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THE CHROMATIC NUMBER OF (P_6, C_4, diamond)-FREE GRAPHS

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Abstract

The diamond is the complete graph on four vertices minus one edge; P_n and C_n denote the path and cycle on n vertices, respectively. We prove that the chromatic number of a $(P_6, C_4, \text{diamond})$-free graph G is no larger than the maximum of 3 and the clique number of G.

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1. Introduction

A graph is an ordered pair $G = (V, E)$, where V is a set and E is a collection of 2-subsets of V. Elements of V are referred to as vertices and elements of E are edges. All our graphs are finite and have no loops or multiple edges. If there is a risk of confusion, then the sets V and E will be denoted as $V(G)$ and $E(G)$, respectively. For classical graph theory, we use the standard notation, following Bondy and Murty [1] and West [19]. If X is a set of vertices in G, denote by $G[X]$ the subgraph of G whose vertex set is X and whose edge set consists of all edges of G which have both ends in X. For any $x \in V(G)$, let $N(x)$ denote the set of all neighbours of x in G and let $d_G(x) := |N(x)|$. The neighbourhood $N(X)$ of a subset $X \subseteq V(G)$ is the set of vertices in $V(G)\backslash X$ which are adjacent to a vertex of X.

A *clique* in a graph is a set of pairwise adjacent vertices and a *stable set* is a set of pariwise nonadjacent vertices. A *k-colouring* of a graph G is a mapping $\varphi : V(G) \to \{1, 2, \ldots, k\}$ such that $\varphi(u) \neq \varphi(v)$ whenever u and v are adjacent in G. Equivalently, a k-colouring of G is a partition of $V(G)$ into k stable sets. A graph is *k-colourable* if it admits a k-colouring. The *chromatic number* of a graph G, denoted by $\chi(G)$, is the minimum number k for which G is k-colourable. The *clique number* of G, denoted by $\omega(G)$, is the size of the largest clique in G. Obviously, $\chi(H) \geq \omega(H)$ for

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any induced subgraph H of G. However, the difference $\chi(H) - \omega(H)$ may be arbitrarily large as there are triangle-free graphs with arbitrarily large chromatic number (see [15]). Furthermore, Erdős [6] showed that for any positive integers k and l there exists a graph G with $\chi(G) > k$ whose shortest cycle has length at least l.

The *complement* $\bar{G}$ of a graph G has the same vertex set as G, and distinct vertices u, v are adjacent in $\bar{G}$ just when they are not adjacent in G. A *hole* of G is an induced subgraph of G which is a cycle of length at least four, and a hole is said to be an odd hole if it has odd length. An *anti-hole* of G is an induced subgraph of G whose complement is a hole in $\bar{G}$. Given a graph with large chromatic number, it is natural to ask whether it must contain induced subgraphs with particular properties. A family $\mathcal{F}$ of graphs is said to be *χ-bounded* if there exists a function f such that $\chi(H) \leq f(\omega(H))$ for every graph H in $\mathcal{F}$. The function f is called a *χ-bounding* function of $\mathcal{F}$. If f is a linear function of ω, then we say that $\mathcal{F}$ is linearly χ-bounded. The notion of χ-bounded families was introduced by Gyárfás [10] in 1987. Since then, it has received considerable attention for $\mathcal{F}$-free graphs. See [17, 18] for further details.

