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A $\mathcal{C}^k$-seeley-extension-theorem for Bastiani's differential calculus

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Abstract. We generalize a classical extension result by Seeley in the context of Bastiani's differential calculus to infinite dimensions. The construction follows Seeley's original approach, but is significantly more involved as not only C^k-maps (for $k \in \mathbb{N} \cup \{\infty\}$) on (subsets of) half spaces are extended, but also continuous extensions of their differentials to some given piece of boundary of the domains under consideration. A further feature of the generalization is that we construct families of extension operators (instead of only one single extension operator) that fulfill certain compatibility (and continuity) conditions. Various applications are discussed as well.

1 Introduction

The extension problem for differentiable maps naturally arises in the context of manifolds with boundary or corners. In the finite-dimensional context, Whitney's extension theorem [20] guarantees more generally the extendability of Whitney jets (families of continuous functions that define formal Taylor expansions) on closed subsets of euclidean spaces. A characterization of closed subsets that admit continuous linear extension operators on C^∞ - Whitney jets was given by Tidten in [18] (see [2] for further investigations). Recent research into Whitney-type extension operators [11, 16] is concerned with generalizations to maps on closed subsets of finite-dimensional manifolds (Whitney germs in [11], and in [16], subsets that satisfy the so-called cusp condition) with values in vector bundles or (infinite-dimensional) manifolds. In [11], the smooth category in the context of the convenient calculus [7] is considered, and in [16], the smooth category within Bastiani's differential calculus [3]. Throughout this paper, we work in Bastiani's setting that is recalled in Section 2.1. We refer to [4, 15] for self-contained introductions into Bastiani's calculus.

Besides Whitney's approach, there is an alternative (significantly simpler) extension construction available that works for maps defined on half spaces. This approach is due to Seeley [17]. He constructs a continuous linear map that extends such smooth maps $(-\infty, 0) \times \mathbb{R}^n \to \mathbb{R}$ ($n \in \mathbb{N}$) to $\mathbb{R} \times \mathbb{R}^n$, whose partial derivatives extend continuously to $(-\infty, 0] \times \mathbb{R}^n$. In this paper, we generalize Seeley's result into several directions:

Let E, F be Hausdorff locally convex vector spaces, and denote the system of continuous seminorms on F by $\mathrm{Sem}(F)$. For $k \in \mathbb{N} \cup \{\infty\}$ and $U \subseteq E$ nonempty open,

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let $C^k(U,F)$ denote the set of all k-times continuously differentiable maps $U \to F$. Let $\Omega(E)$ denote the set of all pairs $(V,\mathfrak{V})$ such that $V \subseteq E$ is nonempty open, and $\mathfrak{V} \subseteq E$ is contained in the closure of V in E with $V \subseteq \mathfrak{V}$. For $-\infty \leq a < b \leq \infty$, let $\mathcal{C}^k_{\mathfrak{V}}((a,b)\times V,F)$ denote the set of all $f \in C^k((a,b)\times V,F)$, such that for each $\ell \in \mathbb{N}$ with $\ell \leq k$, the ℓth differential of f extends to a continuous map

$$\mathrm{Ext}(f,\ell)\colon (a,b]\times\mathfrak{V}\times(\mathbb{R}\times E)^\ell \to F,$$

(we set $(a,b] := (a,\infty)$ if $b=\infty$ holds). Our main result Theorem 3.1 (stated to the full extent in Section 3.1) inter alia implies that, for $-\infty \leq a < \tau < 0$ fixed, there exists a linear (extension) map

$$\mathcal{E}\colon \mathcal{C}^k_{\mathfrak{V}}((a,0)\times V,F) \to \mathcal{C}^k_{\mathfrak{V}}((a,\infty)\times V,F),$$

such that for $f \in \mathcal{C}^k_{\mathfrak{V}}((a,0)\times V,F)$ and $0 \leq \ell \leq k$, we have

$$\begin{aligned}\mathrm{Ext}(\mathcal{E}(f),\ell)|_{(a,0]\times\mathfrak{V}\times(\mathbb{R}\times E)^\ell} &= \mathrm{Ext}(f,\ell),\\ \mathrm{Ext}(\mathcal{E}(f),\ell)|_{[|\tau|,\infty)\times\mathfrak{V}\times(\mathbb{R}\times E)^\ell} &= 0.\end{aligned}$$

For $E=\mathbb{R}^n$, $F=\mathbb{R}$, and $a=-\infty$, this implies Seeley's original theorem from [17]. We mention, but do not present the details at this point, that Theorem 3.1 is formulated more generally in terms of families of extension operators indexed by triples $(E,V,\mathfrak{V})$, where E runs over the class of Hausdorff locally convex vector spaces and $(V,\mathfrak{V}) \in \Omega(E)$ holds (a and τ are thus fixed parameters). Theorem 3.1 additionally contains continuity estimates, as well as compatibility conditions that can be used, e.g., to construct extensions of maps by gluing together local extensions. This is demonstrated in Example 3.10 for the unit ball in a real pre-Hilbert space. In Application 3.6 in Section 3.2, we carry over the extension result (in the form stated above) to quadrants, which is of relevance in the context of (infinite-dimensional) manifolds with corners [8]. Specifically, given $k \in \mathbb{N}$ and $(V,\mathfrak{V}) \in \Omega(E)$, we construct an extension operator for C^k-maps $(a_1,0)\times\cdots\times(a_n,0)\times V \to F$ (with $-\infty \leq a_1,\ldots,a_n < 0$) whose ℓth differential, for $0 \leq \ell \leq k$, extends continuously to $(a_1,0]\times\cdots\times(a_n,0]\times\mathfrak{V}\times(\mathbb{R}^n\times E)^\ell$. We remark that in the convenient setting (for $k=\infty$ and $V=\mathfrak{V}=E=\{0\}$) the existence of a continuous extension operator was already shown in Proposition 24.10 in [7]. The proof given there also works in Bastiani's setting, but still only for $k=\infty$ as the exponential law for smooth mappings is explicitly applied.[1]

We finally want to emphasize that our extension result can also be used to extend C^k-maps on subsets in infinite dimensions that admit a certain kind of geometry. Indeed, we have already mentioned that Example 3.10 covers the (real) pre-Hilbert unit ball. In Application 3.8 in Section 3.3, we consider subsets of Hausdorff locally convex vector spaces that are defined by a particular kind of distance function (e.g., nonzero C^k-seminorms). The (real) pre-Hilbert unit ball is an example for this, but the construction in Example 3.10 differs from the construction in Application 3.8 that gets along without explicit use of the compatibility property admitted by the extension operators.

[1] We refer to [1] for subtleties concerning the exponential law in the nonsmooth category.

A brief outline of the paper is as follows. In Section 2, we fix the notations, recall Bastiani's differential calculus, and provide some elementary facts and definitions concerning locally convex vector spaces (and maps) that we shall need in the main text. In Section 3, we state our main result, Theorem 3.1, and discuss various applications to it. Section 4 is dedicated to the proof of Theorem 3.1.

2 Preliminaries

Let $\mathsf{hlcVect}$ denote the class of Hausdorff locally convex vector spaces, and let $E \in \mathsf{hlcVect}$ be given. We denote the completion of E by $\mathrm{comp}(E) \in \mathsf{hlcVect}$. The system of continuous seminorms on E is denoted by $\mathrm{Sem}(E)$. For $\mathfrak{p} \in \mathrm{Sem}(E)$, we let $\hat{\mathfrak{p}}$ denote the continuous extension of $\mathfrak{p}$ to $\mathrm{comp}(E)$. For a subset $V \subseteq E$, we let $\mathrm{clos}(V) \subseteq E$ denote the closure of V in E. A subset $\mathcal{B} \subseteq E$ is said to be bounded if $\sup\{\mathfrak{p}(X) \mid X \in \mathcal{B}\} < \infty$ holds for each $\mathfrak{p} \in \mathrm{Sem}(E)$. Let $-\infty \leq a < b \leq \infty$ be given:

- For $a = -\infty$, we set $[a, b] := (-\infty, b]$ and $[a, b) := (-\infty, b)$.
- For $b = \infty$, we set $[a, b] := [a, \infty)$ and $(a, b] := (a, \infty)$.
- For $a = -\infty$, $b = \infty$, we set $[a, b] := (-\infty, \infty)$.

Let $k \in \mathbb{N} \cup \{\infty\}$ be given. We write $0 \leq \ell \preceq k$,

- for $k \in \mathbb{N}$ if $\mathbb{N} \ni \ell \leq k$ holds,
- for $k = \infty$ if $\ell \in \mathbb{N}$ holds.

2.1 Bastiani's differential calculus

In this section, we recall Bastiani's differential calculus, see also [3, 4, 12–15]. Let $E, F \in \mathsf{hlcVect}$ be given. A map $f \colon U \to F$, with $U \subseteq E$ open, is said to be differentiable if

$$(D_v f)(x) := \lim_{t \to 0} 1/t \cdot (f(x + t \cdot v) - f(x)) \in F$$

exists for each $x \in U$ and $v \in E$. The map f is said to be k-times differentiable for $k \geq 1$ if

$$D_{v_k,\ldots,v_1} f := D_{v_k}(D_{v_{k-1}}(\cdots(D_{v_1}(f))\cdots)) \colon U \to F$$

is defined for all $v_1, \ldots, v_k \in E$. Implicitly, this means that f is p-times differentiable for each $1 \leq p \leq k$, and we set

$$\mathrm{d}^p_x f(v_1, \ldots, v_p) \equiv \mathrm{d}^p f(x, v_1, \ldots, v_p) := D_{v_p,\ldots,v_1} f(x) \qquad \forall\, x \in U,\ v_1, \ldots, v_p \in E$$

for $p = 1, \ldots, k$. We furthermore define $\mathrm{d}f := \mathrm{d}^1 f$, as well as $\mathrm{d}_x f := \mathrm{d}^1_x f$ for each $x \in U$. The map $f \colon U \to F$ is said to be

- of class C^0 if it is continuous. In this case, we define $\mathrm{d}^0 f := f$.
- of class C^k for $k \geq 1$ if it is k-times differentiable, such that

  $$\mathrm{d}^p f \colon U \times E^p \to F, \qquad (x, v_1, \ldots, v_p) \mapsto D_{v_p,\ldots,v_1} f(x)$$

  is continuous for $p = 0, \ldots, k$. In this case, $\mathrm{d}^p_x f$ is symmetric and p-multilinear for each $x \in U$ and $p = 1, \ldots, k$, see [3].
- of class C^∞ if it is of class C^k for each $k \in \mathbb{N}$.

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Remark 2.1 Let E, F be normed spaces. We define $L^0(E,F) := F$, and let $L^\ell(E,F)$, for $\ell \geq 1$, denote the space of all continuous ℓ-multilinear maps $E^\ell \to F$ equipped with the operator topology.[2] For $k \in \mathbb{N}$ and $U \subseteq E$ non-empty open, we denote the set of all k-times Fréchet differentiable maps $U \to F$ by $\mathcal{F}C^k(U,F)$. Given $f \in \mathcal{F}C^k(U,F)$, we denote its ℓth Fréchet differential, for $0 \leq \ell \leq k$, by $D^{(\ell)} f \colon U \to L^\ell(E,F)$. We recall that $C^{k+1}(U,F) \subseteq \mathcal{F}C^k(U,F) \subseteq C^k(U,F)$ holds [13, 20], with

$$D^{(\ell)} f(x) = \mathrm{d}_x^\ell f \qquad \forall\, x \in U,\ 0 \leq \ell \leq k.$$

In particular, we have $C^\infty(U,F) = \mathcal{F}C^\infty(U,F)$.

We have the following differentiation rules [3].

Proposition 2.2 (a) *A map $f \colon E \supseteq U \to F$ is of class C^k for $k \geq 1$ if and only if df is of class C^{k-1} when considered as a map $E \times E \supseteq U \times E \to F$.*

(b) *Let $f \colon E \to F$ be linear and continuous. Then, f is smooth, with $d_x^1 f = f$ for each $x \in E$, as well as $d^p f = 0$ for each $p \geq 2$.*

(c) *Let $F_1, \ldots, F_m$ be Hausdorff locally convex vector spaces, and let $f_q \colon E \supseteq U \to F_q$ be of class C^k for $k \geq 1$ and $q = 1, \ldots, m$. Then,*

$$f := f_1 \times \cdots \times f_m \colon U \to F_1 \times \cdots \times F_m, \qquad x \mapsto (f_1(x), \ldots, f_m(x))$$

is of class C^k, with $d^p f = d^p f_1 \times \cdots \times d^p f_m$ for $p = 1, \ldots, k$.

(d) *Let $F, \bar{F}, \bar{\bar{F}} \in$ hlcVect, $1 \leq k \leq \infty$, as well as $f \colon F \supseteq U \to \bar{U} \subseteq \bar{F}$ and $\bar{f} \colon \bar{F} \supseteq \bar{U} \to \bar{\bar{U}} \subseteq \bar{\bar{F}}$ be of class C^k. Then, $\bar{f} \circ f \colon U \to \bar{\bar{F}}$ is of class C^k, with*

$$d_x(\bar{f} \circ f) = d_{f(x)} \bar{f} \circ d_x f \qquad \forall\, x \in U.$$

(e) *Let $F_1, \ldots, F_m, E \in$ hlcVect, and $f \colon F_1 \times \cdots \times F_m \supseteq U \to E$ be of class C^0. Then, f is of class C^1 if and only if for each $p = 1, \ldots, m$, the partial derivative*

$$\begin{aligned} \partial_p f \colon U \times F_p \ni &((x_1, \ldots, x_m), v_p) \\ &\mapsto \lim_{t \to 0} 1/t \cdot (f(x_1, \ldots, x_p + t \cdot v_p, \ldots, x_m) - f(x_1, \ldots, x_m)), \end{aligned}$$

exists in E and is continuous. In this case, we have

$$\begin{aligned} df((x_1, \ldots, x_m), v_1, \ldots, v_m) &= \textstyle\sum_{p=1}^m \partial_p f((x_1, \ldots, x_m), v_p), \\ &\Big(= \textstyle\sum_{p=1}^m\ df((x_1, \ldots, x_m), (0, \ldots, 0, v_p, 0, \ldots, 0))\Big), \end{aligned}$$

for each $(x_1, \ldots, x_m) \in U$, and $v_p \in F_p$ for $p = 1, \ldots, m$.

We observe the following.

Corollary 2.3 *Let $F, \bar{F}, \bar{\bar{F}} \in$ hlcVect, $1 \leq k \leq \infty$, as well as $f \colon F \supseteq U \to \bar{U} \subseteq \bar{F}$ and $\bar{f} \colon \bar{F} \supseteq \bar{U} \to \bar{\bar{U}} \subseteq \bar{\bar{F}}$ be of class C^k. Then, for $1 \leq \ell \leq k$ we have*

$$d^\ell(\bar{f} \circ f)(x, v_1, \ldots, v_\ell) = d^\ell \bar{f}(f(x), df(x, v_1), \ldots, df(x, v_\ell)) + \Lambda_f(x, v_1, \ldots, v_\ell),$$

[2] The notations here are adapted to the notations used (Appendices A.2 and A.3) in [20], where the relationships between Bastiani's differentiability concept and Fréchet differentiability are presented in detail.

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where $\Lambda_f\colon U\times F^\ell \to \bar{\bar{F}}$ *is given as a linear combination of maps of the form*

$$U\times F^\ell \ni (x,v_1,\dots,v_\ell) \mapsto d^q\bar{f}(f(x),d^{p_1}f(x,v_1,\dots,v_{p_1}),\dots,d^{p_q}f(x,v_{\ell-p_q+1},\dots,v_\ell)) \in \bar{\bar{F}},$$

such that the following conditions are fulfilled:

- *We have* $1\leq q<\ell$*, as well as* $p_1,\dots,p_q\geq 1$ *with* $p_1+\dots+p_q=\ell$*.*
- *If* $\ell\geq 2$ *holds, then we have* $p_i\geq 2$ *for some* $1\leq i\leq q$*.*

Proof For $\ell=1$, the claim is clear from (d) in Proposition 2.2. Moreover, we obtain from the differentiation rules in Proposition 2.2 that

$$\mathrm{d}^2(\bar{f}\circ f)(x,v_1,v_2)=\mathrm{d}^2\bar{f}(f(x),\mathrm{d}^1f(x,v_1),\mathrm{d}^1f(x,v_2))+\mathrm{d}^1\bar{f}(f(x),\mathrm{d}^2f(x,v_1,v_2))$$

holds, which proves the claim for $\ell=2$. The rest now follows by induction from Proposition 2.2. ■

Let us finally consider the situation, where $f\equiv\gamma\colon U\equiv I\to F$ holds for a nonempty open interval $I\subseteq\mathbb{R}$ (hence, $E\equiv\mathbb{R}$). It is then not hard to see that γ is of class C^k for $k\in\mathbb{N}_{\geq 1}\cup\{\infty\}$ if and only if $\gamma^{(p)}$, inductively defined by $\gamma^{(0)}:=\gamma$ as well as[3]

$$\gamma^{(p)}(t):=\lim_{h\to 0}\tfrac{1}{h}\cdot(\gamma^{(p-1)}(t+h)-\gamma^{(p-1)}(t)) \qquad \forall\, t\in I,\ p=1,\dots,k$$

exists and is continuous for $0\leq p\leq k$. If $D\subseteq\mathbb{R}$ is an arbitrary interval (connected, nonempty and nonsingleton), we let $C^k(D,F)$ ($k\in\mathbb{N}\cup\{\infty\}$) denote the set of all maps $\gamma\colon D\to F$, such that $\gamma=\widetilde{\gamma}|_D$ holds for some $\widetilde{\gamma}\in C^k(I,F)$ with $I\subseteq\mathbb{R}$ an open interval such that $D\subseteq I$. In this case, we set $\gamma^{(p)}:=\widetilde{\gamma}^{(p)}|_D$ for each $0\leq p\leq k$.

2.2 Locally convex vector spaces

In this section, we collect some elementary statements concerning locally convex vector spaces.

2.2.1 Product spaces and continuous maps

Given $F_1,\dots,F_n,F\in$ hlcVect, the Tychonoff topology on $E:=F_1\times\dots\times F_n$ equals the Hausdorff locally convex topology that is generated by the seminorms

$$\max[\mathfrak{q}_1,\dots,\mathfrak{q}_n]\colon E\ni(X_1,\dots,X_n)\mapsto\max(\mathfrak{q}_1(X_1),\dots,\mathfrak{q}_n(X_n)), \tag{2.1}$$

with $\mathfrak{q}_p\in\mathrm{Sem}(F_p)$ for $p=1,\dots,n$. We recall the following statements.

Lemma 2.4 *For each* $\mathfrak{q}\in\mathrm{Sem}(E)$*, there exist* $\mathfrak{q}_p\in\mathrm{Sem}(F_p)$ *for* $p=1,\dots,n$*, with* $\mathfrak{q}\leq\max[\mathfrak{q}_1,\dots,\mathfrak{q}_n]$*.*

Proof Since the seminorms (2.1) form a fundamental system, the claim is clear from Proposition 22.6 in [9], when applied to the identity id_E.[4] ■

[3] We have $\gamma^{(p)}(t)=\mathrm{d}^p_t\gamma(1,\dots,1)$ for $t\in I$ and $p=1,\dots,k$.

[4] Observe $c\cdot\max[\mathfrak{q}_1,\dots,\mathfrak{q}_n]=\max[c\cdot\mathfrak{q}_1,\dots,c\cdot\mathfrak{q}_n]$ with $c\cdot\mathfrak{q}_p\in\mathrm{Sem}(F_p)$ for $p=1,\dots,n$, for each $c>0$.

