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Dynamics of Newton maps

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Abstract. In this paper, we study the dynamics of the Newton maps for arbitrary polynomials. Let p be an arbitrary polynomial with at least three distinct roots, and f be its Newton map. It is shown that the boundary ∂B of any immediate root basin B of f is locally connected. Moreover, ∂B is a Jordan curve if and only if $\deg(f|_B) = 2$. This implies that the boundaries of all components of root basins, for the Newton maps for all polynomials, from the viewpoint of topology, are tame.

Key words: Newton maps, puzzles, local connectivity
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1. *Introduction*

Newton's method, also known as the *Newton–Raphson method* named after Isaac Newton (1642–1727) and Joseph Raphson (1648–1715), is probably the oldest and most famous iterative process to be found in mathematics. The method was first proposed to find successively better approximations to the roots (or zeros) of a real-valued function $p(z)$. Picking an initial point z_0 near a root of p, Newton's method produces an nth approximation of the root via the formula $z_{n+1} = f_p(z_n)$, where

$$f_p(z) = z - \frac{p(z)}{p'(z)}$$

is called the *Newton map* of p. Replacing z_n by z_{n+1} generates a sequence of approximations $\{z_n\}$ which may or may not converge to a root of p.

A brief history of Newton's method, following [A], is as follows. Versions of Newton's method had been in existence for centuries previous to Newton and Raphson. Anticipations of Newton's method are found in an ancient Babylonian iterative method of approximating

the square root of a,

$$z_{n+1} = \frac{1}{2}\left(z_n + \frac{a}{z_n}\right),$$

which is equivalent to Newton's method for the function $f(z) = z^2 - a$. The modern formulation of the method is also attributed to Thomas Simpson (1710–1761) and Joseph Fourier (1768–1830).

By the mid-1800s, several mathematicians had already examined the convergence of Newton's method towards the real roots of an equation $p(z) = 0$, but the investigations of Ernst Schröder (1841–1902) and Arthur Cayley (1821–1895) are distinguished from their predecessors in their consideration of the convergence of Newton's method to the complex roots of $p(z) = 0$.

Schröder and Cayley each studied the convergence of Newton's method for the quadratic polynomials, and both showed that on either side of the perpendicular bisector of the roots, Newton's method converges to the root on that particular side. However, in 1879, Cayley [C] first noticed the difficulties in generalizing Newton's method to cubic polynomials, or general polynomials with at least three distinct roots. In [C], Cayley wrote:

'*The solution is easy and elegant in the case of a quadratic equation, but (Newton's method for) the next succeeding case of the cubic equation appears to present considerable difficulty.*'

The study of Newton's method led to the theory of iterations of holomorphic functions, as initiated by Pierre Fatou and Gaston Julia around the 1920s. Since then, the study of Newton maps became one of the major themes with general interest, both in discrete dynamical system (pure mathematics), and in root-finding algorithm (applied mathematics), see for example [AR, Ba, Be, BFJK1, BFJK2, HSS, Pr, Ro07, Ro08, RWY, Sh, Tan].

Let p be a polynomial with at least two distinct roots (the discussion is trivial when p has only one (possibly multiple) root), and let $\zeta \in \mathbb{C}$ be a root of p. For its Newton map f_p, the *attracting basin* or *root basin* of ζ, denoted by $B(\zeta)$, consists of points z on the Riemann sphere $\widehat{\mathbb{C}}$ whose orbit $\{f_p^n(z); n \in \mathbb{N}\}$ (here g^n means the nth iterate of g) converges to ζ:

$$B(\zeta) = \{z \in \widehat{\mathbb{C}};\ f_p^n(z) \to \zeta \text{ as } n \to +\infty\}.$$

It is well known that $B(\zeta)$ is an open set of $\widehat{\mathbb{C}}$. In the case that p has two distinct (possibly multiple) roots, by quasi-conformal surgery, one can show that $B(\zeta)$ is a quasi-disk and the Julia set $J(f_p)$ is a quasi-circle. So this case is easy.

We say that a polynomial p is *non-trivial* (in the sense of Cayley) if p has at least three distinct roots. A non-trivial polynomial takes the form

$$p(z) = a(z - a_1)^{n_1} \cdots (z - a_d)^{n_d},$$

where $a \in \mathbb{C} - \{0\}$, $d \geq 3$, and $a_1, \ldots, a_d \in \mathbb{C}$ are distinct roots of p, with multiplicities $n_1, \ldots, n_d \geq 1$. This is the general case and the attracting basin $B(\zeta)$ consists of countably many connected components. The one containing ζ is called the *immediate attracting basin* or *immediate root basin*, and is denoted by $B^0(\zeta)$. Przytycki [Pr] showed

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that $B^0(\zeta)$ is a topological disk when p is a non-trivial cubic polynomial. By means of quasi-conformal surgery, Shishikura [**Sh**] proved that the Julia set of the Newton map for any non-trivial polynomial is connected. This result is further generalized to Newton's method for entire functions by Baranski *et al* [**BFJK1**, **BFJK2**]. This implies, in particular, each component of $B(\zeta)$ is a topological disk.

Although $B = B^0(\zeta)$ has a simple topology, its boundary ∂B exhibits rich topological structures. The reason is that the Newton map f_p can have unpredictable dynamics and complicated bifurcations on ∂B. Therefore, for Newton maps, understanding the topology of ∂B makes a fundamental and challenging problem from the view point of dynamical system.

Little progress had been made towards the problem until the ground-breaking work of Roesch. In [**Ro08**], Roesch proved, building on previous works of Head [**He**] and Tan [**Tan**], that ∂B is always a Jordan curve, when p is a non-trivial cubic polynomial and $\deg(f_p|_B) = 2$. The proof is the first successful application of the Branner–Hubbard–Yoccoz puzzle theory to rational maps. The puzzle theory has also been developed by Roesch, Wang, and Yin [**RWY**] to study the local connectivity and rigidity phenomenon in parameter space.

The main result of the paper is to give a complete characterization of ∂B for the Newton maps for all polynomials.

THEOREM 1.1. *Let f_p be the Newton map for any non-trivial polynomial p. Then the boundary ∂B of any immediate root basin B is locally connected. Moreover, ∂B is a Jordan curve if and only if* $\deg(f_p|_B) = 2$.

The theorem implies that the boundary of each component of the root basins is locally connected. Therefore, *the boundaries of all components of root basins, for the Newton maps for all polynomials, from the viewpoint of topology, are tame*. Our argument also has a byproduct: the Julia set of a non-renormalizable Newton map is always locally connected, which generalizes Yoccoz's famous theorem to Newton maps.

Our work extends Roesch's theorem [**Ro08**, Theorem 6] for cubic Newton maps to Newton maps of arbitrary polynomials.

It is distinguished from Roesch's work [**Ro08**] in two ways. First, the invariant graph is different from those in [**Ro08**]. In our work, we construct only one graph adapted to the puzzle theory: the one generated by the *channel graph*, while in [**Ro08**], countably many candidate graphs are provided, and each of them involves very technical construction. Second, each cubic Newton map has only one free critical point, so the puzzle theory in [**Ro08**] is the same as the quadratic case; however, the Newton maps for higher degree non-trivial polynomials can have more free critical points, and the quadratic puzzle theory does not work here. To deal with this general case, we take advantage of recent developments [**KL1**, **KL2**, **KSS**] in multi-critical polynomial dynamics.

1.1. *Organization of the paper.* The paper is organized as follows.

In §2, we present some basic facts for Newton maps.

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In §3, we develop a method to count the number of poles (counting a suitable multiplicity) for Newton maps in certain domains arising from dynamics. This allows us to construct an invariant graph for Newton maps by an inductive procedure in §4. This graph is used to develop the puzzle theory.

In §5, we introduce the Branner–Hubbard–Yoccoz puzzle theory and sketch the idea of the proof, whose details are carried out in the forthcoming sections. The strategy is deeply inspired by the work of Roesch and Yin [**RY**].

To prove the local connectivity of ∂B, for each $z \in \partial B$, we define its *end* $\mathbf{e}(z)$ as the intersection of infinitely many nested puzzle pieces containing z. The main point is to show that $\mathbf{e}(z) \cap \partial B = \{z\}$. For this purpose, we need to treat two cases: the wandering case and the renormalizable case.

In §6, we will show that each wandering end is a singleton. This is based on the dichotomy: a wandering end $\mathbf{e}$ either satisfies the *bounded degree* property or its combinatorial limit set $\omega(\mathbf{e})$ contains a *persistently recurrent* critical end. The treatments for these two cases are different: the former needs to control the number of critical points in long orbits of puzzle pieces, while the latter makes essential use of recent developments in multi-critical polynomial dynamics, especially the *principle nest* construction and its properties [**KL1**, **KL2**, **KSS**].

In §7, we handle the *renormalizable case*. We will show that if $\mathbf{e}(z)$ is periodic and non-trivial, then $\mathbf{e}(z) \cap \partial B = \{z\}$. The main idea is to construct an invariant curve which separates the end $\mathbf{e}(z)$ from B. The construction is natural and less technical (compare [**Ro08**]). The idea is new and can be applied to study other rational maps.

In §8, we complete the proof of the main theorem.

1.2. *Notation.* Throughout the paper, we will use the following notation.

(1) $\widehat{\mathbb{C}}$, $\mathbb{C}$, and $\mathbb{D}$ are the Riemann sphere, the complex plane, and the unit disk, respectively. The boundary of $\mathbb{D}$ is denoted by $\mathbb{S}$.

(2) Let A be a set in $\widehat{\mathbb{C}}$. The closure and the boundary of A are denoted by $\overline{A}$ and ∂A, respectively. We denote by $\mathrm{Comp}(A)$ the collection of all connected components of A. The cardinality of A is $\#A$.

(3) Given two subsets A and B of $\widehat{\mathbb{C}}$, we say that $A \Subset B$ if A is contained in the interior of B.

(4) The Julia set and Fatou set of a rational map f are denoted by $J(f)$ and $F(f)$, respectively.

2. *Preliminaries*

This section collects some basic facts and introduces some notation for Newton maps.

Let p be a complex polynomial, factored as

$$p(z) = a(z - a_1)^{n_1} \cdots (z - a_d)^{n_d},$$

where $a \neq 0$ and $a_1, \ldots, a_d \in \mathbb{C}$ are distinct roots of p, with multiplicities $n_1, \ldots, n_d \geq 1$. In our discussion, we may assume $d \geq 2$.

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Its Newton map f_p fixes each root a_k with multiplier

$$f_p'(a_k) = \frac{p(z)p''(z)}{p'(z)^2}\bigg|_{z=a_k} = \frac{n_k - 1}{n_k}.$$

Therefore, each root a_k of p corresponds to an attracting fixed point of f_p with multiplier $1 - 1/n_k$. It follows from the equation

$$\frac{1}{f_p(z) - z} = -\sum_{k=1}^{d} \frac{n_k}{z - a_k}$$

that the degree of f_p equals d, the number of distinct roots of p. One may also verifies that ∞ is a repelling fixed point of f_p with multiplier

$$\lambda_\infty = \frac{\sum_{k=1}^{d} n_k}{\sum_{k=1}^{d} n_k - 1} = \frac{\deg(p)}{\deg(p) - 1}.$$

From the above discussion, we see that a degree-d Newton map has $d + 1$ distinct fixed points with specific multipliers. However, a well-known theorem of Head states that the fixed points together with the specific multipliers can determine a unique Newton map.

THEOREM 2.1. (Head [**He**]) *A rational map f of degree $d \geq 2$ is the Newton map of a polynomial p if and only if f has $d + 1$ distinct fixed points*

$$a_1, a_2, \ldots, a_d, \infty,$$

such that for each fixed point a_k, the multiplier takes the form

$$f'(a_k) = 1 - 1/n_k \quad \text{with } n_k \in \mathbb{N},\ 1 \leq k \leq d.$$

In this case, the polynomial p has the form $a(z - a_1)^{n_1} \cdots (z - a_d)^{n_d},\ a \neq 0$.

Now, for the Newton map $f = f_p$ of p, let $B(a_k)$ be the root basin of a_k and B_k be the immediate root basin of a_k. Recall that

$$B(a_k) = \{z \in \widehat{\mathbb{C}};\ f^n(z) \to a_k \text{ as } n \to +\infty\}.$$

The attracting basin for all roots is

$$B_f = B(a_1) \cup \cdots \cup B(a_d).$$

We say that f is *post-critically finite* in B_f if there are only finitely many post-critical points in B_f, or equivalently, each critical point in B_f will eventually be iterated to one critical point of a_k.

According to Shishikura [**Sh**], the Julia set of a Newton map f is always connected, or equivalently, all Fatou components of f are simply connected (see Figure 1). By means of quasi-conformal surgery, one can show that f is quasi-conformally conjugate, in a neighborhood of of $\widehat{\mathbb{C}} - B_f$, to a Newton map g which is post-critically finite in its root basin B_g. Because the topology of the Julia set $J(f)$ does not change under this conjugacy, throughout the paper, we pose the following.

Assumption 2.2. The Newton map f is post-critically finite in B_f.

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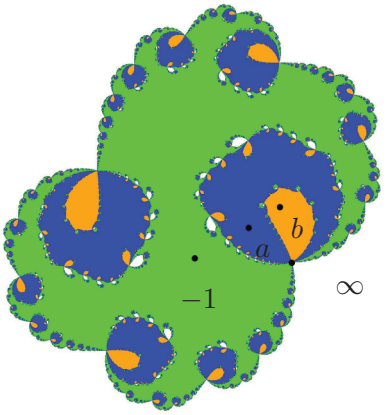

FIGURE 1. Image of the Julia set $J(f)$ under the action of the Möbius map $h(z) = z/(z-1)$, where f is the Newton map for the polynomial $p(z) = (z^2-1)(z-a)(z-b)$ with $a = -1.142 - 2.0477\,i$ and $b = 0.1667 - 3.15485\,i$.

Under Assumption 2.2, if the degree d of f is two, then f is affinely conjugate to z^2. In this case, the collection $\mathrm{Comp}(B_f)$ of all components of B_f consists of only two elements. In other situations, $\mathrm{Comp}(B_f)$ consists of infinitely many elements.

A virtue of Assumption 2.2 is that one can give a natural dynamical parameterization of root basins (see [Mi06]).

LEMMA 2.3. *Assume f is post-critically finite in B_f, then there exist so-called Böttcher maps, $\{\Phi_B\}_{B\in\mathrm{Comp}(B_f)}$, such that for each $B \in \mathrm{Comp}(B_f)$:*

(1) $\Phi_B : B \to \mathbb{D}$ *is a conformal map;*

(2) $\Phi_{f(B)} \circ f \circ \Phi_B^{-1}(z) = z^{d_B}$, *for all $z \in \mathbb{D}$, where $d_B = \deg(f|_B)$.*

In general, for each $B \in \mathrm{Comp}(B_f)$, the Böttcher map Φ_B is not unique. There are $d_B - 1$ choices of Φ_B when $f(B) = B$, and d_B choices of Φ_B when $f(B) \neq B$ and $\Phi_{f(B)}$ is determined. Once we fix a choice of Böttcher maps $\{\Phi_B\}_{B\in\mathrm{Comp}(B_f)}$, we may define the *internal rays*, as follows.

For each $B \in \mathrm{Comp}(B_f)$, the point $\Phi_B^{-1}(0)$ is called the *center* of B, and the Jordan arc

$$R_B(\theta) := \Phi_B^{-1}(\{re^{2\pi i\theta} : 0 < r < 1\})$$

is called the *internal ray* of angle θ in B. According to a well-known landing theorem [Mi06, Theorem 18.10], when θ is rational, the internal ray $R_B(\theta)$ always lands (that is the limit $\lim_{r\to 1^-} \Phi_B^{-1}(re^{2\pi i\theta})$ exists). A number $r \in (0, 1)$ and two rational angles θ_1, θ_2 induce a *sector*:

$$S_B(\theta_1, \theta_2; r) := \Phi_B^{-1}(\{t\, e^{2\pi i\theta} : r < t < 1, \theta_1 < \theta < \theta_2\}),$$

here $\theta_1 < \theta < \theta_2$ means that the angles $\theta_1, \theta, \theta_2$ sit in the circle in the counter clock-wise order.

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3. *Counting number of poles*

In this section, we develop a method to count the number of poles (counting suitable multiplicity) for Newton maps f in certain domains (which arise from dynamics). We will show that in the domains we consider, the number of poles is strictly less than the number of Jordan curves which bound the domain. This fact allows us to construct an invariant graph for Newton maps by an inductive procedure (see §4).

3.1. *Counting number of fixed points.*

By a *graph* we mean a connected and compact subset of $\widehat{\mathbb{C}}$, written as the disjoint union of finitely many points (called *vertices*) and finitely many open Jordan arcs (called *edges*), any two of which touch only at vertices. A graph can contain a loop.

Let G be a graph. For any $z \in G$, let $\nu(G, z)$ be the number of components of $G \setminus \{z\}$. We call z a *cut point* of G if $\nu(G, z) \geq 2$ ($\Longleftrightarrow G \setminus \{z\}$ is disconnected), a *non-cut point* of G if $\nu(G, z) = 1$ ($\Longleftrightarrow G \setminus \{z\}$ is connected). Observe that all components in $\widehat{\mathbb{C}} \setminus G$ are Jordan disks if and only if all $z \in G$ are non-cut points.

In our discussion, by a *Jordan domain* or *Jordan disk*, we mean an open subset of $\widehat{\mathbb{C}}$ whose boundary is a Jordan curve. A *pre-Jordan domain* W means a connected component of $g^{-1}(D)$, where D is a Jordan disk and g is a rational map. Here the boundary ∂D may or may not contain critical values of g. If ∂D contains no critical value of g, then each boundary component of W is a Jordan curve; if ∂D contains at least one critical value of g, then each component of ∂W can be written as a union of finitely many Jordan curves, touching at critical points. In either case, for any component γ of ∂W, the map $g|_\gamma : \gamma \to \partial D$ has a well-defined degree, denoted by $\deg(g|_\gamma)$. It is equal to the number of preimages $g^{-1}(q)$ on γ of a point $q \in \partial D$ which is not a critical value.

One may observe that *for any pre-Jordan domain W, any component V of $h^{-1}(W)$ is again a pre-Jordan domain, here h is a rational map*. To see this, note that W is a component of $g^{-1}(D)$ (D is a Jordan disk) and V is a component of $(g \circ h)^{-1}(D)$, where $g \circ h$ is a rational map.

Let U be a pre-Jordan domain and D be a Jordan disk in $\widehat{\mathbb{C}}$ such that $U \subseteq D$. The *filled closure* of U with respect to D, denoted by $\widehat{U}_D$, is

$$\widehat{U}_D = \overline{U} \cup \bigcup_V \overline{V},$$

where V ranges over all components of $\widehat{\mathbb{C}} \setminus \overline{U}$ with $V \subseteq D$. See Figure 2.

It is easy to verify the following facts.

(1) $\overline{U} \subseteq \widehat{U}_D \subseteq \overline{D}$ and $\widehat{U}_D = \widehat{U}_{D'}$ for any Jordan disk D' containing D.

(2) The filled closure $\widehat{U}_D$ is always a Jordan disk. To see this, if $\widehat{U}_D$ is not a Jordan disk, then $\partial \widehat{U}_D$ is not a Jordan curve, and can be written as a union of finitely many Jordan curves, say $\alpha_1, \ldots, \alpha_k$ with $k \geq 2$, such that the intersection of any two curves is a finite set. These curves enclose mutually disjoint Jordan disks, $D_1, \ldots, D_k$, in $\widehat{\mathbb{C}} - \overline{U}$. Note that $\partial D \subset \overline{D}_j$ for some j, however this will contradict the definition of $\widehat{U}_D$.

(3) $\widehat{U}_D = \overline{U}$ if and only if U is a Jordan disk.

The following fixed point theorem appears in [**RS**, Theorem 4.8].

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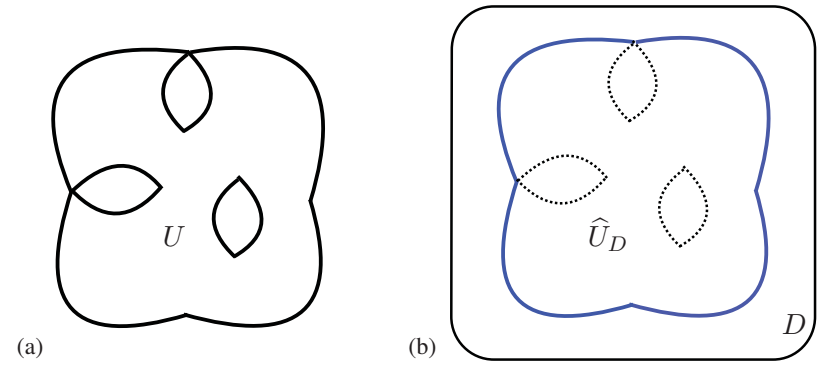

FIGURE 2. An example of pre-Jordan domain U (a) and its filled closure $\widehat{U}_D$ with respect to D (b). Clearly $\widehat{U}_D$ is a Jordan disk bounded by a blue curve.

LEMMA 3.1. *Let $D \subseteq \widehat{\mathbb{C}}$ be a Jordan disk and g be a rational map of degree at least two. Suppose that $g^{-1}(D)$ has a component $U \subseteq D$. If $\partial D \cap \partial U$ contains a fixed point q, we further require that q is repelling and $g(N_q \cap \partial U) \supseteq N_q \cap \partial U$ in a neighborhood N_q of q. Then*

$$\#\mathrm{Fix}(g|_{\overline{U}}) = \deg(g|_{\partial U}).$$

Here the number of fixed points is counted with multiplicity. Recall that the *multiplicity* of a fixed point $z_0 \in \mathbb{C}$ is defined to be the unique integer $m \geq 1$ such that, near z_0,

$$g(z) - z = a_m(z - z_0)^m + a_{m+1}(z - z_0)^{m+1} + \cdots$$

with $a_m \neq 0$. The number m is independent of the choice of coordinates.

As a consequence of Lemma 3.1, one has the following corollary.

COROLLARY 3.2. *Let D be a Jordan disk in $\widehat{\mathbb{C}}$ and g be a rational map of degree at least two. Suppose that $g^{-1}(D)$ has a component $U \subseteq D$. If $\partial D \cap \partial U$ contains a fixed point q, we further require that q is repelling and $g(N_q \cap \partial U) \supseteq N_q \cap \partial U$ in a neighborhood N_q of q. Then*

$$\#\mathrm{Fix}(g|_{\widehat{U}_D}) = \sum_V \deg(g|_{\partial V}),$$

where V runs over all components of $g^{-1}(D)$ such that $V \subseteq \widehat{U}_D$.

