g}\in\Lambda_{t_i(N)}} T^{-\mathbf{g}}(\bigvee_{j=i}^k \alpha_j))(x)}{\lambda_{t_i(N)}}$$
$$- \sum_{i=1}^k (a_0 + \cdots + a_{i-1})^d \lim_{N\to\infty} \frac{I_\mu(\bigvee_{\mathbf{g}\in\Lambda_{t_{i-1}(N)}} T^{-\mathbf{g}}(\bigvee_{j=i}^k \alpha_j))(x)}{\lambda_{t_{i-1}(N)}}$$
$$= \sum_{i=1}^k (a_0 + \cdots + a_i)^d h_\mu\Big(\mathcal{T}, \bigvee_{j=i}^k \alpha_j\Big) - \sum_{i=1}^k (a_0 + \cdots + a_{i-1})^d h_\mu\Big(\mathcal{T}, \bigvee_{j=i}^k \alpha_j\Big)$$
$$= \sum_{i=1}^k c_i h_\mu\Big(\mathcal{T}, \bigvee_{j=i}^k \alpha_j\Big).$$

Then equation (A.1) holds and equation (A.1) implies equation (A.2) obviously. □

The following lemma is similar to [**43**, Theorem 8.3], we omit the proof.

LEMMA A.3. *Let $(X, \mathcal{T})$ be a $\mathbb{Z}^d$-actions TDS. Let $(\alpha_n)_{n=1}^\infty \subset \mathcal{P}_X$ such that diam$(\alpha_n)\to 0$ as $\to \infty$. For every $\mu \in \mathcal{M}(X, \mathcal{T})$, $h_\mu(\mathcal{T}) = \lim_{n\to\infty} h_\mu(\mathcal{T}, \alpha_n)$.*

Proof of Theorem 3.1. Let $\epsilon > 0$, $\mu \in E(X_1, \mathcal{T}_1)$ and $\alpha_i \in \mathcal{P}_{X_i}(\epsilon)$, $i = 1, \ldots, k$. Given $n \in \mathbb{N}$ and $x \in X_1$, by Definition 1.1, we have

$$\bigvee_{i=1}^k \bigvee_{\mathbf{g}\in\Pi_i(n)} T_1^{-\mathbf{g}} \tau_{i-1}^{-1}\alpha_i(x) \subseteq B_n^{\mathbf{a}}(x, \epsilon).$$

Hence, for μ-a.e. $x \in X_1$,

$$\begin{aligned}
\limsup_{n\to\infty} \frac{-\log \mu(B_n^{\mathbf{a}}(x,\epsilon))}{\lambda_n} &\le \limsup_{n\to\infty} \frac{-\log \mu(\bigvee_{i=1}^k \bigvee_{\mathbf{g}\in\Pi_i(n)} T_1^{-\mathbf{g}}\tau_{i-1}^{-1}\alpha_i(x))}{\lambda_n} \\
&= \limsup_{n\to\infty} \frac{I_\mu(\bigvee_{i=1}^k \bigvee_{\mathbf{g}\in\Pi_i(n)} T_1^{-\mathbf{g}}\tau_{i-1}^{-1}\alpha_i(x))}{\lambda_n} = \sum_{i=1}^k c_i h_\mu\bigg(\mathcal{T}_1, \bigvee_{j=i}^k \tau_{j-1}^{-1}\alpha_j\bigg) \\
&= \sum_{i=1}^k c_i h_\mu\bigg(\mathcal{T}_1, \tau_{i-1}^{-1}\bigg(\alpha_i \vee \bigvee_{j=i+1}^k \pi_i^{-1}\circ\cdots\circ\pi_{j-1}^{-1}\alpha_j\bigg)\bigg) \\
&= \sum_{i=1}^k c_i h_{\mu\circ\tau_{i-1}^{-1}}\bigg(\mathcal{T}_i, \alpha_i \vee \bigvee_{j=i+1}^k \pi_i^{-1}\circ\cdots\circ\pi_{j-1}^{-1}\alpha_j\bigg) \\
&\le \sum_{i=1}^k c_i h_{\mu\circ\tau_{i-1}^{-1}}(\mathcal{T}_i) = h_\mu^{\mathbf{a}}(\mathcal{T}_1).
\end{aligned}$$