Lemma 2.5 *Let X be a topological space, and let $\Phi: X \times F_1 \times \cdots \times F_n \to F$ be continuous, such that $\Phi(x, \cdot)$ is n-multilinear for each $x \in X$. Then, for each compact $K \subseteq X$ and each $\mathfrak{p} \in \mathrm{Sem}(F)$, there exist seminorms $\mathfrak{q}_p \in \mathrm{Sem}(F_p)$ for $p = 1, \ldots, n$, as well as $O \subseteq X$ open with $K \subseteq O$, such that*

$$(\mathfrak{p} \circ \Phi)(x, X_1, \ldots, X_n) \leq \mathfrak{q}_1(X_1) \cdot \cdots \cdot \mathfrak{q}_n(X_n) \qquad \forall\, x \in O,\ X_1 \in F_1, \ldots, X_n \in F_n.$$

Proof See, e.g., Corollary 1 in [5]. ∎

2.2.2 The Riemann integral

Let $\gamma \in C^0([r, r'], F)$ be given. We denote the Riemann integral[5] of γ by $\int \gamma(s)\, \mathrm{d}s \in \mathrm{comp}(F)$, and define

$$\int_a^b \gamma(s)\, \mathrm{d}s := \int \gamma|_{[a,b]}(s)\, \mathrm{d}s \qquad \forall\, r \leq a < b \leq r'.$$

The Riemann integral is linear, with

$$\hat{\mathfrak{p}}\Big(\int_a^b \gamma(s)\, \mathrm{d}s \Big) \leq \int_a^b \mathfrak{p}(\gamma(s))\, \mathrm{d}s \qquad \forall\, \mathfrak{p} \in \mathrm{Sem}(F),\ r \leq a < b \leq r'.$$

It follows that the Riemann integral is C^0-continuous, i.e., continuous w.r.t. the seminorms

$$\mathfrak{p}_\infty(\gamma) := \sup\{\mathfrak{p}(\gamma(t)) \mid t \in [a, b]\} \qquad \forall\, \mathfrak{p} \in \mathrm{Sem}(F),\ \gamma \in C^0([a, b], F),\ a < b.$$

For $\gamma \in C^1(I, F)$ ($I \subseteq \mathbb{R}$ an open interval) and $a < b$ with $[a, b] \subseteq I$, we have by [3] that

$$\gamma(b) - \gamma(a) = \int_a^b \gamma^{(1)}(s)\, \mathrm{d}s. \tag{2.2}$$

It is furthermore not hard to see that given $\gamma \in C^0(I, F)$, then for $a < b$ with $[a, b] \subseteq I$ and $\Gamma: [a, b] \ni t \mapsto \int_a^t \gamma(s)\, \mathrm{d}s \in \mathrm{comp}(F)$, we have

$$\Gamma \in C^1([a, b], \mathrm{comp}(F)) \qquad \text{with} \qquad \Gamma^{(1)} = \gamma|_{[a,b]}. \tag{2.3}$$

2.2.3 Harmonic subsets and extensions

Let $\{0\} \neq H \in \mathrm{hlcVect}$, $U \subseteq H$ nonempty open, and $\emptyset \neq A \subseteq U$ closed in U w.r.t. the subspace topology on U. Then, A is said to be **harmonic** if for each $(x, v) \in A \times (H \backslash \{0\})$, there exists $\delta > 0$ as well as $\gamma_\pm: [0, 1) \to H$ continuous at 0 with $\gamma_\pm(0) = 0$, such that[6]

$$\big(x + \gamma_\pm((0,1)) \pm (0, \delta) \cdot v \big) \subseteq U \backslash A. \tag{2.4}$$

Example 2.6 (Harmonic Subsets)

(i) If $A \subseteq U$ is harmonic and $\emptyset \neq B \subseteq A$ closed in U, then $B \subseteq U$ is harmonic.

[5] The Riemann integral can be defined exactly as in the finite-dimensional case; namely, as a limit over Riemann sums. Details can be found, e.g., in Section 2 in [7].

[6] More precisely, this means $(x + \gamma_+((0,1)) + (0, \delta) \cdot v) \subseteq U \backslash A$ as well as $(x + \gamma_-((0,1)) - (0, \delta) \cdot v) \subseteq U \backslash A$.

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(ii) Each nonempty finite subset of U is harmonic.
Proof If $\varnothing \neq A \subseteq U$ is finite, then A is closed in U. For $x \in A$ fixed, there exists $\mathfrak{h} \in \mathrm{Sem}(H)$ with $\mathrm{B}_1(x) := \{y \in H \,|\, \mathfrak{h}(y-x) < 1\} \subseteq U$, such that $A \cap (\mathrm{B}_1(x)\backslash\{x\}) = \varnothing$ holds. For $0 \neq v \in H$ fixed, we set $\delta := \frac{1}{2\max(1,\mathfrak{h}(v))}$ and define $\gamma_\pm\colon [0,1) \ni t \mapsto 0 \in H$. Then, we have

$$\begin{aligned} x + \gamma_\pm \pm \lambda \cdot v &= x \pm \lambda \cdot v \neq x, \\ \mathfrak{h}(x - (x + \gamma_\pm \pm \lambda \cdot v)) &= \lambda \cdot \mathfrak{h}(v) < \delta \cdot \mathfrak{h}(v) < 1, \end{aligned}$$

for all $\lambda \in (0,\delta)$, hence $(x + \gamma_\pm((0,1)) \pm (0,\delta)\cdot v) \subseteq \mathrm{B}_1(x)\backslash\{x\} \subseteq U\backslash A$. ■

(iii) Let $\tilde{H} := H \times F$ with $F \in \mathsf{hlcVect}$, $\varnothing \neq W \subseteq F$ open, and $\tilde{U} := U \times W$. If $A \subseteq U$ is harmonic, then $\tilde{A} := A \times W \subseteq \tilde{U}$ is harmonic.
Proof Let $\tilde{x} \equiv (x,z) \in \tilde{A}$ and $\tilde{v} \equiv (v,u) \in \tilde{H}\backslash\{(0,0)\}$ be given.
- Let $v \neq 0$. We choose $\delta > 0$ and $\gamma_\pm$ as in (2.4). Shrinking $\delta > 0$ if necessary, we can assume $z + (-\delta,\delta)\cdot u \subseteq W$ (as W is open). We set $\tilde{\gamma}_\pm\colon [0,1) \ni t \mapsto (\gamma_\pm(t), 0) \in \tilde{H}$, and obtain
$$\tilde{x} + \tilde{\gamma}_\pm(\lambda) \pm \mu \cdot \tilde{v} = \big(x + \gamma_\pm(\lambda) \pm \mu \cdot v, z \pm \mu \cdot u\big) \in (U\backslash A) \times W = \tilde{U}\backslash\tilde{A},$$
for all $\lambda \in (0,1)$ and $\mu \in (0,\delta)$, hence $\tilde{x} + \tilde{\gamma}_\pm((0,1)) \pm (0,\delta)\cdot\tilde{v} \subseteq \tilde{U}\backslash\tilde{A}$.
- Let $v = 0$. We fix $0 \neq w \in H$, and choose $\delta > 0$ and $\gamma_\pm$ as in (2.4) for $v \equiv w$ there. Shrinking $\delta > 0$ if necessary, we can assume $z + (-\delta,\delta)\cdot u \subseteq W$ (as W is open). We set $\tilde{\gamma}_\pm\colon [0,1) \ni t \mapsto (\gamma_\pm(t) \pm t\cdot\delta\cdot w, 0) \in \tilde{H}$, and obtain (observe $\mu \cdot v = 0$ and $\lambda\cdot\delta \in (0,\delta)$ for $\lambda \in (0,1)$)
$$\tilde{x} + \tilde{\gamma}_\pm(\lambda) \pm \mu \cdot \tilde{v} = \big(x + \gamma_\pm(\lambda) \pm \lambda\cdot\delta\cdot w, z \pm \mu \cdot u\big) \in (U\backslash A) \times W = \tilde{U}\backslash\tilde{A},$$
for all $\lambda \in (0,1)$ and $\mu \in (0,\delta)$. Hence, we have $\tilde{x} + \tilde{\gamma}_\pm((0,1)) \pm (0,\delta)\cdot\tilde{v} \subseteq \tilde{U}\backslash\tilde{A}$. ■

(iv) Let $H = \mathbb{R} \times E$ for $E \in \mathsf{hlcVect}$, $p \in \mathbb{R}$, as well as $U = \mathbb{R} \times V$ with $\varnothing \neq V \subseteq E$ open. Then, $\{p\} \times V \subseteq U$ is harmonic.

Proof $A := \{p\} \subseteq \mathbb{R}$ is harmonic by (ii). The claim thus follows from (iii) (with $H, U \equiv \mathbb{R}$, $F \equiv E$ and $W \equiv V$). ■

(v) If $p \in (0,\infty)$ and $0 \neq \mathfrak{h} \in \mathrm{Sem}(H)$, then $U \cap \mathfrak{h}^{-1}(p) \subseteq U$ is harmonic.

Proof $B := \mathfrak{h}^{-1}(p)$ is closed in H as $\mathfrak{h}$ is continuous, as well as nonempty as $\mathfrak{h} \neq 0$. Hence, $A := U \cap B$ is closed in U. For $z \in B$ and $w \in H$, the reverse triangle inequality yields

(2.5)
$$|\mathfrak{h}(z - \lambda\cdot(z \pm w)) - \mathfrak{h}(\mp\lambda\cdot w)| \leq \mathfrak{h}(z - \lambda\cdot z) = (1-\lambda)\cdot\mathfrak{h}(z) = (1-\lambda)\cdot p,$$

for all $\lambda \in (0,1)$. Let now $x \in A$ and $0 \neq v \in H$ be given:
- Let $\mathfrak{h}(v) = 0$. Then, (2.5) applied to $z = x$ and $w = \mu\cdot v$ for $\mu \in (0,\infty)$ yields
$$\mathfrak{h}(x - \lambda\cdot(x \pm \mu\cdot v)) \leq (1-\lambda)\cdot p < p \qquad \forall\, \lambda \in (0,1),\ \mu \in (0,\infty),$$

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hence $(x - (0,1) \cdot x \pm (0,1) \cdot v) \subseteq H \backslash B$. Since U is open with $x \in U$, there exists $\varepsilon > 0$ with $(x - (0,\varepsilon) \cdot x \pm (0,\varepsilon) \cdot v) \subseteq U$, hence $(x - (0,\varepsilon) \cdot x \pm (0,\varepsilon) \cdot v) \subseteq U \backslash B = U \backslash A$. The condition (2.4) thus holds for $\delta := \varepsilon$ and $\gamma_\pm \equiv \gamma : [0,1) \ni t \mapsto -(t \cdot \varepsilon) \cdot x \in H$.

- Let $\mathfrak{h}(v) > 0$. Since $\mathfrak{h}(x) = p > 0$ holds, there exists (by continuity) $0 < \sigma < \min\left(1, \frac{p}{\mathfrak{h}(v)}\right)$ with

$$\lambda \cdot \mu \cdot \mathfrak{h}(v) < \mathfrak{h}(x - \lambda \cdot (x \pm \mu \cdot v)) \qquad \forall\, 0 < \lambda, \mu < \sigma.$$

Then, given $\lambda, \mu \in (0, \sigma)$, (2.5) applied to $z = x$ and $w = \mu \cdot v$ yields

$$\mathfrak{h}(x - \lambda \cdot (x \pm \mu \cdot v)) \le \lambda \cdot \mu \cdot \mathfrak{h}(v) + (1 - \lambda) \cdot p = p - \lambda \cdot (p - \mu \cdot \mathfrak{h}(v)) < p.$$

We obtain $(x - (0,\sigma^2) \cdot x \pm (0,\sigma^2) \cdot v) \subseteq H \backslash B$. Since U is open with $x \in U$, there exists $0 < \varepsilon < \sigma^2$ with $(x - (0,\varepsilon) \cdot x \pm (0,\varepsilon) \cdot v) \subseteq U$, hence $(x - (0,\varepsilon) \cdot x \pm (0,\varepsilon) \cdot v) \subseteq U \backslash B = U \backslash A$. The condition (2.4) thus holds for $\delta := \varepsilon$ and $\gamma_\pm \equiv \gamma : [0,1) \ni t \mapsto -(t \cdot \varepsilon) \cdot x \in H$. ■

(vi) If $0 \ne \mathfrak{h} \in \mathrm{Sem}(H)$, then $U \cap \mathfrak{h}^{-1}(0) \subseteq U$ is harmonic.

Proof $B := \mathfrak{h}^{-1}(0)$ is closed in H as $\mathfrak{h}$ is continuous. Hence, $A := U \cap B$ is closed in U. The reverse triangle inequality yields (observe $|\mathfrak{h}(z + w) - \mathfrak{h}(w)| \le \mathfrak{h}(z)$ for all $z, w \in H$)

$$\mathfrak{h}(z + w) = \mathfrak{h}(w) \qquad \forall\, z \in B,\ w \in H. \tag{2.6}$$

Since $\mathfrak{h} \ne 0$ holds, there exists some $u \in H \backslash B$. Let now $x \in A$ and $0 \ne v \in H$ be given:

- Let $\mathfrak{h}(v) > 0$. Then, $(x \pm (0,\infty) \cdot v) \subseteq H \backslash B$ holds, by (2.6) applied to $z = x$ and $w = \pm\, \mu \cdot v$ for $\mu \in (0,\infty)$. Since U is open with $x \in U$, there exists $\varepsilon > 0$ with $(x \pm (0,\varepsilon) \cdot v) \subseteq U$, hence $(x \pm (0,\varepsilon) \cdot v) \subseteq U \backslash B = U \backslash A$. Condition (2.4) thus holds for $\delta := \varepsilon$ and $\gamma_\pm : [0,1) \ni t \mapsto 0 \in H$.
- Let $\mathfrak{h}(v) = 0$. We obtain for $t, \mu \in (0,\infty)$ that

$$\begin{aligned} \mathfrak{h}(x + t \cdot (\pm v + \mu \cdot u)) &\overset{(2.6)}{=} \mathfrak{h}(t \cdot (\pm v + \mu \cdot u)) \\ &= t \cdot \mathfrak{h}(\pm v + \mu \cdot u) \overset{(2.6)}{=} t \cdot \mathfrak{h}(\mu \cdot u) = t \cdot \mu \cdot \mathfrak{h}(u) > 0 \end{aligned}$$

holds, hence $(x + (0,\infty) \cdot u \pm (0,\infty) \cdot v) \subseteq H \backslash B$. Since U is open with $x \in U$, there exists $\varepsilon > 0$ with $(x + (0,\varepsilon) \cdot u \pm (0,\varepsilon) \cdot v) \subseteq U \backslash B = U \backslash A$, so that (2.4) holds for $\delta := \varepsilon$ and $\gamma_\pm \equiv \gamma : [0,1) \ni t \mapsto (t \cdot \varepsilon) \cdot u \in H$. ■

Notably, the statement in (iv) also follows from (i), (v), and (vi):

Proof Let $\mathfrak{h} : H \ni (x, v) \mapsto |x| \in [0,\infty)$ for $H = \mathbb{R} \times E$. Then, $0 \ne \mathfrak{h} \in \mathrm{Sem}(H)$ holds, with $\mathfrak{h}^{-1}(p) = \{-p, p\} \times E$. Hence, we have $A := \{-p, p\} \times V = U \cap \mathfrak{h}^{-1}(p)$ for $U = \mathbb{R} \times V$, so that (v) ($p \ne 0$) and (vi) ($p = 0$) show that $A \subseteq U$ is harmonic. By (i), then also $B := \{p\} \times V \subseteq A$ is harmonic, as nonempty and closed in U. ■

We have the following statement.

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Lemma 2.7 *Let $H, F \in \mathsf{hlcVect}$, $U \subseteq H$ nonempty open, $A \subseteq U$ harmonic, and $S \subseteq H$ a subset with $U \subseteq S$. Let $f \in C^k(U \backslash A, F)$ for $k \in \mathbb{N} \cup \{\infty\}$ be given. For each $0 \leq \ell \leq k$, let $\Phi^\ell \colon S \times H^\ell \to F$ be continuous with*

$$\Phi^\ell|_{(U\backslash A)\times H^\ell} = d^\ell f.$$

Then, we have $\tilde{f} := \Phi^0|_U \in C^k(U, F)$, with $d^\ell \tilde{f} = \Phi^\ell|_{U\times H^\ell}$ for all $0 \leq \ell \leq k$.

Proof By definition, we have $\tilde{f} \in C^0(U, F)$ with $\mathrm{d}^0 \tilde{f} = \tilde{f} = \Phi^0|_U$. We thus can assume that there exists $0 \leq q < k$, such that $\tilde{f}$ is of class C^q with $\mathrm{d}^\ell \tilde{f} = \Phi^\ell|_{U\times H^\ell}$ for all $0 \leq \ell \leq q$. The claim then follows by induction once we have shown that[7]

(2.7)
$$\lim_{h\to 0} \tfrac{1}{h} \cdot (\Phi^q(x + h\cdot v, v_1, \ldots, v_q) - \Phi^q(x, v_1, \ldots, v_q)) = \Phi^{q+1}(x, v_1, \ldots, v_q, v)$$

holds for all $x \in A$ and $v_1, \ldots, v_q, v \in H$. To show (2.7), we choose $\delta > 0$ and $\gamma_\pm \colon [0,1) \to H$ as in (2.4), and consider the maps

$$\alpha_\pm \colon [0,1) \times [0,\delta] \ni (\lambda, s) \mapsto \Phi^{q+1}(x + \gamma_\pm(\lambda) \pm s\cdot v, v_1, \ldots, v_q, \pm v) \in F.$$

- By assumption, we have
$$\alpha_\pm(\lambda, s) = \mathrm{d}^{q+1} f(x + \gamma_\pm(\lambda) \pm s\cdot v, v_1, \ldots, v_q, \pm v) \qquad \forall\, \lambda \in (0,1),\ s \in (0,\delta).$$
- By compactness and continuity, we have $\lim_{\lambda\to 0} \mathfrak{p}_\infty(\alpha_\pm(\lambda,\cdot) - \alpha_\pm(0,\cdot)) = 0$ for each $\mathfrak{p} \in \mathsf{Sem}(F)$.

Since the Riemann integral is C^0-continuous (used in the second step), and since Φ^q is continuous (used in the last step), we obtain for $0 < h < \delta$ that (in the fourth step, we apply (2.2) as well as Proposition 2.2.(d))

$$\begin{aligned}
&\pm \int_0^h \Phi^{q+1}(x \pm s\cdot v, v_1, \ldots, v_q, v)\, \mathrm{d}s \\
&\quad = \int_0^h \alpha_\pm(0, s)\, \mathrm{d}s \\
&\quad = \lim_{0<\lambda\to 0} \int_0^h \alpha_\pm(\lambda, s)\, \mathrm{d}s \\
&\quad = \lim_{0<\lambda\to 0} \int_0^h \mathrm{d}^{q+1} f(x + \gamma_\pm(\lambda) \pm s\cdot v, v_1, \ldots, v_q, \pm v)\, \mathrm{d}s \\
&\quad = \lim_{0<\lambda\to 0} \mathrm{d}^q f(x + \gamma_\pm(\lambda) \pm h\cdot v, v_1, \ldots, v_q) \\
&\qquad - \lim_{0<\lambda\to 0} \mathrm{d}^q f(x + \gamma_\pm(\lambda), v_1, \ldots, v_q) \\
&\quad = \lim_{0<\lambda\to 0} \Phi^q(x + \gamma_\pm(\lambda) \pm h\cdot v, v_1, \ldots, v_q) \\
&\qquad - \lim_{0<\lambda\to 0} \Phi^q(x + \gamma_\pm(\lambda), v_1, \ldots, v_q) \\
&\quad = \Phi^q(x \pm h\cdot v, v_1, \ldots, v_q) - \Phi^q(x, v_1, \ldots, v_q)
\end{aligned}$$

holds. Together with (2.3) this implies (2.7). ∎

2.3 Particular mapping spaces

Let $H, F \in \mathsf{hlcVect}$ and $k \in \mathbb{N} \cup \{\infty\}$ be given. Let $\Omega(H)$ denote the set of all pairs $(U, \mathfrak{U})$ that consist of a nonempty open subset $U \subseteq H$, and a subset $\mathfrak{U} \subseteq \mathsf{clos}(U)$ with

[7] Due to the assumptions, (2.7) holds for all $x \in U\backslash A$.