In particular, if $\widehat{U}_D$ contains only one fixed point (counting multiplicity), then U is a Jordan disk ($\Longleftrightarrow \overline{U} = \widehat{U}_D$) and $g : U \to D$ is a homeomorphism.

Proof. Let $V_1 = U, V_2, \ldots, V_n$ (respectively $V_1', \ldots, V_m'$) be all the components of $g^{-1}(D)$ (respectively $g^{-1}(\widehat{\mathbb{C}} \setminus \overline{D})$) in the filled closure $\widehat{U}_D$. Then by definition,

$$\widehat{U}_D = \overline{V}_1 \cup \cdots \cup \overline{V}_n \cup V_1' \cdots \cup V_m'.$$

These V_k' terms are clearly disjoint from fixed points. For distinct $\overline{V}_i, \overline{V}_j$, the intersection $\overline{V}_i \cap \overline{V}_j$ is a finite set because it is contained in the critical set of g. Further, if $\overline{V}_i \cap \overline{V}_j$ contains a fixed point, say q, of g, then q is a critical point and hence a superattracting

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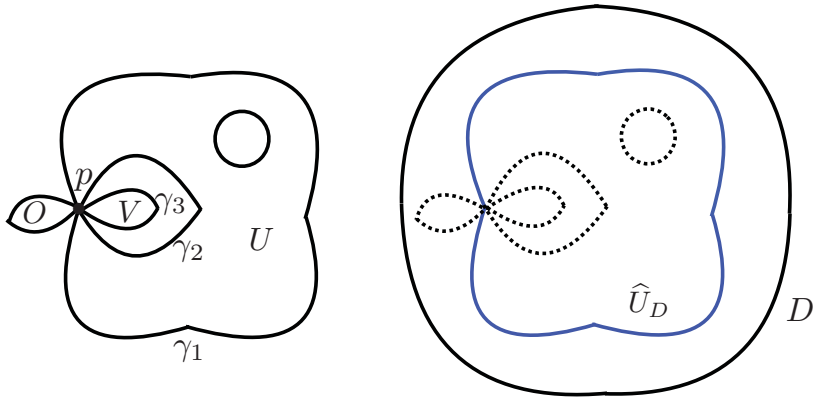

FIGURE 3. In this example, there are three components of $g^{-1}(D)$ contained in D. They are U, V, and O. Their boundaries touch at p. The filled closure $\widehat{U}_D$ contains U, V. Clearly, $\partial U = \gamma_1 \cup \gamma_2$, $\partial V = \gamma_3$. Moreover, ∂U meets p twice and ∂V meets p once, and hence $m(p, \widehat{U}_D) = 2 + 1 = 3$. Note also $\alpha(p) = \gamma_1 \cup \gamma_2 \cup \gamma_3$ and $\nu(\alpha(p), p) = 3 = m(p, \widehat{U}_D) < \deg(g, p) = 4$.

fixed point. Moreover, we have $q \in \partial V_i \cap \partial D$, and this implies that q is also on $\partial U \cap \partial D$. However, this contradicts our assumption. Therefore, there is no fixed point on $\partial V_i \cap \partial V_j$.

It follows from Lemma 3.1 that

$$\#\mathrm{Fix}(g|_{\widehat{U}_D}) = \sum_{1\le k\le n} \#\mathrm{Fix}(g|_{\overline{V}_k}) = \sum_{1\le k\le n} \deg(g|_{\partial V_k}).$$

This equality implies that if $\#\mathrm{Fix}(g|_{\widehat{U}_D}) = 1$, then U is the unique component of $g^{-1}(D)$ in D and $\deg(g|_U) = 1$, which proves the statement. □

*Remark 3.3. (*What is multiplicity?) In Corollary 3.2, the sum

$$\sum_{1\le k\le n} \deg(g|_{\partial V_k})$$

is the cardinality $\#(g^{-1}(q) \cap \widehat{U}_D)$ for (any) $q \in \partial D$, counting multiplicity.

The multiplicity $m(p, \widehat{U}_D)$ of $p \in g^{-1}(q) \cap \widehat{U}_D$ is an integer between 1 and the local degree $\deg(g, p)$. A natural definition is as follows (see Figure 3).

Let $\mathcal{F}(p, \widehat{U}_D)$ consist of those $V \in \{V_1, \ldots, V_n\}$ so that $p \in \partial V$. For each $V \in \mathcal{F}(p, \widehat{U}_D)$, the boundary ∂V has a natural positive orientation so that the region V is on the right if one moves along ∂V in the orientation.

If one moves along ∂V in positive orientation, it is possible that one passes through p once or several times. Let $T(p, \partial V) \ge 1$ be the number of the times that one passes through p. The multiplicity $m(p, \widehat{U}_D)$ is defined by

$$m(p, \widehat{U}_D) = \sum_{V \in \mathcal{F}(p, \widehat{U}_D)} T(p, \partial V).$$

Let $\alpha(p)$ be the component of $g^{-1}(\partial D) \cap \widehat{U}_D$ containing p. One may show

$$m(p, \widehat{U}_D) = \nu(\alpha(p), p).$$

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Hence, we have the identity

$$\sum_{p\in g^{-1}(q)\cap \widehat{U}_D} \nu(\alpha(p), p) = \sum_{p\in g^{-1}(q)\cap \widehat{U}_D} m(p, \widehat{U}_D) = \sum_{1\le k\le n} \deg(g|_{\partial V_k}).$$

3.2. *The inverse image of a Jordan curve.* Let γ be a Jordan curve in $\widehat{\mathbb{C}}$. Its complement $\widehat{\mathbb{C}} - \gamma$ has two components, one is called the *interior part* of γ, denoted by $\mathrm{Int}(\gamma)$, while the other is called the *exterior part* of γ, denoted by $\mathrm{Ext}(\gamma)$. The designation of the interior or exterior parts is arbitrary at this moment.

Let g be a rational map. Suppose there is a component U of $g^{-1}(\mathrm{Ext}(\gamma))$ contained in $\mathrm{Ext}(\gamma)$. Let $\widehat{U}$ be the filled closure of U with respect to $\mathrm{Ext}(\gamma)$. The *inverse image* γ^{-1} of γ, with respect to g, is the Jordan curve

$$\gamma^{-1} = \partial\widehat{U}.$$

One may verify that $g(\gamma^{-1}) = \gamma$ and γ^{-1} is contained in (possibly equal to) a connected component, say α, of $g^{-1}(\gamma)$. Moreover, the degrees of $g|_{\gamma^{-1}}$ and $g|_\alpha$ are well defined, and satisfy

$$\deg(g|_{\gamma^{-1}}) \le \deg(g|_\alpha) \le \deg(g).$$

The equality $\deg(g|_{\gamma^{-1}}) = \deg(g|_\alpha)$ holds if and only if $\gamma^{-1} = \alpha$. Applying the same operation to the new curve γ^{-1}, one gets $\gamma^{-2} = (\gamma^{-1})^{-1}$. Precisely, suppose $\mathrm{Ext}(\gamma^{-1})$ is given (remark: we do not specify $\mathrm{Ext}(\gamma^{-1})$ for the moment, but in §3.3, the choice of $\mathrm{Ext}(\gamma^{-1})$ will be clear), and let V be a component $g^{-1}(\mathrm{Ext}(\gamma^{-1}))$ contained in $\mathrm{Ext}(\gamma^{-1})$ and $\widehat{V}$ be the filled closure of V with respect to $\mathrm{Ext}(\gamma^{-1})$, we set $\gamma^{-2} = (\gamma^{-1})^{-1} = \partial\widehat{V}$. Similarly, for any integer $n \ge 1$, the curve γ^{-n} can be defined inductively:

$$\gamma^{-n} = (\gamma^{-n+1})^{-1},$$

with the property $\mathrm{Ext}(\gamma^{-n}) \subseteq \mathrm{Ext}(\gamma^{-n+1}) \subseteq \cdots \subseteq \mathrm{Ext}(\gamma^{-1})$.

We remark that the only ambiguity in the definition of γ^{-1} occurs when we are choosing the component U of $g^{-1}(\mathrm{Ext}(\gamma))$. There might be several components of $g^{-1}(\mathrm{Ext}(\gamma))$ contained in $\mathrm{Ext}(\gamma)$, and U is not unique. However, the readers do not need to worry about that because in the following discussion, we actually choose some specific component U of $g^{-1}(\mathrm{Ext}(\gamma))$, and there will be no ambiguity then.

3.3. *Counting number of poles.* We say that the Jordan curves $\gamma_1, \ldots, \gamma_n$ with $n \ge 2$ in $\widehat{\mathbb{C}}$ are *independent*, if:

(1) $\gamma_i \cap \gamma_j$ is a finite set (possibly empty), for $i \ne j$;
(2) for any k, there is a component $\mathrm{Int}(\gamma_k)$ of $\widehat{\mathbb{C}} \setminus \gamma_k$, designated as the *interior part* of γ_k, such that the Jordan disks $\mathrm{Int}(\gamma_1), \ldots, \mathrm{Int}(\gamma_n)$ are mutually disjoint (see Figure 4).

Note that when we are saying that the curves $\gamma_1, \ldots, \gamma_n$ are independent, their interior part $\mathrm{Int}(\gamma_k)$ is determined. The other component of $\widehat{\mathbb{C}} \setminus \gamma_k$ is the *exterior part* of γ_k, denoted by $\mathrm{Ext}(\gamma_k)$. Let

$$A(\gamma_1, \ldots, \gamma_n) = \bigcap_{1\le k\le n} \mathrm{Ext}(\gamma_k) = \widehat{\mathbb{C}} - \bigcup_{1\le k\le n} \overline{\mathrm{Int}(\gamma_k)}.$$

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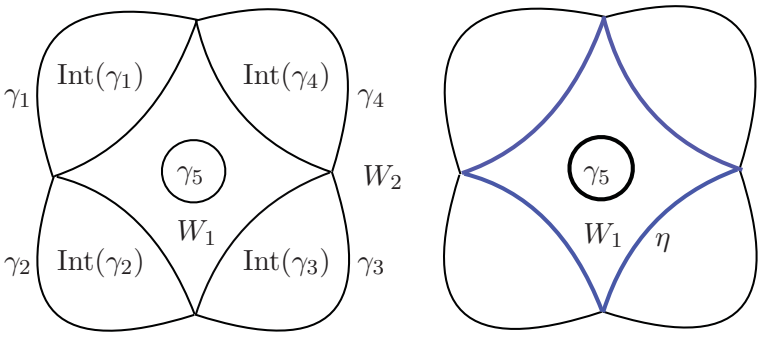

FIGURE 4. An example of independent Jordan curves $\gamma_1, \ldots, \gamma_5$ (a), here $A(\gamma_1, \ldots, \gamma_5) = W_1 \cup W_2$, and W_1 can be written as $A(\gamma_5, \eta)$, where γ_5, η are independent (b).

Clearly $A(\gamma_1, \ldots, \gamma_n)$ is an open set and has finitely many connected components. It is worth observing that for any component W of $A(\gamma_1, \ldots, \gamma_n)$, which is not a Jordan disk, there are independent Jordan curves $\eta_1, \ldots, \eta_m$ for some $m \geq 2$ such that $W = A(\eta_1, \ldots, \eta_m)$, see Figure 4.

PROPOSITION 3.4. *Let g be a rational map with ∞ a repelling fixed point. Let $\gamma_1, \ldots, \gamma_n$ be independent Jordan curves in $\widehat{\mathbb{C}}$ satisfying:*

(a) *$\gamma_i \cap \gamma_j = \{\infty\}$ for any $i \neq j$;*

(b) *all fixed points of g in $\mathbb{C}$ are contained in $\mathrm{Int}(\gamma_1) \cup \cdots \cup \mathrm{Int}(\gamma_n)$;*

(c) *in a neighborhood $N(\infty)$ of ∞, one has*

$$N(\infty) \cap \gamma_k \subseteq g(N(\infty) \cap \gamma_k), \quad \text{for all } 1 \leq k \leq n;$$

(d) *the unbounded component of $g^{-1}(\gamma_k)$ is contained in $\overline{\mathrm{Ext}}(\gamma_k)$.*

Then the unbounded component U_k of $g^{-1}(\mathrm{Ext}(\gamma_k))$ satisfies

$$U_k \subseteq \mathrm{Ext}(\gamma_k).$$

Further, let $\widehat{U}_k$ be the filled closure of U_k with respect to $\mathrm{Ext}(\gamma_k)$ and let $\gamma_k^{-1} = \partial \widehat{U}_k$. Then:

(1) *$\gamma_1^{-1}, \ldots, \gamma_n^{-1}$ are independent Jordan curves with $\overline{\mathrm{Ext}}(\gamma_k^{-1}) = \widehat{U}_k$ (see Figure 5);*

(2) *in each $\widehat{U}_k$, the number of poles (counting multiplicity, see Remark 3.3) equals that of fixed points;*

(3) *$g^{-1}(\mathrm{Ext}(\gamma_k))$ is disjoint from $\mathrm{Ext}(\gamma_k) \setminus \mathrm{Ext}(\gamma_k^{-1})$;*

(4) *the unbounded component of $g^{-1}(\overline{A(\gamma_1, \ldots, \gamma_n)})$ is contained in*

$$\overline{A(\gamma_1^{-1}, \ldots, \gamma_n^{-1})}.$$

Proof. Let α_k be the unbounded component of $g^{-1}(\gamma_k)$. The set α_k is a union of finitely many Jordan curves, touching at finitely many points. Clearly, $\widehat{\mathbb{C}} \setminus \alpha_k$ has finitely many components, and each component of $g^{-1}(\mathrm{Ext}(\gamma_k))$ (respectively $g^{-1}(\mathrm{Int}(\gamma_k))$) is contained in one of them. Write

$$\mathrm{Comp}(\widehat{\mathbb{C}} \setminus \alpha_k) = \{C_{k,\infty}, C_{k,1}, \ldots, C_{k,l}, C'_{k,\infty}, C'_{k,1}, \ldots, C'_{k,l'}\},$$

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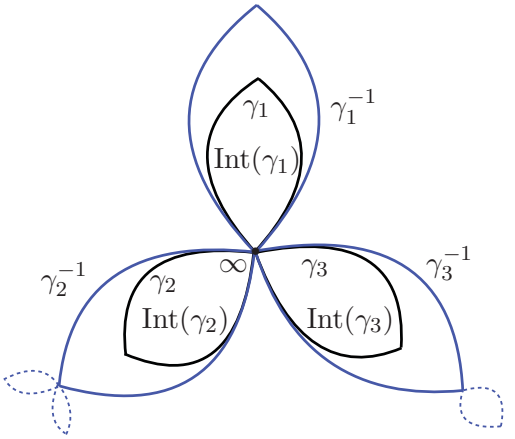

FIGURE 5. Inverse images of independent curves.

where the notation is labeled so that:

(i) $C_{k,\infty}$, $C_{k,\infty'}$ are the only two unbounded components;

(ii) each $C \in \{C_{k,\infty}, C_{k,1}, \ldots, C_{k,l}\}$ (respectively $\{C'_{k,\infty}, C'_{k,1}, \ldots, C'_{k,l'}\}$) contains a component V of $g^{-1}(\mathrm{Ext}(\gamma_k))$ (respectively $g^{-1}(\mathrm{Int}(\gamma_k))$) such that $\partial C \subseteq \partial V$.

The unbounded component U_k of $g^{-1}(\mathrm{Ext}(\gamma_k))$ is contained in $C_{k,\infty}$. By condition (d), either $C_{k,\infty} \subseteq \mathrm{Ext}(\gamma_k)$ or $\mathrm{Int}(\gamma_k) \subseteq C_{k,\infty}$. The latter cannot happen, because locally near ∞, g behaves like $N(\infty) \cap \gamma_k \subseteq g(N(\infty) \cap \gamma_k)$, and globally, g is orientation preserving. Thus $U_k \subseteq C_{k,\infty} \subseteq \mathrm{Ext}(\gamma_k)$.

We will prove the properties (1)–(4), based on the following claim.

$$C_{k,1} \cup \cdots \cup C_{k,l} \subseteq \widehat{U}_k.$$

In fact, if the claim is not true, we have $C_{k,i} \subseteq \widehat{\mathbb{C}} \setminus \widehat{U}_k$ for some i. By condition (d), the filled closure $\widehat{C}_{k,i}$ of $C_{k,i}$ with respect to $\widehat{\mathbb{C}} \setminus \widehat{U}_k$ is disjoint from $\mathrm{Int}(\gamma_k)$, and $\mathrm{Int}(\gamma_k) \subseteq C'_{k,\infty}$. Let $D = \mathrm{Ext}(\gamma_k)$ and U be a component of $g^{-1}(\mathrm{Ext}(\gamma_k))$ contained in $C_{k,i}$. Clearly $U \subseteq D$. Applying Corollary 3.2 to the pair (D, U), we have that $\widehat{C}_{k,i}$ contains at least one fixed point of g. This contradicts condition (b), which completes the proof of the claim.

(1) The following observation

$$\gamma_i^{-1} \cap \gamma_j^{-1} \subseteq g^{-1}(\gamma_i) \cap g^{-1}(\gamma_j) \subseteq g^{-1}(\gamma_i \cap \gamma_j) = g^{-1}(\infty), \quad i \neq j$$

implies that $\gamma_i^{-1} \cap \gamma_j^{-1}$ is a finite set, as it consists of finitely many poles.

Note that $\gamma_i \subseteq \overline{\mathrm{Ext}}(\gamma_k)$ for $i \neq k$, the unbounded component α_i of $g^{-1}(\gamma_i)$ is contained in the unbounded component of $g^{-1}(\overline{\mathrm{Ext}}(\gamma_k))$. Therefore, $\gamma_i^{-1} \subseteq \alpha_i \subseteq \widehat{U}_k$, and there are mutually disjoint interior parts $\mathrm{Int}(\gamma_k^{-1})$ of the curves γ_k^{-1}. This verifies that the curves γ_k^{-1} are independent.

(2) It is an immediate consequence of Corollary 3.2.

(3) It follows from the claim above.

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(4) Because

$$\overline{A(\gamma_1, \ldots, \gamma_n)} \subseteq \overline{\mathrm{Ext}}(\gamma_i) \quad \text{for } 1 \le i \le n,$$

the unbounded component of $g^{-1}(\overline{A(\gamma_1, \ldots, \gamma_n)})$, denoted by E, is contained in that of $g^{-1}(\overline{\mathrm{Ext}}(\gamma_i))$, and therefore $E \subseteq \widehat{U}_i$ by the claim above. Thus $E \subseteq \widehat{U}_1 \cap \cdots \cap \widehat{U}_n = \overline{A(\gamma_1^{-1}, \ldots, \gamma_n^{-1})}$. □

PROPOSITION 3.5. *Let g be a rational map with ∞ a repelling fixed point. Let $\gamma_1, \ldots, \gamma_n$ be independent Jordan curves in $\widehat{\mathbb{C}}$ such that:*

(1) $\infty \in \gamma_1 \cap \cdots \cap \gamma_n$*;*
(2) *all fixed points of g in $\mathbb{C}$ are contained in* $\mathrm{Int}(\gamma_1) \cup \cdots \cup \mathrm{Int}(\gamma_n)$*;*
(3) *in each $\overline{\mathrm{Ext}}(\gamma_k) \setminus \{\infty\}$, the number of poles equals that of fixed points.*

Then the number of poles in $\overline{A(\gamma_1, \ldots, \gamma_n)}$ is $n-1$, strictly less than n.

Here, the number of poles is counted with multiplicity (see Remark 3.3).

Proof. Let a_k (respectively b_k) be the number of poles (respectively fixed points) in $\overline{\mathrm{Ext}}(\gamma_k) \setminus \{\infty\}$. Let $\widetilde{a}_k$ (respectively $\widetilde{b}_k$) be the number of poles (respectively fixed points) in $\mathrm{Int}(\gamma_k) = \widehat{\mathbb{C}} - \overline{\mathrm{Ext}}(\gamma_k)$. Let a be the number of poles in $\overline{H} \setminus \{\infty\}$, where $H = A(\gamma_1, \ldots, \gamma_n)$. All these numbers are counted with multiplicities.

The independent curves $\gamma_1, \ldots, \gamma_n$ decompose $\mathbb{C}$ into several parts. These parts satisfy the following relations:

(1) $\overline{\mathrm{Ext}}(\gamma_k) \setminus \{\infty\} = (\overline{H} \setminus \{\infty\}) \cup \bigcup_{i \ne k} \mathrm{Int}(\gamma_i)$;
(2) $\mathbb{C} = (\overline{H} \setminus \{\infty\}) \cup \bigcup_{1 \le i \le n} \mathrm{Int}(\gamma_1)$.

By counting the number of poles, we have the following identity:

$$\begin{aligned}\sum_{1 \le k \le n} a_k &= \sum_{1 \le k \le n} \Bigl(a + \sum_{i \ne k} \widetilde{a}_i\Bigr) = an + (n-1) \sum_{1 \le i \le n} \widetilde{a}_i \\ &= a + (n-1)\Bigl(a + \sum_{1 \le i \le n} \widetilde{a}_i\Bigr) = a + (n-1)(d-1),\end{aligned}$$

where d is the degree of g. Note that $\overline{H} \setminus \{\infty\}$ is disjoint from the fixed points of g. By counting the number of fixed points in $\mathbb{C}$, we have

$$\sum_{1 \le k \le n} b_k = \sum_{1 \le k \le n} \Bigl(0 + \sum_{i \ne k} \widetilde{b}_i\Bigr) = (n-1) \sum_{1 \le i \le n} \widetilde{b}_i = (n-1)d.$$

By assumption, one has $a_k = b_k$ for all k, which implies that $\sum a_k = \sum b_k$. Therefore, we have $a = n - 1$. The proof is completed. □

4. *Invariant graph*

Let f be a Newton map of degree $d \ge 3$, post-critically finite on B_f. The aim of this section is to prove the existence of an invariant graph for f. Here, a graph G is said to be *invariant* for f if it satisfies

$$f(G) \subseteq G \quad \text{and} \quad f^{-1}(G) \text{ is connected.}$$

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In fact, the existence of an invariant graph is first proven by Drach *et al* [**DMRS**] and Mikulich, Rückert, and Schleicher [**MRS**]. Our work is distinguished from theirs in two aspects.

First, our idea of proof is essentially different from theirs. The construction in Theorem 4.1 is actually inspired by a previous work [**CGZ**]. Analogous to [**CGZ**, Proposition 4.4], it is crucial to see that the point ∞ (respectively each periodic Fatou center in [**CGZ**]) has to be a non-cut point for the graph that is used to construct puzzles in §5 (respectively tiles in [**CGZ**, §5]). This point is the main goal of our construction. In contrast, the proof in [**DMRS**] aims to show that all the poles lie in the unbounded component, say Δ_n, of $f^{-n}(\Delta_0)$ for sufficiently large n; see (4.1) for the definition of *channel graph* Δ_0. Additionally, it seems to be not clear from [**DMRS**] whether ∞ is a cut point of Δ_n or not.