Next we show that for any $\delta > 0$, there exist $\epsilon > 0$ and a measurable set $D \subset X_1$ so that $\mu(D) > 1 - 3\delta$ and

$$\liminf_{n\to\infty} \frac{-\log \mu(B_n^{\mathbf{a}}(x,\epsilon))}{\lambda_n} \ge \min\left\{\frac{1}{\delta}, h_\mu^{\mathbf{a}}(\mathcal{T}_1) - \delta\right\} - [2 + 2(a_1 + \cdots + a_k)^d]\delta$$

for each $x \in D$.

Fix $\delta > 0$. By [43, Lemma 8.5 and Theorem 8.3], we can choose $\beta_i = \{B_1^i, \ldots, B_{v_i}^i\} \in \mathcal{P}_{X_i}$ for $i = 1, \ldots, k$, so that $\mu \circ \tau_{i-1}^{-1}(\partial\beta_i) = 0$ and $\mathrm{diam}(\beta_i)$ are small enough and

$$h_{\mu\circ\tau_{i-1}^{-1}}(\mathcal{T}_i, \beta_i) \ge \begin{cases} \dfrac{1}{c_1\delta} & \text{if } h_{\mu\circ\tau_{i-1}^{-1}}(\mathcal{T}_i) = \infty, \\ h_{\mu\circ\tau_{i-1}^{-1}}(\mathcal{T}_i) - \dfrac{\delta}{c_1 + \cdots + c_k} & \text{otherwise.} \end{cases}$$

Define $\alpha_i \in \mathcal{P}_{X_i}$ recursively for $i = k, k-1, \ldots, 1$ by setting $\alpha_k = \beta_k$ and

$$\alpha_j = \beta_j \vee \pi_j^{-1}(\alpha_{j+1}) \quad \text{for } j = k-1, \ldots, 1.$$

Then:

(1) $\alpha_i \succeq \pi_i^{-1}(\alpha_{i+1})$, that is, $\tau_{i-1}^{-1}\alpha_i \succeq \tau_i^{-1}\alpha_{i+1}$ for $i = 1, \ldots, k-1$;

(2) $\sum_{i=1}^k c_i h_{\mu\circ\tau_{i-1}^{-1}}(\mathcal{T}_i, \alpha_i) \ge \min\{1/\delta, h_\mu^{\mathbf{a}}(\mathcal{T}_1) - \delta\}$;

(3) $\mu \circ \tau_{i-1}^{-1}(\partial\alpha_i) = 0$ for $i = 1, \ldots, k$.

Write $\alpha_i = \{A_1^i, \ldots, A_{u_i}^i\}$ for $i = 1, \ldots, k$. Let $M = \max_{1\le i\le k} u_i$ and $\Psi = \{1, \ldots, M\}$. Given $m \in \mathbb{N}$, for $\mathbf{s} = (s_i)_{i=0}^{m-1}$, $\mathbf{t} = (s_i)_{i=0}^{m-1} \in \Psi^{\{0,\ldots,m-1\}}$, the Hamming distance between $\mathbf{s}$ and $\mathbf{t}$ is defined by

$$\mathrm{Ham}(\mathbf{s}, \mathbf{t}) := \frac{1}{m}\,\mathrm{Card}\{i \in \{0, \ldots, m-1\} : s_i \ne t_i\}.$$