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$U \subseteq \mathfrak{U}$. Let $\mathcal{C}^k_{\mathfrak{U}}(U,F)$ denote the set of all $f \in C^k(U,F)$, such that $\mathrm{d}^\ell f$ extends for $0 \leq \ell \leq k$ to a continuous map $\mathrm{Ext}(f,\ell)\colon \mathfrak{U} \times H^\ell \to F$.

Remark 2.8 Let $1 \leq \ell \leq k$, $(U,\mathfrak{U}) \in \Omega(H)$, and $f \in \mathcal{C}^k_{\mathfrak{U}}(U,F)$ be given. By continuity, the map $\mathrm{Ext}(f,\ell)(z,\cdot)\colon H^\ell \to F$ is necessarily ℓ-multilinear and symmetric for each fixed $z \in \mathfrak{U}$. Thus, given $\mathfrak{p} \in \mathsf{Sem}(F)$ and $\mathrm{K} \subseteq \mathfrak{U}$ compact, Lemma 2.5 provides $\mathfrak{h} \in \mathsf{Sem}(H)$ as well as $O \subseteq H$ open with $\mathrm{K} \subseteq O$, such that

$$(\mathfrak{p} \circ \mathrm{Ext}(f,\ell))(z,\underline{w}) \leq \mathfrak{h}(w_1)\cdot\dots\cdot\mathfrak{h}(w_\ell)$$

holds for all $z \in O \cap \mathfrak{U}$ and $\underline{w} = (w_1,\dots,w_\ell) \in H^\ell$. ∎

We have the following corollary to Lemma 2.7.

Corollary 2.9 *Let $H,F \in \mathsf{hlcVect}$, $(U,\mathfrak{U}) \in \Omega(H)$, $A \subseteq U$ harmonic, and $f \in C^k(U\backslash A,F)$ for $k \in \mathbb{N} \cup \{\infty\}$ be given. For each $0 \leq \ell \leq k$, let $\Phi^\ell\colon \mathfrak{U} \times H^\ell \to F$ be continuous with*

$$\Phi^\ell|_{(U\backslash A)\times H^\ell} = d^\ell f.$$

Then, we have $\tilde{f} := \Phi^0|_U \in \mathcal{C}^k_{\mathfrak{U}}(U,F)$, with $Ext(\tilde{f},\ell) = \Phi^\ell$ for all $0 \leq \ell \leq k$.

Proof Set $S \equiv \mathfrak{U}$ in Lemma 2.7. ∎

Corollary 2.3 provides the following statement.

Lemma 2.10 *Let $H,\bar{H},F \in \mathsf{hlcVect}$, $O \subseteq H$, $\bar{O} \subseteq \bar{H}$ both nonempty open, and $\psi \in C^k(O,\bar{O})$ be fixed. Let $(U,\mathfrak{U}) \in \Omega(H)$ with $\mathfrak{U} \subseteq O$ be given, as well as $(\bar{U},\bar{\mathfrak{U}}) \in \Omega(\bar{H})$ with $\psi(U) \subseteq \bar{U}$ and $\psi(\mathfrak{U}) \subseteq \bar{\mathfrak{U}}$. Then, for $f \in \mathcal{C}^k_{\bar{\mathfrak{U}}}(\bar{U},F)$ we have $f\circ\psi|_U \in \mathcal{C}^k_{\mathfrak{U}}(U,F)$. Specifically, the following assertions hold:*

(i) *We have $Ext(f\circ\psi|_U,0) = Ext(f,0)\circ\psi|_{\mathfrak{U}}$.*
(ii) *For $1 \leq \ell \leq k$, we have*

$$\begin{aligned} &Ext(f\circ\psi|_U,\ell)(x,v_1,\dots,v_\ell)\\ &\quad = Ext(f,\ell)(\psi(x),d\psi(x,v_1),\dots,d\psi(x,v_\ell)) + \Lambda_\psi(x,v_1,\dots,v_\ell),\end{aligned}$$

where $\Lambda_\psi\colon U \times H^\ell \to F$ is given as a linear combination of maps of the form

$$\begin{aligned} &(x,v_1,\dots,v_\ell)\\ &\quad \mapsto Ext(f,q)(\psi(x),d^{p_1}\psi(x,v_1,\dots,v_{p_1}),\dots,d^{p_q}\psi(x,v_{\ell-p_q+1},\dots,v_\ell)),\end{aligned}$$

such that the following conditions are fulfilled:

- *We have $1 \leq q < \ell$, as well as $p_1,\dots,p_q \geq 1$ with $p_1+\dots+p_q = \ell$.*
- *If $\ell \geq 2$ holds, then we have $p_i \geq 2$ for some $1 \leq i \leq q$.*

Proof Part (i) is clear from the continuity properties of the involved maps. Now, we have $f\circ\psi \in C^k(U,F)$, as ψ is of class C^k. Moreover, ψ is defined on $\mathfrak{U} \subseteq O$ with $\psi(\mathfrak{U}) \subseteq \bar{\mathfrak{U}}$. Part (ii) is thus clear from Corollary 2.3, as well as from continuity of the occurring differentials and their extensions. ∎

For $\mathrm{K} \subseteq \mathfrak{U}$ compact, $\mathscr{B} \subseteq H$ bounded, $\mathfrak{p} \in \mathsf{Sem}(F)$, $f \in \mathcal{C}^k_{\mathfrak{U}}(U,F)$, we define

$$\mathfrak{p}^0_{\mathrm{K}} \equiv \mathfrak{p}[0]_{\mathrm{K}\times\mathscr{B}}(f) := \sup\{\mathfrak{p}(\mathrm{Ext}(f,0)(z)) \mid z \in \mathrm{K}\},$$

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$$\mathfrak{p}[\ell]_{\mathrm{K}\times\mathscr{B}}(f) := \sup\{\mathfrak{p}(\mathrm{Ext}(f,\ell)(z,\underline{w})) \,|\, z \in \mathrm{K},\ \underline{w} \in \mathscr{B}^\ell\} \qquad \forall\, 1 \le \ell \le k,$$

$$\mathfrak{p}^{\mathrm{s}}_{\mathrm{K}\times\mathscr{B}}(f) := \max\left(0 \le \ell \le \mathrm{s} \,\middle|\, \mathfrak{p}[\ell]_{\mathrm{K}\times\mathscr{B}}(f)\right) \qquad \forall\, 0 \le \mathrm{s} \le k. \tag{2.8}$$

Finally assume $H = P \times E$ with $P, E \in \mathsf{hlcVect}$. Then,

$$(U, \mathfrak{U}) := (W \times V, \mathfrak{W} \times \mathfrak{V}) \in \Omega(H) \quad \text{holds for all}$$
$$(W, \mathfrak{W}) \in \Omega(P) \quad \text{and} \quad (V, \mathfrak{V}) \in \Omega(E).$$

In the following, we will rather denote

$$\mathcal{C}^k_{\mathfrak{V}}(W \times V, F) := \mathcal{C}^k_{\mathfrak{U}}(W \times V, F),$$

as it will be clear from the context, which $\mathfrak{W} \subseteq \mathrm{clos}(W)$ has to be assigned to some given $W \subseteq P$.

3 Statement of the results

In this section, we state our main result Theorem 3.1, and discuss several applications. Theorem 3.1 is proven in Section 4.

3.1 Statement of the main result

Let $F \in \mathsf{hlcVect}$ and $k \in \mathbb{N} \cup \{\infty\}$ be fixed. For each $E \in \mathsf{hlcVect}$, we set $H[E] := \mathbb{R} \times E$, and define[8]

$$\mathcal{C}^k_{\mathfrak{V}}((a,b) \times V, F) := \mathcal{C}^k_{(a,b]\times\mathfrak{V}}((a,b) \times V, F) \qquad \forall\ -\infty \le a < b \le \infty, \tag{3.1}$$

for each $(V, \mathfrak{V}) \in \Omega(E)$. For a bounded subset $\mathcal{B} \subseteq E$, we set

$$\mathscr{B}(\mathcal{B}) := \{(1,0)\} \cup (0 \times \mathcal{B}) \subseteq H[E]. \tag{3.2}$$

Let $R \subseteq \mathbb{R}$ be a subset, and $\mathcal{W} \subseteq E$ a linear subset.

- For each $x \in E$ and $\ell \in \mathbb{N}$, we define
$$\mathcal{W}(R, x, \ell)\colon R \times (\mathbb{R} \times \mathcal{W})^\ell \hookrightarrow H[E] \times H[E]^\ell, \qquad (t, \underline{w}) \mapsto ((t, x), \underline{w}),$$
hence $\mathcal{W}(R, x, 0)\colon R \ni t \mapsto (t, x) \in H[E]$.
- Given $\bar{E} \in \mathsf{hlcVect}$, $\bar{x} \in \bar{E}$, $\ell \in \mathbb{N}$, and a linear map $\Upsilon\colon \mathcal{W} \to \bar{E}$, we define
$$\mathcal{W}_\Upsilon(R, \bar{x}, \ell)\colon R \times (\mathbb{R} \times \mathcal{W})^\ell \hookrightarrow H[\bar{E}] \times H[\bar{E}]^\ell,$$
$$(t, \underline{w}) \mapsto ((t, \bar{x}), (\mathrm{id}_{\mathbb{R}} \times \Upsilon)^\ell(\underline{w})),$$
hence $\mathcal{W}_\Upsilon(R, \bar{x}, 0)\colon R \ni t \mapsto (t, \bar{x}) \in H[\bar{E}]$.

Our main result states the following.

Theorem 3.1 *Let $-\infty \le a < \tau < b < \infty$ be fixed. There exist linear (extension) maps*

$$\mathcal{E}_{a,\tau,b}(E, V, \mathfrak{V})\colon \mathcal{C}^k_{\mathfrak{V}}((a,b) \times V, F) \to \mathcal{C}^k_{\mathfrak{V}}((a,\infty) \times V, F),$$

for $E \in \mathsf{hlcVect}$ and $(V, \mathfrak{V}) \in \Omega(E)$, such that the following conditions are fulfilled:

[8] Observe that, according to our conventions concerning intervals, we have $(a,b] = (a,b)$ if $b = \infty$ holds.

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(1) *For* $E \in \mathsf{hlcVect}$, $(V, \mathfrak{V}) \in \Omega(E)$, $f \in \mathcal{C}^k_{\mathfrak{V}}((a,b) \times V, F)$, $0 \le \ell \le k$, *we have*

$$\begin{aligned} Ext(\mathcal{E}_{a,\tau,b}(E,V,\mathfrak{V})(f),\ell)|_{(a,b]\times\mathfrak{V}\times H[E]^\ell} &= Ext(f,\ell),\\ Ext(\mathcal{E}_{a,\tau,b}(E,V,\mathfrak{V})(f),\ell)|_{[2b-\tau,\infty)\times\mathfrak{V}\times H[E]^\ell} &= 0. \end{aligned}$$

(2) *There exist constants* $\{C_s\}_{0\le s\le k} \subseteq [1,\infty)$, *such that the following assertions hold for each* $E \in \mathsf{hlcVect}$, $(V,\mathfrak{V}) \in \Omega(E)$, $t \in (b,\infty)$, $x \in \mathfrak{V}$, $\mathfrak{p} \in \mathsf{Sem}(F)$, *and* $f \in \mathcal{C}^k_{\mathfrak{V}}((a,b)\times V, F)$:
- *We have* $\mathfrak{p}(Ext(\mathcal{E}_{a,\tau,b}(E,V,\mathfrak{V})(f),0)(t,x)) \le C_0 \cdot \mathfrak{p}^0_{[\tau,b]\times\{x\}}(f)$.
- *For* $1 \le \ell \le s \le k$, $\mathcal{B} \subseteq E$ *bounded, and* $\underline{w} = ((\lambda_1, X_1), \ldots, (\lambda_\ell, X_\ell)) \in (\mathbb{R}\times\mathcal{B})^\ell$ *we have*

$$\begin{aligned} &\mathfrak{p}(Ext(\mathcal{E}_{a,\tau,b}(E,V,\mathfrak{V})(f),\ell)((t,x),\underline{w}))\\ &\quad \le C_s \cdot \max(1, |\lambda_1|, \ldots, |\lambda_\ell|)^\ell \cdot \mathfrak{p}^s_{[\tau,b]\times\{x\}\times\mathscr{B}(\mathcal{B})}(f). \end{aligned}$$

(3) *Let* $E, \bar{E} \in \mathsf{hlcVect}$, $\mathcal{W} \subseteq E$ *a linear subspace,* $\Upsilon\colon \mathcal{W} \to \bar{E}$ *a linear map,* $(V,\mathfrak{V}) \in \Omega(E)$, $(\bar{V}, \bar{\mathfrak{V}}) \in \Omega(\bar{E})$, *as well as*

$$\begin{gathered} f \in \mathcal{C}^k_{\mathfrak{V}}((a,b)\times V, F), \qquad \bar{f} \in \mathcal{C}^k_{\bar{\mathfrak{V}}}((a,b)\times \bar{V}, F),\\ x \in \mathfrak{V}, \qquad \bar{x} \in \bar{\mathfrak{V}}, \qquad 0 \le s \le k. \end{gathered}$$

Then, the first line implies the second line:

$$\begin{aligned} &Ext(f,\ell) \circ \mathcal{W}([\tau,b],x,\ell)\\ &\quad = Ext(\bar{f},\ell) \circ \mathcal{W}_\Upsilon([\tau,b],\bar{x},\ell) \qquad \forall\, 0 \le \ell \le s,\\ &\quad\ Ext(\mathcal{E}_{a,\tau,b}(E,V,\mathfrak{V})(f),s) \circ \mathcal{W}([\tau,\infty),x,s)\\ &\quad = Ext(\mathcal{E}_{a,\tau,b}(\bar{E},\bar{V},\bar{\mathfrak{V}})(\bar{f}),s) \circ \mathcal{W}_\Upsilon([\tau,\infty),\bar{x},s). \end{aligned}$$

Remark 3.2 The extension operator in Theorem 3.1 and the constants $\{C_s\}_{0\le s\le k}$ in Part (2), only depend on the choice of some fixed $\rho \in C^\infty(\mathbb{R},\mathbb{R})$ with

$$|\rho| \le 1, \qquad \rho|_{(-\infty,\tau]} = 0, \qquad \rho|_{[v,b]} = 1,$$

for some $\tau < v < b$. Specifically, see (4.1) for the case $a = -\infty$ and $b = 0$ as well as (4.3) for an ad hoc definition of the extension $\tilde{f} \in \mathcal{C}^k_{\mathfrak{V}}((-\infty,\infty)\times V, F)$ of some given $f \in \mathcal{C}^k_{\mathfrak{V}}((-\infty,0)\times V,F)$. See also (4.27) and (4.29) for the definition of the constants $\{C_s\}_{0\le s\le k}$ via the constants (4.2), i.e.,

$$M_p = \sup\left\{ \left|\rho^{(j)}(t)\right| \;\middle|\; t \in [\tau,0],\ 0 \le j \le p \right\} \qquad \forall\, p \in \mathbb{N}.$$

Remark 3.3 Let E, F be normed spaces, and recall the definitions made in Remark 2.1. Given $(V,\mathfrak{V}) \in \Omega(E)$, $a < b$ and $k \in \mathbb{N}$, let $\mathcal{F}\mathcal{C}^k_{\mathfrak{V}}((a,b)\times V,F)$ denote the set of all $f \in \mathcal{F}C^k((a,b)\times V,F)$, such that $D^{(\ell)}f$ extends for $0 \le \ell \le k$ to a continuous map $\mathcal{F}\mathrm{Ext}(f,\ell)\colon (a,b]\times\mathfrak{V} \to L^\ell(E,F)$. Seeley already mentioned in [17] that his construction also works for smooth $\mathbb{R}$-valued functions defined on half Banach spaces. Expectably, the same holds true for the construction made in Section 4, then leading to extension operators

$$\mathcal{F}\mathcal{E}_{a,\tau,b}(E,V,\mathfrak{V})\colon \mathcal{F}\mathcal{C}^k_{\mathfrak{V}}((a,b)\times V,F) \to \mathcal{F}\mathcal{C}^k_{\mathfrak{V}}((a,\infty)\times V,F),$$

for $a < \tau < b$, $(V, \mathfrak{V}) \in \Omega(E)$, $k \in \mathbb{N}$ that admit properties analogous to that in Theorem 3.1. We will not provide the details in this paper, but mention that Theorem 3.1 together with Remark 2.1 already provides the extension operators[9]

$$\mathcal{E}_{a,\tau,b}(E, V, V)|_{\mathcal{F}\mathcal{C}^{\infty}_{V}((a,b)\times V,F)} : \mathcal{F}\mathcal{C}^{\infty}_{V}((a,b) \times V, F) \to C^{\infty}((a,\infty) \times V, F) = \mathcal{F}C^{\infty}((a,\infty) \times V, F)$$

for $V \subseteq E$ non-empty open and $a < \tau < b$. ■

Remark 3.4 The second point in Theorem 3.1 shows that the extension operators constructed admit considerable continuity properties. Seeley already mentioned in [17] that his extension operator is continuous in many functional topologies. Expectably, the same holds true for their infinite-dimensional counterparts. However, it would go far beyond the scope of this article to investigate all possible continuity properties of the extension operators provided here—they have to be extracted on demand from the explicit construction performed in Section 4. At this point, we only want to emphasize the following:

- The second estimate in Theorem 3.1.(2) can be sharpened if $\lambda_j = 0$ holds for $j = 1, \ldots, \ell$. Specifically, on the right side of this estimate, the set $\mathscr{B}(\mathcal{B})$ then can just be replaced by $\{0\} \times \mathcal{B}$.
- Let $0 \le s \le k$, $x \in \mathfrak{V}$, $f \in \mathcal{C}^k_{\mathfrak{V}}((a,b) \times V, F)$ be given. Then, Theorem 3.1.(2) shows

$$\begin{aligned} & \mathrm{Ext}(f,\ell)|_{[\tau,b]\times\{x\}\times H[E]^{\ell}} = 0 \qquad \forall\, 0 \le \ell \le s, \\ \Longrightarrow \quad & \mathrm{Ext}(\mathcal{E}_{a,\tau,b}(E,V,\mathfrak{V})(f),\ell)|_{[\tau,\infty)\times\{x\}\times H[E]^{\ell}} = 0 \qquad \forall\, 0 \le \ell \le s. \end{aligned}$$

- Let $f, g \in \mathcal{C}^k_{\mathfrak{V}}((a,b) \times V, F)$ be given, such that

$$\mathrm{C} := \mathrm{clos}(\{z \in (a,b) \times \mathfrak{V} \mid \mathrm{Ext}(f,0)(z) \neq \mathrm{Ext}(g,0)(z)\})$$

  is compact. Then, $\mathrm{C} \subseteq [c,b] \times \mathrm{K}$ holds for certain $-\infty < c \le b$ as well as $\mathrm{K} \subseteq \mathfrak{V}$ compact. Then, $\widetilde{\mathrm{C}} := \mathrm{C} \cup ([b, 2b - \tau] \times \mathrm{K})$ is compact, and the parts (1) and (3) of Theorem 3.1 imply

$$\mathrm{clos}(\{z \in (a,b) \times V \mid \mathrm{Ext}(\mathcal{E}_{a,\tau,b}(E,V,\mathfrak{V})(f),0)(z) \neq \mathrm{Ext}(\mathcal{E}_{a,\tau,b}(E,V,\mathfrak{V})(g),0)(z)\}) \subseteq \widetilde{\mathrm{C}}.$$

  This might be of relevance, e.g., in the context of spaces of smooth mappings $f: M \to N$ between manifolds M, N (N possibly infinite-dimensional), where the $\mathscr{D}$-topology [10] (called very strong topology in [6]) is refined to the $\mathscr{F}\mathscr{D}$-topology [11] (called fine very strong topology in [6]) by additionally considering the classes defined by the equivalence relation

$$f \sim g \qquad \Longleftrightarrow \qquad \mathrm{clos}(\{x \in M \mid f(x) \neq g(x)\}) \subseteq M \quad \text{is compact}$$

  with $f, g \in C^{\infty}(M, N)$. ■

We close this section with the following summarizing corollary to Lemma 2.4 and Remark 2.8 that we shall need for our estimates in Section 4.3.