Second, our graph is different from theirs. The graph G constructed in Theorem 4.1 is a strict subset of Δ_n. It has very good properties: points in G except some strictly pre-periodic Fatou centers are non-cut points; see Proposition 4.8. Therefore, it is well adapted to construct puzzles; see §5.1. In contrast, for the graph Δ_n, many iterated preimages of ∞ are cut points. That is the reason why we develop a different proof and construct a different graph.

4.1. *Channel graph.* For any immediate root basin B of f, there are exactly $d_B - 1$ fixed internal rays in B:

$$R_B(j/(d_B-1)), \quad 0 \le j \le d_B - 2 \text{ where } d_B = \deg(f|_B).$$

Each of these fixed internal rays must land at a fixed point on ∂B. Because ∞ is the unique fixed point of f on its Julia set, all fixed internal rays land at the common point ∞.

The *channel graph* of f, denoted by Δ_0, is defined by

$$\Delta_0 = \bigcup_{B} \bigcup_{j=1}^{d_B-1} \overline{R_B(j/(d_B-1))}, \tag{4.1}$$

where B ranges over all immediate root basins in $\{B_1, \ldots, B_d\}$. Clearly $f(\Delta_0) = \Delta_0$. Figure 6 illustrates all possible channel graphs when $d = 4$. Some graphs which look like channel graphs but in fact are fake ones are given in Figure 7.

4.2. *Invariant graph.* The main result in this section is the following.

THEOREM 4.1. *Let f be a Newton map which is post-critically finite on B_f. Then there exists an invariant graph G such that:*

(1) $f^N(G) = \Delta_0$ *for some integer* $N \ge 1$*;*

(2) ∞ *is a non-cut point with respect to* G.

The idea of the proof. Let us sketch the idea, so that the readers can have a rough picture of the proof. For each $k \ge 1$, let $C_k = f^{-k}(\Delta_0) \setminus f^{-k+1}(\Delta_0)$. From Theorem 4.1(1), one may easily imagine that G is actually a union of some suitable iterated preimages of Δ_0.

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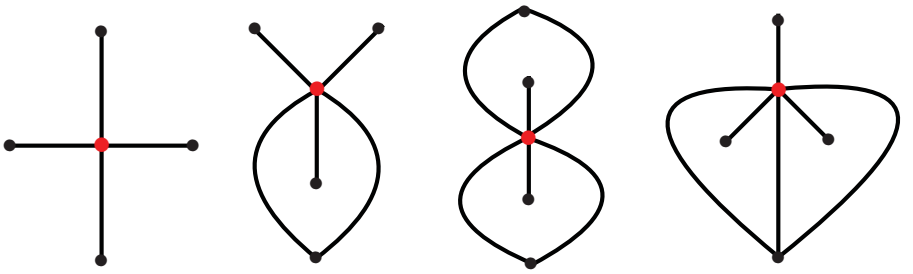

FIGURE 6. All possible channel graphs when $d = 4$. The red dot is ∞ and the black dots are the centers of immediate root basins.

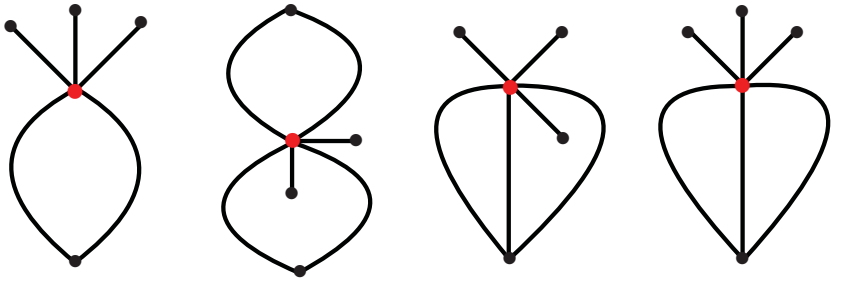

FIGURE 7. Caution: these graphs are not channel graphs for $d = 4$, they are fake ones. Want to know why? See Fact 4.2.

These iterated preimages are chosen in an inductive fashion. First, we extend the graph $\widetilde{\Delta}_0 := \Delta_0$ to a larger one $\widetilde{\Delta}_1$, by adding a suitable subset of C_1. Inductively, at step k, we will get an extension of the graph $\widetilde{\Delta}_k$ from $\widetilde{\Delta}_{k-1}$ by adding a subset of $C_1 \cup \cdots \cup C_k$.

The choice of the subset of $C_1 \cup \cdots \cup C_k$ is delicate, we actually choose a suitable subset such that either its endpoint is a pole, or some endpoint of the iterated preimage is a pole. This dichotomy is guaranteed by the shrinking lemma (see Lemma 4.11). The heart of the proof is to show that any subset of this kind can touch another one at some pole. This will be based on the counting number of poles (Propositions 3.4 and 3.5) in the preceding section. However one cannot apply these results directly.

To compensate for the situation, we need to make a modification G_k of the graph $\widetilde{\Delta}_k$ in each step. For these G_k terms, we can apply Propositions 3.4 and 3.5 successfully. For this technical reason, in our discussion, we actually focus on the construction of G_k (whose modification yields $\widetilde{\Delta}_k$), and the graphs $\widetilde{\Delta}_k$ do not appear directly in the proof.

Then Theorem 4.1(2) can guide each step of the proof. To construct a graph G so that ∞ is a non-cut point, we construct a sequence of modified graphs G_k so that

$$\infty \in G_0 \subseteq G_1 \subseteq G_2 \subseteq \ldots,$$

here, the graph G_0 is a modified version of the channel graph Δ_0, and G_{k+1} is constructed inductively so that the difference set $G_{k+1} \setminus G_k$ is the union of finitely many Jordan arcs,

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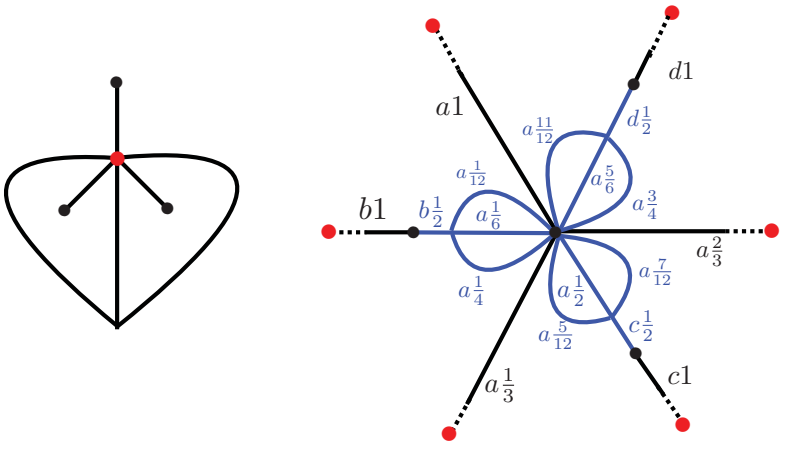

FIGURE 8. A type of channel graph Δ_0 (a) and its resulting invariant graph G (b). Here the red dots represent the same point ∞. The notation 'wt' means the internal ray of angle t starting from the center $w \in \{a, b, c, d\}$.

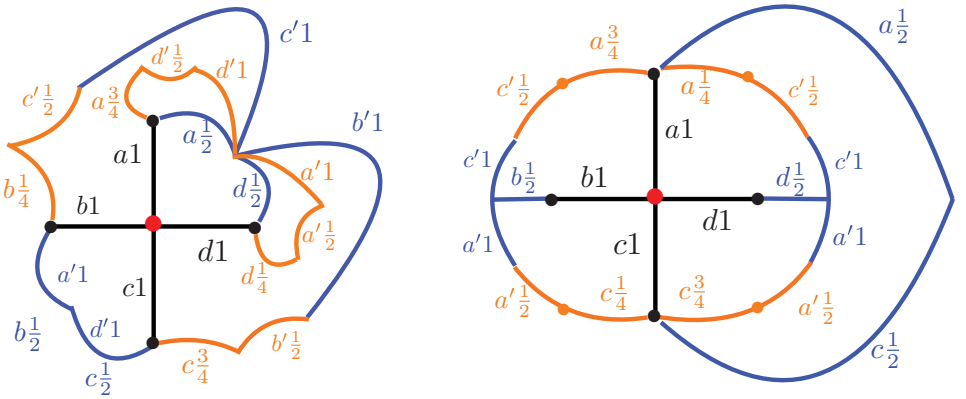

FIGURE 9. Two invariant graphs for two different maps. Their channel graphs are the same, but the combinations are different. The notation 'wt' means the internal ray of angle t starting from the center $w \in \{a, b, c, d\} \cup \{a', b', c', d'\}$, here x' is a preimage of x. The blue/orange parts are the subsets of the first/second preimages of the channel graph.

and that

$$\nu(G_{k+1}, \infty) < \nu(G_k, \infty) \quad \text{if } \nu(G_k, \infty) \geq 2.$$

The property $d = \nu(G_0, \infty) > \nu(G_1, \infty) > \nu(G_2, \infty) > \dots$ implies that after finitely many steps, the procedure will terminate at a graph G_ℓ with $\nu(G_\ell, \infty) = 1$, which is equivalent to say that ∞ is a non-cut point for G_ℓ. Finally, a suitable modification of G_ℓ yields the required graph G. See Figures 8 and 9.

Proof of Theorem 4.1. The proof proceeds in six steps, as follows.

Step 1: from Δ_0 to G_0. The aim of this step is to modify Δ_0 to a new graph G_0, such that G_0 is disjoint from the d attracting fixed points.

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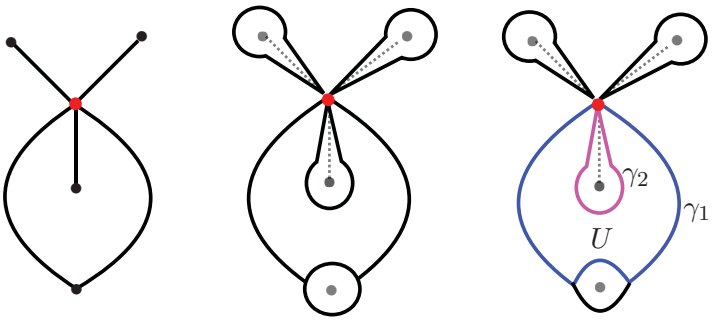

FIGURE 10. The channel graph Δ_0 (a), the modified graph G_0 (b), and a non-trivial component U of $\widehat{\mathbb{C}} \setminus G_0$ (c). This U can be written as $A(\gamma_1, \gamma_2)$, where γ_1 is the blue curve and γ_2 is the purple one.

Consider an immediate root basin B of f. Recall that $\Phi_B : B \to \mathbb{D}$ is a Böttcher map, satisfying that $\Phi_B(z)^{d_B} = \Phi_B(f(z))$. Fix a number $r \in (0, 1)$.

If $d_B \geq 3$, let

$$\Delta_B = \{\infty\} \cup \Phi_B^{-1}(\{[r, 1)e^{2\pi ik/(d_B-1)}; 0 \leq k \leq d_B - 2\}) \cup \Phi_B^{-1}(r\mathbb{S}).$$

If $d_B = 2$, take a small angle $\theta_0 \in (0, 1/2)$ and define two arcs $\alpha_\pm$ in B by

$$\alpha_\pm = \Phi_B^{-1}(\{e^{s(\log r \pm 2\pi i\theta_0)}; 0 < s < 1\}).$$

Clearly, $\alpha_\pm$ connect ∞ to $\Phi_B^{-1}(re^{\pm 2\pi i\theta_0})$, and $\alpha_\pm \subseteq f(\alpha_\pm)$. For convenience, in this case, we say that the arcs $\alpha_\pm$ are *tangent* to the internal ray $\Phi_B^{-1}((0, 1))$ at ∞. The preimages $f^{-n}(\alpha_\pm)$ are also said to be *tangent* to internal rays $f^{-n}(\Phi_B^{-1}((0, 1)))$ at some points in $f^{-n}(\infty)$. We set

$$\Delta_B = \{\infty\} \cup \alpha_+ \cup \alpha_- \cup \Phi_B^{-1}(\{r\, e^{2\pi it};\ \theta_0 \leq t \leq -\theta_0\}).$$

Finally, let

$$G_0 = \bigcup_B \Delta_B,$$

where the union is taken over all immediate root basins B of f. Clearly G_0 avoids all centers of the immediate root basins. See Figure 10.

Step 2: from G_0 to G_1. For a finite graph $\Gamma \subseteq \widehat{\mathbb{C}}$ with $\infty \in \Gamma$, its complement $\widehat{\mathbb{C}} \setminus \Gamma$ has finitely many components. There are two kinds of unbounded ones. An unbounded component U of $\widehat{\mathbb{C}} \setminus \Gamma$ is called *trivial* if $\nu(\partial U, \infty) = 1$ (equivalently, ∞ is a non-cut point of ∂U); *non-trivial* if $\nu(\partial U, \infty) \geq 2$ (i.e., ∞ is a cut point of ∂U).

For the graph G_0 given by Step 1, the following fact is non-trivial.

FACT 4.2. *An unbounded component U of $\widehat{\mathbb{C}} \setminus G_0$ is trivial (that is, $\partial U \setminus \{\infty\}$ is connected) if and only if $\partial U = \Delta_B$ with $d_B = 2$.*

Proof. The '$\Longleftarrow$' part is obvious. We need to show the '$\Longrightarrow$' part. If it is not true, then there are two possibilities for U:

(1) ∂U intersects at least two root basins; or

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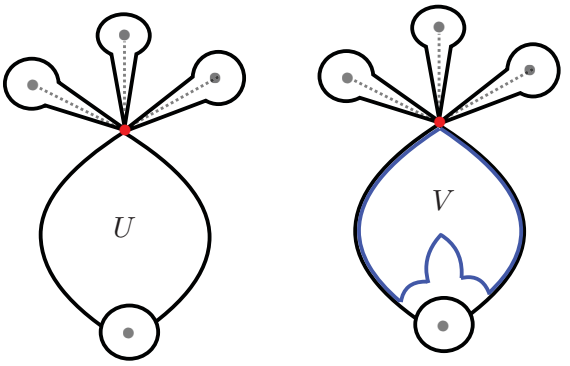

FIGURE 11. This figure shows why the first graph in Figure 7 is not a channel graph, and it is used in the proof of Fact 4.2. Here V is a connected component of $f^{-1}(U)$ (in general, V is not necessarily a topological disk).

(2) ∂U is a component of $\widehat{\mathbb{C}} - \Delta_B$ with $d_B \geq 3$ (see Figures 10 and 11).

The former case implies that $\partial U \setminus \{\infty\}$ has at least two components. Hence it is impossible.

In the following, we consider the latter case. Note that $U \subseteq f(U)$ and the image $f(U \cap B)$ covers $U \cap B$ twice. One may also observe that there is a component V of $f^{-1}(U)$, contained in U, such that ∂V contains two sections of fixed internal rays. Note that $\overline{V}$ contains only one fixed point, namely ∞. However, by Lemma 3.1, one has

$$\#\mathrm{Fix}(f|_{\overline{V}}) = \deg(f|_{\partial V}).$$

This gives a contradiction, because $\#\mathrm{Fix}(f|_{\overline{V}}) = 1$ and $\deg(f|_{\partial V}) \geq 2$. □

Fact 4.2 has the following interesting corollary.

FACT 4.3. *Under the assumption $d = \deg(f) \geq 3$, there are at least two immediate root basins B with $d_B = 2$. As a consequence, trivial and non-trivial components both exist in* $\mathrm{Comp}(\widehat{\mathbb{C}} \setminus G_0)$.

Proof. This fact is obvious if $d_B = 2$ for all immediate root basins B. So we may assume $d_B \geq 3$ for some B. In this case, $\widehat{\mathbb{C}} - \Delta_B$ has at $d_B - 1 \geq 2$ unbounded components. To prove the fact, we will show each unbounded component of $\widehat{\mathbb{C}} - \Delta_B$ contains an immediate root basin B'' with $d_{B''} = 2$.

In fact, if some unbounded component of $\widehat{\mathbb{C}} - \Delta_B$, say U, contains no immediate root basin B' with $d_{B'} = 2$, then there are two cases:

(1) there is no immediate root basin completely contained in U; or
(2) all immediate root basins B'' in U satisfy that $d_{B''} \geq 3$.

In the former case, U is trivial. However, this contradicts Fact 4.2.

In the latter case, we can find an immediate root basin $B'' \subseteq U$, which is an innermost one, and such that some unbounded component V of $\widehat{\mathbb{C}} - \Delta_{B''}$ is also trivial (this is an easy observation). This again contradicts Fact 4.2. □

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The idea of this step is to take pullbacks of the boundaries of non-trivial unbounded components of $\widehat{\mathbb{C}} \setminus G_0$.

Let Q_0 be a non-trivial unbounded component of $\widehat{\mathbb{C}} \setminus G_0$. Such a component can be written as $Q_0 = A(\gamma_1, \dots, \gamma_n)$ where $n \geq 2$ and $\gamma_1, \dots, \gamma_n$ are independent Jordan curves. One may verify that the curves $\gamma_1, \dots, \gamma_n$ satisfy the conditions (a)–(d) in Proposition 3.4. Indeed, by the construction in Step 1, the conditions (a)–(c) are satisfied, we only need to check that the unbounded component α_k of $f^{-1}(\gamma_k)$ is contained in $\overline{\mathrm{Ext}}(\gamma_k)$. To see this, note that either $\alpha_k \subseteq \overline{\mathrm{Int}}(\gamma_k)$ or $\alpha_k \subseteq \overline{\mathrm{Ext}}(\gamma_k)$; the former cannot happen, the reason is that γ_k contains a section of some equipotential curve $\Phi_B^{-1}(r\partial\mathbb{D})$ (for some immediate root basin B) and α_k contains a section of the equipotential curve $\Phi_B^{-1}({}^{d_B}\sqrt{r}\partial\mathbb{D})$, which can not be completely contained in $\overline{\mathrm{Int}}(\gamma_k)$.

We then apply Proposition 3.4 to the independent curves $\gamma_1, \dots, \gamma_n$, and obtain the new independent Jordan curves $\gamma_1^{-1}, \dots, \gamma_n^{-1}$. For these new curves, observe that:

(1) for each k, the curve γ_k^{-1} contains at least one pole of f in $\mathbb{C}$;
(2) one has $\gamma_i^{-1} \cap \gamma_j^{-1} \subseteq f^{-1}(\infty)$, for $i \neq j$.

In the following, we will show that at least two curves of $\gamma_1^{-1}, \dots, \gamma_n^{-1}$ have a common pole. In fact, if this is not true, then the set $\overline{A(\gamma_1^{-1}, \dots, \gamma_n^{-1})}$ contains at least n distinct poles. However, by Proposition 3.4, the curves $\gamma_1^{-1}, \dots, \gamma_n^{-1}$ satisfy the assumptions in Proposition 3.5. Then by Proposition 3.5, the number of poles in $\overline{A(\gamma_1^{-1}, \dots, \gamma_n^{-1})}$ is exactly $n-1$ (counting multiplicity). This is a contradiction.

Finally, let us define three curve families Γ_0, Γ_1^*, Γ_1, and a new graph G_1 by

$$\Gamma_0 = \bigcup_{Q_0} \{\gamma_1, \dots, \gamma_n\},\ \Gamma_1^* = \Gamma_1 = \bigcup_{Q_0} \{\gamma_1^{-1}, \dots, \gamma_n^{-1}\},\ G_1 = \bigcup_{\gamma \in \Gamma_1^*} \gamma,$$

where Q_0 ranges over all non-trivial unbounded components of $\widehat{\mathbb{C}} \setminus G_0$. The existence of common poles for the curves γ_k^{-1}, implies that

$$\nu(G_1, \infty) < \nu(G_0, \infty) = d.$$

Note that we have the inclusion

$$f(\Gamma_1) := \{f(\gamma); \gamma \in \Gamma_1\} \subseteq \Gamma_0, \quad f(G_1) \subseteq G_0.$$

Step 3: from G_1 to G_2. The idea of the proof is similar to that of Step 2: taking pullbacks of the boundaries of non-trivial unbounded components of $\widehat{\mathbb{C}} \setminus G_1$, until some pullback hits a pole. We remark that this step actually reveals the general case of the pullback procedure. The shrinking lemma is involved to deal with the difficulty arising here.

Note that if $\nu(G_1, \infty) = 1$, then ∞ is a non-cut point of G_1, and hence there is nothing to do in this step. So we may assume that $\nu(G_1, \infty) \geq 2$, and this case happens if and only if there exists a non-trivial unbounded component, say Q_1, of $\widehat{\mathbb{C}} \setminus G_1$.

Note that Q_1 is contained in some $Q_0 = A(\gamma_1, \dots, \gamma_n)$ in Step 2, and that Q_0 is decomposed by the curves $\gamma_1^{-1}, \dots, \gamma_n^{-1}$ into several parts. Because Q_1 is non-trivial, it can be written as

$$Q_1 = A(\alpha_1, \dots, \alpha_m),$$

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where $\alpha_1, \ldots, \alpha_m$ are independent Jordan curves. The set $\{\alpha_1, \ldots, \alpha_m\}$ can be decomposed into two disjoint subsets $\Gamma_1(Q_1)$ and $\Xi(Q_1)$, such that:

(1) each curve $\lambda \in \Gamma_1(Q_1)$ comes from Γ_1, namely $\Gamma_1(Q_1) \subseteq \Gamma_1$;
(2) each curve $\eta \in \Xi(Q_1)$ is *new*, that is, composed of several sections, each section is a part of a curve in Γ_1.

Note that each curve $\lambda \in \Gamma_1(Q_1) \subseteq \Gamma_1$ must contain a pole in $\mathbb{C}$. Each curve $\eta \in \Xi(Q_1)$ must also contain a pole in $\mathbb{C}$, because if two curves in Γ_1 intersect at a point other than ∞, then this point is a pole.

FACT 4.4. $\Xi(Q_1) \neq \emptyset$. *In other words, at least one curve among α_k is new.*

Proof. If not, then $\Gamma_1(Q_1) = \{\alpha_1, \ldots, \alpha_m\} \subseteq \Gamma_1$, and $\alpha_i \cap \alpha_j = \{\infty\}$ for $i \neq j$. Therefore, the number of poles in $\overline{Q}_1 = \overline{A(\alpha_1, \ldots, \alpha_m)}$ is at least m.