For $\mathbf{s} \in \Psi^{\{0,\ldots,m-1\}}$ and $0 < \tau \leq 1$, let $Q(\mathbf{s}, \tau)$ be the total number of those $\mathbf{t} \in \Psi^{\{0,\ldots,m-1\}}$ satisfying $\mathrm{Ham}(\mathbf{s}, \mathbf{t}) \leq \tau$. Then

$$Q_m(\tau) := \max_{\mathbf{s}\in\Psi^{\{0,\ldots,m-1\}}} Q(\mathbf{s}, \tau) \leq \binom{m}{\lceil m\delta_1\rceil} M^{\lceil m\delta_1\rceil}.$$

By the Stirling formula, there exists a small $\delta_1 > 0$ and a positive $C := C(\delta, M) > 0$ so that

$$\binom{m}{\lceil m\delta_1\rceil} M^{\lceil m\delta_1\rceil} \leq e^{\delta m + C} \tag{A.3}$$

for all $m \in \mathbb{N}$.

For $\eta > 0$, set

$$U_\eta^i(\alpha_i) := \{x \in X_1 : B(\tau_{i-1}x, \eta) \nsubseteq \alpha_i(\tau_{i-1}x)\}, \quad i = 1, \ldots, k.$$

Then $\bigcap_{\eta>0} U_\eta^i(\alpha_i) = \tau_{i-1}^{-1}(\partial\alpha_i)$ and therefore $\mu(U_\eta^i(\alpha_i)) \to \mu(\tau_{i-1}^{-1}(\partial\alpha_i))$ as $\eta \downarrow 0$. We can choose $\epsilon > 0$ so that $\mu(U_\eta^i(\alpha_i)) < \delta_1$ for any $0 < \eta \leq \epsilon$ and $i = 1, \ldots, k$. Notice that $\sum_{i=1}^k c_i = (a_1 + \cdots + a_k)^d$, by Birkhoff's ergodic theorem, for μ-a.e. $x \in X_1$, we have

$$\lim_{n\to\infty} \frac{1}{\lambda_{t_k(n)}} \sum_{i=1}^k \sum_{\mathbf{g}\in\Pi_i(n)} \chi_{U_\epsilon^i(\alpha_i)}(T_1^{\mathbf{g}}x) = \frac{1}{(a_1 + \cdots + a_k)^d} \sum_{i=1}^k c_i \mu(U_\epsilon^i(\alpha_i)) < \delta_1.$$

Thus, there exists $\ell_0 \in \mathbb{N}$ large enough so that $\mu(A_\ell) > 1 - \delta$ for any $\ell \geq \ell_0$, where

$$A_\ell := \bigg\{x \in X_1 : \frac{1}{\lambda_{t_k(n)}} \sum_{i=1}^k \sum_{\mathbf{g}\in\Pi_i(n)} \chi_{U_\epsilon^i(\alpha_i)}(T_1^{\mathbf{g}}x) \leq \delta_1 \text{ for all } n \geq \ell\bigg\}.$$

Since $\tau_0^{-1}\alpha_1 \succeq \tau_1^{-1}\alpha_2 \succeq \cdots \succeq \tau_{k-1}^{-1}\alpha_k$, by Lemma A.2, we have

$$\begin{aligned}\lim_{n\to\infty} & \frac{-\log \mu(\bigvee_{i=1}^k \bigvee_{\mathbf{g}\in\Pi_i(n)} T_1^{-\mathbf{g}}\tau_{i-1}^{-1}\alpha_i(x))}{\lambda_n} \\ &= \sum_{i=1}^k c_i h_\mu(\mathcal{T}_1, \tau_{i-1}^{-1}\alpha_i) = \sum_{i=1}^k c_i h_{\mu\circ\tau_{i-1}^{-1}}(\mathcal{T}_i, \alpha_i)\end{aligned}$$

for μ-a.e. $x \in X_1$. Then there exists $\ell_1 \in \mathbb{N}$ large enough so that $\mu(B_\ell) > 1 - \delta$ for any $\ell \geq \ell_1$, where B_ℓ is the set of all points $x \in X_1$ so that