We say that a graph G contains a graph H if H is isomorphic to an induced subgraph of G. A graph G is *H-free* if it does not contain H. For a family $\mathcal{F}$ of graphs, G is *$\mathcal{F}$-free* if G is H-free for every $H \in \mathcal{F}$; when $\mathcal{F}$ has two elements H_1 and H_2, we simply write G is (H_1, H_2)-free instead of $\{H_1, H_2\}$-free. If $\mathcal{F}$ is a finite family of graphs, and if C is the class of $\mathcal{F}$-free graphs which is χ-bounded, then by a classical result of Erdős [6], at least one member of $\mathcal{F}$ is a forest (see also [10]). A graph G is *perfect* if $\chi(H) = \omega(H)$ for each induced subgraph H of G. A chordless cycle of length $2k + 1, k \geq 2$, satisfies $3 = \chi > \omega = 2$, and its complement satisfies $k + 1 = \chi > \omega = k$. These graphs are therefore *imperfect*. The strong perfect graph theorem [4] says that the class of graphs without odd holes or odd anti-holes is linearly χ-bounded and the χ-bounding function is the identity function $f(x) = x$. If we only forbid odd holes, then the resulting class remains χ-bounded, but the best known χ-bounding function is not linear [17]. In recent years, there has been an ongoing project led by Scott and Seymour that aims to determine the existence of χ-bounding functions for classes of graphs without holes of various lengths (see the recent survey [18]).

Let P_n, C_n and K_n denote the path, cycle and complete graph on n vertices, respectively. Gyárfás [10] showed that the class of P_t-free graphs is χ-bounded. Gravier *et al.* [9] improved Gyárfás's bound slightly by proving that every P_t-free graph G satisfies $\chi(G) \leq (t-2)^{\omega(G)-1}$. It is well known that every P_4-free graph is perfect. The preceding result implies that every P_5-free graph G satisfies $\chi(G) \leq 3^{\omega(G)-1}$. The problem of determining whether the class of P_5-free graphs admits a polynomial χ-bounding function remains open, and it is remarked in [14] (without proof) that the known χ-bounding functions f for this class of graphs satisfy $c(\omega^2/\log\omega) \leq f(\omega) \leq 2^{\omega}$. So the recent focus is on obtaining χ-bounding functions for some classes of P_5-free graphs. Chudnovsky and Sivaraman [5] showed that every (P_5, C_5)-free graph G satisfies $\chi(G) \leq 2^{\omega(G)-1}$, and that every (P_5, bull)-free graph G satisfies $\chi(G) \leq \binom{\omega(G)+1}{2}$. Schiermeyer [16] showed that every (P_5, H)-free graph G satisfies $\chi(G) \leq \omega(G)^2$, for some special graphs H. Char and Karthick [3] showed that every

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(P_5, 4-wheel)-free graph G satisfies $\chi(G) \le \frac{3}{2}\omega(G)$. Gaspers and Huang in [7] proved that every (P_6, C_4)-free graph G has $\chi(G) \le \frac{3}{2}\omega(G)$. This $\frac{3}{2}$ bound was improved recently by Karthick and Maffray [12] to $\chi(G) \le \frac{5}{4}\omega(G)$. Karthick and Maffray [11] also showed that every (P_5, diamond)-free graph G satisfies $\chi(G) \le \omega(G) + 1$, where the diamond is the complete graph on four vertices minus one edge. For the family of (P_6, diamond)-free graphs, Karthick and Mishra [13] showed that every (P_6, diamond)-free graph G satisfies $\chi(G) \le 2\omega(G) + 5$. In the same paper, they proved that every (P_6, diamond, K_4)-free graph is 6-colourable. In 2021, Cameron *et al.* [2] improved the χ-bounding function of (P_6, diamond)-free graphs to $\omega(G) + 3$. In a recent paper [8], Goedgebeur *et al.* proved that every (P_6, diamond)-free graph G satisfies $\chi(G) \le \max\{6, \omega(G)\}$.

We investigate the chromatic number of (P_6, C_4, diamond)-free graphs. We do this by reducing the problem to imperfect (P_6, C_4, diamond)-free graphs via the strong perfect graph theorem, dividing the imperfect graphs into several cases and giving a proper colouring for each case. More precisely, the result is stated in the following theorem.

THEOREM 1.1. *Let G be a* (P_6, C_4, *diamond*)*-free graph. Then* $\chi(G) \le \max\{3, \omega(G)\}$.