However, applying Proposition 3.4 to the curves $f(\alpha_1), \ldots, f(\alpha_m)$, we see that the curves $\alpha_1, \ldots, \alpha_m$ satisfy the assumptions in Proposition 3.5. Then by Proposition 3.5, the number of poles in $\overline{A(\alpha_1, \ldots, \alpha_m)}$ is exactly $m - 1$ (counting multiplicity). This is a contradiction. □

We may write

$$\Gamma_1(Q_1) = \{\lambda_1, \ldots, \lambda_r\}, \ \Xi(Q_1) = \{\eta_1, \ldots, \eta_s\}.$$

CLAIM. *The Jordan curves $f(\lambda_1), \ldots, f(\lambda_r), \eta_1, \ldots, \eta_s$ are independent, and satisfy the conditions of Proposition 3.4.*

Proof. By the definition of Q_1, we see that $\lambda_1, \ldots, \lambda_r, \eta_1, \ldots, \eta_s$ are in different components of $\widehat{\mathbb{C}} \setminus Q_1$. So the bound mutually disjoint components are $\mathrm{Int}(\lambda_1), \ldots, \mathrm{Int}(\lambda_r), \mathrm{Int}(\eta_1), \ldots, \mathrm{Int}(\eta_s)$. Because $f(\lambda_k) \subset \overline{\mathrm{Int}}(\lambda_k)$ (we have proven this when we deal with Q_0), we see immediately that $f(\lambda_1), \ldots, f(\lambda_r), \eta_1, \ldots, \eta_s$ are independent, and they satisfy the conditions of Proposition 3.4. □

Applying Proposition 3.4 to these curves, for each η_j, one gets η_j^{-1}. Moreover, the curves $\lambda_1, \ldots, \lambda_r, \eta_1^{-1}, \ldots, \eta_s^{-1}$ are independent.

If one of the resulting curves η_j^{-1}, say η_k^{-1}, is disjoint from poles in $\mathbb{C}$, then it is exactly the unbounded component of $f^{-1}(\eta_k)$, and:

(1) η_k^{-1} intersects each of $\lambda_1, \ldots, \lambda_r, \eta_j^{-1}$, $j \neq k$, only at ∞;
(2) $f : \eta_k^{-1} \to \eta_k$ is one-to-one.

For any integer $l \geq 1$, one may define η_k^{-l-1} inductively by

$$\eta_k^{-l-1} = (\eta_k^{-l})^{-1}$$

as long as the curves

$$f(\lambda_1), \ldots, f(\lambda_r), \eta_1, \ldots, \eta_{k-1}, \eta_k^{-l}, \eta_{k+1}, \ldots, \eta_s,$$

are independent, and $\eta_k^{-1}, \ldots, \eta_k^{-l}$ are disjoint from poles in $\mathbb{C}$. In this case, the curves $\lambda_1, \ldots, \lambda_r, \eta_1^{-1}, \ldots, \eta_{k-1}^{-1}, \eta_k^{-l-1}, \eta_{k+1}^{-1}, \ldots, \eta_s^{-1}$ are independent.

To continue our discussion, we need the following crucial fact.

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LEMMA 4.5. *For each curve $\eta \in \Xi(Q_1) = \{\eta_1, \dots, \eta_s\}$, there is a minimal integer $N = N_\eta \geq 1$, such that η^{-N} contains a pole of f in $\mathbb{C}$.*

Proof. If it is not true for $\eta = \eta_k$, then for any $j \geq 1$, the curves $\lambda_1, \dots, \lambda_r, \eta_1^{-1}, \dots, \eta_{k-1}^{-1}, \eta_k^{-j}, \eta_{k+1}^{-1}, \dots, \eta_s^{-1}$ are independent, and the domains

$$H_j = A(\lambda_1, \dots, \lambda_r, \eta_1^{-1}, \dots, \eta_{k-1}^{-1}, \eta_k^{-j}, \eta_{k+1}^{-1}, \dots, \eta_s^{-1})$$

satisfy

$$H_1 \supseteq H_2 \supseteq \cdots \supseteq H_j \supseteq \cdots .$$

In particular, we have

$$\mathrm{Int}(\eta_k^{-1}) \subseteq \mathrm{Int}(\eta_k^{-j}) \quad \text{and} \quad \mathrm{Int}(\lambda_1) \cup \cdots \cup \mathrm{Int}(\lambda_r) \subseteq \mathrm{Ext}(\eta_k^{-j}).$$

This implies that the spherical diameters $\mathrm{diam}(\eta_k^{-j})$ with $j \geq 1$ are uniformly bounded from below and above.

To get a contradiction, we will show $\mathrm{diam}(\eta_k^{-j}) \to 0$ as $j \to \infty$. Note that all Jordan curves η_k^{-j} traverse two distinct immediate root basins, say B', B''. We may decompose η_k^{-j} into three segments $\beta_j, \beta_j', \beta_j''$:

(1) β_j' (respectively β_j'') is the intersection of η_k^{-j} with the closure of some fixed internal ray of B' (respectively B''), it takes the form $\overline{\phi_{B'}^{-1}((r^{1/d_{B'}^j}, 1)e^{2\pi ik/(d_{B'}-1)})}$ for some k;

(2) $\beta_j = \eta_k^{-j} \setminus (\beta_j' \cup \beta_j'')$.

The observation $\bigcap_j \beta_j' = \bigcap_j \beta_j'' = \{\infty\}$ implies that $\mathrm{diam}(\beta_j') \to 0$ and $\mathrm{diam}(\beta_j'') \to 0$ as $j \to \infty$. It remains to prove

$$\mathrm{diam}(\beta_j) \to 0 \quad \text{as} \quad j \to \infty.$$

Note that $f : \beta_{j+1} \to \beta_j$ is a homeomorphism. By the construction of β_j, there is a large integer $n_0 > 0$ such that $\beta_0 \cap \beta_{n_0} = \emptyset$. It follows that $\beta_{jn_0} \cap \beta_{(j+1)n_0} = \emptyset$ for all $j \geq 0$.

Choose a large integer $n_0 > 0$ such that $\beta_0 \cap \beta_{n_0} = \emptyset$, and β_{n_0} has no intersection with the postcritical set. Choose a disk neighborhood U_0 of $\overline{\beta_{n_0}}$ such that $f^{n_0}(U_0) \cap U_0 = \emptyset$. Then by pulling back U_0 via f^{kn_0}, we get a disk neighborhood U_k of β_{kn_0}. It follows that there exists a sequence of open sets $\{U_j\}$ with $\beta_{jn_0} \Subset U_j$ such that $f^{n_0}(U_{j+1}) = U_j$ and $U_{j+1} \cap U_0 = \emptyset$ for $j \geq 0$. By the shrinking lemma (see Lemma 4.11), one has $\mathrm{diam}(\beta_{jn_0}) \to 0$ as $j \to \infty$. The shrinking property $H_j \supseteq H_{j+1}$ implies that $\mathrm{diam}(\beta_j) \to 0$ as $j \to \infty$. This gives a contradiction. □

Remark 4.6. In Lemma 4.5, it may happen that for some $N \geq 1$,

$$\eta^{-N} \cap f^{-1}(\infty) = \eta \cap f^{-1}(\infty).$$

In other words, the poles in η^{-N} are already contained in η, and hence not new. Figure 12 gives such an example (in this example $\eta = \eta_2$, $N = 1$).

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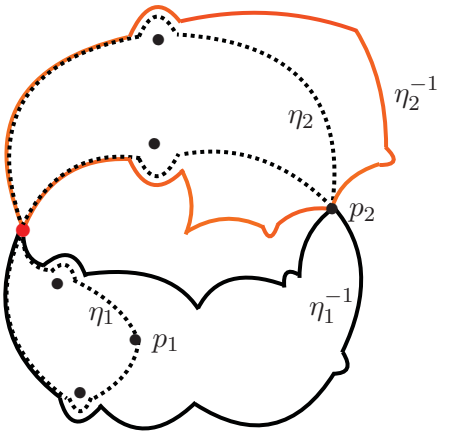

FIGURE 12. In this example, the black dashed curves η_1, η_2 contain the poles p_1, p_2, respectively. Here, η_1^{-1} is the black curve containing the new pole p_2, and η_2^{-1} is the orange curve containing the same pole p_2 as η_2 does. The pole p_2 is also a critical point.

By Lemma 4.5, for each η_k, there exists a minimal integer $N_k \geq 1$ such that $\eta_k^{-N_k}$ contains a pole. One may verify further that the Jordan curves

$$f(\lambda_1), \ldots, f(\lambda_r), \eta_1^{-l_1}, \ldots, \eta_s^{-l_s}$$

with $0 \leq l_1 < N_1, \ldots, 0 \leq l_s < N_s$ are independent and satisfy the conditions (a)–(d) in Proposition 3.4. Applying Proposition 3.4 to the curves

$$f(\lambda_1), \ldots, f(\lambda_r), \eta_1^{-N_1+1}, \ldots, \eta_s^{-N_s+1},$$

we get the following independent curves

$$\lambda_1, \ldots, \lambda_r, \eta_1^{-N_1}, \ldots, \eta_s^{-N_s},$$

each of which contains a pole in $\mathbb{C}$. Again Proposition 3.5 implies that at least two of these curves contain a common pole in $\mathbb{C}$. We remark that each η_k passes through exactly two immediate root basins B', B'', and so do the curves η_k^{-j}, $1 \leq j \leq N_k$; these Jordan curves overlap on an invariant subarc in $\{\infty\} \cup B' \cup B''$.

Let us define two families of Jordan curves

$$\Gamma_2^* = \bigcup_{Q_1} \{\eta_1^{-1}, \ldots, \eta_1^{-N_1}, \ldots, \eta_s^{-1}, \ldots, \eta_s^{-N_s}\}, \quad \Gamma_2 = \bigcup_{Q_1} \{\eta_1^{-N_1}, \ldots, \eta_s^{-N_s}\},$$

where Q_1 ranges over all non-trivial unbounded components of $\widehat{\mathbb{C}} \setminus G_1$. Now we get a new graph G_2, which is an extension of G_1:

$$G_2 = G_1 \bigcup \bigcup_{\gamma \in \Gamma_2^*} \gamma.$$

Observe that $f(G_2) \subseteq G_0 \cup G_2$. The construction and the existence of common poles for the curves $\lambda_1, \ldots, \lambda_r, \eta_1^{-N_1}, \ldots, \eta_s^{-N_s}$ imply that

$$\nu(G_2, \infty) < \nu(G_1, \infty).$$

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Step 4: from G_k to G_{k+1}, an induction procedure. Suppose for some $k \geq 2$, we have constructed the graphs $G_1, \ldots, G_k$ and the curve families $\Gamma_1^*, \Gamma_1 \ldots, \Gamma_k^*, \Gamma_k$, inductively in the following way:

$$\Gamma_l^* = \bigcup_{Q_{l-1}} \{\eta_1^{-1}, \ldots, \eta_1^{-N_1}, \ldots, \eta_s^{-1}, \ldots, \eta_s^{-N_s}\},$$

$$\Gamma_l = \bigcup_{Q_{l-1}} \{\eta_1^{-N_1}, \ldots, \eta_s^{-N_s}\}, \ G_l = G_{l-1} \bigcup \bigcup_{\gamma \in \Gamma_l^*} \gamma,$$

where Q_{l-1} is taken over all non-trivial unbounded components of $\widehat{\mathbb{C}} \setminus G_{l-1}$, and that $f(G_l) \subseteq G_0 \cup G_l$ and $\nu(G_l, \infty) < \nu(G_{l-1}, \infty)$, for $2 \leq l \leq k$.

If $\nu(G_k, \infty) = 1$, then the step is done. If $\nu(G_k, \infty) \geq 2$, we consider each non-trivial unbounded component Q_k of $\widehat{\mathbb{C}} \setminus G_k$. Write Q_k as $A(\delta_1, \ldots, \delta_t)$ and compare the curves $\delta \in \{\delta_1, \ldots, \delta_t\}$ with the curves in $\Gamma_1 \cup \cdots \cup \Gamma_k$, there are two possibilities: either

(1) $\delta \in \Gamma_1 \cup \cdots \cup \Gamma_k$; or

(2) δ is *new*, that is, $\delta \notin \Gamma_1 \cup \cdots \cup \Gamma_k$. In this case, δ is composed of several sections, and each section is a part of a curve in $\Gamma_1 \cup \cdots \Gamma_k$.

Let $\Xi(Q_k)$ be the collection of new curves. For each $\eta \in \Xi(Q_k)$, by the same argument as Lemma 4.5, there is a minimal integer $N_\eta \geq 1$ such that η^{-N_η} meets a pole in $\mathbb{C}$. By Proposition 3.5, at least two curves of $\{\eta^{-N_\eta}; \eta \in \Xi(Q_k)\}$ share a common pole. Similarly as above, we get a new graph G_{k+1} and two curve families $\Gamma_{k+1} \subseteq \Gamma_{k+1}^*$:

$$\Gamma_{k+1}^* = \bigcup_{Q_k} \bigcup_{\eta \in \Xi(Q_k)} \{\eta^{-1}, \ldots, \eta^{-N_\eta}\}, \quad \Gamma_{k+1} = \bigcup_{Q_k} \bigcup_{\eta \in \Xi(Q_k)} \{\eta^{-N_\eta}\},$$

$$G_{k+1} = G_k \bigcup \bigcup_{\gamma \in \Gamma_{k+1}^*} \gamma,$$

where Q_k is taken over all the non-trivial unbounded component Q_k of $\widehat{\mathbb{C}} \setminus G_k$.

The resulting graph G_{k+1} satisfies

$$\nu(G_{k+1}, \infty) < \nu(G_k, \infty), \quad f(G_{k+1}) \subseteq G_0 \cup G_{k+1}.$$

After finitely many steps, we have $\nu(G_\ell, \infty) = 1$ for some minimal integer $\ell \geq 1$. Then ∞ is a non-cut point for the graph G_ℓ, and $f(G_\ell) \subseteq G_0 \cup G_\ell$.

Step 5: from G_ℓ to G, a natural modification. By construction, all points in $G_\ell \cap J(f)$ are iterated preimages of ∞, and

$$f(G_\ell \cap J(f)) \subseteq G_\ell \cap J(f), \quad f^N(G_\ell \cap J(f)) = \{\infty\}$$

for some large integer $N \geq 1$. Note that for any $0 \leq k \leq \ell$, the graph G_k is a union of some curves in

$$\Gamma = \Gamma_0 \cup \Gamma_1^* \cup \Gamma_2^* \cup \Gamma_3^* \cup \cdots \cup \Gamma_\ell^*.$$

To give a natural modification of G_ℓ, it suffices to define the modification of each curve $\delta \in \Gamma$. This goes in the following way.

Let $B \in \mathrm{Comp}(B_f)$ with $B \cap \delta \neq \emptyset$, then $B \cap \delta$ consists of finitely many components. Suppose B is eventually iterated to the immediate root basin B_0. Note that each component

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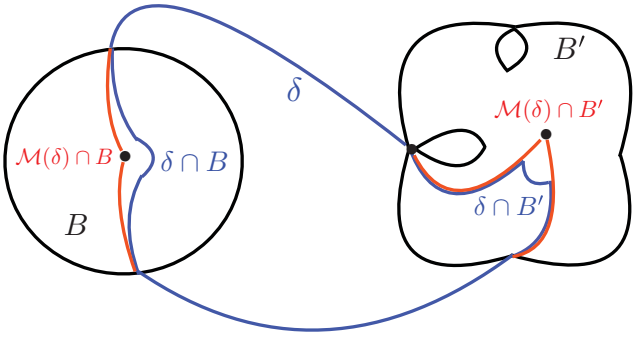

FIGURE 13. This figure shows how to modify a curve arc by arc. Here, $B\cap\delta$ is tangent to two internal rays near ∂B, while $B'\cap\delta$ is equal to two internal rays near $\partial B'$.

σ of $B\cap\delta$ is an open arc, and near the boundary ∂B, σ is either tangent to (if $d_{B_0}=2$) or equal to (if $d_{B_0}\geq 3$) two internal rays (see Figure 13), say $R_B(\alpha)$, $R_B(\beta)$. We define the modification $\mathcal{M}(\sigma)$ of σ by

$$\mathcal{M}(\sigma)=R_B(\alpha)\cup R_B(\beta)\cup\{c_B\},$$

where c_B is the center of B (it is possible that $\alpha=\beta$). We then set

$$\mathcal{M}(\delta)=\overline{\bigcup_B\bigcup_\sigma\mathcal{M}(\sigma)},$$

where B ranges over all components $B\in\mathrm{Comp}(B_f)$ with $B\cap\delta\neq\emptyset$ and σ is taken over all components of $B\cap\delta$.

By the law $\mathcal{M}(\delta_1\cup\delta_2)=\mathcal{M}(\delta_1)\cup\mathcal{M}(\delta_2)$, we obtain the modification of the graphs G_k, which satisfy

$$\Delta_0=\mathcal{M}(G_0)\subseteq\mathcal{M}(G_1)\subseteq\cdots\subseteq\mathcal{M}(G_\ell)\subseteq f^{-N}(\Delta_0).$$

Let $G=\mathcal{M}(G_\ell)$. Clearly one has $f^N(G)=\Delta_0$. Moreover,

$$f(G)=\mathcal{M}(f(G_\ell))\subseteq\mathcal{M}(G_0\cup G_\ell)=\mathcal{M}(G_\ell)=G.$$

$$\nu(G,\infty)=\nu(G_\ell,\infty)=1.$$

(See Figure 14.)

Step 6: $f^{-1}(G)$ is connected. To prove the connectivity of $f^{-1}(G)$, we need to investigate some properties of G and G_ℓ (given in Step 4) first.

FACT 4.7. *Each component of $\widehat{\mathbb{C}}\setminus G_\ell$ is a Jordan disk.*

Proof. It is equivalent to show that $\nu(G_\ell,z)=1$ for all $z\in G_\ell$. Clearly this is true for $z=\infty$ by the construction of G_ℓ. For $z\in G_\ell-\{\infty\}$, note that $G_\ell=\bigcup_{\delta\in\Gamma\setminus\Gamma_0}\delta$ (here Γ,Γ_0 are defined in Step 5) and $\infty\in\bigcap_{\delta\in\Gamma\setminus\Gamma_0}\delta$. The observation $G_\ell\setminus\{z\}=\bigcup_{\delta\in\Gamma\setminus\Gamma_0}(\delta\setminus\{z\})$ and $\infty\in\bigcap_{\delta\in\Gamma\setminus\Gamma_0}(\delta\setminus\{z\})$ imply that $G_\ell\setminus\{z\}$ is connected, and hence z is not a cut point of G_ℓ. □

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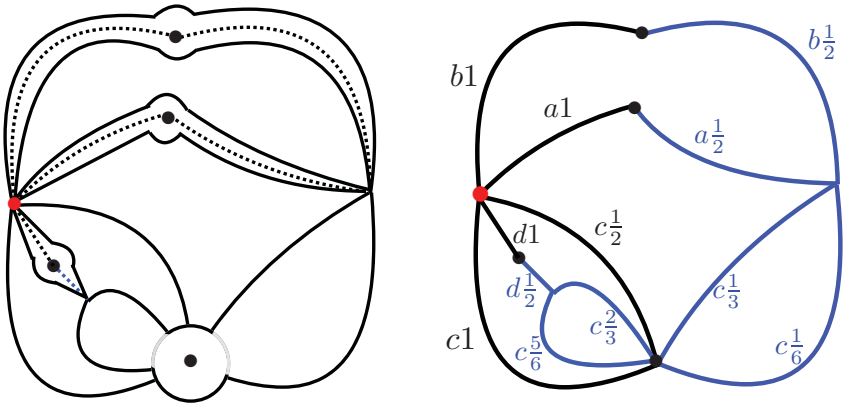

FIGURE 14. This figure shows how to get the graph G (a) by a natural modification of G_ℓ (b; $\ell = 2$) in Step 5.

PROPOSITION 4.8. *The graph G satisfies:*

(1) *any point in $G \cap J(f)$ is not a cut point of G;*
(2) *the center of any immediate root basin is a non-cut point of G;*
(3) *for any immediate root basin B, the intersection $G \cap \overline{B}$ is connected. In other words, any Julia point in $\overline{B} \cap J(f)$ is linked to the center of B by an internal ray in G.*

Proof. It is worth observing that

$$G = \mathcal{M}(G_\ell) = \bigcup_{\delta \in \Gamma \setminus \Gamma_0} \mathcal{M}(\delta).$$

(1) For any $z \in G \cap J(f)$ and $z \neq \infty$, the facts

$$G \setminus \{z\} = \bigcup_{\delta \in \Gamma \setminus \Gamma_0} (\mathcal{M}(\delta) \setminus \{z\}), \quad \infty \in \bigcap_{\delta \in \Gamma \setminus \Gamma_0} (\mathcal{M}(\delta) \setminus \{z\})$$

imply that $G \setminus \{z\}$ is connected, and hence z is not a cut point of G.

(2) Recall that each curve $\delta \in \Gamma_2^* \cup \Gamma_3^* \cup \cdots \cup \Gamma_\ell^*$ starts at an immediate root basin B' and terminates at a different one B''. Each curve $\delta \in \Gamma_1^*$ will be connected to another immediate root basin by another curve $\beta \in \Gamma_1 \cup \Gamma_2 \cup \cdots \cup \Gamma_\ell$. It follows that after the modification, the centers of immediate root basins are not cut points with respect to G.

(3) Note that each curve $\delta \in \Gamma_2^* \cup \Gamma_3^* \cup \cdots \cup \Gamma_\ell^*$ meets exactly two different immediate root basins. By construction, if $\delta \cap B \neq \emptyset$ for some immediate root basin B, then $\mathcal{M}(\delta) \cap \overline{B}$ is the closure of the union of two internal rays. (This implies, in particular, that $\mathcal{M}(\delta) \cap \overline{B}$ has no isolated point.)

Note also for $\delta \in \Gamma_1$, the intersection $\mathcal{M}(\delta) \cap \overline{B}$ is the closure of the union of two (if $d_B = 2$) or four (if $d_B > 2$, see Figure 11) internal rays. Therefore,

$$G \cap \overline{B} = \bigcup_{\delta \in \Gamma \setminus \Gamma_0} (\mathcal{M}(\delta) \cap \overline{B})$$

is the closure of the union of finitely many internal rays, and hence connected. □

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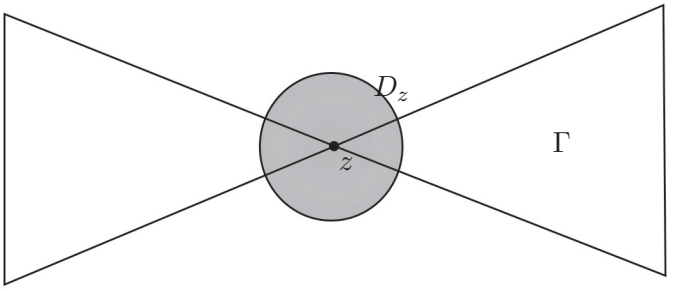

FIGURE 15. The graph Γ is the union of boundaries of the two symmetric triangles with only a cut point z as shown above. Take a closed disk D_z around the point z such that $D_z \cap \Gamma$ is a star-like tree. The new graph $\partial(\Gamma \cup D_z)$ $(= (\Gamma \setminus D_z) \cup \partial D_z)$ has no cut points. In general, if a graph Γ possesses finitely many cut points $z_1, \ldots, z_n$, then the new graph $\partial(\Gamma \cup \bigcup_{1 \le i \le n} D_{z_i})$ has no cut points, where D_{z_i} are pairwise disjoint closed disks associated to z_i.