[9] It is straightforward from Remark 2.1 that $\mathcal{F}\mathcal{C}^{\infty}_{\mathfrak{V}}((a,b) \times V, F) \subseteq \mathcal{C}^{\infty}_{\mathfrak{V}}((a,b) \times V, F)$ holds for each $(V, \mathfrak{V}) \in \Omega(E)$, i.e., in particular for $\mathfrak{V} = V$.

Corollary 3.5 *Let $E \in \mathsf{hlcVect}$, $(V,\mathfrak{V}) \in \Omega(E)$, $-\infty \leq a < c < d \leq b < \infty$, $f \in \mathcal{C}^k_{\mathfrak{V}}((a,b)\times V, F)$, $\mathfrak{p} \in \mathsf{Sem}(F)$, $1 \leq \ell \leq k$, and $K \subseteq \mathfrak{V}$ be compact. There exist $\widetilde{C}_\ell \geq 1$, $\mathfrak{q} \in \mathsf{Sem}(E)$, and $U \subseteq E$ open with $K \subseteq U$, such that*[10]

$$(\mathfrak{p} \circ \mathit{Ext}(f,\ell))(z,\underline{w}) \leq \widetilde{C}_\ell \cdot \max[|\cdot|,\mathfrak{q}](w_1)\cdot\dots\cdot\max[|\cdot|,\mathfrak{q}](w_\ell)$$

holds for all $z \in [c,d]\times(U\cap\mathfrak{V})$ and $\underline{w} = (w_1,\dots,w_\ell) \in H[E]^\ell$.

Proof According to[11] Lemma 2.4 and Remark 2.8, there exist $\widetilde{C}_\ell \geq 1$, $\mathfrak{q} \in \mathsf{Sem}(E)$, as well as $O \subseteq H[E]$ open with $[c,d]\times \mathrm{K} \subseteq O$, such that

$$(\mathfrak{p} \circ \mathrm{Ext}(f,\ell))(z,\underline{w}) \leq \widetilde{C}_\ell \cdot \max[|\cdot|,\mathfrak{q}](w_1)\cdot\dots\cdot\max[|\cdot|,\mathfrak{q}](w_\ell) \tag{3.3}$$

holds for all $z \in O \cap ((a,b]\times\mathfrak{V})$ and $\underline{w} = (w_1,\dots,w_\ell) \in H[E]^\ell$. By compactness, there exists $U \subseteq E$ open with $\mathrm{K} \subseteq U$, such that $[c,d]\times U \subseteq O$ holds. Then, (3.3) holds for each $z \in [c,d]\times(U\cap\mathfrak{V})$ and $\underline{w} = (w_1,\dots,w_\ell) \in H[E]^\ell$, which proves the claim. ■

3.2 Multiple variables

Let $F \in \mathsf{hlcVect}$ and $k \in \mathbb{N}\cup\{\infty\}$ be fixed. For $n \geq 1$ and $E \in \mathsf{hlcVect}$, we define $H[E,n] := \mathbb{R}^n \times E$. Given $\underline{a} = (a_1,\dots,a_n)$, $\underline{\tau} = (\tau_1,\dots,\tau_n)$, $\underline{b} = (b_1,\dots,b_n)$ with $-\infty \leq a_i < \tau_i < b_i \leq \infty$ for $i = 1,\dots,n$, we set

$$\begin{aligned} \mathcal{Q}(\underline{a},\underline{b}) &:= (a_1,b_1)\times\cdots\times(a_n,b_n) \\ \text{and}\quad \breve{\mathcal{Q}}(\underline{a},\underline{b}) &:= (a_1,b_1]\times\cdots\times(a_n,b_n]. \end{aligned}$$

If $b_1,\dots,b_n = \infty$ holds, we also denote $\underline{b} = \underline{\infty}$, and observe that then $\mathcal{Q}(\underline{a},\underline{b}) = \breve{\mathcal{Q}}(\underline{a},\underline{b})$ holds according to our conventions concerning intervals. For each $(V,\mathfrak{V}) \in \Omega(E)$, we set

$$\mathcal{C}^k_{\mathfrak{V}}(\mathcal{Q}(\underline{a},\underline{b})\times V, F) := \mathcal{C}^k_{\breve{\mathcal{Q}}(\underline{a},\underline{b})\times\mathfrak{V}}(\mathcal{Q}(\underline{a},\underline{b})\times V, F).$$

Theorem 3.1 provides the following statement.

Application 3.6 Let $n \geq 1$, $E \in \mathsf{hlcVect}$, and $(V,\mathfrak{V}) \in \Omega(E)$. Let $\underline{a} = (a_1,\dots,a_n)$, $\underline{\tau} = (\tau_1,\dots,\tau_n)$, $\underline{b} = (b_1,\dots,b_n)$ be given with $-\infty \leq a_i < \tau_i < b_i < \infty$ for $i = 1,\dots,n$. There exists a linear (extension) map

$$\mathcal{E}_{\underline{a},\underline{\tau},\underline{b}}(E,V,\mathfrak{V})\colon \mathcal{C}^k_{\mathfrak{V}}(\mathcal{Q}(\underline{a},\underline{b})\times V,F) \to \mathcal{C}^k_{\mathfrak{V}}(\mathcal{Q}(\underline{a},\underline{\infty})\times V,F),$$

that admits the following two properties:

(a) For $f \in \mathcal{C}^k_{\mathfrak{V}}(\mathcal{Q}(\underline{a},\underline{b})\times V,F)$ and $0 \leq \ell \leq k$, we have

$$\mathrm{Ext}(\mathcal{E}_{\underline{a},\underline{\tau},\underline{b}}(E,V,\mathfrak{V})(f),\ell)|_{\breve{\mathcal{Q}}(\underline{a},\underline{b})\times\mathfrak{V}\times H[E,n]^\ell} = \mathrm{Ext}(f,\ell).$$

[10] See (2.1) for the Definition of the seminorms on the right side.

[11] Additionally observe that for each (continuous) seminorm $\mathfrak{h}$ on $\mathbb{R}$, we have $\mathfrak{h}(x) = |x|\cdot\mathfrak{h}(1) \leq \max(1,\mathfrak{h}(x))\cdot|x|$ for all $x \in \mathbb{R}$.

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(b) Let $\underline{y} = (y_1, \ldots, y_n) \in \mathcal{Q}(\underline{a}, \underline{\infty})$ be given, with $y_i \geq 2b_i - \tau_i$ for some $1 \leq i \leq n$. Then,

$$\operatorname{Ext}(\mathcal{E}_{\underline{a},\underline{\tau},\underline{b}}(E, V, \mathfrak{V})(f), \ell)|_{\{\underline{y}\}\times\mathfrak{V}\times H[E,n]^\ell} = 0$$

holds for each $f \in \mathcal{C}^k_{\mathfrak{V}}(\mathcal{Q}(\underline{a}, \underline{b}) \times V, F)$ and $0 \leq \ell \leq k$.

Proof According to Theorem 3.1, we can assume that the claim holds for some $n \geq 1$. Let thus $-\infty \leq a_0, \ldots, a_n, \tau_0, \ldots, \tau_n, b_0, \ldots, b_n < \infty$ be given with $a_i < \tau_i < b_i$ for $i = 0, \ldots, n$, and define

$$\underline{a} := (a_1, \ldots, a_n), \qquad \underline{a}_0 := (a_0, \ldots, a_n),$$
$$\underline{\tau} := (\tau_1, \ldots, \tau_n), \qquad \underline{\tau}_0 := (\tau_0, \ldots, \tau_n),$$
$$\underline{b} := (b_1, \ldots, b_n), \qquad \underline{b}_0 := (b_0, \ldots, b_n).$$

Let $E \in \mathsf{hlcVect}$ and $(V, \mathfrak{V}) \in \Omega(E)$ be given. We set $\widetilde{E} := \mathbb{R} \times E$, $\widehat{E} := \mathbb{R}^n \times E$, as well as

$$(\widetilde{V}, \widetilde{\mathfrak{V}}) := ((a_0, b_0) \times V, (a_0, b_0] \times \mathfrak{V}) \in \Omega(\widetilde{E}),$$
$$(\widehat{V}, \widehat{\mathfrak{V}}) := (\mathcal{Q}(\underline{a}, \underline{\infty}) \times V, \mathcal{Q}(\underline{a}, \underline{\infty}) \times \mathfrak{V}) \in \Omega(\widehat{E}).$$

The induction hypotheses provides the extension operator

$$\mathcal{E}_{\underline{a},\underline{\tau},\underline{b}}(\widetilde{E}, \widetilde{V}, \widetilde{\mathfrak{V}}) \colon \underbrace{\mathcal{C}^k_{\widetilde{\mathfrak{V}}}(\mathcal{Q}(\underline{a}, \underline{b}) \times \widetilde{V}, F)}_{\cong \mathcal{C}^k_{\mathfrak{V}}(\mathcal{Q}(\underline{a}_0, \underline{b}_0) \times V, F)} \to \mathcal{C}^k_{\widetilde{\mathfrak{V}}}(\mathcal{Q}(\underline{a}, \underline{\infty}) \times \widetilde{V}, F)$$
$$\cong \mathcal{C}^k_{\widehat{\mathfrak{V}}}((a_0, b_0) \times \widehat{V}, F). \tag{3.4}$$

Theorem 3.1 provides the extension operator

$$\mathcal{E}_{a_0,\tau_0,b_0}(\widehat{E}, \widehat{V}, \widehat{\mathfrak{V}}) \colon \mathcal{C}^k_{\widehat{\mathfrak{V}}}((a_0, b_0) \times \widehat{V}, F) \to \mathcal{C}^k_{\widehat{\mathfrak{V}}}((a_0, \infty) \times \widehat{V}, F)$$
$$\cong \mathcal{C}^k_{\mathfrak{V}}(\mathcal{Q}(\underline{a}_0, \underline{\infty}) \times V, F). \tag{3.5}$$

We consider the linear map

$$\mathcal{E}_{\underline{a}_0,\underline{\tau}_0,\underline{b}_0}(E, V, \mathfrak{V}) := \mathcal{E}_{a_0,\tau_0,b_0}(\widehat{E}, \widehat{V}, \widehat{\mathfrak{V}}) \circ \mathcal{E}_{\underline{a},\underline{\tau},\underline{b}}(\widetilde{E}, \widetilde{V}, \widetilde{\mathfrak{V}}),$$

so that under the identifications made we have

$$\mathcal{E}_{\underline{a}_0,\underline{\tau}_0,\underline{b}_0}(E, V, \mathfrak{V}) \colon \mathcal{C}^k_{\mathfrak{V}}(\mathcal{Q}(\underline{a}_0, \underline{b}_0) \times V, F) \to \mathcal{C}^k_{\mathfrak{V}}(\mathcal{Q}(\underline{a}_0, \underline{\infty}) \times V, F).$$

Let now $f \in \mathcal{C}^k_{\mathfrak{V}}(\mathcal{Q}(\underline{a}_0, \underline{b}_0) \times V, F)$ and $\underline{y} = (y_0, \ldots, y_n) \in \mathcal{Q}(\underline{a}_0, \underline{b}_0)$ be given. The induction hypotheses provides the following statements:

- Up to the identifications in (3.4), we have

$$\operatorname{Ext}(\mathcal{E}_{\underline{a},\underline{\tau},\underline{b}}(\widetilde{E}, \widetilde{V}, \widetilde{\mathfrak{V}})(f), \ell)|_{(a_0,b_0]\times(\check{\mathcal{Q}}(\underline{a},\underline{b})\times\mathfrak{V})\times H[\widetilde{E},1]^\ell} = \operatorname{Ext}(f, \ell) \qquad \forall\, 0 \leq \ell \leq k.$$

- Let $y_i \geq 2b_i - \tau_i$ for some $1 \leq i \leq n$, as well as $y_0 \in (a_0, b_0)$. Then, up to the identifications in (3.4), we have

$$\operatorname{Ext}(\mathcal{E}_{\underline{a},\underline{\tau},\underline{b}}(\widetilde{E}, \widetilde{V}, \widetilde{\mathfrak{V}})(f), \ell)((y_0, ((y_1, \ldots, y_n), \cdot)), \cdot) = 0 \qquad \forall\, 0 \leq \ell \leq k. \tag{3.6}$$

Theorem 3.1.(1) (for $\mathcal{E}_{a_0,\tau_0,b_0}(\widehat{E},\widehat{V},\widehat{\mathfrak{V}})$), provides the following statements:

- For $0 \le \ell \le k$, we have (under the identification in (3.4))

$$\begin{aligned}&\mathrm{Ext}(\mathcal{E}_{a_0,\tau_0,b_0}(\widehat{E},\widehat{V},\widehat{\mathfrak{V}})(\mathcal{E}_{\underline{a},\underline{\tau},\underline{b}}(\widetilde{E},\widetilde{V},\widetilde{\mathfrak{V}})(f)),\ell)|_{(a_0,b_0]\times\widehat{\mathfrak{V}}\times H[\widehat{E},1]^\ell}\\ &\quad= \mathrm{Ext}(\mathcal{E}_{\underline{a},\underline{\tau},\underline{b}}(\widetilde{E},\widetilde{V},\widetilde{\mathfrak{V}})(f),\ell).\end{aligned}$$

- If $y_0 \ge 2b_0 - \tau_0$ holds, then for $0 \le \ell \le k$, we have

(3.7)
$$\mathrm{Ext}(\mathcal{E}_{a_0,\tau_0,b_0}(\widehat{E},\widehat{V},\widehat{\mathfrak{V}})(\mathcal{E}_{\underline{a},\underline{\tau},\underline{b}}(\widetilde{E},\widetilde{V},\widetilde{\mathfrak{V}})(f)),\ell)((y_0,((y_1,\dots,y_n),\cdot)),\cdot) = 0.$$

We obtain for $0 \le \ell \le k$ (under the identification in (3.5) in the first step) that

$$\begin{aligned}&\mathrm{Ext}(\mathcal{E}_{\underline{a}_0,\underline{\tau}_0,\underline{b}_0}(E,V,\mathfrak{V})(f),\ell)|_{\check{\mathcal{Q}}(\underline{a}_0,\underline{b}_0)\times\mathfrak{V}\times H[E,n+1]^\ell}\\ &\quad= \mathrm{Ext}(\mathcal{E}_{a_0,\tau_0,b_0}(\widehat{E},\widehat{V},\widehat{\mathfrak{V}})(\mathcal{E}_{\underline{a},\underline{\tau},\underline{b}}(\widetilde{E},\widetilde{V},\widetilde{\mathfrak{V}})(f)),\ell)|_{(a_0,b_0]\times(\check{\mathcal{Q}}(\underline{a},\underline{b})\times\mathfrak{V})\times H[\widehat{E},1]^\ell}\\ &\quad= \mathrm{Ext}(\mathcal{E}_{\underline{a},\underline{\tau},\underline{b}}(\widetilde{E},\widetilde{V},\widetilde{\mathfrak{V}})(f),\ell)|_{(a_0,b_0]\times(\check{\mathcal{Q}}(\underline{a},\underline{b})\times\mathfrak{V})\times H[\widehat{E},1]^\ell}\\ &\quad= \mathrm{Ext}(f,\ell)\end{aligned}$$

holds, which proves Part (a). Finally, assume that $y_i \ge 2b_i - \tau_i$ holds for some $1 \le i \le n$. Then, Theorem 3.1.(2) together with (3.6) shows

$$\mathfrak{p}(\mathrm{Ext}(\mathcal{E}_{a_0,\tau_0,b_0}(\widehat{E},\widehat{V},\widehat{\mathfrak{V}})(\mathcal{E}_{\underline{a},\underline{\tau},\underline{b}}(\widetilde{E},\widetilde{V},\widetilde{\mathfrak{V}})(f)),\ell)((y_0,((y_1,\dots,y_n),\cdot)),\cdot) \le 0$$

for each $\mathfrak{p} \in \mathsf{Sem}(F)$ and $0 \le \ell \le k$. Together with (3.7), this proves Part (b). ∎

Remark 3.7 Domains as considered in Application 3.6 occur, e.g., in the context of manifolds with corners as (open subsets of) quadrants in Hausdorff locally convex vector spaces. Specifically, let $H \in \mathsf{hlcVect}$, and $\mathcal{L}_1,\dots,\mathcal{L}_n\colon H \to \mathbb{R}$ with $n \ge 1$ be linearly independent continuous linear maps. Consider the closed subspace $E := \ker(\mathcal{L}_1) \cap \cdots \cap \ker(\mathcal{L}_n) \subseteq H$, and let $\underline{a} := (-\infty,\dots,-\infty)$, $\underline{b} := (0,\dots,0)$ (both n-times). Then, we have $H \cong \mathbb{R}^n \times E$, and the corresponding open and closed quadrants $\mathrm{Q} \subseteq H$ and $\check{\mathrm{Q}} \subseteq H$, respectively, are given by

$$\begin{aligned}\mathrm{Q} &:= \{X \in H \,|\, \mathcal{L}_p(X) < 0 \quad\text{for}\quad p = 1,\dots,n\} \cong \mathcal{Q}(\underline{a},\underline{b}) \times E,\\ \check{\mathrm{Q}} &:= \{X \in H \,|\, \mathcal{L}_p(X) \le 0 \quad\text{for}\quad p = 1,\dots,n\} \cong \check{\mathcal{Q}}(\underline{a},\underline{b}) \times E.\end{aligned}$$

Proof Let $e_1,\dots,e_n \in H$ be linearly independent with $\mathcal{L}_i(e_j) = \delta_{ij}$ for $1 \le i,j \le n$, and set $H \supseteq W := \langle e_1,\dots,e_n\rangle \cong \mathbb{R}^n$. Then,

$$\mathcal{P}\colon H \ni X \mapsto \textstyle\sum_{p=1}^n \mathcal{L}_p(X)\cdot e_p \in W$$

is a continuous projection operator, with $\mathcal{P}(H) \cong \mathbb{R}^n$, $\mathcal{P}(\mathrm{Q}) \cong \mathcal{Q}(\underline{a},\underline{b})$, $\mathcal{P}(\check{\mathrm{Q}}) \cong \check{\mathcal{Q}}(\underline{a},\underline{b})$ as topological spaces. Moreover, the following maps are continuous, linear, and inverse to each other:

$$\begin{aligned}\Xi\colon &\quad H \to W \times E, &&X \mapsto (\mathcal{P}(X), X - \mathcal{P}(X)),\\ \Xi^{-1}\colon &W \times E \to H, &&(Z,Y) \mapsto Z + Y.\end{aligned}$$

Since $\Xi(H) \cong \mathbb{R}^n \times E$, $\Xi(\mathrm{Q}) \cong \mathcal{Q}(\underline{a},\underline{b}) \times E$, and $\Xi(\check{\mathrm{Q}}) \cong \check{\mathcal{Q}}(\underline{a},\underline{b}) \times E$ are homeomorphic, the claim follows. ∎

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3.3 Particular subsets in infinite dimensions

Let $P, E \in$ hlcVect, $(V, \mathfrak{V}) \in \Omega(E)$, $0 < \tau < 1$, $k \in \mathbb{N} \cup \{\infty\}$ be fixed, and set $H := P \times E$. Let $\xi: P \to [0, \infty)$ be a continuous map that admits the following properties:

(1) ξ is of class C^k on $\mathcal{W} := \xi^{-1}((0, \infty))$,
(2) $\xi(Z) \neq 0$ for some $Z \in P$,
(3) $\xi(\lambda \cdot Z) = \lambda \cdot \xi(Z)$ for each $\lambda \in [0, \infty)$ and $Z \in P$, and
(4) $\mathcal{A} := \xi^{-1}(1) \subseteq \mathcal{W}$ is harmonic.