We end this section by setting up the notation that we will be using. Let X and Y be any two subsets of $V(G)$. We write $[X, Y]$ to denote the set of edges that have one end in X and other end in Y. We say that X is complete to Y or $[X, Y]$ is *complete* if every vertex in X is adjacent to every vertex in Y; and X is *anti-complete* to Y if $[X, Y] = \emptyset$. If X is a singleton, say $\{u\}$, we simply write u is complete (anti-complete) to Y instead of writing $\{u\}$ is complete (anti-complete) to Y.

2. (P_6, C_4, diamond)-free graphs

One of the most celebrated theorems in graph theory is the strong perfect graph theorem [4].

THEOREM 2.1. *A graph is perfect if and only if it does not contain an odd hole or an odd anti-hole as an induced subgraph.*

Karthick and Maffray [12] proved the following lemma.

LEMMA 2.2. *Let G be any* (P_6, C_4)*-free graph. Then* $\chi(G) \le \lceil \frac{5}{4}\omega(G) \rceil$.

We first study the structure of imperfect (P_6, C_4, diamond)-free graphs. Since a P_6-free graph contains no hole of length at least 7, and a diamond-free graph contains no anti-hole of length at least 7, by Theorem 2.1, we have the following result.

LEMMA 2.3. *Every imperfect* (P_6, C_4, *diamond*)*-free graph contains an induced* C_5.

Let $G = (V, E)$ be an imperfect (P_6, C_4, diamond)-free graph that contains an induced C_5. Denote the vertex set of this C_5 by $\mathcal{P} := \{u_1, u_2, u_3, u_4, u_5\}$ and its edge

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set by $\{u_1u_2, u_2u_3, u_3u_4, u_4u_5, u_5u_1\}$. Define the sets.

$$\mathcal{N}_1 := \{u \in V(G)\backslash\mathcal{P} : N(u) \cap \mathcal{P} \neq \emptyset\} \quad \text{and} \quad \mathcal{N}_2 := V(G)\backslash(\mathcal{N}_1 \cup \mathcal{P}).$$

It is straightforward to see that $V(G) = \mathcal{P} \cup \mathcal{N}_1 \cup \mathcal{N}_2$.

From now on, every subscript is taken modulo 5. Since G is diamond-free and C_4-free, we may assume that each vertex in $\mathcal{N}_1$ is either adjacent to exactly one vertex in $\mathcal{P}$ or exactly two consecutive vertices in $\mathcal{P}$. That is, $\mathcal{N}_1$ can be partitioned into two subsets

$$A_i := \{u \in \mathcal{N}_1 : N(u) \cap \mathcal{P} = \{u_i\}\} \quad \text{and} \quad B_{i,i+1} := \{u \in \mathcal{N}_1 : N(u) \cap \mathcal{P} = \{u_i, u_{i+1}\}\}.$$

Let $A := \bigcup_{i=1}^{5} A_i$ and $B := \bigcup_{i=1}^{5} B_{i,i+1}$ so that $N(\mathcal{P}) = A \cup B$ and $V(G) = \mathcal{P} \cup A \cup B \cup \mathcal{N}_2$.

We now claim that $\mathcal{N}_2$ is empty. For otherwise, suppose that there is a vertex $z \in \mathcal{N}_2$. Then z has a neighbour $x \in A \cup B$ since G is connected. Without loss of generality, we may assume that x is adjacent to u_i, but adjacent to none of u_{i+2}, u_{i+3} and u_{i+4}. Then $\{z, x, u_i, u_{i+2}, u_{i+3}, u_{i+4}\}$ induces a P_6. However, this is a contradiction and so $V(G) = \mathcal{P} \cup A \cup B$.

We next observe a few useful properties of the sets A and B before proceeding with the proof of the theorem.