$$\frac{-\log \mu(\bigvee_{i=1}^k \bigvee_{\mathbf{g}\in\Pi_i(n)} T_1^{-\mathbf{g}}\tau_{i-1}^{-1}\alpha_i(x))}{\lambda_n} \geq \sum_{i=1}^k c_i h_{\mu\circ\tau_{i-1}^{-1}}(\mathcal{T}_i, \alpha_i) - \delta \tag{A.4}$$

for all $n \geq \ell$. Fix $\ell \geq \max\{\ell_0, \ell_1\}$. Setting $E = A_\ell \cap B_\ell$, then $\mu(E) \geq 1 - 2\delta$. For $x \in X_1$ and $n \in \mathbb{N}$, the unique element

$$C(n, x) = (C_{\mathbf{g}}(n, x))_{\mathbf{g}\in\Lambda_{t_k(n)}}$$

in $\Psi^{\{0,1,\ldots,\lambda_{t_k(n)}-1\}}$ satisfying $T_1^{\mathbf{g}}x \in \tau_{i-1}^{-1}(A^i_{C_{\mathbf{g}}(n,x)})$ for $\mathbf{g} \in \Pi_i(n)$, $i = 1, \ldots, k$ is called the $(\{\alpha_i\}_{i=1}^k, \mathbf{a}; n)$-name of x. Since each point in one atom A of $\bigvee_{\mathbf{g}\in\Pi_i(n)} T_1^{-\mathbf{g}}\tau_{i-1}^{-1}\alpha_i(x)$

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has the same $(\{\alpha_i\}_{i=1}^k, \mathbf{a}; n)$-name, we define

$$C(n, A) := C(n, x)$$

for any $x \in A$, which is called the $(\{\alpha_i\}_{i=1}^k, \mathbf{a}; n)$-name of A.

If $y \in B_n^{\mathbf{a}}(x, \epsilon)$, then for $i = 1, \ldots, k$ and $\mathbf{g} \in \Pi_i(n)$, either $T_1^{\mathbf{g}}x$ and $T_1^{\mathbf{g}}y$ belong to the same element of $\tau_{i-1}^{-1}\alpha_i$ or $T_1^{\mathbf{g}}x \in U_\epsilon^i(\alpha_i)$. Thus, if $x \in E$, $n \geq \ell$ and $y \in B_n^{\mathbf{a}}(x, \epsilon)$, then $\mathrm{Ham}(C(n, x), C(n, y)) \leq \delta_1$, that is, the Hamming distance between $(\{\alpha_i\}_{i=1}^k, \mathbf{a}; n)$-name of x and y does not exceed δ_1. Moreover, $B_n^{\mathbf{a}}(x, \epsilon)$ is contained in the set of points y whose $(\{\alpha_i\}_{i=1}^k, \mathbf{a}; n)$-name is δ_1-close to $(\{\alpha_i\}_{i=1}^k, \mathbf{a}; n)$-name of x. More precisely, for $x \in E$, $n \geq \ell$,

$$\begin{aligned} B_n^{\mathbf{a}}(x, \epsilon) &\subset \{y \in X_1 : \mathrm{Ham}(C(n, x), C(n, y)) \leq \delta_1\} \\ &= \bigg\{A \in \bigvee_{i=1}^k \bigg(\bigvee_{\mathbf{g}\in\Pi_i(n)} T_1^{-\mathbf{g}}\tau_{i-1}^{-1}\alpha_i\bigg) : \mathrm{Ham}(C(n, A), C(n, y)) \leq \delta_1\bigg\} =: \Omega_n(x). \end{aligned} \tag{A.5}$$

In addition, by equation (A.3),

$$\mathrm{Card}\ \Omega_n(x) \leq \binom{\lambda_{t_k(n)}}{\lceil \lambda_{t_k(n)}\delta_1 \rceil} M^{\lceil \lambda_{t_k(n)}\delta_1 \rceil} \leq e^{\delta\lambda_{t_k(n)}+C}. \tag{A.6}$$