Remark 4.9.

(1) Proposition 4.8 implies that the only possible cut points in G are the centers of strictly pre-periodic components $B \in \mathrm{Comp}(B_f)$. These points are finitely many.
(2) Let Γ be a graph with finitely many cut points. There is a natural way to produce a new graph from Γ such that it has no cut points; see Figure 15.

To prove the connectivity of $f^{-1}(G)$, it is equivalent to show that each component of $f^{-1}(\widehat{\mathbb{C}} \setminus G)$ is simply connected.

To this end, let $\mathcal{D} = \bigcup_B \overline{\Phi_B^{-1}(\mathbb{D}_{1/2})}$ and $X = \widehat{\mathbb{C}} \setminus \mathcal{D}$, where the union is taken over all $B \in \mathrm{Comp}(B_f)$ such that $B \cap G \neq \emptyset$, and $\Phi_B : B \to \mathbb{D}$ is the Böttcher map of B. By Proposition 4.8 and Remark 4.9, the graph $\widetilde{G} := \partial(G \cup \mathcal{D})$ has no cut points. Then each component of $X \setminus G$, which serves as a component of $\widehat{\mathbb{C}} \setminus \widetilde{G}$, is a Jordan disk. To show that each component of $f^{-1}(\widehat{\mathbb{C}} \setminus G)$ is simply connected, it is equivalent to show that each component of $f^{-1}(X \setminus G)$ is simply connected.

In fact, this is an important property for puzzle pieces, so we have restated it as Proposition 5.1, and the proof can be found there. (The slight difference is we deal with $f^{-1}(X \setminus G)$ instead of $f^{-1}(\widehat{\mathbb{C}} \setminus G)$, in order to apply Corollary 3.2.)

This completes proof of Step 6, and hence the whole proof of Theorem 4.1. □

Remark 4.10. One may assume the number N in Theorem 4.1 is minimal, in the sense that $G \subseteq f^{-N}(\Delta_0)$ and $G \nsubseteq f^{-N+1}(\Delta_0)$. This minimal N can not be controlled by the degree of f, even in the cubic case.

In fact, we can show the following. For any integer $n \ge 1$, there is a post-critically finite cubic Newton map f, for which the invariant graph G constructed in Theorem 4.1 satisfies that

$$G \subseteq f^{-n}(\Delta_0), \quad G \nsubseteq f^{-n+1}(\Delta_0).$$

The proof is based on the deeper understanding of the parameter space [**RWY**]. Because we will not use this fact in the paper, we skip its proof.

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4.3. *Appendix: Shrinking lemma revisited.* At the end of this section, we prove a version of the shrinking lemma (see [**LM**, **TY**] for its original form), which plays an important role in the proof of Lemma 4.5.

LEMMA 4.11. *Let f be a rational map. Let $\{(E_n, U_n)\}_{n\geq 0}$ be a sequence of subsets in $\widehat{\mathbb{C}}$, such that, for all $n \geq 0$:*

(1) *$E_n \Subset U_n$ with E_n full continua and U_n open sets (a set is said to be full if its complement is connected);*
(2) *$f(E_{n+1}) = E_n$, $f(U_{n+1}) = U_n$;*
(3) *$U_{n+1} \cap U_0 = \emptyset$.*

Then the spherical diameter of E_n converges to zero as $n \to \infty$.

Proof. First observe that the sets U_n are pair-wisely disjoint. If not, assume $U_{n_1} \cap U_{n_2} \neq \emptyset$ for some $0 \leq n_1 < n_2$. Then we have $\emptyset \neq f^{n_1}(U_{n_1} \cap U_{n_2}) \subseteq U_0 \cap U_{n_2-n_1}$, which contradicts (3). Thus by ignoring finitely many pairs (E_n, U_n), one may assume that $\bigcup_{n\geq 0} U_n$ does not contain the critical values of f. Because E_0 is full and $E_0 \Subset U_0$, we can choose a topological disk D_0 such that $E_0 \Subset D_0 \Subset U_0$. Then for each n, the unique component D_n of $(f^n|_{U_n})^{-1}(D_0)$, which contains E_n, is a topological disk. Moreover, the map $f^n : D_n \to D_0$ is conformal, whose inverse is denoted by g_n. Then $\{g_n\}$ forms a normal family.

We claim that the limit map g_∞ of any convergent subsequence $\{g_{n_k}\}$ is a constant map. If not, then $g_\infty(D_0)$ is an open subset of $\widehat{\mathbb{C}}$. Therefore, for any sufficiently large integers $k \neq k'$, the images $g_{n_k}(D_0)(= D_{n_k})$ and $g_{n_{k'}}(D_0)(= D_{n_{k'}})$ will overlap, which is impossible.

Finally, if $\lim_n \operatorname{diam}(E_n) \to 0$ is not true, then there is a constant $\epsilon > 0$ and a subsequence $\{E_{l_k}\}$ with $\operatorname{diam}(E_{l_k}) \geq \epsilon$. This is impossible, because by passing to a further subsequence, the maps g_{l_k} converge uniformly on E_0 to a constant. □

5. *Branner–Hubbard–Yoccoz puzzle*

In this section, we develop the Branner–Hubbard–Yoccoz puzzle theory for Newton maps, using the invariant graph given by the preceding section.

5.1. *Puzzles and ends.* Let G be the graph given by Theorem 4.1. Recall that $\Phi_B : B \to \mathbb{D}$ is the Böttcher map of $B \in \operatorname{Comp}(B_f)$. Let

$$X = \widehat{\mathbb{C}} \setminus \bigcup_B \overline{\Phi_B^{-1}(\mathbb{D}_{1/2})},$$

where the union is taken over all $B \in \operatorname{Comp}(B_f)$ such that $B \cap G \neq \emptyset$. Clearly $f^{-1}(X) \subseteq X$. For any integer $n \geq 0$, let $\mathcal{P}_n$ be the collection of all connected components of $f^{-n}(X \setminus G)$. An element $P \in \mathcal{P}_n$ is called a *puzzle piece* of depth (or level) $n \geq 0$. Note that two distinct puzzle pieces P, Q are either disjoint (that is, $P \cap Q = \emptyset$) or nested (that is, $P \subseteq Q$ or $Q \subseteq P$).

An important fact about puzzle pieces is as follows.

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PROPOSITION 5.1. *Let P, Q be two puzzle pieces with $Q = f(P)$. Then we have the following two implications:*

(1) *Q is a Jordan disk $\Longrightarrow$ P is a Jordan disk;*

(2) *$P \subseteq Q \Longrightarrow \infty \in \partial P \cap \partial Q$ and $f : P \to Q$ is conformal.*

Proof. Let $l \geq 1$ be the depth of P. Note that there is a unique puzzle piece of depth $l-1$, say S, containing P. Let S_0 be the puzzle piece of depth 0 containing P. By Proposition 4.8 and Remark 4.9, S_0 is a Jordan disk. Note that the only possible fixed point in $\overline{S_0}$ is ∞ by the construction. Thus the filled closure $\widehat{P}_{S_0}$ of P with respect to S_0 contains at most one fixed point, which can only be ∞ on its boundary.

To prove the two implications, we discuss the relation of Q and S.

Case 1: $Q = S$ or equivalently $P \subseteq Q$. Then $\widehat{P}_{S_0} = \widehat{P}_Q$. If $\partial Q \cap \partial P$ contains a fixed point q, then $q = \infty$, and in a neighborhood N_q of q, we have $N_p \cap \partial P = \Delta_0 \cap N_q \cap \partial P$. The f-invariance of the channel diagram Δ_0 implies that $f(N_q \cap \partial P) \supseteq N_q \cap \partial P$ in this situation. Then, by applying Corollary 3.2 to the case $(D, U) = (Q, P)$, we have

$$1 \geq \#\mathrm{Fix}(f|_{\widehat{P}_Q}) = \sum_{V \subseteq \widehat{P}_Q, V \in \mathrm{Comp}(f^{-1}(Q))} \deg(f|_{\partial V}) \geq \deg(f|_{\partial P}) \geq 1.$$

This implies that $\infty \in \partial P \cap \partial Q$, $\widehat{P}_S = \overline{P}$, and $f : P \to Q$ is conformal. In this case, we also have the first implication.

Case 2: $Q \neq S$ or equivalently $Q \cap S = \emptyset$. In this case, we only need to prove 1. Assume that Q is a Jordan disk. If P is not a Jordan disk, then $\widehat{P}_{S_0} \setminus \overline{P}$ is non-empty, furthermore, it contains at least a component V of $f^{-1}(W)$ with $W := \widehat{\mathbb{C}} \setminus \overline{Q}$. Clearly $V \subseteq W$. Applying Corollary 3.2 to the case $(D, U) = (W, V)$, we know that the filled closure $\widehat{V}_W (\subseteq \widehat{P}_S \subseteq S)$ contains fixed points, which must be ∞. Therefore, we have

$$\infty \in \partial V \cap \partial P \cap \partial S \cap \partial Q.$$

However, the local behavior of f near ∞ implies that in a neighborhood $N(\infty)$ of ∞, we have $P \cap N(\infty) \subseteq f(P \cap N(\infty))$. It follows that $Q = S$. This is a contradiction. □

LEMMA 5.2. *The puzzle pieces satisfy the following properties:*

(1) *each puzzle piece is a Jordan disk;*

(2) *for any puzzle piece P and any immediate root basin B, the intersection $\overline{P} \cap \partial B$ is connected (caution: if $B \in \mathrm{Comp}(B_f)$ is not an immediate root basin, then $\overline{P} \cap \partial B$ might be disconnected);*

(3) *for any puzzle piece P, the intersection $\overline{P} \cap J(f)$ is connected.*

Proof. (1) By Proposition 4.8 and Remark 4.9, each puzzle piece of depth 0 is a Jordan disk. By Proposition 5.1 and induction, all puzzle pieces are Jordan disks.

(2) By Proposition 4.8, the set $B \cap P$ is some sector $S_B(\theta, \theta'; r)$, and $\overline{P} \cap \partial B = \bigcap_{0<s<1} \overline{S_B(\theta, \theta'; s)}$, which is connected.

(3) By Proposition 4.8, if $B \in \mathrm{Comp}(B_f)$ satisfies $B \cap P \neq \emptyset$ and $B \not\subseteq P$, then $B \cap P$ is the union of finitely many sectors $S_B(\theta, \theta'; r)$ (the reason is that the center c_B of B might be a cut point. In this case, $\overline{P} \cap \partial B$ is a union of finitely many connected set). Note that $J(f) \cap \overline{S_B(\theta, \theta'; r)}$ is connected, because $J(f) \cap \overline{S_B(\theta, \theta'; r)} = \bigcap_{0<s<1} \overline{S_B(\theta, \theta'; s)}$.

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We aim to the show that any two points $z_1, z_2 \in \overline{P} \cap J(f)$ are contained in a connected subset $C \subseteq J(f) \cap \overline{P}$. Let γ be a Jordan arc in P connecting z_1 and z_2. Then $\gamma \cap F(f)$ consists of countably many open segments $\{\gamma_i\}_{i\in\Lambda}$. For each γ_i, if there is $B_0 \in \mathrm{Comp}(B_f)$ so that $\gamma_i \subseteq B_0 \subseteq P$, we set $C_i = \partial B_0$; otherwise, γ_i is contained in some sector $S_B(\theta, \theta'; r)$, and we set $C_i = J(f) \cap \overline{S_B(\theta, \theta'; r)}$. The set $C = (\gamma \cap J(f)) \cup (\bigcup_{i\in\Lambda} C_i)$ is a connected subset of $J(f) \cap \overline{P}$ connecting z_1 and z_2. □

It is worth observing that the number of unbounded puzzle pieces of depth n is independent of n. This number is $d_0 = \sum_B (\deg(f|_B) - 1)$, where the sum is taken over all immediate root basins B. Let $\mathcal{P}_n^\infty = \{P_{n,1}^\infty, \ldots, P_{n,d_0}^\infty\}$ be the set of all unbounded puzzle pieces of depth n, numbered in the way that $P_{n+1,k}^\infty \subseteq P_{n,k}^\infty$, for any $n \geq 0$ and $1 \leq k \leq d_0$. Clearly, the sets

$$Y_n(\infty) = \overline{P_{n,1}^\infty} \cup \cdots \cup \overline{P_{n,d_0}^\infty}, \quad n \geq 0$$

are closed neighborhoods of ∞. The *grand orbit* of ∞ is denoted by

$$\Omega_f = \bigcup_{k\geq 0} f^{-k}\{\infty\}.$$

For any $z \in \Omega_f$, let us define

$$\mathcal{P}_n^z = \{P \in \mathcal{P}_n; z \in \overline{P}\}, \quad Y_n(z) = \bigcup_{P\in\mathcal{P}_n^z} \overline{P}.$$

For any point $z \in \widehat{\mathbb{C}} - B_f \cup \Omega_f$, its orbit avoids the graph G; therefore, the puzzle piece of depth $k \geq 0$ containing z is well defined, and is denoted by $P_k(z)$. For $z \in \Omega_f$, let $P_k(z)$ be the interior of $Y_k(z)$. In this way, for all $z \in \widehat{\mathbb{C}} - B_f$ and all $k \geq 0$, the piece $P_k(z)$ is well defined.

For any $z \in \widehat{\mathbb{C}} - B_f$, the *end* of z, denoted by $\mathbf{e}(z)$, is defined by

$$\mathbf{e}(z) = \bigcap_{k\geq 0} \overline{P_k(z)}.$$

PROPOSITION 5.3. *For any $z \in \widehat{\mathbb{C}} - B_f$ and any integer $k \geq 0$, there is an integer $n_k = n_k(z) > 0$ with the property:*

$$P_{k+n_k}(z) \Subset P_k(z).$$

This implies, in particular, that $\mathbf{e}(z) = \{z\}$ for any $z \in \Omega_f$.

Proof. We first consider $z \in \Omega_f$. In this case, there is an integer $N \geq 0$ with $f^N(\mathbf{e}(z)) = \mathbf{e}(\infty)$. To show the statement, it suffices to show $\mathbf{e}(\infty) = \{\infty\}$.

By Proposition 5.1, for each $n \geq 1$, the map $f^n : P_n(\infty) \to P_0(\infty)$ is conformal, and the boundaries $\partial Y_n(\infty)$, $\partial Y_0(\infty)$ are Jordan curves. Therefore, $f^n : \partial Y_n(\infty) \to \partial Y_0(\infty)$ is a homeomorphism. We claim that $Y_N(\infty) \Subset Y_0(\infty)$ for some large N. In fact, if $\partial Y_n(\infty) \cap \partial Y_0(\infty) \neq \emptyset$ for all $n \geq 1$, then the relation $Y_{n+1}(\infty) \subseteq Y_n(\infty)$ implies that

$$\partial Y_{n+1}(\infty) \cap \partial Y_0(\infty) \subseteq \partial Y_n(\infty) \cap \partial Y_0(\infty).$$

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Therefore,

$$\bigcap \partial Y_n(\infty) \neq \emptyset \quad \text{and} \quad \bigcap \partial Y_n(\infty) \subseteq \Omega_f \subseteq J(f).$$

To see this, $\partial Y_n(\infty)$ is bounded by some internal rays and sections of equipotential curves of the form $G_B^{-1}(r_n)$, where $G_B : B \to [-\infty, 0)$ is the Green function in B. The potential $r_n \to 0^-$ as $n \to +\infty$. Hence, the intersection $\bigcap \partial Y_n(\infty)$ is contained in the Julia set $J(f)$. Because $\partial Y_n(\infty) \cap J(f) \subseteq \Omega_f$ for any n, we have

$$\bigcap \partial Y_n(\infty) = \bigcap (\partial Y_n(\infty) \cap J(f)) \subseteq \Omega_f.$$

Take $p \in \bigcap \partial Y_n(\infty)$ and suppose $f^{n_0}(p) = \infty$. Clearly $p \neq \infty$. This contradicts the fact that $f^{n_0} : \partial Y_{n_0}(\infty) \to \partial Y_0(\infty)$ is a homeomorphism.

By the claim and applying the Schwarz lemma to the inverse of $f^N : Y_N(\infty) \to Y_0(\infty)$, we have that $\mathbf{e}(\infty) = \bigcap_k Y_{Nk}(\infty) = \{\infty\}$.

For those $z \in \widehat{\mathbb{C}} - (B_f \cup \Omega_f)$, the idea of the proof is same as above. If there is an integer $k_0 \geq 0$, such that $\partial P_{k_0}(z) \cap \partial P_{k_0+l}(z) \neq \emptyset$ for all $l > 0$, then the nested property (that is, $P_{k_0+l+1}(z) \subseteq P_{k_0+l}(z)$) gives that

$$\partial P_{k_0}(z) \cap \partial P_{k_0+l+1}(z) \subseteq \partial P_{k_0}(z) \cap \partial P_{k_0+l}(z).$$

Therefore,

$$\emptyset \neq \bigcap_{l \geq 1} (\partial P_{k_0}(z) \cap \partial P_{k_0+l}(z)) = \bigcap_{l \geq 0} \partial P_{k_0+l}(z) \subseteq \Omega_f \cap J(f).$$

It follows that the puzzle pieces $\{P_{k_0+l}(z)\}_{l \geq 0}$ have a common boundary point ξ with $f^m(\xi) = \infty$ for some $m \geq 0$. Applying the f^m-action on these puzzle pieces, we get

$$\infty \in \overline{P_{k_0-m+l}(f^m(z))} \subseteq Y_{k_0-m+l}(\infty) \quad \text{for all } l \geq m.$$

This gives that $\infty \in \mathbf{e}(f^m(z)) \subseteq \mathbf{e}(\infty)$. By the proven fact $\mathbf{e}(\infty) = \{\infty\}$, we have $f^m(z) = \infty$. This contradicts the assumption $z \in \widehat{\mathbb{C}} - (B_f \cup \Omega_f)$. □

We collect some facts about ends as follows:

(1) $\mathbf{e}(z)$ is either a singleton or a full continuum in $\widehat{\mathbb{C}}$;

(2) $f(\mathbf{e}(z)) = \mathbf{e}(f(z))$;

(3) for any $z' \in \widehat{\mathbb{C}} - B_f$ with $z' \neq z$, based on the proven fact $\mathbf{e}(q) = \{q\}$ for any $q \in \Omega_f$ (see Proposition 5.3), we have that either

$$\mathbf{e}(z') = \mathbf{e}(z) \quad \text{or} \quad \mathbf{e}(z') \cap \mathbf{e}(z) = \emptyset;$$

(4) by Lemma 5.2, $\mathbf{e}(z) = \{z\}$ implies the local connectivity of $J(f)$ at z. For any immediate root basin B and any $z \in \partial B$, the fact $\mathbf{e}(z) \cap \partial B = \{z\}$ implies the local connectivity of ∂B at z.

Let $\mathcal{E} = \{\mathbf{e}(z); z \in \widehat{\mathbb{C}} - B_f\}$ be the collection of all ends. An end is *trivial* if it is a singleton. An end $\mathbf{e}$ is called *critical* if it contains a critical point of f. The *orbit* $\mathrm{orb}(\mathbf{e})$ of an end $\mathbf{e} \in \mathcal{E}$ is $\mathrm{orb}(\mathbf{e}) = \{f^k(\mathbf{e})\}_{k \geq 0}$.

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An end $\mathbf{e}$ is *preperiodic* if $f^{m+n}(\mathbf{e}) = f^m(\mathbf{e})$ for some $m \geq 0, n \geq 1$. In particular, $\mathbf{e}$ is called *periodic* if $m = 0$. If there is no such m, n, then $\mathbf{e}$ is called *wandering*. In this case, its orbit $\mathrm{orb}(\mathbf{e})$ has infinitely many elements.

For each end $\mathbf{e} = \mathbf{e}(z)$ with $z \in \widehat{\mathbb{C}} - B_f$, let $P_n(\mathbf{e}) = P_n(z)$. It follows from Proposition 5.3 that $P_n(\mathbf{e})$ is the puzzle piece of depth n containing $\mathbf{e}$.

Let $(\mathbf{e}_k)_{k\in\mathbb{N}}$ be a sequence of wandering ends with distinct entries $\mathbf{e}_k$, the *combinatorial accumulation set* $\mathcal{A}((\mathbf{e}_k)_{k\in\mathbb{N}})$ consists of the ends $\mathbf{e}' \in \mathcal{E}$, such that for any integer $n > 0$, the index set $\{k \in \mathbb{N};\ \mathbf{e}_k \subseteq P_n(\mathbf{e}')\}$ is infinite.

LEMMA 5.4. $\mathcal{A}((\mathbf{e}_k)_{k\in\mathbb{N}}) \neq \emptyset$.

Proof. For any $n \geq 0$, recall that the collection $\mathcal{P}_n$ of puzzle pieces of depth n is a finite set. We define the index set I_n and the puzzle piece $P_n \in \mathcal{P}_n$ inductively as follows. First, there is a puzzle piece $P_0 \in \mathcal{P}_0$ such that

$$I_0 = \{k \in \mathbb{N};\ \mathbf{e}_k \subseteq P_0\}$$

is an infinite set. Suppose that we have constructed the infinite index set I_j and the puzzle piece $P_j \in \mathcal{P}_j$ for $0 \leq j \leq \ell$, satisfying that

$$I_0 \supseteq \cdots \supseteq I_\ell,\ \ P_0 \supseteq \cdots \supseteq P_\ell.$$

Then one can find $P_{\ell+1} \in \mathcal{P}_{\ell+1}$ with $P_{\ell+1} \subseteq P_\ell$, such that the index set

$$I_{\ell+1} = \{k \in I_\ell;\ \mathbf{e}_k \subseteq P_{\ell+1}\}$$

is an infinite set. Now let us define $\mathbf{e}' = \bigcap_n \overline{P_n}$.