We consider the sets:

- $\mathcal{O} := \xi^{-1}((0,1))$,
- $\check{\mathcal{O}} := \xi^{-1}((0,1])$,
- $\mathcal{U} := \xi^{-1}((1,\infty))$,
- $\mathcal{T} := \xi^{-1}([2-\tau,\infty))$,
- $\mathcal{V} := \xi^{-1}((2-\tau,\infty))$,
- $\mathcal{J} := \mathcal{W}\backslash\mathcal{A}$, and
- $A := \mathcal{A} \times V$.

These sets are nonempty by (2) and (3); and, by continuity of ξ, the sets $\mathcal{W}, \mathcal{O}, \mathcal{U}, \mathcal{V}, \mathcal{J}$ are open. Moreover, $A \subseteq \mathcal{W} \times V$ is harmonic by Example 2.6.(iii), and the condition (3) implies:

– $\check{\mathcal{O}} \subseteq \operatorname{clos}(\mathcal{O})$, hence $(\mathcal{O}, \check{\mathcal{O}}) \in \Omega(P)$ and $(\mathcal{O} \times V, \check{\mathcal{O}} \times \mathfrak{V}) \in \Omega(H)$,
– $\mathcal{T} \subseteq \operatorname{clos}(\mathcal{V})$, and
– $\mathcal{J} \subseteq \operatorname{clos}(\mathcal{W})$, hence $(\mathcal{J}, \mathcal{W}) \in \Omega(P)$ and $(\mathcal{J} \times V, \mathcal{W} \times \mathfrak{V}) \in \Omega(H)$.

We define

$$\begin{aligned} \mathcal{C}^k_{\mathfrak{V}}(\mathcal{O} \times V, F) &:= \mathcal{C}^k_{\check{\mathcal{O}} \times \mathfrak{V}}(\mathcal{O} \times V, F), \\ \mathcal{C}^k_{\mathfrak{V}}(\mathcal{W} \times V, F) &:= \mathcal{C}^k_{\mathcal{W} \times \mathfrak{V}}(\mathcal{W} \times V, F). \end{aligned}$$

In this section, we prove the following statement.

Application 3.8 There exists a linear (extension) map

$$\mathcal{E}: \mathcal{C}^k_{\mathfrak{V}}(\mathcal{O} \times V, F) \to \mathcal{C}^k_{\mathfrak{V}}(\mathcal{W} \times V, F),$$

such that for all $f \in \mathcal{C}^k_{\mathfrak{V}}(\mathcal{O} \times V, F)$ and $0 \leq \ell \leq k$, we have

$$\operatorname{Ext}(\mathcal{E}(f), \ell)|_{\check{\mathcal{O}} \times \mathfrak{V} \times H^\ell} = \operatorname{Ext}(f, \ell) \qquad \text{and} \qquad \operatorname{Ext}(\mathcal{E}(f), \ell)|_{\mathcal{T} \times \mathfrak{V} \times H^\ell} = 0. \tag{3.8}$$

Remark 3.9 Application 3.8 holds in the same form if $\mathcal{O}$ is replaced by $\mathcal{S} := \xi^{-1}([0,1))$, $\check{\mathcal{O}}$ is replaced by $\check{\mathcal{S}} := \xi^{-1}([0,1])$, and $\mathcal{W}$ is replaced by P.

Proof We have $(\mathcal{S}, \check{\mathcal{S}}) \in \Omega(P)$ by continuity of ξ as well as by (2) and (3). Let now $f \in \mathcal{C}^k_{\mathfrak{V}}(\mathcal{S} \times V, F)$ be given. Then, $f|_{\mathcal{O} \times V} \in \mathcal{C}^k_{\mathfrak{V}}(\mathcal{O} \times V, F)$ holds by Lemma 2.10 (with $O \equiv P$ and $\psi \equiv \mathrm{id}_P$). Hence, we have $\mathcal{E}(f|_{\mathcal{O} \times V}) \in \mathcal{C}^k_{\mathfrak{V}}(\mathcal{W} \times V, F)$, with $\mathcal{E}$ as in Application 3.8. We define

$$\tilde{\mathcal{E}}(f): P \times V \to F, \qquad (Z, x) \mapsto \begin{cases} \mathcal{E}(f|_{\mathcal{O} \times V})(Z, x) & \text{for} \quad Z \neq 0, \\ f(Z, x) & \text{for} \quad Z = 0. \end{cases}$$

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- By construction, we have $\tilde{\mathcal{E}}(f)|_{\mathcal{W}\times V} = \mathcal{E}(f|_{\mathcal{O}\times V})|_{\mathcal{W}\times V}$, as well as $\tilde{\mathcal{E}}(f)|_{\mathcal{S}\times V} = f$ by $\tilde{\mathcal{E}}(f)|_{\mathcal{O}\times V} = \mathcal{E}(f|_{\mathcal{O}\times V})|_{\mathcal{O}\times V} \overset{(3.8)}{=} f|_{\mathcal{O}\times V}$ and $\tilde{\mathcal{E}}(f)|_{\{0\}\times V} = f|_{\{0\}\times V}$.

  Since $\mathcal{W}\times V, \mathcal{S}\times V$ are open with $(\mathcal{W}\times V)\cup(\mathcal{S}\times V) = P\times V$, the map $\tilde{\mathcal{E}}(f)$ is of class C^k with

$$(\mathrm{d}^\ell\tilde{\mathcal{E}}(f))|_{\mathcal{W}\times V\times H^\ell} = \mathrm{d}^\ell\mathcal{E}(f|_{\mathcal{O}\times V}) = \mathrm{Ext}(\mathcal{E}(f|_{\mathcal{O}\times V}),\ell)|_{\mathcal{W}\times V\times H^\ell} \qquad \forall\, 0\le\ell\le k,$$
$$(\mathrm{d}^\ell\tilde{\mathcal{E}}(f))|_{\mathcal{S}\times V\times H^\ell} = \mathrm{d}^\ell f = \mathrm{Ext}(f,\ell)|_{\mathcal{S}\times V\times H^\ell} \qquad \forall\, 0\le\ell\le k.$$

  Then, continuity implies $\tilde{\mathcal{E}}(f)\in\mathcal{C}^k_{\mathfrak{V}}(P\times V,F)$, with (observe $\xi^{-1}(0)\subseteq\check{\mathcal{S}}$)

$$\mathrm{Ext}(\tilde{\mathcal{E}}(f),\ell)|_{\mathcal{W}\times\mathfrak{V}\times H^\ell} = \mathrm{Ext}(\mathcal{E}(f|_{\mathcal{O}\times V}),\ell) \qquad \text{and}$$
$$\mathrm{Ext}(\tilde{\mathcal{E}}(f),\ell)|_{\check{\mathcal{S}}\times\mathfrak{V}\times H^\ell} = \mathrm{Ext}(f,\ell),$$

  for $0\le\ell\le k$. We thus have the linear (extension) map

$$\tilde{\mathcal{E}}\colon \mathcal{C}^k_{\mathfrak{V}}(\mathcal{S}\times V,F)\to\mathcal{C}^k_{\mathfrak{V}}(P\times V,F), \qquad f\mapsto\tilde{\mathcal{E}}(f).$$

- By construction, we have $\tilde{\mathcal{E}}(f)|_{\mathcal{T}\times V} = \mathcal{E}(f|_{\mathcal{O}\times V})|_{\mathcal{T}\times V} \overset{(3.8)}{=} 0$. Since $\mathcal{T}\times V$ is open, we obtain

$$(\mathrm{d}^\ell\tilde{\mathcal{E}}(f))|_{\mathcal{T}\times V\times H^\ell} = 0 \quad\overset{\text{continuity}}{\Longrightarrow}\quad \mathrm{Ext}(\tilde{\mathcal{E}}(f),\ell)|_{\mathcal{T}\times\mathfrak{V}\times H^\ell} = 0$$

  for $0\le\ell\le k$. ■

Example 3.10 Let $0\neq\xi\in\mathrm{Sem}(P)$ be of class C^k on $\mathcal{W}$. Then, (1)–(3) are evident, and (4) holds by Example 2.6.(v). For instance.

(a) Let $P=\mathbb{R}^2$ and $\xi\colon\mathbb{R}^2\ni(x,y)\mapsto|x|\in[0,\infty)$. Then, ξ is smooth on $\mathcal{W}=\{(x,y)\in\mathbb{R}^2\mid x\neq 0\}$.

(b) Let $(\mathscr{H},\langle\cdot,\cdot\rangle)$ be a real or complex[12] pre-Hilbert space, and set $P:=\mathscr{H}$ as well as $\xi(\cdot):=\sqrt{\langle\cdot,\cdot\rangle}$. Then, ξ is smooth on $\mathcal{W}=P\backslash\{0\}$ by Proposition 2.2. We mention that in the real case, an extension operator can also be obtained by explicit application of Theorem 3.1.(3). More details are provided in Appendix A. ■

Let $\rho\in C^\infty(\mathbb{R},\mathbb{R})$ be given with[13]

$$\mathrm{id}_{(0,1)} < \rho|_{(0,1)} < 1, \qquad \rho(1)=1, \qquad \mathrm{id}_{(1,\infty)} < \rho|_{(1,\infty)}. \tag{3.9}$$

- We consider the smooth map $\eta\colon\mathbb{R}\times H\ni(t,Z,x)\mapsto(t\cdot Z,x)\in H$. Lemma 2.10 and (3) imply

$$\mathcal{N}\colon\mathcal{C}^k_{\check{\mathcal{O}}\times\mathfrak{V}}(\mathcal{O}\times V,F)\to\mathcal{C}^k_{(0,1]\times\check{\mathcal{O}}\times\mathfrak{V}}((0,1)\times\mathcal{O}\times V,F), \qquad f\mapsto f\circ\eta|_{(0,1)\times\mathcal{O}\times V}.$$

- Theorem 3.1 provides the extension operator $\hat{\mathcal{E}}\equiv\mathcal{E}_{0,\tau,1}(H,\mathcal{O}\times V,\check{\mathcal{O}}\times\mathfrak{V})$, hence

$$\hat{\mathcal{E}}\colon\mathcal{C}^k_{(0,1]\times\check{\mathcal{O}}\times\mathfrak{V}}((0,1)\times\mathcal{O}\times V,F)\to\mathcal{C}^k_{(0,\infty)\times\check{\mathcal{O}}\times\mathfrak{V}}((0,\infty)\times\mathcal{O}\times V,F).$$

[12] Also in the complex case, differentiability is meant w.r.t. the real structure on $\mathscr{H}$, i.e., we do not consider complex differentiability (holomorphicity) at this point.

[13] For instance, choose $\rho\colon\mathbb{R}\ni x\mapsto x+(x-1)^2\in\mathbb{R}$.

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- We consider the C^k-map (recall (1), (3) and (3.9))

$$\mu\colon \mathcal{W}\times E \to \mathbb{R}\times H, \qquad (Z,x)\mapsto \big((\rho\circ\xi)(Z),(\rho\circ\xi)(Z)^{-1}\cdot Z,x\big).$$

By construction, we have $\eta\circ\mu = \mathrm{id}_{\mathcal{W}\times E}$, and furthermore for $Y\subseteq E$:
  - $\mu(\mathcal{W}\times Y)\subseteq (0,\infty)\times\breve{\mathcal{O}}\times Y$,
  - $\mu(\mathcal{O}\times Y)\subseteq \ (0,1)\times\mathcal{O}\times Y$, and
  - $\mu(\mathcal{U}\times Y)\subseteq (1,\infty)\times\mathcal{O}\times Y$.

Let now $f\in\mathcal{C}^k_{\breve{\mathcal{O}}\times\mathfrak{V}}(\mathcal{O}\times V,F)$ be given. We set

$$\chi := (\hat{\mathcal{E}}\circ\mathcal{N})(f) = \hat{\mathcal{E}}(f\circ\eta)\in\mathcal{C}^k_{(0,\infty)\times\breve{\mathcal{O}}\times\mathfrak{V}}((0,\infty)\times\mathcal{O}\times V,F),$$

and

$$\alpha := \chi\circ\mu|_{\mathcal{J}\times V}.$$

- We have $\alpha\in\mathcal{C}^k_{\mathcal{W}\times\mathfrak{V}}(\mathcal{J}\times V,F)$ by Lemma 2.10, because

$$\mu(\mathcal{J}\times V)\subseteq(0,\infty)\times\mathcal{O}\times V \qquad\text{and}\qquad \mu(\mathcal{W}\times\mathfrak{V})\subseteq(0,\infty)\times\breve{\mathcal{O}}\times\mathfrak{V} \qquad\text{holds.}$$

- Since $\mu(\mathcal{O}\times V)\subseteq(0,1)\times\mathcal{O}\times V$ holds, we have by Theorem 3.1.(1)

$$\alpha|_{\mathcal{O}\times V} = (\hat{\mathcal{E}}\circ\mathcal{N})(f)\circ\mu|_{\mathcal{O}\times V} = \mathcal{N}(f)\circ\mu|_{\mathcal{O}\times V} = (f\circ\eta\circ\mu)|_{\mathcal{O}\times V} = f. \tag{3.10}$$

We consider the continuous maps

$$\Phi^\ell := \mathrm{Ext}(\alpha,\ell)\colon \mathcal{W}\times\mathfrak{V}\times H^\ell\to F \qquad \forall\, 0\le\ell\le k,$$

and proceed as follows:

- By construction, we have

$$\Phi^\ell|_{\mathcal{J}\times V\times H^\ell} = \mathrm{Ext}(\alpha,\ell)|_{\mathcal{J}\times V\times H^\ell} = \mathrm{d}^\ell\alpha \qquad \forall\, 0\le\ell\le k. \tag{3.11}$$

Corollary 2.9 (with $f\equiv\alpha$, $A\equiv\mathcal{A}\times V$, $U\equiv\mathcal{W}\times V$, $\mathfrak{U}\equiv\mathcal{W}\times\mathfrak{V}$, i.e., $U\backslash A=\mathcal{J}\times V$) thus shows

$$\tilde{f} := \Phi^0|_{\mathcal{W}\times V}\in\mathcal{C}^k_{\mathcal{W}\times\mathfrak{V}}(\mathcal{W}\times V,F)$$
$$\text{with}\qquad \mathrm{Ext}(\tilde{f},\ell)=\Phi^\ell \qquad\text{for all}\qquad 0\le\ell\le k.$$

- We obtain from (3.10) and (3.11) that

$$\tilde{f}|_{\mathcal{O}\times V} = (\Phi^0|_{\mathcal{W}\times V})|_{\mathcal{O}\times V} \overset{(3.11)}{=} \mathrm{Ext}(\alpha,0)|_{\mathcal{O}\times V} = \alpha|_{\mathcal{O}\times V} \overset{(3.10)}{=} f$$

holds. Since $\mathcal{O}\times V$ is open, we obtain

$$\mathrm{d}^\ell\tilde{f}|_{\mathcal{O}\times V\times H^\ell} = \mathrm{d}^\ell(\tilde{f}|_{\mathcal{O}\times V}) = \mathrm{d}^\ell f = \mathrm{Ext}(f,\ell)|_{\mathcal{O}\times V\times H^\ell} \qquad \forall\, 0\le\ell\le k,$$

so that continuity yields

$$\mathrm{Ext}(\tilde{f},\ell)|_{\breve{\mathcal{O}}\times\mathfrak{V}\times H^\ell} = \mathrm{Ext}(f,\ell) \qquad \forall\, 0\le\ell\le k. \tag{3.12}$$

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- We obtain from (3.11)

$$\tilde{f}|_{\mathcal{V}\times V} = (\Phi^0|_{\mathcal{W}\times V})|_{\mathcal{V}\times V} \overset{(3.11)}{=} \mathrm{Ext}(\alpha, 0)|_{\mathcal{V}\times V} = \alpha|_{\mathcal{V}\times V} = \chi \circ \mu|_{\mathcal{V}\times V}$$
$$= \hat{\mathcal{E}}(\mathcal{N}(f)) \circ \mu|_{\mathcal{V}\times V}.$$

Since $\rho|_{(1,\infty)} > \mathrm{id}_{(1,\infty)}$ holds, Theorem 3.1.(1) yields

$$\tilde{f}(Z, x) = \hat{\mathcal{E}}(\mathcal{N}(f))\big((\rho \circ \xi)(Z), (\rho \circ \xi)(Z)^{-1} \cdot Z, x\big) = 0 \qquad \forall\, Z \in \mathcal{V},\ x \in V.$$

Since $\mathcal{V} \times V$ is open, this implies $\mathrm{d}^\ell \tilde{f}|_{\mathcal{V}\times V\times H^\ell} = 0$ for all $0 \le \ell \le k$, so that continuity implies

$$\text{(3.13)} \qquad \mathrm{Ext}(\tilde{f}, \ell)|_{\mathcal{T}\times\mathfrak{V}\times H^\ell} = 0 \qquad \forall\, 0 \le \ell \le k.$$

We are ready for the proof of Application 3.8:

Proof of Application 3.8 Obviously, the assignment

$$\mathcal{E}\colon \mathcal{C}^k_{\mathfrak{V}}(\mathcal{O} \times V, F) \ni f \mapsto \tilde{f} \in \mathcal{C}^k_{\mathcal{W}\times\mathfrak{V}}(\mathcal{W} \times V, F)$$

is linear; and the rest is clear from (3.12) and (3.13). ∎

3.4 Partially constant maps and parametrizations

Let $E, F \in \mathsf{hlcVect}$, $k \in \mathbb{N} \cup \{\infty\}$, and $S \equiv \{S_\alpha\}_{\alpha\in I}$ be a family of disjoint subsets of E with $E = \bigcup_{\alpha\in I} S_\alpha$. For $-\infty \le a < b \le \infty$, we define

$$C^k(a, b, S) := \{f \in C^k((a, b) \times E, F) \mid f|_{\{t\}\times S_\alpha} \text{ is constant for each } t \in (a, b) \text{ and } \alpha \in I\}$$
$$\mathcal{C}^k(a, b, S) := C^k(a, b, S) \cap \mathcal{C}^k_{(a,b]\times E}((a, b) \times E, F).$$

Let now $-\infty \le a < \tau < b < \infty$ be fixed. Theorem 3.1 provides the extension operator

$$\mathcal{E} \equiv \mathcal{E}_{a,\tau,b}(E, E, E)\colon \mathcal{C}^k_{(a,b]\times E}((a, b) \times E, F) \to C^k((a, \infty) \times E, F).$$

Theorem 3.1.(3) (for s ≡ 0) implies

$$\text{(3.14)} \qquad \mathcal{E}_S := \mathcal{E}|_{\mathcal{C}^k(a,b,S)}\colon \mathcal{C}^k(a, b, S) \to C^k(a, \infty, S).$$

We can apply this in the following way. Let $H \in \mathsf{hlcVect}$, and $\psi \in C^k((a, \infty) \times E, H)$ an open map, such that the following conditions are fulfilled:

(a) $\psi|_{\{t\}\times S_\alpha}$ is constant for each $t \in (a, \infty)$ and $\alpha \in I$.
(b) For each $z \in \mathrm{im}[\psi]$, we have $\psi^{-1}(z) = \{t(z)\} \times S_{\alpha(z)}$, for certain $t(z) \in (a, \infty)$ and $\alpha(z) \in I$.
(c) For each $z \in \mathrm{im}[\psi]$, there exist $U_z \subseteq (a, \infty) \times E$ and $W_z \subseteq \mathrm{im}[\psi]$ open with $z \in W_z$, such that $\psi|_{U_z}\colon U_z \to W_z$ is a C^k-diffeomorphism, i.e., we have $(\psi|_{U_z})^{-1} \in C^k(W_z, U_z)$.