For $n \in \mathbb{N}$, denote by E_n the sets of points $x \in E$ so that there exists an element $A \in \bigvee_{i=1}^k(\bigvee_{\mathbf{g}\in\Pi_i(n)} T_1^{-\mathbf{g}}\tau_{i-1}^{-1}\alpha_i)$ with

$$\mu(A) > \exp\bigg\{\bigg(-\sum_{i=1}^k c_i h_{\mu\circ\tau_{i-1}^{-1}}(\mathcal{T}_i, \alpha_i) + [2 + (a_1 + \cdots + a_k)^d]\delta\bigg)\lambda_n\bigg\}$$

and $\mathrm{Ham}(C(n, x), C(n, A)) \leq \delta_1$. Obviously, if $x \in E \setminus E_n$, for each $A \in \bigvee_{i=1}^k(\bigvee_{\mathbf{g}\in\Pi_i(n)} T_1^{-\mathbf{g}}\tau_{i-1}^{-1}\alpha_i)$ with $\mathrm{Ham}(C(n, x), C(n, A)) \leq \delta_1$, one has

$$\mu(A) \leq \exp\bigg\{\bigg(-\sum_{i=1}^k c_i h_{\mu\circ\tau_{i-1}^{-1}}(\mathcal{T}_i, \alpha_i) + [2 + (a_1 + \cdots + a_k)^d]\delta\bigg)\lambda_n\bigg\}. \tag{A.7}$$

In the following, we wish to estimate $\mu(E_n)$ for $n \geq \ell$.

Let $n \geq \ell$. Set

$$\begin{aligned} \mathcal{F}_n := \bigg\{A \in \bigvee_{i=1}^k \bigg(\bigvee_{\mathbf{g}\in\Pi_i(n)} T_1^{-\mathbf{g}}\tau_{i-1}^{-1}\alpha_i\bigg) &: \mu(A) \\ &> \exp\bigg\{\bigg(-\sum_{i=1}^k c_i h_{\mu\circ\tau_{i-1}^{-1}}(\mathcal{T}_i, \alpha_i) + [2 + (a_1 + \cdots + a_k)^d]\delta\bigg)\lambda_n\bigg\}\bigg\}. \end{aligned}$$

Since $\mu(X_1) = 1$, we have

$$\mathrm{Card}\ \mathcal{F}_n \leq \exp\bigg\{\bigg(\sum_{i=1}^k c_i h_{\mu\circ\tau_{i-1}^{-1}}(\mathcal{T}_i, \alpha_i) - [2 + (a_1 + \cdots + a_k)^d]\delta\bigg)\lambda_n\bigg\}.$$

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Additionally, by the definition of E_n, there exists $A \in \mathcal{F}_n$ with $\mathrm{Ham}(C(n,x), C(n, A)) \le \delta_1$. That is to say, the $(\{\alpha_i\}_{i=1}^k, \mathbf{a}; n)$-name of A is δ_1-close to the $(\{\alpha_i\}_{i=1}^k, \mathbf{a}; n)$-name of

$$\bigvee_{i=1}^{k}\bigg(\bigvee_{\mathbf{g}\in\Pi_i(n)} T_1^{-\mathbf{g}}\tau_{i-1}^{-1}\alpha_i\bigg)(x).$$

Denote by $\mathcal{G}_n$ the set of all elements $B \in \bigvee_{i=1}^k(\bigvee_{\mathbf{g}\in\Pi_i(n)} T_1^{-\mathbf{g}}\tau_{i-1}^{-1}\alpha_i)$ satisfying

$$\mu(B) \le \exp\bigg\{\bigg(\sum_{i=1}^k c_i h_{\mu\circ\tau_{i-1}^{-1}}(\mathcal{T}_i, \alpha_i) - \delta\bigg)\lambda_n\bigg\}$$

and $\mathrm{Ham}(C(n, B), C(n, A)) \le \delta_1$ for some $A \in \mathcal{F}_n$. Then