To finish, we show $\mathbf{e}' \in \mathcal{E}$, which implies that $\mathbf{e}' \in \mathcal{A}((\mathbf{e}_k)_{k\in\mathbb{N}})$. To this end, we discuss two cases. If $\mathbf{e}' \cap \Omega_f \neq \emptyset$, we take $z \in \mathbf{e}' \cap \Omega_f \neq \emptyset$, then the fact $\{z\} \subseteq \mathbf{e}' = \bigcap_n \overline{P_n} \subseteq \bigcap_n Y_n(z) = \{z\}$ (by Proposition 5.3) implies that $\mathbf{e}' = \{z\} = \mathbf{e}(z)$. If $\mathbf{e}' \cap \Omega_f = \emptyset$, we take $z \in \mathbf{e}'$, then $\mathbf{e}' = \bigcap_n \overline{P_n} = \bigcap_n \overline{P_n(z)} = \mathbf{e}(z)$. In either case, we have $\mathbf{e}' \in \mathcal{E}$, which completes the proof. □

The *combinatorial limit set* $\omega(\mathbf{e})$ of a wandering end $\mathbf{e} \in \mathcal{E}$ is defined by

$$\omega(\mathbf{e}) = \mathcal{A}((f^k(\mathbf{e}))_{k\in\mathbb{N}}).$$

One may verify that $\omega(\mathbf{e})$ satisfies the following properties:

(1) $\omega(f(\mathbf{e})) = \omega(\mathbf{e})$;
(2) $f(\omega(\mathbf{e})) \subseteq \omega(\mathbf{e})$;
(3) for any wandering end $\mathbf{e}' \in \omega(\mathbf{e})$, we have $\omega(\mathbf{e}') \subseteq \omega(\mathbf{e})$.

The first two follow from the definition of $\omega(\mathbf{e})$. We only verify the third one. Let $\mathbf{e}' \in \omega(\mathbf{e})$ be a wandering end, and take $\mathbf{e}'' \in \omega(\mathbf{e}')$. By definition, for any $n \geq 0$, the index set $J_n = \{k \in \mathbb{N};\ f^k(\mathbf{e}') \subseteq P_n(\mathbf{e}'')\}$ is infinite. For $k \in J_n$, note that $f^k(\mathbf{e}') \in \omega(\mathbf{e})$, this implies that the index set $\{t \in \mathbb{N};\ f^t(\mathbf{e}) \subseteq P_n(f^k(\mathbf{e}')) = P_n(\mathbf{e}'')\}$ is infinite. Therefore, $\mathbf{e}'' \in \omega(\mathbf{e})$.

PROPOSITION 5.5. *Let $L > 0$ be an integer with $Y_L(\infty) \Subset Y_0(\infty)$. Let $\mathbf{e}$ be a wandering end with $\mathbf{e} \subseteq Y_L(\infty)$, then there is an (minimal) integer $s = s(\mathbf{e}) \geq 0$ with the*

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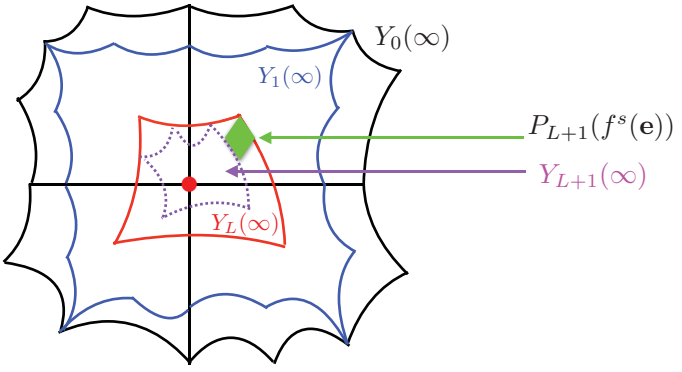

FIGURE 16. The puzzle pieces around ∞ (red dot). It may happen that $\partial Y_1(\infty) \cap \partial Y_0(\infty) \neq \emptyset$. Here $Y_L(\infty) \Subset Y_0(\infty)$ for some $L \geq 1$. Assume $\mathbf{e} \subseteq Y_L(\infty)$, then $P_{L+1}(f^s(\mathbf{e})) \subseteq Y_L(\infty) \setminus Y_{L+1}(\infty)$ for some minimal integer $s \geq 0$. Moreover $P_{L+1}(f^s(\mathbf{e})) \Subset P_0(f^s(\mathbf{e}))$.

following property:

$$P_{L+1}(f^s(\mathbf{e})) \Subset P_0(f^s(\mathbf{e})) \in \mathcal{P}_0^\infty$$

and $f^s : P_{L+s+1}(\mathbf{e}) \to P_{L+1}(f^s(\mathbf{e}))$ *is conformal.*

Proof. We first claim that $\mathbf{e} \cap \partial Q = \emptyset$ for any puzzle piece Q. Note that $\partial Q \cap J(f) \subset \Omega_f$ and $\partial Q \cap F(f) \subset B_f$. By definition, we have $\mathbf{e} \cap B_f = \emptyset$. If $\mathbf{e} \cap \partial Q \neq \emptyset$, then $\mathbf{e}$ contains a preperiodic point in Ω_f by Proposition 5.3. It is impossible, as $\mathbf{e}$ is wandering.

Recall that $Y_k(\infty) = \bigcup_j \overline{P_{k,j}^\infty}$ and $P_k(\infty)$ is the interior of $Y_k(\infty)$. By Propositions 5.1 and 5.3, for any $k \geq 0$, the map $f : P_{k+1}(\infty) \to P_k(\infty)$ is one-to-one. The assumption

$$\mathbf{e} \subseteq Y_L(\infty) = \bigcup_{s \geq 0} (Y_{L+s}(\infty) \setminus Y_{L+s+1}(\infty))$$

implies that $\mathbf{e} \subseteq Y_{L+s}(\infty) \setminus Y_{L+s+1}(\infty)$ for some integer $s \geq 0$. Then we can find an index j with $\mathbf{e} \subseteq P_{L+s,j}^\infty \setminus P_{L+s+1,j}^\infty$. Because for any $k \geq 0$, the map $f : P_{k,j}^\infty \setminus P_{k+1,j}^\infty \to P_{k-1,j}^\infty \setminus P_{k,j}^\infty$ is a homeomorphism, we have that $f^s(\mathbf{e}) \subseteq P_{L,j}^\infty \setminus P_{L+1,j}^\infty$. Hence, $P_{L+1}(f^s(\mathbf{e}))$ is bounded.

Because each unbounded puzzle piece is bounded by the fixed internal rays, their iterated preimages, and equipotential curves, we see that $\overline{P_{L+1}(f^s(\mathbf{e}))}$ is disjoint from $\Delta_0 \cup \partial Y_0(\infty)$ (see Figure 16). This implies that

$$P_{L+1}(f^s(\mathbf{e})) \Subset P_0(f^s(\mathbf{e})) = P_{0,j}^\infty \in \mathcal{P}_0^\infty. \qquad \square$$

5.2. *Strategy of the proof.* To prove our main Theorem 1.1, it suffices to show that for any immediate root basin B, we have

$$\mathbf{e}(z) \cap \partial B = \{z\} \quad \text{for all } z \in \partial B. \tag{$*$}$$

To this end, we first need to classify all ends in $\mathcal{E}$ into two types: wandering ones and preperiodic ones, which are denoted by $\mathcal{E}_{\mathrm{w}}$ and $\mathcal{E}_{\mathrm{pp}}$, respectively.

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The set $\mathcal{E}_{\mathrm{w}}$ of wandering ends has a further decomposition:

$$\mathcal{E}_{\mathrm{w}} = \mathcal{E}_{\mathrm{w}}^{\mathrm{pp}} \sqcup \mathcal{E}_{\mathrm{w}}^{\mathrm{nr}} \sqcup \mathcal{E}_{\mathrm{w}}^{\mathrm{r}},$$

where

$\mathcal{E}_{\mathrm{w}}^{\mathrm{pp}} = \{\mathbf{e} \in \mathcal{E}_{\mathrm{w}}; \mathcal{E}_{\mathrm{pp}} \cap \omega(\mathbf{e}) \neq \emptyset\}$;
$\mathcal{E}_{\mathrm{w}}^{\mathrm{nr}} = \{\mathbf{e} \in \mathcal{E}_{\mathrm{w}}; \mathcal{E}_{\mathrm{pp}} \cap \omega(\mathbf{e}) = \emptyset$ and $\omega(\mathbf{e}) \neq \omega(\mathbf{e}')$ for some $\mathbf{e}' \in \omega(\mathbf{e})\}$;
$\mathcal{E}_{\mathrm{w}}^{\mathrm{r}} = \{\mathbf{e} \in \mathcal{E}_{\mathrm{w}}; \mathcal{E}_{\mathrm{pp}} \cap \omega(\mathbf{e}) = \emptyset$ and $\omega(\mathbf{e}) = \omega(\mathbf{e}')$ for all $\mathbf{e}' \in \omega(\mathbf{e})\}$.

The proof of the statement $(*)$ will be carried out in the following two sections. In §6, we prove a stronger fact that any wandering end is a singleton. In §7, we prove that for any pre-periodic end $\mathbf{e}$, the intersection $\mathbf{e} \cap \partial B$ is either empty or a singleton. These two cases cover all situations.

In the rest of the paper, let $\mathcal{E}_{\mathrm{crit}} \subseteq \mathcal{E}$ be the collection of all critical ends. Set $\kappa = \#\mathcal{E}_{\mathrm{crit}}$. Recall that d is the degree of the Newton map f.

6. *Wandering ends are trivial*

In this section, we show that any wandering end is a singleton. The proof is based on the following dichotomy: for any wandering end $\mathbf{e}$, either

(1) $\mathbf{e}$ satisfies the *bounded degree* property; or
(2) $\omega(\mathbf{e})$ contains a *persistently recurrent* critical end.

The treatments of these two situations are different.

6.1. *Bounded degree property implies triviality of ends.*

Definition 6.1. An end $\mathbf{e}$ is said to have a bounded degree (BD for short) property if there exist puzzle pieces $\{P_{n_k}(\mathbf{e})\}$, with $n_k \to \infty$ as $k \to \infty$, and an integer D, such that

$$\deg(f^{n_k} : P_{n_k}(\mathbf{e}) \to P_0(f^{n_k}(\mathbf{e}))) \leq D \quad \text{for all } k \geq 1. \tag{$\star$}$$

PROPOSITION 6.2. *A wandering end* $\mathbf{e}$ *with BD property is trivial.*

Proof. By assumption, there is a sequence of puzzle pieces $\{P_{n_k}(\mathbf{e})\}$ satisfying $(\star)$. The combinatorial accumulation set $\mathcal{A}((f^{n_k}(\mathbf{e}))_{k\in\mathbb{N}})$ of the sequence $(f^{n_k}(\mathbf{e}))_{k\in\mathbb{N}}$ satisfies

$$\emptyset \neq \mathcal{A}((f^{n_k}(\mathbf{e}))_{k\in\mathbb{N}}) \subseteq \omega(\mathbf{e}).$$

Take $\mathbf{e}_0 \in \mathcal{A}((f^{n_k}(\mathbf{e}))_{k\in\mathbb{N}})$, note that for any $n \geq 0$, the index set $\{k \in \mathbb{N}; f^{n_k}(\mathbf{e}) \subseteq P_n(\mathbf{e}_0)\}$ is infinite.

To prove the proposition, we need to discuss two cases:

Case 1: $\mathbf{e}_0 \notin \{\mathbf{e}(z); z \in \Omega_f\}$. In this case, by Proposition 5.3, there is an integer $L_0 > 0$ such that $P_{L_0}(\mathbf{e}_0) \Subset P_0(\mathbf{e}_0)$. By passing to a subsequence, we may assume $f^{n_k}(\mathbf{e}) \subseteq P_{L_0}(\mathbf{e}_0)$ for all $k \geq 1$. By pulling back the triple $(f^{n_k}(\mathbf{e}), P_{L_0}(\mathbf{e}_0), P_0(\mathbf{e}_0))$ along the orbit $\mathbf{e} \mapsto f(\mathbf{e}) \mapsto \cdots \mapsto f^{n_k}(\mathbf{e})$, we get the non-degenerate annuli $P_{n_k}(\mathbf{e}) \setminus \overline{P_{L_0+n_k}(\mathbf{e})}$, whose moduli satisfy

$$\mathrm{mod}(P_{n_k}(\mathbf{e}) \setminus \overline{P_{L_0+n_k}(\mathbf{e})}) \geq \frac{1}{D}\mathrm{mod}(P_0(\mathbf{e}_0) \setminus \overline{P_{L_0}(\mathbf{e}_0)}), \quad \text{for all } k \geq 1.$$

This implies that $\mathbf{e} = \bigcap \overline{P_k(\mathbf{e})}$ is a singleton.

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Case 2: $\mathbf{e}_0 \in \{\mathbf{e}(z); z \in \Omega_f\}$. In this case, replacing $(f^{n_k}(\mathbf{e}))_{k\in\mathbb{N}}$ by the new sequence $(f^{n_k+l}(\mathbf{e}))_{k\in\mathbb{N}}$ (here $l \geq 0$ is some integer) if necessary, we may assume $\mathbf{e}_0 = \mathbf{e}(\infty)$. Recall that $Y_n(\infty) = \bigcup_k \overline{P^{\infty}_{n,k}}$ and $P_n(\infty)$ is the interior of $Y_n(\infty)$. Let $L > 0$ be an integer with $Y_L(\infty) \Subset Y_0(\infty)$.

By choosing a subsequence of $\{n_k\}_k$, we may assume that

$$f^{n_k}(\mathbf{e}) \subseteq Y_L(\infty) \quad \text{and} \quad \deg(f^{n_k} : P_{n_k}(\mathbf{e}) \to P^{\infty}_{0,m}) \leq D$$

with $P^{\infty}_{0,m} \in \mathcal{P}^{\infty}_0$ and $P^{\infty}_{0,m} = P_0(f^{n_k}(\mathbf{e}))$, for all $k \in \mathbb{N}$.

For each k, the assumption $f^{n_k}(\mathbf{e}) \subseteq Y_L(\infty) = \bigcup_{s\geq 0}(Y_{L+s}(\infty) \setminus Y_{L+s+1}(\infty))$ implies that there is a unique integer $s_k \geq 0$ such that $f^{n_k}(\mathbf{e}) \subseteq Y_{L+s_k}(\infty) \setminus Y_{L+s_k+1}(\infty)$. The behavior of f near ∞ gives that $P_0(f^{n_k+j}(\mathbf{e})) \Subset P^{\infty}_{0,m}$ for all $0 \leq j \leq s_k$. By Proposition 5.5, we have

$$P_{L+1}(f^{n_k+s_k}(\mathbf{e})) \Subset P_0(f^{n_k+s_k}(\mathbf{e})) \in \mathcal{P}^{\infty}_0.$$

We factor the map $f^{n_k+s_k} : P_{s_k+n_k}(\mathbf{e}) \to P_0(f^{s_k+n_k}(\mathbf{e}))$ as

$$P_{n_k+s_k}(\mathbf{e}) \xrightarrow{f^{n_k}} P_{s_k}(f^{n_k}(\mathbf{e})) \xrightarrow{f^{s_k}} P_0(f^{s_k+n_k}(\mathbf{e}))(= P^{\infty}_{0,m}).$$

The first factor has degree at most D. For the second factor, note that $f^{n_k}(\mathbf{e}) \subseteq Y_{L+s_k}(\infty) \subseteq Y_{s_k}(\infty)$, this implies that $P_{s_k}(f^{n_k}(\mathbf{e})) = P^{\infty}_{s_k,m}$. Hence, the map $f^{s_k} : P_{s_k}(f^{n_k}(\mathbf{e})) \to P_0(f^{s_k+n_k}(\mathbf{e}))$ is conformal. So the degree of $f^{n_k+s_k} : P_{s_k+n_k}(\mathbf{e}) \to P_0(f^{s_k+n_k}(\mathbf{e}))$ is bounded above by D.

By pulling back the pair $(P_{L+1}(f^{n_k+s_k}(\mathbf{e})), P_0(f^{n_k+s_k}(\mathbf{e}))$ along the orbit $\mathbf{e} \mapsto f(\mathbf{e}) \mapsto \cdots \mapsto f^{n_k+s_k}(\mathbf{e})$ by $f^{n_k+s_k}$, we get the annuli $A_k = P_{s_k+n_k}(\mathbf{e}) \setminus \overline{P_{L+1+n_k+s_k}(\mathbf{e})}$, whose moduli have a uniform lower bound

$$\begin{aligned}\operatorname{mod}(A_k) &\geq \frac{1}{D}\operatorname{mod}(P_0(f^{n_k+s_k}(\mathbf{e})) \setminus \overline{P_{L+1}(f^{n_k+s_k}(\mathbf{e}))}) \\ &\geq \frac{1}{D}\min\{\operatorname{mod}(P^{\infty}_{0,m} \setminus \overline{Q}); Q \in \mathcal{P}_{L+1}, Q \Subset P^{\infty}_{0,m}\}.\end{aligned}$$

This implies that $\mathbf{e} = \bigcap \overline{P_k(\mathbf{e})}$ is a singleton. □

Let $\mathbf{e}$ be a wandering end and P be a puzzle piece. The *first entry time* of $\mathbf{e}$ into P, denoted by $r_{\mathbf{e}}(P)$, is the minimal integer $k \geq 1$ such that $f^k(\mathbf{e}) \subseteq P$. If no such integer exists, we set $r_{\mathbf{e}}(P) = \infty$. If $r_{\mathbf{e}}(P) \neq \infty$, we denote by $L_{\mathbf{e}}(P)$ the unique puzzle piece containing $\mathbf{e}$ such that $f^{r_{\mathbf{e}}(P)}(L_{\mathbf{e}}(P)) = P$. Clearly, if $P \in \mathcal{P}_k$ for some k, then $L_{\mathbf{e}}(P) \in \mathcal{P}_{k+r_{\mathbf{e}}(P)}$.

LEMMA 6.3. *Let* $\mathbf{e}$ *be a wandering end and* P *be a puzzle piece. Suppose that the first entry time* $r = r_{\mathbf{e}}(P)$ *is finite, then:*

(1) *the* r *puzzle pieces* $L_{\mathbf{e}}(P), \ldots, f^{r-1}(L_{\mathbf{e}}(P))$ *are pair-wisely disjoint;*
(2) *the degree of* $f^r : L_{\mathbf{e}}(P) \to f^r(L_{\mathbf{e}}(P)) = P$ *is at most* d^K*;*
(3) *any puzzle piece* Q *containing* $\mathbf{e}$ *such that* $f^s(Q) = P$ *for some* $s \geq 1$ *is contained in* $L_{\mathbf{e}}(P)$.

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Proof. Write $Q_k = f^k(L_{\mathbf{e}}(P))$ for $0 \le k \le r-1$. (1) If $Q_{k_1} \cap Q_{k_2} \neq \emptyset$ for some $k_1 < k_2$, then $Q_{k_1} \subseteq Q_{k_2}$. By pulling back (Q_{k_1}, Q_{k_2}) along the orbit $Q_0 \mapsto \cdots \mapsto Q_{r-1}$, we get the pairs $(Q_{k_1-1}, Q_{k_2-1}), \ldots, (Q_0, Q_{k_2-k_1})$. It follows that $\mathbf{e} \subseteq Q_0 \subseteq Q_{k_2-k_1}$ and $f^{r-(k_2-k_1)}(Q_{k_2-k_1}) = Q$. This obviously contradicts the definition of first entry time.

(2) It is a direct consequence of 1, because each critical end appears in the orbit $Q_0 \mapsto \cdots \mapsto Q_{r-1}$ at most once.

(3) If it is not true, we have $s < r$ and $f^s(\mathbf{e}) \subseteq P$. This contradicts the definition of the first entry time r. □

PROPOSITION 6.4. *Any end* $\mathbf{e} \in \mathcal{E}_{\mathrm{w}}^{\mathrm{pp}}$ *satisfies the BD property.*

Proof. Let $\mathbf{e} \in \mathcal{E}_{\mathrm{w}}^{\mathrm{pp}}$. The fact $f(\omega(\mathbf{e})) \subseteq \omega(\mathbf{e})$ implies that $\omega(\mathbf{e})$ contains at least a periodic end, say $\mathbf{e}_0$. Let p be the period of $\mathbf{e}_0$. Observe that $p = 1$ if and only if $\mathbf{e}_0 = \mathbf{e}(\infty)$.

Case 1: $\mathbf{e}_0 \neq \mathbf{e}(\infty)$. Let N be a large integer so that $P_N(f^k(\mathbf{e}_0)) \setminus f^k(\mathbf{e}_0)$ contains no critical points of f, for all $0 \le k < p$. Let $A_n(\mathbf{e}_0) = P_n(\mathbf{e}_0) \setminus \overline{P_{n+1}(\mathbf{e}_0)}$ for all $n \ge 0$. By the choice of N, for any $n \ge N$, any puzzle piece Q in $A_n(\mathbf{e}_0)$ will be mapped, by some f^k, into a puzzle piece in $A_N(\mathbf{e}_0) \cup \cdots \cup A_{N+p-1}(\mathbf{e}_0)$ conformally (because the choice of N guarantees that there is no critical points along the orbit of Q).

For each $n > N$, let r_n be the first entry time of $\mathbf{e}$ into $P_n(\mathbf{e}_0)$. Clearly $r_n \to \infty$ as $n \to \infty$, and the degree of $f^{r_n} : L_{\mathbf{e}}(P_n(\mathbf{e}_0)) \to P_n(\mathbf{e}_0)$ is at most d^κ (by Lemma 6.3). Note that $f^{r_n}(\mathbf{e}) \subseteq P_n(\mathbf{e}_0)$ and $f^{r_n}(\mathbf{e}) \neq \mathbf{e}_0$, there is a unique integer $s_n \ge 0$ so that $f^{r_n}(\mathbf{e}) \subseteq A_{n+s_n}(\mathbf{e}_0)$. It follows that $P_{n+s_n+1}(f^{r_n}(\mathbf{e})) \subseteq A_{n+s_n}(\mathbf{e}_0)$. So there is a minimal integer $t_n \ge 0$ satisfying that $f^{t_n}(P_{n+s_n+1}(f^{r_n}(\mathbf{e}))) = P_{n+s_n-t_n+1}(f^{r_n+t_n}(\mathbf{e})) \subseteq A_{n+s_n-t_n}(\mathbf{e}_0) \in \{A_N(\mathbf{e}_0), \ldots, A_{N+p-1}(\mathbf{e}_0)\}$, here

$$N < n + s_n - t_n + 1 \le N + p \quad \text{for all } n \ge N.$$

We factor the map $f^{r_n+t_n} : P_{n+s_n+r_n+1}(\mathbf{e}) \to P_{n+s_n-t_n+1}(f^{r_n+t_n}(\mathbf{e}))$ as

$$P_{n+s_n+r_n+1}(\mathbf{e}) \xrightarrow{f^{r_n}} P_{n+s_n+1}(f^{r_n}(\mathbf{e})) \xrightarrow{f^{t_n}} P_{n+s_n-t_n+1}(f^{r_n+t_n}(\mathbf{e})).$$

The former has degree at most d^κ, while the latter is conformal. Therefore, by choosing a subsequence of n terms so that $n + s_n - t_n + 1$ equals a constant, we see that $\mathbf{e}$ satisfies the BD property.