M1. For any $v \in V(G)$, $N(v)$ induces a P_3-free graph, so each $G[A_i]$ is the disjoint union of complete graphs for all $i \in [5]$. This follows directly from the fact that G is diamond-free.

M2. The set A_i is anti-complete to A_{i+1} for all $i \in [5]$. For if $a_1 \in A_i$ and $a_2 \in A_{i+1}$ are adjacent, then $\{a_1, a_2, u_i, u_{i+1}\}$ induces a C_4 and $\{a_1, a_2, u_{i+1}, u_{i+2}, u_{i+3}, u_{i+4}\}$ induces a P_6, which is a contradiction.

M3. The set A_i is complete to A_{i+2} for all $i \in [5]$. For if $a_1 \in A_i$ and $a_2 \in A_{i+2}$ are not adjacent, then $\{a_1, a_2, u_{i-2}, u_{i-1}, u_i, u_{i+2}\}$ induces a P_6, which is a contradiction.

M4. Each $G[B_{i,i+1}]$ is a clique for all $i \in [5]$. For if $b_1, b_2 \in B_{i,i+1}$ are not adjacent, then $\{b_1, b_2, u_i, u_{i+1}\}$ induces a diamond, which is a contradiction.

M5. The set $B = B_{i,i+1} \cup B_{i+2,i+3}$ for some i. It suffices to show that for each i at least one of $B_{i,i+1}, B_{i-1,i}$ is empty. Suppose the contrary. Let $b_1 \in B_{i,i+1}$ and $b_2 \in B_{i-1,i}$. Then, either $\{b_1, b_2, u_i, u_{i+1}\}$ induces a diamond if $b_1b_2 \in E$ or $\{b_1, b_2, u_{i-1}, u_{i+1}, u_{i+2}, u_{i+3}\}$ induces a P_6 if $b_1b_2 \notin E$, which is a contradiction.

M6. The set $B_{i,i+1}$ is anti-complete to $A_i \cup A_{i+1}$ for all $i \in [5]$. By symmetry, it suffices to show that $B_{i,i+1}$ is anti-complete to A_i. If $a \in A_i$ and $b \in B_{i,i+1}$ are adjacent, then $\{a, b, u_i, u_{i+1}\}$ induces a diamond, which is a contradiction.

M7. Either $B_{i,i+1} = \emptyset$ or $A_{i-1} \cup A_{i+2} = \emptyset$ for all $i \in [5]$. To the contrary, assume that $a \in A_{i+2}$ and $b \in B_{i,i+1}$. If a and b are adjacent, then $\{a, b, u_{i+1}, u_{i+2}\}$ induces a C_4, which is a contradiction. If a and b are not adjacent, then $\{a, b, u_i, u_{i+2}, u_{i+3}, u_{i+4}\}$ induces a P_6, which is a contradiction. The case with $a \in A_{i-1}$ is symmetric.

M8. If A_i contains an edge, then $A_{i+2} = A_{i+3} = B_{i+1,i+2} = B_{i-2,i-1} = \emptyset$ for all $i \in [5]$. Suppose that A_i contains an edge a_1a_2. If there is a vertex x in $A_{i+2} \cup A_{i+3}$, then

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x is adjacent to a_1 and a_2 by M3. Then $\{x, a_1, a_2, u_i\}$ induces a diamond, which is a contradiction. Since $A_i \neq \emptyset$, it follows that $B_{i+1,i+2} = B_{i-2,i-1} = \emptyset$ by M7.

M9. If $A_i \neq \emptyset$, then each of $B_{i+1,i+2} = B_{i-2,i-1} = \emptyset$ for all $i \in [5]$. This follows directly from M7.

M10. The set $B_{i,i+1}$ is anti-complete to $B_{i+2,i+3}$ for all $i \in [5]$. For if $b_1 \in B_{i,i+1}$ and $b_2 \in B_{i+2,i+3}$ are such that b_1 and b_2 are adjacent, then $\{b_1, b_2, u_{i+1}, u_{i+2}\}$ induces a C_4, which is a contradiction.