$$E_n \subset \{B : B \in \mathcal{G}_n\}. \tag{A.8}$$

Fix $A \in \mathcal{F}_n$, the total number of $B \in \bigvee_{i=1}^k(\bigvee_{\mathbf{g}\in\Pi_i(n)} T_1^{-\mathbf{g}}\tau_{i-1}^{-1}\alpha_i)$, whose $(\{\alpha_i\}_{i=1}^k, \mathbf{a}; n)$-name is δ_1-close to the $(\{\alpha_i\}_{i=1}^k, \mathbf{a}; n)$-name of A, is upper bounded by $\binom{\lambda_{t_k(n)}}{\lceil\lambda_{t_k(n)}\delta_1\rceil} M^{\lceil\lambda_{t_k(n)}\delta_1\rceil} \le e^{\delta\lambda_{t_k(n)}+C}$. Then

$$\begin{aligned}\mathrm{Card}\,\mathcal{G}_n &\le e^{\delta\lambda_{t_k(n)}+C}\,\mathrm{Card}\,\mathcal{F}_n\\ &\le \exp\bigg\{\bigg(\sum_{i=1}^k c_i h_{\mu\circ\tau_{i-1}^{-1}}(\mathcal{T}_i, \alpha_i) - [2 + (a_1 + \cdots + a_k)^d]\delta\bigg)\lambda_n + \delta\lambda_{t_k(n)} + C\bigg\}.\end{aligned}$$

In addition, combining equation (A.8) and the definition of $\mathcal{G}_n$, we obtain

$$\begin{aligned}\mu(E_n) &\le \bigg\{\bigg(-\sum_{i=1}^k c_i h_{\mu\circ\tau_{i-1}^{-1}}(\mathcal{T}_i, \alpha_i) + \delta\bigg)\lambda_n\bigg\}\,\mathrm{Card}\,\mathcal{G}_n\\ &\le \exp\bigg\{-\delta\lambda_n + C + [\lambda_{t_k(n)} - (a_1 + \cdots + a_k)^d\lambda_n]\delta\bigg\}.\end{aligned}$$

Notice that $\lim_{n\to\infty}((\lambda_{t_k(n)} - (a_1 + \cdots + a_k)^d\lambda_n)/\lambda_n) = 0$. Thus, $\mu(E_n) \le e^{-\delta\lambda_n+C+o(\lambda_n)}$ when n is large enough. So we can choose $\ell_2 \ge \ell$ such that $\sum_{n=\ell_2}^{\infty}\mu(E_n) < \delta$. Then $\mu(\bigcup_{n\ge\ell_2} E_n) < \delta$. Setting $D = E \setminus \bigcup_{n\ge\ell_2} E_n$, we have $\mu(D) > 1 - 3\delta$. For $x \in D$ and $n \ge \ell_2$, since $x \in E_n$, combining equations (A.5), (A.6) and the definition of E_n, one has

$$\mu(B_n^{\mathbf{a}}(x,\epsilon)) \le e^{\delta\lambda_{t_k(n)}+C}\exp\bigg\{\bigg(-\sum_{i=1}^k c_i h_{\mu\circ\tau_{i-1}^{-1}}(\mathcal{T}_i, \alpha_i) + [2 + (a_1 + \cdots + a_k)^d]\delta\bigg)\lambda_n\bigg\}.$$

Thus, for $x \in D$,

$$\begin{aligned}\liminf_{n\to\infty}\frac{-\log\mu(B_n^{\mathbf{a}}(x,\epsilon))}{\lambda_n} &\ge \sum_{i=1}^k c_i h_{\mu\circ\tau_{i-1}^{-1}}(\mathcal{T}_i, \alpha_i) - [2 + 2(a_1 + \cdots + a_k)^d]\delta\\ &\ge \min\bigg\{\frac{1}{\delta}, h_\mu^{\mathbf{a}}(\mathcal{T}_1) - \delta\bigg\} - [2 + 2(a_1 + \cdots + a_k)^d]\delta. \qquad \square\end{aligned}$$

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