Case 2: $\mathbf{e}_0 = \mathbf{e}(\infty)$. In this case, for any $n \ge 0$, the index set $\{k \in \mathbb{N};\ f^k(\mathbf{e}) \subseteq P_n(\infty)\}$ is infinite, implying that for some j independent of n, the index set $\{k \in \mathbb{N};\ f^k(\mathbf{e}) \subseteq P_{n,j}^\infty\}$ is infinite. For each $n \ge 1$, let r_n be the first entry time of $\mathbf{e}$ into $P_{n,j}^\infty$. Then the degree of $f^{r_n} : L_{\mathbf{e}}(P_{n,j}^\infty) \to P_{n,j}^\infty$ has upper bound d^κ. By postcomposing the conformal map $f^n : P_{n,j}^\infty \to P_{0,j}^\infty$, we see that the degree of $f^{n+r_n} : L_{\mathbf{e}}(P_{n,j}^\infty) \to P_{0,j}^\infty$ is uniformly bounded by d^κ. Therefore, $\mathbf{e}$ also satisfies the BD property in this case. □

PROPOSITION 6.5. *Any end* $\mathbf{e} \in \mathcal{E}_{\mathrm{w}}^{\mathrm{nr}}$ *satisfies the BD property.*

Proof. By definition, there exists $\mathbf{e}' \in \omega(\mathbf{e})$ with $\omega(\mathbf{e}') \neq \omega(\mathbf{e})$. As is pointed out before that $\omega(\mathbf{e}')$ is a proper subset of $\omega(\mathbf{e})$, hence there is an end $\mathbf{e}_0 \in \omega(\mathbf{e}) \setminus \omega(\mathbf{e}')$. For sufficiently large N, we have $\mathrm{orb}(\mathbf{e}') \cap P_N(\mathbf{e}_0) = \emptyset$.

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For any n, let r_n be the first entry time of $\mathbf{e}$ into $P_n(\mathbf{e}')$. Then $L_{\mathbf{e}}(P_n(\mathbf{e}')) = P_{n+r_n}(\mathbf{e})$ and the degree of $f^{r_n} : L_{\mathbf{e}}(P_n(\mathbf{e}')) \to P_n(\mathbf{e}')$ is bounded above by d^κ (by Lemma 6.3). Note that $f^{r_n}(\mathbf{e}) \neq \mathbf{e}'$, otherwise, it would happen that $\omega(\mathbf{e}') = \omega(\mathbf{e})$, which is impossible. It then follows that $r_n \to \infty$ as $n \to \infty$.

Because $\mathbf{e}_0 \in \omega(\mathbf{e})$, the orbit of $f^{r_n}(\mathbf{e})$ will meet $P_N(\mathbf{e}_0)$. Let s_n be the first entry time s_n of $f^{r_n}(\mathbf{e})$ into $P_N(\mathbf{e}_0)$. Then $L_{f^{r_n}(\mathbf{e})}(P_N(\mathbf{e}_0)) = P_{N+s_n}(f^{r_n}(\mathbf{e}))$ and the map $f^{s_n} : L_{f^{r_n}(\mathbf{e})}(P_N(\mathbf{e}_0)) \to P_N(\mathbf{e}_0)$ has degree at most d^κ.

Note that both $L_{f^{r_n}(\mathbf{e})}(P_N(\mathbf{e}_0))$ and $P_n(\mathbf{e}')$ contain $f^{r_n}(\mathbf{e})$. We claim that $L_{f^{r_n}(\mathbf{e})}(P_N(\mathbf{e}_0))$ is a proper subset of $P_n(\mathbf{e}')$. Because, otherwise, one has $P_n(\mathbf{e}') \subseteq L_{f^{r_n}(\mathbf{e})}(P_N(\mathbf{e}_0))$. This would imply $\mathrm{orb}(\mathbf{e}') \cap P_N(\mathbf{e}_0) \neq \emptyset$, which contradicts our assumption on $P_N(\mathbf{e}_0)$.

Then we pull back $L_{f^{r_n}(\mathbf{e})}(P_N(\mathbf{e}_0)) = P_{N+s_n}(f^{r_n}(\mathbf{e}))$ along the orbit $\mathbf{e} \mapsto \cdots \mapsto f^{r_n}(\mathbf{e})$ by f^{r_n}, and get the puzzle piece $P_{N+r_n+s_n}(\mathbf{e})$ containing $\mathbf{e}$. Further, the degree of the map

$$f^{r_n+s_n} : P_{N+r_n+s_n}(\mathbf{e}) \to P_N(\mathbf{e}_0)$$

is at most $d^{2\kappa}$. This implies that $\mathbf{e}$ has BD property. □

6.2. *The case* $\mathbf{e} \in \mathcal{E}_{\mathrm{w}}^{\mathrm{r}}$. In this part, we will show that any $\mathbf{e} \in \mathcal{E}_{\mathrm{w}}^{\mathrm{r}}$ is trivial. By definition of $\mathcal{E}_{\mathrm{w}}^{\mathrm{r}}$, each end $\mathbf{e}' \in \omega(\mathbf{e})$ is wandering, satisfying that

$$\omega(\mathbf{e}') = \omega(\mathbf{e}) \quad \text{and} \quad \mathbf{e}' \in \omega(\mathbf{e}').$$

A wandering end $\mathbf{e}'$ with the property $\mathbf{e}' \in \omega(\mathbf{e}')$ is called *combinatorially recurrent*. Clearly, all ends in $\omega(\mathbf{e})$ are combinatorially recurrent.

We first discuss an easy case (Lemma 6.6). Recall that $\mathcal{E}_{\mathrm{crit}}$ is the set of all critical ends. Let $\mathbf{c} \in \mathcal{E}_{\mathrm{crit}} \cap \mathcal{E}_{\mathrm{w}}$ be a critical wandering end. A puzzle piece $P_{n+k}(\mathbf{c})$ with $k \geq 1$ is called a *successor* of $P_n(\mathbf{c})$ if:

(1) $f^k(P_{n+k}(\mathbf{c})) = P_n(\mathbf{c})$; and
(2) each critical end appears at most once along the orbit

$$P_{n+k}(\mathbf{c}) \mapsto P_{n+k-1}(f(\mathbf{c})) \mapsto \cdots \mapsto P_{n+1}(f^{k-1}(\mathbf{c})).$$

By definition, if $P_{n+k}(\mathbf{c})$ is a successor of $P_n(\mathbf{c})$, then

$$\deg(f^k : P_{n+k}(\mathbf{c}) \to P_n(\mathbf{c})) \leq d^\kappa,$$

here, recall that $\kappa = \#\mathcal{E}_{\mathrm{crit}}$.

LEMMA 6.6. *An end* $\mathbf{e} \in \mathcal{E}_{\mathrm{w}}^{\mathrm{r}}$ *is trivial, if it satisfies one of the following:*

(1) $\omega(\mathbf{e}) \cap \mathcal{E}_{\mathrm{crit}} = \emptyset$;
(2) *some piece* $P_{n_0}(\mathbf{c})$ *of* $\mathbf{c} \in \omega(\mathbf{e}) \cap \mathcal{E}_{\mathrm{crit}}$ *has infinitely many successors.*

Proof. We will show that $\mathbf{e}$ satisfies the BD property, then the triviality of $\mathbf{e}$ follows from Proposition 6.2.

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(1) The assumption $\omega(\mathbf{e}) \cap \mathcal{E}_{\text{crit}} = \emptyset$ implies that there is an integer $N > 0$ such that the index set

$$\left\{k \geq 0;\ f^k(\mathbf{e}) \subseteq \bigcup_{\mathbf{c}\in\mathcal{E}_{\text{crit}}} P_N(\mathbf{c})\right\}$$

is finite. Let m be the cardinality of this index set. One sees that for each $k \geq 1$, the degree of $f^k : P_{N+k}(\mathbf{e}) \to P_N(f^k(\mathbf{e}))$ is bounded above by d^m.

(2) Let $\{P_{n_k}(\mathbf{c})\}_{k\geq 1}$ be all successors of $P_{n_0}(\mathbf{c})$ with $n_1 < n_2 < \cdots \to \infty$. By the assumption that $\mathbf{c} \in \omega(\mathbf{e})$, for each $k \geq 1$, there is a well defined first entry time r_k of $\mathbf{e}$ into $P_{n_k}(\mathbf{c})$. Then $L_{\mathbf{e}}(P_{n_k}(\mathbf{c})) = P_{n_k+r_k}(\mathbf{e})$ and the degree of $f^{n_k+r_k-n_0} : P_{n_k+r_k}(\mathbf{e}) \to P_{n_0}(\mathbf{c})$ is bounded above by

$$\deg(f^{r_k} : P_{n_k+r_k}(\mathbf{e}) \to P_{n_k}(\mathbf{c})) \cdot \deg(f^{n_k-n_0} : P_{n_k}(\mathbf{c}) \to P_{n_0}(\mathbf{c})) \leq d^{2\kappa}.$$

We see that $\mathbf{e}$ satisfies the BD property in both cases. □

By Lemma 6.6, we only need to discuss the ends $\mathbf{e} \in \mathcal{E}^{\text{r}}_{\text{w}}$ satisfying that $\omega(\mathbf{e}) \cap \mathcal{E}_{\text{crit}} \neq \emptyset$ and that for any $\mathbf{c} \in \omega(\mathbf{e}) \cap \mathcal{E}_{\text{crit}}$, and any $n \geq 0$, the puzzle piece $P_n(\mathbf{c})$ has finitely many successors. To show the triviality of ends, we first discuss the critical case.

A critical end $\mathbf{c} \in \mathcal{E}_{\text{crit}} \cap \mathcal{E}^{\text{r}}_{\text{w}}$ is called *persistently recurrent* in the combinatorial sense, if it satisfies:

(1) $\mathbf{c} \in \omega(\mathbf{c})$; and
(2) for any $\mathbf{c}' \in \omega(\mathbf{c}) \cap \mathcal{E}_{\text{crit}}$ and any $k \geq 1$, the puzzle piece $P_k(\mathbf{c}')$ has only finitely many successors.

We first choose a large integer $L_0 > 0$ so that $P_{L_0}(\mathbf{c}) \Subset P_0(\mathbf{c})$ and:

(1) for any different $\mathbf{c}_1, \mathbf{c}_2 \in \mathcal{E}_{\text{crit}} \cap \omega(\mathbf{c})$, one has $P_{L_0}(\mathbf{c}_1) \cap P_{L_0}(\mathbf{c}_2) \neq \emptyset$;
(2) for any $\mathbf{c}_1, \mathbf{c}_2 \in \mathcal{E}_{\text{crit}}$, we have the implication

$$\mathbf{c}_1 \notin \omega(\mathbf{c}_2) \Longrightarrow \mathbf{c}_1 \cap \bigcup_{k\geq 1} P_{L_0}(f^k(\mathbf{c}_2)) = \emptyset.$$

Let $[\mathbf{c}] = \omega(\mathbf{c}) \cap \mathcal{E}_{\text{crit}}$ and $\text{orb}([\mathbf{c}]) = \bigcup_{\mathbf{c}'\in[\mathbf{c}]} \bigcup_{k\geq 0} f^k(\mathbf{c}')$. The persistent recurrence of $\mathbf{c}$ allows one to construct the *principal nest*, whose significant properties are summarized as follows.

THEOREM 6.7. *Assume* $\mathbf{c}$ *is persistently recurrent and* $L_0 > 0$ *is chosen as above. Then there is a nest of* $\mathbf{c}$*-puzzle pieces*

$$Q_0(\mathbf{c}) \supset Q_1(\mathbf{c}) \supset Q'_1(\mathbf{c}) \supset Q_2(\mathbf{c}) \supset Q'_2(\mathbf{c}) \supset \ldots,$$

where each puzzle piece is a suitable pull back of $Q_0(\mathbf{c}) = P_{L_0}(\mathbf{c})$ *by some iterate of* f*, satisfying the following properties.*

(1) *There exist integers* $D_0 > 0$, $n_j > m_j \geq 1$ *for all* $j \geq 1$, *so that*

$$f^{m_j} : Q'_j(\mathbf{c}) \to Q_j(\mathbf{c}), \quad f^{n_j} : Q_{j+1}(\mathbf{c}) \to Q_j(\mathbf{c})$$

are proper maps of degree $\leq D_0$, *and* $f^{n_j}(Q'_{j+1}(\mathbf{c})) \subseteq Q'_j(\mathbf{c})$.

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(2) *The gap d_j of the depths between Q_j and Q'_j satisfies*

$$d_j \to +\infty \quad \text{as } j \to +\infty.$$

(3) *For all $j \geq 1$,*

$$(Q_j(\mathbf{c}) - \overline{Q'_j}(\mathbf{c})) \cap \mathrm{orb}([\mathbf{c}]) = \emptyset.$$

(4) *We have the following asymptotic lower bound of moduli,*

$$\liminf_{j \to +\infty} \mathrm{mod}(Q_j(\mathbf{c}) - \overline{Q'_j}(\mathbf{c})) > 0.$$

The construction of the principal nest is attributed to Kahn and Lyubich [**KL1**] in the unicritical case, and to Kozlovski, Shen and van Strien [**KSS**] in the multicritical case. The complex bounds are proven by Kahn and Lyubich [**KL1**, **KL2**] (unicritical case), and by Kozlovski and van Strien [**KS**] and Qiu and Yin [**QY**] independently (multicritical case). The interested readers may see these references for a detailed construction of the nest and the proof of its properties. We remark that in our setting, the annuli $Q_j(\mathbf{c}) - \overline{Q'_j}(\mathbf{c})$ might be degenerate for the first few indices j. However, because of the growth of the gaps d_j, the annuli $Q_j(\mathbf{c}) - \overline{Q'_j}(\mathbf{c})$ will be non-degenerate when j is large enough. That is the reason why we use the term 'asymptotic lower bound' instead of 'uniform lower bound' in Theorem 6.7(4).

PROPOSITION 6.8. *The end $\mathbf{e} \in \mathcal{E}^{\mathrm{r}}_{\mathrm{w}}$ is trivial, if $\omega(\mathbf{e}) \cap \mathcal{E}_{\mathrm{crit}}$ contains a persistently recurrent end* $\mathbf{c}$.

Proof. Let $(Q_j(\mathbf{c}), Q'_j(\mathbf{c}))$ be the puzzle pieces of principal nest given by Theorem 6.7. For each $j \geq 1$, let r_j be the first entry time of $\mathbf{e}$ into $Q'_j(\mathbf{c})$. Let $T'_j(\mathbf{e}) = L_{\mathbf{e}}(Q'_j(\mathbf{c}))$ and $T_j(\mathbf{e})$ be the component of $f^{-r_j}(Q_j(\mathbf{c}))$ containing $\mathbf{e}$. Then Theorem 6.7(3) implies that

$$\deg(f^{r_j}|_{T'_j(\mathbf{e})}) = \deg(f^{r_j}|_{T_j(\mathbf{e})}) \leq d^{\kappa}.$$

Hence, by Theorem 6.7(4) and letting μ be the asymptotic lower bound of moduli, for all large j, we have

$$\mathrm{mod}(T_j(\mathbf{e}) \setminus \overline{T'_j(\mathbf{e})}) \geq \mathrm{mod}(Q_k(\mathbf{c}) \setminus \overline{Q'_j(\mathbf{c})})/d^{\kappa} \geq \mu/d^{\kappa}.$$

It follows that $\mathbf{e}$ is trivial. □

7. *The renormalizable case*

As we have seen in the previous section, wandering ends are always trivial. However, preperiodic ends can be non-trivial (see Lemma 7.1). Nevertheless, the intersection of such an end and the boundary of an immediate root basin is trivial. The aim of this section is to prove this statement.

We first introduce the *renormalization* of Newton maps. We say that the Newton map f is *renormalizable* if there exist an integer $p \geq 1$ and two multi-connected domains U, V with $U \Subset V \subseteq \mathbb{C}$ such that $f^p : U \to V$ is a proper mapping with a connected *filled Julia set* $K(f^p|_U) = \bigcap_{k \geq 0} f^{-kp}(U)$. The triple (f^p, U, V) is called a *renormalization* of f.

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Note that in the definition, we assume $\infty \notin V$ to exclude the existence of a f-fixed point in $K(f^p|_U)$.

In the definition, one may further require that U, V are topological disks, and this kind of renormalization is called *P-renormalization* (here 'P' refers to 'polynomial-like'). For Newton maps, we have

$$f \text{ is renormalizable} \Longleftrightarrow f \text{ is P-renormalizable.}$$

To see this, we only need to show the '$\Longrightarrow$' part. Suppose that (f^p, U, V) is a renormalization of f, with U, V multiconnected and $K(f^p|_U)$ connected. The assumption $\infty \notin K(f^p|_U)$ implies that $K(f^p|_U)$ is disjoint from the boundary of puzzle pieces, and hence contained in a periodic end $\mathbf{e} \in \mathcal{E}$, which satisfies $f^p(\mathbf{e}) = \mathbf{e}$. Consider the map $f^{\ell p} : P_{\ell p}(\mathbf{e}) \to P_0(\mathbf{e})$, choose an integer $\ell > 0$ so that $P_{\ell p}(\mathbf{e}) \Subset P_0(\mathbf{e})$, we see that $(f^{\ell p}, P_{\ell p}(\mathbf{e}), P_0(\mathbf{e}))$ is a P-renormalization of f.

Because of this equivalence, when we are discussing the renormalizations of Newton maps, we always require that U, V are topological disks.

Periodic ends are closely related to renormalizations.

LEMMA 7.1. *Let* $\mathbf{e}$ *be a periodic end, with period* $p \geq 1$.

(1) *If none of* $\mathbf{e}, \ldots, f^{p-1}(\mathbf{e})$ *is critical, then* $\mathbf{e}$ *is a singleton.*

(2) *If some end of* $\mathbf{e}, \ldots, f^{p-1}(\mathbf{e})$ *is critical, then* f *is renormalizable. In this case,* $\mathbf{e}$ *is the filled Julia set of a renormalization.*

Proof. Choose a large integer $N > 0$ so that

$$(P_N(\mathbf{e}) \cup \cdots \cup P_N(f^{p-1}(\mathbf{e}))) \setminus (\mathbf{e} \cup \cdots \cup f^{p-1}(\mathbf{e}))$$

contains no critical point of f. By Proposition 5.3, there is an integer $\ell > 0$ so that $P_{N+\ell p}(\mathbf{e}) \Subset P_N(\mathbf{e})$. If none of $\mathbf{e}, \ldots, f^{p-1}(\mathbf{e})$ is critical, then $f^{\ell p} : P_{N+\ell p}(\mathbf{e}) \to P_N(\mathbf{e})$ is conformal. Applying the Schwarz lemma to its inverse, we see that $\mathbf{e}$ is singleton. If some end of $\mathbf{e}, \ldots, f^{p-1}(\mathbf{e})$ is critical, then $(f^{\ell p}, P_{N+\ell p}(\mathbf{e}), P_N(\mathbf{e}))$ is a renormalization of f. In this case, the filled Julia set $K(f^{\ell p}|_{P_{N+\ell p}(\mathbf{e})}) = \bigcap_{k\geq 1} P_{N+k\ell p}(\mathbf{e}) = \mathbf{e}$. □

The main result of this section is the following.

PROPOSITION 7.2. *For any preperiodic end* $\mathbf{e} \in \mathcal{E}_{\mathrm{pp}}$ *and any immediate root basin* $B \in \mathrm{Comp}(B_f)$*, the intersection* $\mathbf{e} \cap \overline{B}$ *is either empty or a singleton.*

Proof. It suffices to treat the periodic case. We may assume $\mathbf{e}$ is non-trivial, of period $p > 1$ (note that $p = 1$ if and only if $\mathbf{e} = \mathbf{e}(\infty) = \{\infty\}$), and $\mathbf{e} \cap \overline{B} \neq \emptyset$ for some immediate root basin B. The idea of the proof is to construct a Jordan curve separating $\mathbf{e}$ from B.

By Proposition 5.3, one can find two puzzle pieces Q_1 and $Q_0 = f^{n_0 p}(Q_1)$, such that $\mathbf{e} \Subset Q_1 \Subset Q_0$. Assume the depths of Q_1, Q_0 are large enough so that all critical points of $g := f^{n_0 p} : Q_1 \to Q_0$ are contained in $\mathbf{e}$. Let $d_{\mathbf{e}} = \deg(g|_{Q_1})$, then $d_{\mathbf{e}} \geq 2$, otherwise, g is conformal and the Schwarz lemma would imply that $\mathbf{e}$ is trivial.

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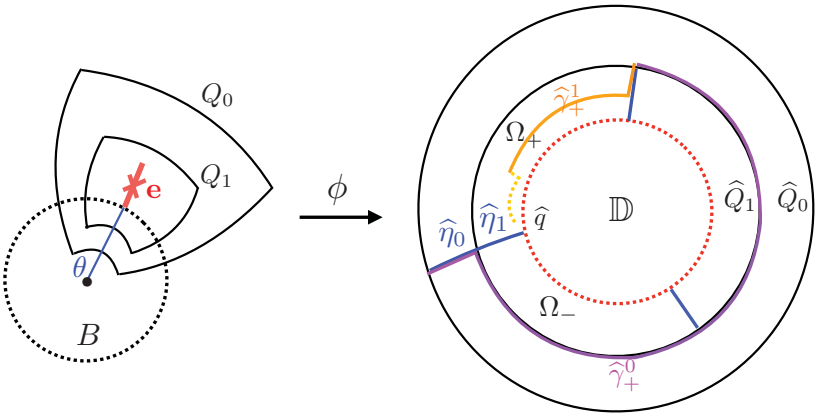

FIGURE 17. Some domains and construction of curves converging to $\widehat{q}$.

Write $Q_k = g^{-k}(Q_0)$ for $k \geq 1$. By Proposition 4.8, for all $k \geq 0$, there exist $\alpha_k, \beta_k \in \mathbb{R}/\mathbb{Z}$ with $\alpha_k < \beta_k$, and $r_k \in (0, 1)$ such that

$$\overline{Q}_k \cap \overline{B} = \overline{Q_k \cap B} = \overline{S_B(\alpha_k, \beta_k; r_k)}.$$

Because $f|_B$ is conjugate to $z \mapsto z^{d_B}$ on $\mathbb{D}$, we have

$$\alpha_k \leq \alpha_{k+1} < \cdots < \beta_{k+1} \leq \beta_k, \quad |\beta_{k+1} - \alpha_{k+1}| = |\beta_k - \alpha_k|/d_B^{n_0 p}.$$

Therefore, the sequences $\{\alpha_k\}$ and $\{\beta_k\}$ have a common limit $\theta = \lim\alpha_k = \lim\beta_k$. The internal ray $R_B(\theta)$ of B is invariant under g, and hence lands at a g-fixed point $q \in \mathbf{e} \cap \partial B$.