3. Proof of Theorem 1.1

In this section, we show that every $(P_6, C_4, \text{diamond})$-free graph G is $(\omega(G) + 1)$-colourable and G is $\omega(G)$-colourable if $\omega \geq 3$. The following lemma can be verified routinely.

LEMMA 3.1 (Cameron *et al.* [2]). *Let G be a graph that can be partitioned into two cliques X and Y such that the edges between X and Y form a matching. If* $\max\{|X|, |Y|\} \leq$ *k for some integer $k \geq 2$, then G is k-colourable.*

To prove Theorem 1.1, we shall use induction on the number of vertices in G. The proof follows the pretty idea presented in [2]. Two nonadjacent vertices x and y in a graph G are *comparable* if $N(x) \subseteq N(y)$ or $N(y) \subseteq N(x)$. The major work lies in proving the following auxiliary theorem.

THEOREM 3.2. *Let G be a connected $(P_6, C_4, \text{diamond})$-free graph without clique cutsets and comparable vertices. Then $\chi(G) \leq \max\{3, \omega(G)\}$.*

PROOF. Let $G = (V, E)$ be a graph satisfying the assumptions of the theorem. In what follows, we let ω denote the clique number of a graph under consideration. If $\omega \leq 2$, then the theorem follows from Lemma 2.2. Therefore, we can assume that $\omega \geq 3$. Aiming for a contradiction, we assume that G is imperfect and hence it contains an induced C_5 by Lemma 2.3, say $\mathcal{P} := \{u_1, u_2, u_3, u_4, u_5\}$ (in order). Define the sets $\mathcal{P}, A, B, A_i$ and $B_{i,i+1}$ for each $i \in \{1, 2, 3, 4, 5\}$ as before. By M5, we may assume that $B = B_{2,3} \cup B_{4,5}$. The idea is to colour $\mathcal{P} \cup A \cup B_{2,3} \cup B_{4,5}$ using exactly ω colours. We consider several cases. In each case, we give a desired colouring explicitly. In the following, when we say that we colour a set, say X, with a certain colour a, we mean that we colour each vertex in X with that colour a. We now proceed by considering the following cases.

Case 1. A_1 contains an edge. By M8, $A_3 = A_4 = B_{2,3} = B_{4,5} = \emptyset$. Since $B_{2,3} = B_{4,5} = \emptyset$, B is empty, that is, $V(G) = \mathcal{P} \cup A$. Furthermore, A_1 is anti-complete to $A_2 \cup A_5$ by M2, and A_2 and A_5 are complete to each other by M3. Now we can colour $\mathcal{P} \cup A$ as follows.

(i) A_2 contains an edge (so that $A_5 = \emptyset$ by M8).

- Colour $\mathcal{P} := u_1, u_2, u_3, u_4, u_5$ with colours $1, 2, 1, 2, 3$ in order.
- Colour each component of A_1 with colours in $\{2, 3, \ldots, \omega\}$.
- Colour each component of A_2 with colours in $\{1, 3, 4, \ldots, \omega\}$.

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(ii) A_2 is stable.

- Colour $\mathcal{P} := u_1, u_2, u_3, u_4, u_5$ with colours 2, 1, 2, 3, 1 in order.
- Colour each component of A_1 with colours in $\{1, 3, 4, \dots, \omega\}$.
- If A_5 contains an edge, then $A_2 = \emptyset$ by M8 and we colour each component of A_5 with colours in $\{2, 3, \dots, \omega\}$. Otherwise, colour A_5 with colour 2 if $A_5 \neq \emptyset$ and colour A_2 with colour 3 if $A_2 \neq \emptyset$.