Let $U := \psi((a, b) \times E)$ and $\mathfrak{U} := \psi((a, b] \times E)$.

- Since ψ is continuous and open, we have $(U, \mathfrak{U}) \in \Omega(H)$.
- Let $f \in \mathcal{C}^k_{\mathfrak{U}}(U, F)$ be fixed. By Lemma 2.10 and (a), we have $g := f \circ \psi|_{(a,b)\times E} \in \mathcal{C}^k(a, b, S)$, hence $\tilde{g} := \mathcal{E}_S(g) \in C^k(a, \infty, S)$ by (3.14).

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- We fix $\iota\colon \mathrm{im}[\psi] \to (a,\infty)\times E$ with $\iota(z)\in\psi^{-1}(z)$ for each $z\in\mathrm{im}[\psi]$, and set
$$\tilde{f}\colon \mathrm{im}[\psi]\to F, \qquad z\mapsto \tilde{g}(\iota(z)).$$
This is defined by (b) and $\tilde{g}\in C^k(a,\infty,S)$. In particular, for each $z\in\mathrm{im}[\psi]$, we have $\tilde{f}|_{W_z}=\tilde{g}\circ(\psi|_{U_z})^{-1}$, which shows that $\tilde{f}$ is of class C^k.

We obtain the linear extension map

$$\tilde{\mathcal{E}}\colon \mathcal{C}^k_{\mathfrak{U}}(U,F)\to C^k(\mathrm{im}[\psi],F), \qquad f\mapsto \tilde{f}. \tag{3.15}$$

We consider the following example.

Example 3.11 Let $E=\mathbb{R}$, $H:=\mathbb{R}^2$, $a=0$, $b=1$, $I:=[0,2\pi)$, $S_\alpha:=\{\alpha+2\pi\cdot\mathbb{Z}\}$ for $\alpha\in I$, and

$$\psi\colon(0,\infty)\times\mathbb{R}\to\mathbb{R}^2\backslash\{0\}, \qquad (t,\varphi)\mapsto(t\cdot\cos(\varphi),t\cdot\sin(\varphi)).$$

According to the above definitions, we have ($\|\cdot\|$ denotes the euclidean norm on $\mathbb{R}^2$)

$$U=\{x\in\mathbb{R}^2\mid 0<\|x\|<1\}, \qquad \mathfrak{U}=\{x\in\mathbb{R}^2\mid 0<\|x\|\le 1\}, \qquad \mathrm{im}[\psi]=\mathbb{R}^2\backslash\{0\}.$$

Then, (3.15) provides the linear extension map $\tilde{\mathcal{E}}\colon\mathcal{C}^k_{\mathfrak{U}}(U,F)\to C^k(\mathbb{R}^2\backslash\{0\},F)$. Let

$$\mathscr{D}:=\{x\in\mathbb{R}^2\mid \|x\|<1\} \qquad\text{and}\qquad \breve{\mathscr{D}}:=\{x\in\mathbb{R}^2\mid\|x\|\le 1\}.$$

We obtain a linear extension map $\hat{\mathcal{E}}\colon\mathcal{C}^k_{\breve{\mathscr{D}}}(\mathscr{D},F)\to C^k(\mathbb{R}^2,F)$ if we set

$$\hat{\mathcal{E}}(f)\colon\mathbb{R}^2\to F, \qquad z\mapsto\begin{cases}\tilde{\mathcal{E}}(f|_U)(z) & \text{for}\quad z\neq 0,\\ f(z) & \text{for}\quad z=0,\end{cases}$$

for each $f\in\mathcal{C}^k_{\breve{\mathscr{D}}}(\mathscr{D},F)$. ■

4 The proof of Theorem 3.1

In this section, we prove Theorem 3.1. For this, we let $F\in\mathsf{hlcVect}$ and $k\in\mathbb{N}\cup\{\infty\}$ be fixed, and recall the definitions made in the beginning of Section 3.1. We make the following simplifications to our argumentation:

- It suffices to prove Theorem 3.1 for the case $a=-\infty$, as the general case then follows by cutoff arguments. Specifically, let $-\infty<a<\tau<b<\infty$ be given, and fix $a<\kappa<\kappa'<\tau$ as well as $\rho\in C^\infty(\mathbb{R},\mathbb{R})$ with
$$\rho|_{(-\infty,\kappa]}=0 \qquad\text{and}\qquad \rho|_{[\kappa',\infty)}=1.$$
For each $E\in\mathsf{hlcVect}$ and $(V,\mathfrak{V})\in\Omega(E)$, we define the linear map $\xi(E,V,\mathfrak{V})\colon\mathcal{C}^k_{\mathfrak{V}}((a,b)\times V,F)\to\mathcal{C}^k_{\mathfrak{V}}((-\infty,b)\times V,F)$ by
$$\xi(E,V,\mathfrak{V})(f)(t,x):=\begin{cases}0 & \text{for}\quad (t,x)\in(-\infty,a]\times V,\\ \rho(t)\cdot f(t,x) & \text{for}\quad (t,x)\in\quad (a,b)\times V,\end{cases}$$

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for $f \in \mathcal{C}^k_{\mathfrak{V}}((a,b) \times V, F)$. We obtain extension operators as in Theorem 3.1 if for $E \in$ hlcVect, $(V, \mathfrak{V}) \in \Omega(E)$, $f \in \mathcal{C}^k_{\mathfrak{V}}((a,b) \times V, F)$, we set

$$\mathcal{E}_{a,\tau,b}(E, V, \mathfrak{V})(f)(t,x) := \begin{cases} f(t,x) & \text{for} \quad (t,x) \in (a,b) \times V, \\ \mathcal{E}_{-\infty,\tau,b}(E, V, \mathfrak{V})(\xi(E, V, \mathfrak{V})(f))(t,x) & \text{for} \quad (t,x) \in [b,\infty) \times V. \end{cases}$$

- To simplify the notations, in the following we restrict to the case $b = 0$. The case $b \neq 0$ follows in the same way, and can alternatively be obtained from the statement for $b = 0$ via application of translations.

For the rest of this section, let thus $\tau \in (-\infty, 0)$ be fixed (i.e., we have $a = -\infty$ and $b = 0$). We choose $\tau < v < 0$ and $\rho \in C^\infty(\mathbb{R}, \mathbb{R})$, such that

$$|\rho| \leq 1, \qquad \rho|_{(-\infty,\tau]} = 0, \qquad \rho|_{[v,0]} = 1$$
$$\text{holds, hence} \qquad \rho^{(j)}|_{[v,0]} = 0 \quad \text{for} \quad j \geq 1, \tag{4.1}$$

and define the constants

$$M_p := \sup\left\{ \left|\rho^{(j)}(t)\right| \;\middle|\; t \in [\tau, 0],\ 0 \leq j \leq p \right\} \geq 1 \qquad \forall\, p \in \mathbb{N}. \tag{4.2}$$

According to [17], there exists a sequence $\{c_n\}_{n \in \mathbb{N}} \subseteq \mathbb{R}$ with

(i) $\sum_{j=0}^\infty c_j \cdot (-2^j)^q = 1$ for each $q \in \mathbb{N}$ and
(ii) $\sum_{j=0}^\infty |c_j| \cdot (2^j)^q < \infty$ for each $q \in \mathbb{N}$.

Given some $f \in \mathcal{C}^k_{\mathfrak{V}}((-\infty,0) \times V, F)$, its extension will be defined (see (4.12) in Section 4.2) in analogy to [17] by

$$\tilde{f}(t,x) := \begin{cases} \mathrm{Ext}(f,0)(t,x) & \text{for} \quad (t,x) \in (-\infty, 0] \times V, \\ \sum_{j=0}^\infty c_j \cdot \rho(-2^j \cdot t) \cdot f(-2^j \cdot t, x) & \text{for} \quad (t,x) \in (0,\infty) \times V. \end{cases} \tag{4.3}$$

The sum in the second line is locally finite as ρ is zero on $(-\infty, \tau]$, hence $\tilde{f}$ is defined and of class C^k on $(\mathbb{R}\backslash\{0\}) \times V$. We basically will have to show that $\tilde{f}$ is of class C^k on whole $\mathbb{R} \times V$, and that its ℓth differential extends continuously to $\mathbb{R} \times \mathfrak{V} \times H[E]^\ell$ for each $0 \leq \ell \leq k$. For this, we need to construct these extensions explicitly, which will be done in analogy to the definition of $\tilde{f}$. For our argumentation, we shall need the following corollary to Lemma 2.7 (Corollary 2.9) and Example 2.6.(iv).

Corollary 4.1 *Let $E \in$ hlcVect, $(V, \mathfrak{V}) \in \Omega(E)$, as well as $f_- \in C^k((-\infty,0) \times V, F)$ and $f_+ \in C^k((0,\infty) \times V, F)$ be given. Assume furthermore that for each $0 \leq \ell \leq k$, there exists a continuous map $\Phi^\ell \colon \mathbb{R} \times \mathfrak{V} \times H[E]^\ell \to F$ that restricts to $d^\ell f_\pm$. Then, we have*

$$\mathcal{C}^k_{\mathfrak{V}}(\mathbb{R} \times V, F) \ni \tilde{f} := \Phi^0|_{\mathbb{R}\times V}.$$
$$d^\ell \tilde{f} = \Phi^\ell|_{\mathbb{R}\times V \times H[E]^\ell} \qquad \forall\, 0 \leq \ell \leq k.$$
$$Ext(\tilde{f}, \ell) = \Phi^\ell \qquad \forall\, 0 \leq \ell \leq k.$$

(the third line implies the second line).

Proof Let $U := \mathbb{R} \times V \subseteq H := H[E]$, $A := \{0\} \times V \subseteq U$, $\mathfrak{U} := \mathbb{R} \times \mathfrak{V} \subseteq H$,

$$f\colon U\backslash A \to F, \qquad (t,x) \mapsto \begin{cases} f_-(t,x) & \text{for} \quad (t,x) \in (-\infty, 0) \times V, \\ f_+(t,x) & \text{for} \quad (t,x) \in \ (0,\infty) \times V, \end{cases}$$

observe that A is harmonic by Example 2.6.(iv) and apply Corollary 2.9. ∎

4.1 Elementary facts and definitions

For $E \in$ hlcVect, we define $\mathbf{1} := (1,0) \in H[E]$ as well as $\mathbf{1}_p := (\mathbf{1},\dots,\mathbf{1}) \in H[E]^p$ for $p \geq 1$, and consider the maps

$$\begin{aligned} \boldsymbol{\lambda} &:= \mathrm{pr}_1\colon & H[E] &\to \mathbb{R}, & (\lambda, X) &\mapsto \lambda, \\ \boldsymbol{\chi} &:= \mathrm{pr}_1 \times 0\colon & H[E] &\to H[E], & (\lambda, X) &\mapsto (\lambda, 0), \\ \boldsymbol{\omega} &:= 0 \times \mathrm{pr}_2\colon & H[E] &\to H[E], & (\lambda, X) &\mapsto (0, X). \end{aligned}$$

We furthermore define the following:

- For $1 \leq \ell \leq k$ and $1 \leq j \leq \ell$, we set

$$\begin{aligned} \boldsymbol{\lambda}_{\ell,j}&\colon H[E]^\ell \to \mathbb{R}, & (w_1,\dots,w_\ell) &\mapsto \boldsymbol{\lambda}(w_j), \\ \boldsymbol{\chi}_{\ell,j}&\colon H[E]^\ell \to H[E], & (w_1,\dots,w_\ell) &\mapsto \boldsymbol{\chi}(w_j), \\ \boldsymbol{\omega}_{\ell,j}&\colon H[E]^\ell \to H[E], & (w_1,\dots,w_\ell) &\mapsto \boldsymbol{\omega}(w_j). \end{aligned}$$

- For $1 \leq \ell \leq k$, $p \geq 1$, and $1 \leq z_1,\dots,z_p \leq \ell$, we set

$$\boldsymbol{\Lambda}_{\ell,z_1,\dots,z_p}(\underline{w}) := \boldsymbol{\lambda}_{\ell,z_1}(\underline{w}) \cdot \dots \cdot \boldsymbol{\lambda}_{\ell,z_p}(\underline{w}) \qquad \forall\, \underline{w} \in H[E]^\ell.$$

  It helps to simplify the notations, in the following just to denote $\boldsymbol{\Lambda}_{\ell,z_1,\dots,z_p}(\underline{w}) := 1$ for the case that $p = 0$ holds.

- For $1 \leq \ell \leq k$ and $0 \leq p \leq \ell$, we let $\mathrm{I}_{\ell,p}$ denote the set of all

$$\underline{\sigma} = (z_1,\dots,z_p,o_1,\dots,o_{\ell-p}) \in \{1,\dots,\ell\}^\ell,$$

  such that the following conditions are fulfilled:[14]
  - $z_i < z_{i+1}$ for $1 \leq i \leq p-1$,
  - $o_j < o_{j+1}$ for $1 \leq j \leq \ell - p - 1$, and
  - $z_i \neq o_j$ for $1 \leq i \leq p$ and $1 \leq j \leq \ell - p$.

Let $V \subseteq E$ be nonempty open, $\Gamma \in C^k((0,\infty) \times V, F)$, and $1 \leq \ell \leq k$. By symmetry (and multilinearity) of the ℓth differential, we have

$$\begin{aligned} &\mathrm{d}^\ell\Gamma((t,x),\underline{w}) \\ &\quad= \textstyle\sum_{p=0}^{\ell} \sum_{\underline{\sigma}\in \mathrm{I}_{\ell,p}} \mathrm{d}^\ell\Gamma((t,x), \boldsymbol{\omega}_{\ell,o_1}(\underline{w}),\dots,\boldsymbol{\omega}_{\ell,o_{\ell-p}}(\underline{w}), \boldsymbol{\chi}_{\ell,z_1}(\underline{w}),\dots,\boldsymbol{\chi}_{\ell,z_p}(\underline{w})) \\ &\quad= \textstyle\sum_{p=0}^{\ell} \sum_{\underline{\sigma}\in \mathrm{I}_{\ell,p}} D_{\boldsymbol{\chi}_{\ell,z_1}(\underline{w}),\dots,\boldsymbol{\chi}_{\ell,z_p}(\underline{w})}\, \mathrm{d}^{\ell-p}\Gamma((t,x), \boldsymbol{\omega}_{\ell,o_1}(\underline{w}),\dots,\boldsymbol{\omega}_{\ell,o_{\ell-p}}(\underline{w})) \\ &\quad= \textstyle\sum_{p=0}^{\ell} \sum_{\underline{\sigma}\in \mathrm{I}_{\ell,p}} \underbrace{\boldsymbol{\lambda}_{\ell,z_1}(\underline{w}) \cdot \dots \cdot \boldsymbol{\lambda}_{\ell,z_p}(\underline{w})}_{=\boldsymbol{\Lambda}_{\ell,z_1,\dots,z_p}(\underline{w})} \\ &\qquad\qquad \cdot\, \partial_t^p\big(\mathrm{d}^{\ell-p}\Gamma((t,x), \boldsymbol{\omega}_{\ell,o_1}(\underline{w}),\dots,\boldsymbol{\omega}_{\ell,o_{\ell-p}}(\underline{w}))\big), \end{aligned} \tag{4.4}$$

[14] We thus have $I_{\ell,0} = (o_1,\dots,o_\ell) = (1,\dots,\ell)$ and $I_{\ell,\ell} = (z_1,\dots,z_\ell) = (1,\dots,\ell)$.

for each $t \in (0, \infty)$, $x \in V$, $\underline{w} \in H[E]^\ell$.

Similarly, if $f \in \mathcal{C}^k_{\mathfrak{V}}((-\infty, 0) \times V, F)$ holds, we obtain for $q = 0, \dots, \ell$ (recall Remark 2.8)

$$\begin{aligned} &\mathrm{Ext}(f,\ell)((t,x),\underline{w}) \\ &= \textstyle\sum_{p=0}^{\ell} \sum_{\underline{\sigma}\in \mathrm{I}_{\ell,p}} \mathbf{\Lambda}_{\ell,z_1,\dots,z_p}(\underline{w}) \cdot \mathrm{Ext}(f,\ell)((t,x),\boldsymbol{\omega}_{\ell,o_1}(\underline{w}),\dots,\boldsymbol{\omega}_{\ell,o_{\ell-p}}(\underline{w}),\mathbf{1}_p), \end{aligned} \tag{4.5}$$

for each $t \in (-\infty, 0)$, $x \in \mathfrak{V}$, $\underline{w} \in H[E]^\ell$.

4.2 Construction of the extension operators

Let $E \in \mathsf{hlcVect}$, $(V, \mathfrak{V}) \in \Omega(E)$, $f \in \mathcal{C}^k_{\mathfrak{V}}((-\infty, 0) \times V, F)$, and $\boldsymbol{\varsigma} \le -1$ be given.