In the following, we show $\mathbf{e} \cap \partial B = \{q\}$. To this end, let $\eta_\varepsilon = R_B(\theta) \cap Q_\varepsilon$ with $\varepsilon \in \{0, 1\}$. Let $\phi : \widehat{\mathbb{C}} \setminus \mathbf{e} \to \widehat{\mathbb{C}} \setminus \overline{\mathbb{D}}$ be a Riemann mapping, and denote

$$(\widehat{\eta}_\varepsilon, \widehat{B}, \widehat{Q}_\varepsilon) = (\phi(\eta_\varepsilon), \phi(B), \phi(Q_\varepsilon \setminus \mathbf{e})).$$

Then $\widehat{g} = \phi \circ g \circ \phi^{-1} : \widehat{Q}_1 \to \widehat{Q}_0$ is a covering map between annuli, of degree $d_{\mathbf{e}}$. By the Schwarz reflection principle, we may assume that $\widehat{g}$ is holomorphic in a neighborhood of $\partial\mathbb{D}$. By [**Mi06**, Corollary 17.10] the arc $\widehat{\eta}_\varepsilon$ lands at a point, say $\widehat{q}$, on $\partial\mathbb{D}$. Because the arc $\widehat{\eta}_\varepsilon$ is evidently $\widehat{g}$-invariant, the point $\widehat{q}$ is $\widehat{g}$-fixed. See Figure 17.

Let Ω_+, Ω_- be the two components of $\widehat{g}^{-1}(\widehat{Q}_0 \setminus \widehat{\eta}_0)$ such that $\widehat{\eta}_1 \subseteq \partial\Omega_+ \cap \partial\Omega_-$. Clearly, Ω_+, Ω_- are Jordan disks.

CLAIM 1. *The map $\widehat{g}$ has exactly one fixed point on $\overline{\Omega}_+$ (or $\overline{\Omega}_-$). This fixed point is $\widehat{q} \in \partial\Omega_+ \cap \partial\Omega_-$.*

Proof. Let $\Omega_+^*, \Omega_-^*, \widehat{\eta}_1^*$ be the reflection part of $\Omega_+, \Omega_-, \widehat{\eta}_1$ with respect to the circle $\partial\mathbb{D}$. Let Y be the interior of the set

$$\Omega_+^* \cup \Omega_-^* \cup \widehat{\eta}_1^* \cup \Omega_+ \cup \Omega_- \cup \widehat{\eta}_1 \cup \partial\mathbb{D}.$$

Clearly, Y is an open topological disk. The Schwarz reflection principle guarantees that $\widehat{g}$ can be defined in Y and $Y \Subset \widehat{g}(Y)$. Let X be the component of $\widehat{g}^{-1}(Y)$ containing $\widehat{q}$.

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One may verify that $X \Subset Y$ and $\widehat{g} : X \to Y$ is conformal. Applying the Schwarz lemma to $\widehat{g}|_X^{-1} : Y \to X$, we conclude that $\widehat{g}$ has exactly one repelling fixed point on X. This fixed point is $\widehat{q} \in \partial\Omega_+ \cap \partial\Omega_-$. □

Let $\Omega_\varepsilon^0 = \overline{\Omega}_\varepsilon \setminus \overline{\widehat{\eta}_1}$ for $\varepsilon \in \{\pm\}$. We consider the bijections

$$\widehat{g}_\varepsilon = \widehat{g}|_{\Omega_\varepsilon^0} : \Omega_\varepsilon^0 \to \overline{\widehat{Q}}_0, \quad \varepsilon \in \{\pm\}.$$

One may verify that $\widehat{g}_\varepsilon$ is conformal in the interior of Ω_ε^0.

CLAIM 2. *For each $\varepsilon \in \{\pm\}$, let us define a sequence of closed Jordan arcs*

$$\widehat{\gamma}_\varepsilon^0 = \overline{(\phi(\partial Q_1) \setminus \Omega_\varepsilon^0) \cup (\widehat{\eta}_0 \setminus \widehat{\eta}_1)}, \quad \text{and} \quad \widehat{\gamma}_\varepsilon^k = \widehat{g}_\varepsilon^{-k}(\widehat{\gamma}_\varepsilon^0), \ k \geq 1.$$

Then $\widehat{\gamma}_\varepsilon = \bigcup_{k\geq 1} \widehat{\gamma}_\varepsilon^k$ is a Jordan arc in Ω_ε^0, satisfying that:

(1) *$\widehat{\gamma}_\varepsilon$ is disjoint from $\overline{\mathbb{D}}$;*
(2) *$\widehat{\gamma}_\varepsilon$ is disjoint from the closure of $\widehat{B}$;*
(3) *$\widehat{\gamma}_\varepsilon$ converges to the $\widehat{g}$-fixed point $\widehat{q}$.*

Proof. We only prove the case $\varepsilon = +$, the other is similar.

(1) It suffices to note that $\mathbf{e}$ has no intersection with $\partial Q_1 \cup \overline{(\eta_0 \setminus \eta_1)}$.

(2) Note that

$$\widehat{\gamma}_+ \cap \overline{\widehat{B}} = \emptyset \Longleftrightarrow \widehat{\gamma}_+^1 \cap \overline{\widehat{B}} = \emptyset \Longleftrightarrow \phi^{-1}(\widehat{\gamma}_+^1) \cap \overline{B} = \emptyset.$$

By Proposition 4.8,

$$\phi^{-1}(\widehat{\gamma}_+^1) \subseteq \overline{Q}_1 \setminus \overline{B} = \overline{Q}_1 \setminus \overline{S_B(\alpha_1, \beta_1; r_1)} \Longrightarrow \phi^{-1}(\widehat{\gamma}_+^1) \cap \overline{B} = \emptyset.$$

(3) Note that $\widehat{\gamma}_+^2 \Subset Y$ and $\widehat{g}^{-1} : Y \to X$ is strictly contracting, we conclude that $\widehat{\gamma}_+$ converges to the $\widehat{g}$-fixed point $\widehat{q}$. □

CLAIM 3. *For each $\varepsilon \in \{\pm\}$, the curve $\gamma_\varepsilon = \phi^{-1}(\widehat{\gamma}_\varepsilon)$ satisfies that $\gamma_\varepsilon \cap (\mathbf{e} \cup \overline{B}) = \emptyset$ and converges to the g-fixed point q.*

Proof. By Claim 2, we see that γ_ε is disjoint from $\mathbf{e} \cup \overline{B}$. Let

$$V = \phi^{-1}(Y \setminus \overline{\mathbb{D}}), \quad U = \phi^{-1}(X \setminus \overline{\mathbb{D}}).$$

Clearly, V is a topological disk, $V \subseteq Q_1$ and $q \in \partial V$, and $g : U \to V$ is conformal. Let $h = g|_U^{-1} : V \to U$. Because the ray $R_B(\theta)$ converges to q, the family of maps $\{h^k\}_{k\in\mathbb{N}}$ converge uniformly on $\overline{R_B(\theta) \cap (Q_0 \setminus Q_1)}$ to the boundary point q. By Denjoy–Wolff's theorem (see [**D**, **W**]), the maps $\{h^k\}_{k\in\mathbb{N}}$ converge uniformly on any compact subset of V, in particular on $\gamma_+^2 = \phi^{-1}(\widehat{\gamma}_+^2) \Subset V$, to the boundary point q. Hence, γ_+ converges to q.

Similar argument works for γ_-. □

Now we define the Jordan curve by

$$\gamma = \begin{cases} \gamma_+ \cup \gamma_- \cup \{q\} \cup (\partial Q_1 \setminus \partial V) & \text{if } d_{\mathbf{e}} \geq 3, \\ (\gamma_+ \cup \gamma_- \cup \{q\}) \setminus g^{-1}(\eta_0 \setminus \overline{\eta_1}) & \text{if } d_{\mathbf{e}} = 2. \end{cases}$$

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Then the sets $\overline{B} \setminus \{q\}$ and $\mathbf{e} \setminus \{q\}$ are in different components of $\widehat{\mathbb{C}} - \gamma$. It follows that $\mathbf{e} \cap \overline{B} = \{q\}$, which completes the proof. □

8. *Proof of the main theorem*

In this section, we will complete the proof of Theorem 1.1. At the end, we give some concluding remarks.

8.1. *Proof of Theorem 1.1.*

To prove the local connectivity of ∂B, it is equivalent to show that for any immediate root basin $B \in \mathrm{Comp}(B_f)$, and any $z \in \partial B$, the intersection $\mathbf{e}(z) \cap \partial B$ is a singleton.

This actually follows from the decomposition

$$\mathcal{E} = \mathcal{E}_{\mathrm{pp}} \sqcup \mathcal{E}_{\mathrm{w}}^{\mathrm{pp}} \sqcup \mathcal{E}_{\mathrm{w}}^{\mathrm{nr}} \sqcup \mathcal{E}_{\mathrm{w}}^{\mathrm{r}}$$

and §§6 and 7.

It remains to show that ∂B is a Jordan curve iff $d_B = \deg(f|_B) = 2$. In fact, if $d_B \geq 3$, then there are $d_B - 1 \geq 2$ internal rays in B, landing at ∞, so ∂B is not a Jordan curve. If $d_B = 2$, it follows from Lemma 8.1 and Corollaries 8.2, 8.3 (see below) that ∂B is a Jordan curve.

LEMMA 8.1. *Let $B \in \mathrm{Comp}(B_f)$. If two different internal rays $R_B(\theta_1)$, $R_B(\theta_2)$ land at the same point, then*

$$f(R_B(\theta_1)) \neq f(R_B(\theta_2)).$$

Proof. We need to discuss two cases: $f(B) = B$ and $f(B) \neq B$.

Case 1: $f(B) = B$. In this case, $f|_B$ is conjugate to the map $z^{d_B}|_{\mathbb{D}}$. To discuss the relative position of the internal rays, we need to consider the angle tupling map on the circle. Let $m_{d_B} : t \mapsto d_B t \pmod{\mathbb{Z}}$ be the angle tupling map on $\mathbb{R}/\mathbb{Z}$. Note that $S_0 := \{0/(d_B - 1), \ldots, (d_B - 2)/(d_B - 1)\}$ is the set of fixed points of m_{d_B}. The components of $\mathbb{R}/\mathbb{Z} \setminus S_0$ are denoted by $I_k = (k/(d_B - 1), (k + 1)/(d_B - 1))$, $0 \leq k \leq d_B - 2$.

First, note that the statement is true when one of θ_1, θ_2 is in S_0. In the following, we assume $\theta_1, \theta_2 \notin S_0$. We will prove by contradiction. If $f(R_B(\theta_1)) = f(R_B(\theta_2))$, the fact that $\bigcup_{\theta \in S_0} \overline{R_B(\theta)}$ divides $\overline{B}$ into $d_B - 1$ parts implies that one of them contains $R_B(\theta_1)$, $R_B(\theta_2)$, together with their common landing point z. Without loss of generality, we assume

$$0 < \theta_1 < \theta_2 < 1/(d_B - 1).$$

The assumption implies that $\theta_1, \theta_2 \in I_0$. Consider the action of m_{d_B} on the open arc I_0. Let $S_1 = f^{-1}(S_0) \cap I_0$. Then $S_1 = \{1/d_B(d_B - 1), \ldots, (d_B - 1)/d_B(d_B - 1)\}$. Because m_{d_B} is injective on S_1, the assumption $f(R_B(\theta_1)) = f(R_B(\theta_2))$ implies that $\theta_1, \theta_2 \notin S_1$.

The set $I_0 \setminus S_1$ consists of d_B components:

$$J_k = \left(\frac{k-1}{d_B(d_B-1)}, \frac{k}{d_B(d_B-1)} \right), \quad 1 \leq k \leq d_B.$$

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Note that on each J_k, the map m_{d_B} is one-to-one. Thus, θ_1, θ_2 belong to distinct J_k terms. Because $m_{d_B}(J_1) = m_{d_B}(J_{d_B}) = I_0$, we conclude that $\theta_1 \in J_1, \theta_2 = \theta_1 + (1/d_B) \in J_{d_B}$. For $k \in \{1, d_B\}$, we denote by $\theta_{1,k}, \theta_{2,k} \in J_k$ such that $m_{d_B}(\theta_{1,k}) = \theta_1, m_{d_B}(\theta_{2,k}) = \theta_2$, then we have

$$\theta_{1,1} = \frac{\theta_1}{d_B}, \quad \theta_{2,1} = \frac{1}{d_B}\left(\theta_1 + \frac{1}{d_B}\right), \quad \theta_{1,d_B} = \theta_{1,1} + \frac{1}{d_B}, \quad \theta_{2,d_B} = \theta_{2,1} + \frac{1}{d_B}.$$

It is easy to see that $\theta_{1,1} < \theta_1 < \theta_{2,1} < \theta_{1,d_B} < \theta_2 < \theta_{2,d_B}$. It follows that $R_B(\theta_{2,1}) \cup R_B(\theta_{1,d_B}) \subseteq S_B(\theta_1, \theta_2; 0)$.

Let W be the component of $\widehat{\mathbb{C}} - \overline{R_B(\theta_1) \cup R_B(\theta_2)}$ such that $\infty \notin W$. Clearly, W contains no fixed point, because W is disjoint from the channel graph Δ_0 which contains all fixed points of f. By the above discussion, there is a component V of $f^{-1}(W)$, such that V contains $S_B(\theta_{1,1}, \theta_{2,1}; 0)$ (or $S_B(\theta_{1,d_B}, \theta_{2,d_B}; 0)$). The facts

$$R_B(\theta_1) \subseteq S_B(\theta_{1,1}, \theta_{2,1}; 0) \quad \text{and} \quad \partial V \cap J(f) \subseteq f^{-1}(q)$$

imply that ∂V contains the common landing point q of $R_B(\theta_1)$, $R_B(\theta_2)$. Because $f(\partial V \cap J(f)) \subseteq \partial W \cap J(f) = \{q\}$, we see that q is a fixed point of f, which is necessarily ∞. This contradicts the assumption $\theta_1, \theta_2 \notin S_0$.

Case 2: $f(B) \neq B$. Assume $f(R_B(\theta_1)) = f(R_B(\theta_2))$. Let $U \subseteq f(B)$ be a Jordan disk, whose boundary passes through two endpoints of $f(R_B(\theta_1))$. Let $D = \widehat{\mathbb{C}} \setminus \overline{U}$.

Let W be the component of $\widehat{\mathbb{C}} - \overline{R_B(\theta_1) \cup R_B(\theta_2)}$ such that $\overline{W} \cap \Delta_0 = \emptyset$. Then $\overline{W}$ contains no fixed points of f, because all fixed points of f are contained in the channel graph Δ_0. Clearly $\widehat{\mathbb{C}} \setminus f(R_B(\theta_1)) \subseteq f(W)$ and $W \subseteq D$. There is a component V of $f^{-1}(D)$ contained in W. In particular, $\overline{V}$ contains no fixed point of f. By Corollary 3.2, there is at least one fixed point in $\overline{V}$. This is a contradiction. □

COROLLARY 8.2. *For any $B \in \mathrm{Comp}(B_f)$ and any $z \in \partial B$, let $\mu_B(z)$ be the number of internal rays in B landing at z. Then we have*

$$\mu_B(z) \leq \mu_{f(B)}(f(z)) \quad \textit{for all } z \in \partial B.$$

In particular,

$$\mu_B(z) \begin{cases} = 1 & \textit{if } z \in \partial B \setminus \Omega_f, \\ \leq d_{f^\ell(B)} - 1 & \textit{if } z \in \partial B \cap \Omega_f, \end{cases}$$

where $\ell \in \mathbb{N}$ is chosen so that $f^\ell(B)$ is fixed.

Proof. By Lemma 8.1, one has

$$\mu_B(z) \leq \mu_{f(B)}(f(z)) \quad \text{for all } B \in \mathrm{Comp}(B_f), \text{ for all } z \in \partial B.$$

For $z \in \partial B \cap \Omega_f$, let $\ell \in \mathbb{N}$ be chosen so that $f^\ell(z) = \infty$ and $f^\ell(B)$ fixed, then

$$\mu_B(z) \leq \mu_{f^\ell(B)}(\infty) = d_{f^\ell(B)} - 1.$$

To prove $\mu_B(z) = 1$ for $z \in \partial B \setminus \Omega_f$, it suffices to consider the fixed case: $f(B) = B$. In this case, for any $z \in \partial B \setminus \Omega_f$, if $\mu_B(z) \geq 2$, then there are two internal rays $R_B(t_1)$, $R_B(t_2)$, with $t_1 < t_2$, landing at z. It follows that $R_B(t_1)$, $R_B(t_2)$ are contained in

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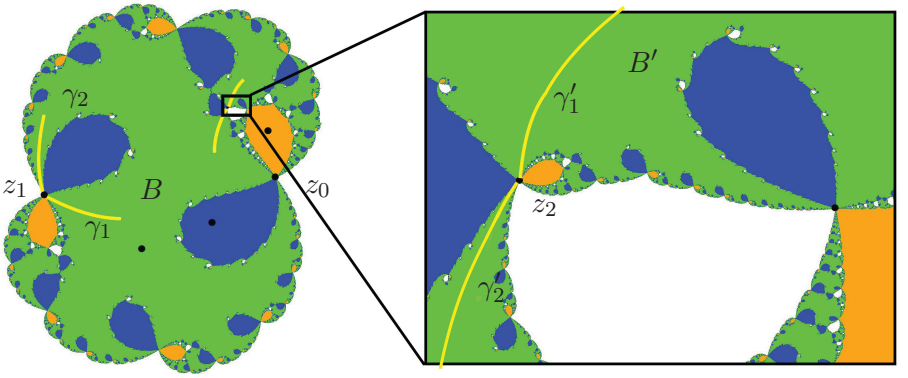

FIGURE 18. This degree four Newton map f sends points $z_2 \mapsto z_1 \mapsto z_0 = \infty$ and Fatou components $B' \mapsto B \mapsto B$. As shown above, z_1 has two non-homotopic accesses γ_1, γ_2 from B, while z_2 has one access γ_2' from B and another access γ_1' from B', here $f(\gamma_k') = \gamma_k$, $k \in \{1, 2\}$.

the same component of $\widehat{\mathbb{C}} - \Gamma_B$, where $\Gamma_B = \bigcup_{0 \le k \le d_B - 2} \overline{R_B(k/(d_B - 1))}$. This implies that

$$0 < t_2 - t_1 < 1/(d_B - 1).$$

It follows that for all $k \ge 0$, the two rays $R_B(d_B^k t_1)$, $R_B(d_B^k t_2)$ land at the common point $f^k(z)$. However, the assumption $z \in \partial B \setminus \Omega_f$ implies for $k_0 \ge 1$, satisfying that

$$d_B^{k_0}(t_2 - t_1) > 1/(d_B - 1) \ge d_B^{k_0 - 1}(t_2 - t_1),$$

the rays $R_B(d_B^{k_0} t_1)$, $R_B(d_B^{k_0} t_2)$ are contained in different components of $\widehat{\mathbb{C}} - \Gamma_B$, and hence can not land at the same point. This is a contradiction. □

As a consequence of Corollary 8.2, if $d_{f^\ell(B)} = 2$, we have $\mu_B(z) = 1$ for all $z \in \partial B$. This fact can be stated in the following form.

COROLLARY 8.3. *For any $B \in \mathrm{Comp}(B_f)$ which is eventually iterated to an immediate root basin B_0 with $d_{B_0} = 2$, the boundary ∂B is a Jordan curve.*

We remark that for Corollary 8.2, when $f(B) = B$ and $d_B \ge 3$, it can happen that for some $z \in \partial B \cap \Omega_f$, the strict inequality

$$\mu_B(z) < d_B - 1$$

holds. Figure 18 provides such an example. In fact, we have an even more interesting example.

Example 8.4. It can also happen that for some $B \in \mathrm{Comp}(B_f)$ which is eventually iterated to an immediate root basin B_0 with $d_{B_0} > 2$, and such that $B \ne B_0$, the boundary ∂B is a Jordan curve.

Figure 19 gives an example of degree five Newton map f, with an immediate root basin B_0 such that $d_{B_0} = 3$. For this example, the boundary of any $B \in \mathrm{Comp}(B_f) \setminus \{B_0\}$ is a Jordan curve.

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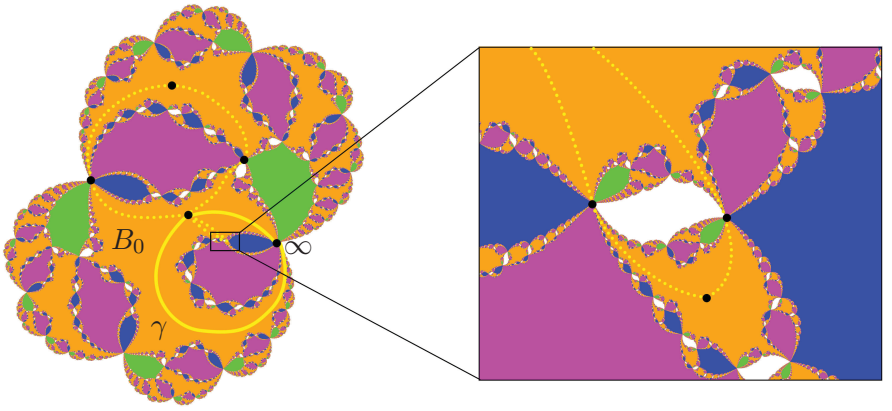

FIGURE 19. There are two internal rays in B_0 converging to ∞. The union of their closures gives a Jordan curve γ. Its preimage $f^{-1}(\gamma)$ consists of three Jordan curves: one is γ; the other two are mapped onto γ by degree two, and hence each encloses a critical point.

8.2. *Concluding remarks.* There are two by-products of our whole proof.

COROLLARY 8.5. *The following hold.*

(1) *The Julia set $J(f)$ of a non-renormalizable Newton map f is locally connected.*
(2) *A wandering continuum $E \subseteq J(f)$ of the Newton map f will eventually be iterated into the filled Julia set of a renormalization. (A continuum (compact set, which is connected and non-singleton) E is called wandering under f, if $f^m(E) \cap f^n(E) = \emptyset$ for all $0 \leq m < n$.)*

Proof. To see (1), it suffices to observe that for a non-renormalizable Newton map f, each periodic end is a singleton (by Lemma 7.1). Combining §6, we see that all possible type of ends are trivial.

To see (2), note that $\infty \notin E$, which implies that E is contained in some end $\mathbf{e}$. If $\mathbf{e}$ is wandering, then it is trivial by §6. This is impossible because E is a continuum. So $\mathbf{e}$ is preperiodic. By Lemma 7.1, for some $k \geq 0$, the end $f^k(\mathbf{e})$ is periodic and equal to a filled Julia set of a renormalization. □

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