- We define the C^k-map and its C^0-extension:

$$\Gamma[f,\boldsymbol{\varsigma}](t,x) := \rho(\boldsymbol{\varsigma}\cdot t)\cdot f(\boldsymbol{\varsigma}\cdot t, x) \qquad \forall\, (t,x) \in (0,\infty)\times V, \tag{4.6}$$

$$\Psi^0[f,\boldsymbol{\varsigma}](t,x) := \rho(\boldsymbol{\varsigma}\cdot t)\cdot \mathrm{Ext}(f,0)(\boldsymbol{\varsigma}\cdot t, x) \qquad \forall\, (t,x) \in (0,\infty)\times \mathfrak{V}. \tag{4.7}$$

- For $1 \le \ell \le k$, we obtain from (4.4) that

$$\begin{aligned} &\mathrm{d}^\ell\Gamma[f,\boldsymbol{\varsigma}]((t,x),\underline{w}) \\ &\quad= \textstyle\sum_{p=0}^{\ell} \sum_{\underline{\sigma}\in \mathrm{I}_{\ell,p}} \mathbf{\Lambda}_{\ell,z_1,\dots,z_p}(\underline{w}) \\ &\qquad \cdot \partial_t^p\big(\rho(\boldsymbol{\varsigma}\cdot t)\cdot\, \mathrm{d}^{\ell-p} f((\boldsymbol{\varsigma}\cdot t,x),\boldsymbol{\omega}_{\ell,o_1}(\underline{w}),\dots,\boldsymbol{\omega}_{\ell,o_{\ell-p}}(\underline{w}))\big) \\ &\quad= \textstyle\sum_{p=0}^{\ell} \sum_{\underline{\sigma}\in \mathrm{I}_{\ell,p}} \sum_{q=0}^{p} \binom{p}{q}\cdot \boldsymbol{\varsigma}^p \cdot \mathbf{\Lambda}_{\ell,z_1,\dots,z_p}(\underline{w})\cdot \rho^{(q)}(\boldsymbol{\varsigma}\cdot t) \\ &\qquad\qquad \cdot \mathrm{d}^{\ell-q} f((\boldsymbol{\varsigma}\cdot t,x),\boldsymbol{\omega}_{\ell,o_1}(\underline{w}),\dots,\boldsymbol{\omega}_{\ell,o_{\ell-p}}(\underline{w}),\mathbf{1}_{p-q}) \end{aligned} \tag{4.8}$$

  holds for all $t \in (0, \infty)$, $x \in V$, $\underline{w} \in H[E]^\ell$.
- For $1 \le \ell \le k$ and $q = 0, \dots, \ell$, we define the continuous map

$$\begin{aligned} &\mathbf{\Theta}_{\ell,q}[f,\boldsymbol{\varsigma}]((t,x),\underline{w}) \\ &\quad := \textstyle\sum_{p=q}^{\ell} \sum_{\underline{\sigma}\in \mathrm{I}_{\ell,p}} \binom{p}{q}\cdot \boldsymbol{\varsigma}^p \cdot \mathbf{\Lambda}_{\ell,z_1,\dots,z_p}(\underline{w})\cdot \rho^{(q)}(\boldsymbol{\varsigma}\cdot t) \\ &\qquad\qquad \cdot \mathrm{Ext}(f,\ell-q)((\boldsymbol{\varsigma}\cdot t,x),\boldsymbol{\omega}_{\ell,o_1}(\underline{w}),\dots,\boldsymbol{\omega}_{\ell,o_{\ell-p}}(\underline{w}),\mathbf{1}_{p-q}), \end{aligned} \tag{4.9}$$

  for all $t \in (0, \infty)$, $x \in \mathfrak{V}$, and $\underline{w} \in H[E]^\ell$. Then by (4.8), the map

$$\Psi^\ell[f,\boldsymbol{\varsigma}] := \textstyle\sum_{q=0}^{\ell} \mathbf{\Theta}_{\ell,q}[f,\boldsymbol{\varsigma}] \in C^0((0,\infty)\times\mathfrak{V}\times H[E]^\ell, F) \tag{4.10}$$

  continuously extends $\mathrm{d}^\ell\Gamma[f, \boldsymbol{\varsigma}]$ for $1 \le \ell \le k$, i.e.,

$$\Psi^\ell[f,\boldsymbol{\varsigma}]|_{(0,\infty)\times V\times H[E]^\ell} = \mathrm{d}^\ell\Gamma[f,\boldsymbol{\varsigma}] \qquad \forall\, 1 \le \ell \le k. \tag{4.11}$$

For $E \in \mathsf{hlcVect}$, $(V, \mathfrak{V}) \in \Omega(E)$, $f \in \mathcal{C}^k_{\mathfrak{V}}((-\infty, 0) \times V, F)$, we define the maps

$$\begin{aligned} f_- &:= f, \\ f_+ &:= \textstyle\sum_{j=0}^{\infty} c_j \cdot\ \Gamma[f,-2^j], \\ \Phi[f]_+^\ell &:= \textstyle\sum_{j=0}^{\infty} c_j \cdot \Psi^\ell[f,-2^j] \qquad \forall\, 0 \le \ell \le k. \end{aligned}$$

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We have the following statement.

Lemma 4.2 *Let* $E \in \mathsf{hlcVect}$, $(V, \mathfrak{V}) \in \Omega(E)$, $f \in \mathcal{C}^k_{\mathfrak{V}}((-\infty, 0) \times V, F)$ *be given. Then,* $f_+ \in C^k((0,\infty) \times V, F)$ *holds, as well as*

$$\Phi[f]^\ell_+ \in C^0((0,\infty) \times \mathfrak{V} \times H[E]^\ell, F) \qquad \forall\, 0 \le \ell \le k.$$

Moreover, $\Phi[f]^\ell_+$ *restricts to* $\mathrm{d}^\ell f_+$ *for* $0 \le \ell \le k$, *with* $\Phi[f]^\ell_+|_{[-\tau,\infty)\times\mathfrak{V}\times H[E]^\ell} = 0$.

Proof Let $s \in (0,\infty)$ as well as $0 < \varepsilon < s$ be given, and set $I := (s-\varepsilon, s+\varepsilon)$. There exists $N \in \mathbb{N}$, such that $-2^j \cdot I \subseteq (-\infty, \tau)$ holds for each $j \ge N$. Since $\rho|_{(-\infty,\tau]} = 0$, we have (the first line implies the second line)

$$\begin{aligned} f_+|_{I\times V} &= \textstyle\sum_{j=0}^N c_j \cdot \quad \Gamma[f, -2^j]|_{I\times V}, \\ \mathrm{d}^\ell f_+|_{I\times V\times H[E]^\ell} &= \textstyle\sum_{j=0}^N c_j \cdot \mathrm{d}^\ell \Gamma[f, -2^j]|_{I\times V\times H[E]^\ell}, \\ \Phi[f]^\ell_+|_{I\times\mathfrak{V}\times H[E]^\ell} &= \textstyle\sum_{j=0}^N c_j \cdot \quad \Psi^\ell[f, -2^j]|_{I\times\mathfrak{V}\times H[E]^\ell}, \end{aligned}$$

for $0 \le \ell \le k$. Thus, $\Phi[f]^\ell_+$ is defined and continuous for $0 \le \ell \le k$, f_+ is defined and of class C^k, and $\Phi[f]^\ell_+$ restricts to $\mathrm{d}^\ell f_+$ for $0 \le \ell \le k$ by (4.11). Since $-2^j \cdot [-\tau, \infty) \subseteq (-\infty, \tau]$ holds for each $j \in \mathbb{N}$ (with $\rho|_{(-\infty,\tau]} = 0$), we have $\Phi[f]^\ell_+|_{[\tau,\infty)\times\mathfrak{V}\times H[E]^\ell} = 0$ for $0 \le \ell \le k$. ∎

For $E \in \mathsf{hlcVect}$, $(V, \mathfrak{V}) \in \Omega(E)$, $f \in \mathcal{C}^k_{\mathfrak{V}}((-\infty,0)\times V, F)$, and $0 \le \ell \le k$, we define the map $\Phi[f]^\ell : \mathbb{R} \times \mathfrak{V} \times H[E]^\ell \to F$ by

$$\begin{aligned} \Phi[f]^\ell|_{(-\infty,0]\times\mathfrak{V}\times H[E]^\ell} &:= \mathrm{Ext}(f, \ell) \qquad \text{as well as} \\ \Phi[f]^\ell|_{(0,\infty)\times\mathfrak{V}\times H[E]^\ell} &:= \Phi[f]^\ell_+. \end{aligned}$$

In Section 4.3, we prove the following statement.

Lemma 4.3 *Let* $E \in \mathsf{hlcVect}$, $(V, \mathfrak{V}) \in \Omega(E)$, $f \in \mathcal{C}^k_{\mathfrak{V}}((-\infty, 0) \times V, F)$ *be given. Then,* $\Phi[f]^\ell$ *is continuous for each* $0 \le \ell \le k$.

Together with Lemma 4.2 and Corollary 4.1, Lemma 4.3 implies[15]

$$\begin{aligned} \mathcal{C}^k_{\mathfrak{V}}(\mathbb{R}\times V, F) \ni \tilde{f} &:= \Phi[f]^0|_{\mathbb{R}\times V}, \\ \mathrm{d}^\ell \tilde{f} &= \Phi[f]^\ell|_{\mathbb{R}\times V\times H[E]^\ell} \qquad 0 \le \ell \le k, \\ \mathrm{Ext}(\tilde{f}, \ell) &= \Phi[f]^\ell \qquad 0 \le \ell \le k. \end{aligned} \tag{4.12}$$

For $E \in \mathsf{hlcVect}$ and $(V, \mathfrak{V}) \in \Omega(E)$, we define the map

$$\mathcal{E}_{-\infty,\tau,0}(E, V, \mathfrak{V}) : \mathcal{C}^k_{\mathfrak{V}}((-\infty,0)\times V, F) \to \mathcal{C}^k_{\mathfrak{V}}(\mathbb{R}\times V, F), \qquad f \mapsto \tilde{f}. \tag{4.13}$$

We observe the following:

- It is clear from the construction that (4.13) is a linear map, with

$$\begin{aligned} \mathrm{Ext}(\mathcal{E}_{-\infty,\tau,0}(E, V, \mathfrak{V})(f), \ell)|_{(-\infty,0]\times\mathfrak{V}\times H[E]^\ell} &= \mathrm{Ext}(f, \ell), \\ \mathrm{Ext}(\mathcal{E}_{-\infty,\tau,0}(E, V, \mathfrak{V})(f), \ell)|_{[-\tau,\infty)\times\mathfrak{V}\times H[E]^\ell} &= 0, \end{aligned}$$

[15] Notably, this coincides with $\tilde{f}$ as defined in (4.3).

for each $f \in \mathcal{C}^k_{\mathfrak{V}}((-\infty,0)\times V,F)$ and $0 \le \ell \le k$ (for the second line use the last statement in Lemma 4.2).

- Let $E,\bar{E} \in$ hlcVect, $\mathcal{W} \subseteq E$ a linear subspace and $\Upsilon\colon \mathcal{W} \to \bar{E}$ a linear map. Let $(V,\mathfrak{V}) \in \Omega(E)$, $(\bar{V},\bar{\mathfrak{V}}) \in \Omega(\bar{E})$, $x \in \mathfrak{V}$, $\bar{x} \in \bar{\mathfrak{V}}$, $f \in \mathcal{C}^k_{\mathfrak{V}}(\mathbb{R}\times V,F)$, $\bar{f} \in \mathcal{C}^k_{\bar{\mathfrak{V}}}(\mathbb{R}\times \bar{V},F)$, and $0 \le \mathrm{s} \le k$ be given with

$$\mathrm{Ext}(f,\ell)\circ \mathcal{W}([\tau,0],x,\ell) = \mathrm{Ext}(\bar{f},\ell)\circ \mathcal{W}_{\Upsilon}([\tau,0],\bar{x},\ell) \qquad \forall\, 0 \le \ell \le \mathrm{s}.$$

  Then, it is clear from the construction that

$$\Psi^{\mathrm{s}}[f,-2^j]\circ \mathcal{W}((0,\infty),x,\mathrm{s}) = \Psi^{\mathrm{s}}[\bar{f},-2^j]\circ \mathcal{W}_{\Upsilon}((0,\infty),\bar{x},\mathrm{s})$$

  holds for each $j \in \mathbb{N}$, hence

$$\begin{aligned}&\mathrm{Ext}(\mathcal{E}_{-\infty,\tau,0}(E,V,\mathfrak{V})(f),\mathrm{s})\circ \mathcal{W}([\tau,\infty),x,\mathrm{s})\\ &\quad= \mathrm{Ext}(\mathcal{E}_{-\infty,\tau,0}(\bar{E},\bar{V},\bar{\mathfrak{V}})(\bar{f}),\mathrm{s})\circ \mathcal{W}_{\Upsilon}([\tau,\infty),\bar{x},\mathrm{s}).\end{aligned}$$

To establish Theorem 3.1, it thus remains to prove Lemma 4.3 (see Section 4.3), as well as the continuity estimates in Part (2) of Theorem 3.1 (see Section 4.4).

4.3 The proof of Lemma 4.3

Let $E \in$ hlcVect, $(V,\mathfrak{V}) \in \Omega(E)$, $f \in \mathcal{C}^k_{\mathfrak{V}}((-\infty,0)\times V,F)$, $x \in \mathfrak{V}$, $\mathfrak{p} \in \mathrm{Sem}(F)$ be given. The following estimates hold for each $\varsigma \le -1$:

(a) Since $\mathrm{Ext}(f,0)$ is continuous, and since $[\tau,0]$ is compact, there exists $C_0 \ge 1$ and a neighbourhood $U_x \subseteq \mathfrak{V}$ of x, with

$$\mathfrak{p}(\mathrm{Ext}(f,0)(t,x')) \le C_0 \qquad \forall\, t \in [\tau,0],\ x' \in U_x. \tag{4.14}$$

We obtain from (4.1), (4.7), and (4.14) that

$$\mathfrak{p}(\Psi^0[f,\varsigma](t,x')) \le C_0 \qquad \forall\, t \in (0,\infty),\ x' \in U_x. \tag{4.15}$$

(b) Let $1 \le \ell \le k$ and $\underline{w} = (w_1,\dots,w_\ell) \in H[E]^\ell$ be given.
  - According to Point (a) and Corollary 3.5, there exists a neighborhood $U_x \subseteq \mathfrak{V}$ of x, $\widetilde{C}_\ell \ge 1$, and $\mathfrak{q} \in \mathrm{Sem}(E)$, such that we have

$$\mathfrak{p}(\mathrm{Ext}(f,0)(t,x')) \le \widetilde{C}_\ell \qquad \forall\, t \in [\tau,0],\ x' \in U_x, \tag{4.16}$$

    as well as

$$\mathfrak{p}(\mathrm{Ext}(f,q)((t,x'),\underline{w}')) \le \widetilde{C}_\ell \cdot \max[|\cdot|,\mathfrak{q}](w_1')\cdot\dots\cdot \max[|\cdot|,\mathfrak{q}](w_q'), \tag{4.17}$$

    for each $t \in [\tau,0]$, $x' \in U_x$, $1 \le q \le \ell$, and $\underline{w}' = (w_1',\dots,w_q') \in H[E]^q$.
  - We obtain for $0 \le q \le \ell$ from (4.1), (4.2), (4.9), (4.16), (4.17) that

$$\begin{aligned}&\mathfrak{p}\big(\mathbf{\Theta}_{\ell,q}[f,\varsigma]((t,x'),\underline{w}')\big)\\ &\quad\le |\varsigma|^\ell \cdot (\ell+1)! \cdot \max(|\mathrm{I}_{\ell,0}|,\dots,|\mathrm{I}_{\ell,\ell}|)\\ &\qquad\cdot M_\ell \cdot \widetilde{C}_\ell \cdot \max(1,\max[|\cdot|,\mathfrak{q}](w_1'),\dots,\max[|\cdot|,\mathfrak{q}](w_\ell'))^\ell\end{aligned}$$

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holds for each $t \in (0,\infty)$, $x' \in U_x$, and $\underline{w}' = (w'_1, \ldots, w'_\ell) \in H[E]^\ell$. We define

$$Q_\ell := (\ell+1) \cdot (\ell+1)! \cdot \max(|\mathrm{I}_{\ell,0}|, \ldots, |\mathrm{I}_{\ell,\ell}|) \cdot M_\ell \cdot \widetilde{C}_\ell \geq \widetilde{C}_\ell, \tag{4.18}$$

and obtain for $t \in (0,\infty)$, $x' \in U_x$, $\underline{w}' = (w'_1, \ldots, w'_\ell) \in H[E]^\ell$ from (4.10) that

$$\begin{aligned} &\mathfrak{p}\big(\Psi^\ell[f,\varsigma]((t,x'),\underline{w}')\big) \\ &\quad \leq |\varsigma|^\ell \cdot Q_\ell \cdot \max(1, \max[|\cdot|,\mathfrak{q}](w'_1), \ldots, \max[|\cdot|,\mathfrak{q}](w'_\ell))^\ell. \end{aligned} \tag{4.19}$$

- We define $C_\ell := Q_\ell \cdot \max(1, \max[|\cdot|,\mathfrak{q}](w_1)+1, \ldots, \max[|\cdot|,\mathfrak{q}](w_\ell)+1)^\ell$, as well as

  $$O_{\underline{w}} := \{(\underline{w}'_1, \ldots, \underline{w}'_\ell) \in H[E]^\ell \mid \max[|\cdot|,\mathfrak{q}](\underline{w}'_p - \underline{w}_p) < 1 \;\text{ for }\; p = 1, \ldots, \ell\}.$$

  We have by (4.17), (4.18) (used for the first line), and (4.19) (used for the second line) that

  $$\begin{aligned} &\mathfrak{p}(\mathrm{Ext}(f,\ell)((0,x),\underline{w})) \leq C_\ell, \\ &\mathfrak{p}(\Psi^\ell[f,\varsigma]((t,x'),\underline{w}')) \leq C_\ell \cdot |\varsigma|^\ell \qquad \forall\, t \in (0,\infty),\ x' \in U_x,\ \underline{w}' \in O_{\underline{w}}. \end{aligned} \tag{4.20}$$

We are ready for the proof of Lemma 4.3.

Proof of Lemma 4.3 Let $E \in \mathsf{hlcVect}$, $(V,\mathfrak{V}) \in \Omega(E)$, $f \in \mathcal{C}^k_{\mathfrak{V}}((-\infty,0) \times V, F)$, $x \in \mathfrak{V}$, $\mathfrak{p} \in \mathsf{Sem}(F)$, and $\varepsilon > 0$ be given. We discuss the cases $\ell = 0$ and $1 \leq \ell \leq k$ separately:

- Let $\ell = 0$. We choose $C_0 \geq 1$ and $U_x \subseteq \mathfrak{V}$ as in (a). By Property (ii), there exists $N \in \mathbb{N}$ with $\sum_{j=N+1}^\infty |c_j| < \frac{\varepsilon}{4C_0}$. We obtain from (4.14), (4.15) and the triangle inequality that

  $$\textstyle\sum_{j=N+1}^\infty |c_j| \cdot \mathfrak{p}\big(\Psi^0[f,-2^j](t,x') - \mathrm{Ext}(f,0)(0,x)\big) < \frac{\varepsilon}{2} \qquad \forall\, t \in (0,\infty),\ x' \in U_x.$$

  Since $\sum_{j=0}^\infty c_j = 1$ holds by Property (i), the triangle inequality yields

  $$\begin{aligned} &\mathfrak{p}\big(\textstyle\sum_{j=0}^\infty c_j \cdot \Psi^0[f,-2^j](t,x') - \mathrm{Ext}(f,0)(0,x)\big) \\ &\qquad = \mathfrak{p}\big(\textstyle\sum_{j=0}^\infty c_j \cdot \Psi^0[f,-2^j](t,x') - \sum_{j=0}^\infty c_j \cdot \mathrm{Ext}(f,0)(0,x)\big) \\ &\qquad \leq \mathfrak{p}\big(\textstyle\sum_{j=0}^N c_j \cdot \Psi^0[f,-2^j](t,x') - \sum_{j=0}^N c_j \cdot \mathrm{Ext}(f,0)(0,x)\big) \\ &\qquad\quad + \textstyle\sum_{j=N+1}^\infty |c_j| \cdot \mathfrak{p}\big(\Psi^0[f,-2^j](t,x') - \mathrm{Ext}(f,0)(0,x)\big) \\ &\qquad < \textstyle\sum_{j=0}^N |c_j| \cdot \mathfrak{p}\big(\Psi^0[f,-2^j](t,x') - \mathrm{Ext}(f,0)(0,x)\big) + \frac{\varepsilon}{2}, \end{aligned} \tag{4.21}$$

  for $t \in (0,\infty)$ and $x' \in U_x$. We observe the following:
  - By (4.1) and (4.7), we have for $0 \leq j \leq N$:

    $$\Psi^0[f,-2^j](t,x') = \mathrm{Ext}(f,0)(-2^j \cdot t, x') \qquad \forall\, t \in (0, |v|/2^N),\ x' \in \mathfrak{V}.$$

  - Since $\mathrm{Ext}(f,0)$ is continuous, we can shrink $U_x \subseteq \mathfrak{V}$ around x and fix $0 < \delta < |v|/2^N$, such that

    $$\mathfrak{p}(\mathrm{Ext}(f,0)(-2^j \cdot t, x') - \mathrm{Ext}(f,0)(0,x)) < \tfrac{\varepsilon}{2\cdot(N+1)\cdot\max(1,|c_0|,\ldots,|c_N|)}$$

    holds for $j = 0, \ldots, N$, for all $t \in (0,\delta)$ and $x' \in U_x$.

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Combining both points with (4.21), we obtain

$$\mathfrak{p}\big(\textstyle\sum_{j=0}^{\infty} c_j \cdot \Psi^0[f,-2^j](t,x') - \mathrm{Ext}(f,0)(0,x)\big) < \varepsilon \qquad \forall\, t \in (0,\delta),\ x' \in U_x.$$

- Let $1 \le \ell \le k$ and $\underline{w} \in H[E]^\ell$ be fixed. We choose $C_\ell \ge 1$, $U_x \subseteq \mathfrak{V}$, and $O_{\underline{w}} \subseteq H[E]^\ell$ as in (b), and define

$$\Xi[j] := \sum_{p=0}^{\ell} \sum_{\underline{\sigma}\in \mathrm{I}_{\ell,p}} (-2^j)^p \cdot \boldsymbol{\Lambda}_{\ell,z_1,\dots,z_p}(\underline{w}) \cdot \mathrm{Ext}(f,\ell)((0,x), \boldsymbol{\omega}_{\ell,o_1}(\underline{w}),\dots,\boldsymbol{\omega}_{\ell,o_{\ell-p}}(\underline{w}), \mathbf{1}_p),$$

for each $j \in \mathbb{N}$. We observe the following:

  - Given $\Delta > 0$, Property (i) provides some $N_\Delta \in \mathbb{N}$ with

$$\big| \textstyle\sum_{j=0}^{N} c_j \cdot ((-2^j)^p - 1)\big| < \Delta \qquad \forall\, N \ge N_\Delta,\ p = 0,\dots,\ell.$$

  By (4.5), there thus exists some $\tilde{N} \in \mathbb{N}$ with

$$\mathfrak{p}\big(\textstyle\sum_{j=0}^{N} c_j \cdot \Xi[j] - \sum_{j=0}^{N} c_j \cdot \mathrm{Ext}(f,\ell)((0,x),\underline{w})\big) < \tfrac{\varepsilon}{3} \qquad \forall\, N \ge \tilde{N}. \tag{4.22}$$

  - By Property (ii), there exists some $N \ge \tilde{N}$ with

$$\textstyle\sum_{j=N+1}^{\infty} |c_j| \cdot (2^j)^q < \frac{\varepsilon}{6C_\ell} \qquad \forall\, q = 0,\dots,\ell.$$

  We obtain from (4.20) and the triangle inequality that

$$\begin{aligned}&\textstyle\sum_{j=N+1}^{\infty} |c_j| \cdot \mathfrak{p}\big(\Psi^\ell[f,-2^j]((t,x'),\underline{w}') - \mathrm{Ext}(f,\ell)((0,x),\underline{w})\big)\\ &\qquad \le C_\ell \cdot \textstyle\sum_{j=N+1}^{\infty} |c_j| \cdot ((2^j)^\ell + 1) < \frac{\varepsilon}{3}\end{aligned} \tag{4.23}$$

  holds for all $t \in (0,\infty)$, $x' \in U_x$, $\underline{w}' \in O_{\underline{w}}$.

  - By (4.1), (4.9), (4.10), for $0 \le j \le N$, $t \in (0, |v|/2^N)$, $x' \in \mathfrak{V}$, and $\underline{w}' \in H[E]^\ell$ we have

$$\begin{aligned}\Psi^\ell[f,-2^j]&((t,x'),\underline{w}')\\ &= \Theta_{\ell,0}[f,-2^j]((-2^j\cdot t,x'),\underline{w}')\\ &= \textstyle\sum_{p=0}^{\ell} \sum_{\underline{\sigma}\in\mathrm{I}_{\ell,p}} (-2^j)^p \cdot \boldsymbol{\Lambda}_{\ell,z_1,\dots,z_p}(\underline{w}')\\ &\qquad \cdot \mathrm{Ext}(f,\ell)((-2^j\cdot t,x'), \boldsymbol{\omega}_{\ell,o_1}(\underline{w}'),\dots,\boldsymbol{\omega}_{\ell,o_{\ell-p}}(\underline{w}'),\mathbf{1}_p).\end{aligned}$$

  Since $\mathrm{Ext}(f,\ell)$ is continuous, we can shrink $U_x \subseteq \mathfrak{V}$ around x as well as $O_{\underline{w}}$ around $\underline{w}$, and furthermore fix $0 < \delta < |v|/2^N$, such that

$$\mathfrak{p}\big(\Psi^\ell[f,-2^j]((2^j\cdot t,x'),\underline{w}') - \Xi[j]\big) < \tfrac{\varepsilon}{3(N+1)\cdot\max(1,|c_0|,\dots,|c_N|)}$$

  holds for $t \in (0,\delta)$, $x' \in U_x$, $\underline{w}' \in O_{\underline{w}}$, and $j = 0,\dots,N$. We obtain

$$\begin{aligned}&\mathfrak{p}\big(\textstyle\sum_{j=0}^{N} c_j \cdot \Psi^\ell[f,-2^j]((t,x'),\underline{w}') - \sum_{j=0}^{N} c_j\cdot \Xi[j]\big)\\ &\qquad \le \textstyle\sum_{j=0}^{N} |c_j| \cdot \mathfrak{p}\big(\Psi^\ell[f,-2^j]((t,x'),\underline{w}') - \Xi[j]\big) < \frac{\varepsilon}{3},\end{aligned} \tag{4.24}$$

  for all $t \in (0,\delta)$, $x' \in U_x$, $\underline{w}' \in O_{\underline{w}}$.

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Since $\sum_{j=0}^{\infty} c_j = 1$ holds by Property (i), the triangle inequality together with (4.22)–(4.24) yields

$$
\begin{aligned}
&\mathfrak{p}\big(\textstyle\sum_{j=0}^{\infty} c_j \cdot \Psi^{\ell}[f,-2^j]((t,x'),\underline{w}') - \mathrm{Ext}(f,\ell)((0,x),\underline{w})\big)\\
&\quad = \mathfrak{p}\big(\textstyle\sum_{j=0}^{\infty} c_j \cdot \Psi^{\ell}[f,-2^j]((t,x'),\underline{w}') - \sum_{j=0}^{\infty} c_j \cdot \mathrm{Ext}(f,\ell)((0,x),\underline{w})\big)\\
&\quad \le \mathfrak{p}\big(\textstyle\sum_{j=0}^{N} c_j \cdot \Psi^{\ell}[f,-2^j]((t,x'),\underline{w}') - \sum_{j=0}^{N} c_j \cdot \mathrm{Ext}(f,\ell)((0,x),\underline{w})\big)\\
&\qquad + \textstyle\sum_{j=N+1}^{\infty} |c_j| \cdot \mathfrak{p}\big(\Psi^{\ell}[f,-2^j]((t,x'),\underline{w}') - \mathrm{Ext}(f,\ell)((0,x),\underline{w})\big)\\
&\quad \le \mathfrak{p}\big(\textstyle\sum_{j=0}^{N} c_j \cdot \Psi^{\ell}[f,-2^j]((t,x'),\underline{w}') - \sum_{j=0}^{N} c_j \cdot \Xi[j]\big)\\
&\qquad + \mathfrak{p}\big(\textstyle\sum_{j=0}^{N} c_j \cdot \Xi[j] - \sum_{j=0}^{N} c_j \cdot \mathrm{Ext}(f,\ell)((0,x),\underline{w})\big)\\
&\qquad + \textstyle\sum_{j=N+1}^{\infty} |c_j| \cdot \mathfrak{p}\big(\Psi^{\ell}[f,-2^j]((t,x'),\underline{w}') - \mathrm{Ext}(f,\ell)((0,x),\underline{w})\big)\\
&\quad < \varepsilon,
\end{aligned}
$$

for all $t \in (0,\delta)$, $x' \in U_x$, $\underline{w}' \in O_{\underline{w}}$. ∎

4.4 The proof of Theorem 3.1.(2)

Let $E \in \mathsf{hlcVect}$, $(V,\mathfrak{V}) \in \Omega(E)$, $t \in (-\infty,0)$, $x \in \mathfrak{V}$, $\mathcal{B} \subseteq E$ bounded, $\mathfrak{p} \in \mathrm{Sem}(F)$, and $f \in \mathcal{C}^k_{\mathfrak{V}}((-\infty,0)\times V,F)$. We recall (3.2) as well as the seminorms in (2.8). The following estimates hold for each $\varsigma \le -1$:

- By (4.1) and (4.7), we have

$$\mathfrak{p}(\Psi^0[f,\varsigma](t,x)) \le \mathfrak{p}^0_{[\tau,0]\times\{x\}}(f). \tag{4.25}$$

- Let $1 \le \mathrm{s} \le k$. Then, for $1 \le \ell \le \mathrm{s}$ and $0 \le q \le \ell$, we have (recall (4.1), (4.2), (4.9), (4.25))

$$
\begin{aligned}
\mathfrak{p}\big(\boldsymbol{\Theta}_{\ell,q}[f,\varsigma]((t,x),\underline{w})\big) \le (\ell+1)! \cdot |\varsigma|^{\ell} \cdot \max(|\mathrm{I}_{\ell,0}|,\dots,|\mathrm{I}_{\ell,\ell}|)&\\
\cdot M_\ell \cdot \max(1,|\boldsymbol{\lambda}_{\ell,1}(\underline{w})|,\dots,|\boldsymbol{\lambda}_{\ell,\ell}(\underline{w})|)^{\ell}&\\
\cdot \mathfrak{p}^{\mathrm{s}}_{[\tau,0]\times\{x\}\times\mathscr{B}(\mathcal{B})}(f),&
\end{aligned}
\tag{4.26}
$$

for each $\underline{w} \in (\mathbb{R}\times\mathcal{B})^{\ell}$. We define

$$Q_{\mathrm{s}} := (\mathrm{s}+1)\cdot(\mathrm{s}+1)!\cdot M_{\mathrm{s}} \cdot \max\big(|\mathrm{I}_{\ell,p}|\,\big|\, 1 \le \ell \le \mathrm{s},\ 0 \le p \le \ell\big) \ge 1, \tag{4.27}$$

and obtain for $1 \le \ell \le \mathrm{s}$ from (4.10) and (4.26) that

$$
\begin{aligned}
&\mathfrak{p}\big(\Psi^{\ell}[f,\varsigma]((t,x),\underline{w})\big) \le |\varsigma|^{\ell}\cdot Q_{\mathrm{s}}\\
&\quad \cdot \max(1,|\boldsymbol{\lambda}_{\ell,1}(\underline{w})|,\dots,|\boldsymbol{\lambda}_{\ell,\ell}(\underline{w})|)^{\ell}\cdot \mathfrak{p}^{\mathrm{s}}_{[\tau,0]\times\{x\}\times\mathscr{B}(\mathcal{B})}(f)
\end{aligned}
\tag{4.28}
$$

holds for each $\underline{w} \in (\mathbb{R}\times\mathcal{B})^{\ell}$.

We are ready for the proof of Theorem 3.1.(2).

Proof of Theorem 3.1.(2) Let $E \in \mathsf{hlcVect}$, $(V,\mathfrak{V}) \in \Omega(E)$, $t \in (-\infty,0)$, $x \in \mathfrak{V}$, $\mathcal{B} \subseteq E$ bounded, $\mathfrak{p} \in \mathrm{Sem}(F)$, and $f \in \mathcal{C}^k_{\mathfrak{V}}((0,\infty)\times V,F)$.

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- Let s = 0. By Property (ii) we have $C_0 := \sum_{j=0}^{\infty} |c_j| < \infty$. We obtain from (4.25) and the triangle inequality that

$$\mathfrak{p}\big(\mathrm{Ext}(\mathcal{E}_{-\infty,\tau,0}(E,V,\mathfrak{V})(f),0)(t,x)\big) \leq C_0 \cdot \mathfrak{p}_{[\tau,0]\times\{x\}}(f).$$

- Let $1 \leq \mathrm{s} \leq k$. We choose $Q_\mathrm{s} \geq 1$ as in (4.27), and define

$$C_\mathrm{s} := Q_\mathrm{s} \cdot \max_{1\leq \ell \leq \mathrm{s}} \Big(\sum_{j=0}^{\infty} |c_j| \cdot (2^j)^\ell \Big) \overset{\mathrm{ii)}}{<} \infty. \tag{4.29}$$

  Then $C_\mathrm{s} \geq 1$ holds, as we have $Q_\mathrm{s} \geq 1$ as well as $\sum_{j=0}^{\infty} |c_j| \cdot (2^j)^\ell \geq \sum_{j=0}^{\infty} c_j = 1$ for $1 \leq \ell \leq \mathrm{s}$ by Property (i). We obtain from (4.28) that

$$\begin{aligned}\mathfrak{p}(\mathrm{Ext}\big(\mathcal{E}_{-\infty,\tau,0}(E,V,\mathfrak{V})(f),\ell\big)((t,x),\underline{w})\big)\\ \leq C_\mathrm{s} \cdot \max(1, |\boldsymbol{\lambda}_{\ell,1}(\underline{w})|, \ldots, |\boldsymbol{\lambda}_{\ell,\ell}(\underline{w})|)^\ell \cdot \mathfrak{p}^\mathrm{s}_{[\tau,0]\times\{x\}\times\mathscr{B}(\mathcal{B})}(f),\end{aligned}$$

  holds for each $1 \leq \ell \leq \mathrm{s}$ and $\underline{w} \in (\mathbb{R}\times\mathcal{B})^\ell$. ■

Appendix A.1. Some details to Example 3.10.(b)

Let $(\mathscr{H}, \langle\cdot,\cdot\rangle)$ be a real pre-Hilbert space, $E \in \mathsf{hlcVect}$ and $H = \mathscr{H} \times E$. We set $\xi(\cdot) := \sqrt{\langle\cdot,\cdot\rangle}$, fix $0 < \tau < 1$, and define $\mathcal{S}$ and $\check{\mathcal{S}}$ as in Remark 3.9. Given $Z \in \mathcal{A} = \xi^{-1}(\{1\})$, we set

$$\begin{aligned} Z_\perp &:= \{X \in \mathscr{H} \mid \langle Z, X\rangle = 0\},\\ \mathscr{D}(Z) &:= \{X \in Z_\perp \mid \xi(X) < 1\},\\ \mathscr{C}(Z) &:= \{X \in \mathscr{H} \mid \xi(X) > 0 \quad \wedge \quad \langle Z, X\rangle > 0\}.\end{aligned}$$

The following maps are smooth and inverse to each other:

$$\psi_Z\colon \quad \mathscr{C}(Z) \to (0,\infty)\times\mathscr{D}(Z), \qquad X \mapsto \Big(\xi(X), \tfrac{1}{\xi(X)}\cdot X - \tfrac{1}{\xi(X)}\cdot\langle X, Z\rangle\cdot Z\Big)$$

$$\phi_Z\colon (0,\infty)\times\mathscr{D}(Z) \to \mathscr{C}(Z) \qquad (t,Y) \mapsto t\cdot\Big(Y + \sqrt{1-\xi(Y)^2}\cdot Z\Big).$$

Now, given $g \in \mathcal{C}^k_{\check{\mathcal{S}}\times\mathfrak{V}}(\mathcal{S}\times V, F)$, the same arguments as in Remark 3.9 show that it suffices to construct

$$\text{an extension} \quad \tilde{f} \in \mathcal{C}^k_{\mathfrak{V}}(\mathcal{W}\times V, F) \quad \text{of the restriction}$$
$$f := g|_{\mathcal{O}\times V} \in \mathcal{C}^k_{\check{\mathcal{O}}\times\mathfrak{V}}(\mathcal{O}\times V, F) \tag{4.30}$$

in order to obtain an extension $\tilde{g} \in \mathcal{C}^k_{\mathfrak{V}}(\mathscr{H}\times V, F)$ of g. For this, we proceed as follows:

- We have by Lemma 2.10

$$f_Z := f \circ (\phi_Z|_{(0,1)\times\mathscr{D}(Z)} \times \mathrm{id}_V) \in \mathcal{C}^k_{(0,1]\times\mathscr{D}(Z)\times\mathfrak{V}}((0,1)\times\mathscr{D}(Z)\times V, F)$$

  with $\mathrm{Ext}(f_Z, 0) = \mathrm{Ext}(f,0)\circ(\phi_Z|_{(0,1]\times\mathscr{D}(Z)} \times \mathrm{id}_{\mathfrak{V}})$.
- First applying the extension operator $\mathcal{E}_{0,\tau,1}(Z_\perp\times E, \mathscr{D}(Z)\times V, \mathscr{D}(Z)\times\mathfrak{V})$ from Theorem 3.1, and then composing with $\psi_Z \times \mathrm{id}_V$, we obtain (from Lemma 2.10) an extension

$$\tilde{f}_Z \in \mathcal{C}^k_{\mathscr{C}(Z)\times\mathfrak{V}}(\mathscr{C}(Z)\times V, F) \qquad \text{of the restriction} \qquad f|_{(\mathcal{O}\cap\mathscr{C}(Z))\times V}.$$

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- Given $Z' \in \mathcal{A}$ and $X \in \mathscr{C}(Z) \cap \mathscr{C}(Z')$, then the definitions (and continuity) ensure that for $Y := (\mathrm{pr}_2 \circ \psi_Z)(X)$ and $Y' := (\mathrm{pr}_2 \circ \psi_{Z'})(X)$, we have

$$\begin{aligned} \mathrm{Ext}(f_Z, 0)(t, Y, x) &= \mathrm{Ext}(f, 0)(t/\xi(X), X, x) \\ &= \mathrm{Ext}(f_{Z'}, 0)(t, Y', x) \qquad \forall\, t \in [\tau, 1],\ x \in \mathfrak{V}. \end{aligned}$$

Theorem 3.1.(3) implies $\tilde{f}_Z|_{((0,\infty)\cdot X)\times V} = \tilde{f}_{Z'}|_{((0,\infty)\cdot X)\times V}$, and we conclude

$$\tilde{f}_Z|_{\mathscr{C}(Z)\cap\mathscr{C}(Z')} = \tilde{f}_{Z'}|_{\mathscr{C}(Z)\cap\mathscr{C}(Z')}.$$

Since $U := \mathscr{C}(Z) \cap \mathscr{C}(Z')$ is open, we obtain for $0 \leq \ell \leq k$:

$$\begin{aligned} & \mathrm{d}^\ell \tilde{f}_Z|_{U\times V\times H^\ell} = \mathrm{d}^\ell \tilde{f}_{Z'}|_{U\times V\times H^\ell} \\ \overset{\text{continuity}}{\Longrightarrow}\quad & \mathrm{Ext}(\tilde{f}_Z, \ell)|_{U\times\mathfrak{V}\times H^\ell} = \mathrm{Ext}(\tilde{f}_{Z'}, \ell)|_{U\times\mathfrak{V}\times H^\ell}. \end{aligned}$$

It follows that the maps $\{\tilde{f}_Z\}_{Z\in\mathcal{A}}$ glue together to an extension $\tilde{f} \in \mathfrak{C}^k_{\mathfrak{V}}(\mathcal{W} \times V, F)$ of (4.30).

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