We note that this colouring is well defined. Since the components of A_1 and A_2 are cliques of size at most $\omega - 1$, every vertex is coloured with some colour. We now show that this is an ω-colouring of $\mathcal{P} \cup A$. Observe first that each trivial component of A_1 is coloured with colour 2. By M1, the colouring is proper on $\mathcal{P} \cup A$. This proves that the colouring is a proper colouring.

Case 2. A_1 is stable but not empty. By M8, there are no edges in A_3 and A_4. By M9, $B_{2,3} = B_{4,5} = \emptyset$, that is, $V(G) = \mathcal{P} \cup A$. If both A_2 and A_5 are stable sets or both A_2 and A_5 are empty, then $\omega = 2$, which is a contradiction. If A_2 is stable but not empty, then A_5 contains no edges by M8, which is a contradiction to $\omega \geq 3$. Therefore, it follows from M2 that the following gives an ω-colouring of $\mathcal{P} \cup A$.

(i) A_2 contains an edge (so that $A_4 = A_5 = \emptyset$ by M8).

- Colour $\mathcal{P} := u_1, u_2, u_3, u_4, u_5$ with colours 2, 1, 2, 1, 3 in order.
- Colour A_1 and A_3 with colours 1 and 3, respectively.
- Colour each component of A_2 with colours in $\{2, 3, \dots, \omega\}$.

(ii) A_2 is empty. (Note that A_5 must contains an edge in this case since $\omega \geq 3$, and hence $A_3 = \emptyset$ by M8.)

- Colour $\{u_1, u_2, u_3, u_4, u_5\}$ with colours 2, 1, 2, 3, 1 in order.
- Colour A_1 and A_4 with colour 1 and 2 (if $A_4 \neq \emptyset$), respectively.
- Colour each component of A_5 with colours in $\{2, 3, \dots, \omega\}$.

By M2 and M3, it is easily verified that the colouring is proper.

Case 3. A_1 is empty. In this case, we further consider the following two subcases.

Subcase 3.1. A_2 contains an edge. By M8, $A_4 = A_5 = \emptyset$. By M9, $A_3 \neq \emptyset$ and $B_{4,5} \neq \emptyset$ cannot occur simultaneously. That is, either A_3 is empty or $B_{4,5}$ is empty.

If $A_3 \neq \emptyset$, then $B_{4,5} = \emptyset$ by M9. That is, $V(G) = \mathcal{P} \cup A_2 \cup A_3 \cup B_{2,3}$. Consider the following colouring of $\mathcal{P} \cup A_2 \cup A_3 \cup B_{2,3}$.

- Colour $\mathcal{P} := u_1, u_2, u_3, u_4, u_5$ with colours 1, 2, 1, 2, 3 in order.
- Colour each component of A_2 with colours in $\{1, 3, 4, \dots, \omega\}$.
- Colour each component of A_3 with colours in $\{2, 3, \dots, \omega\}$.
- Colour vertices in $B_{2,3}$ with colours in $\{3, 4, \dots, \omega\}$.

By M4, $|B_{2,3}| \leq \omega - 2$. An argument similar to that in Case 1 shows that the above is a proper ω-colouring of $\mathcal{P} \cup A_2 \cup A_3 \cup B_{2,3}$.

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Suppose now that A_3 is empty. That is, $V(G) = \mathcal{P} \cup A_2 \cup B_{2,3} \cup B_{4,5}$. Since G is diamond-free, the edges (if there are any) between $B_{4,5}$ and each component of A_2 form a matching. Consider the following colouring of $\mathcal{P} \cup A \cup B_{2,3} \cup B_{4,5}$.

- Colour $\mathcal{P} := u_1, u_2, u_3, u_4, u_5$ with colours $3, 1, 2, 1, 2$ in order.
- Colour each component of A_2 with colours in $\{2, 3, \ldots, \omega\}$. By Lemma 3.1, there exists an $(\omega - 2)$-colouring of $B_{4,5}$ with colours in $\{3, 4, \ldots, \omega\}$ by permuting colours in A_2 (if necessary).
- By M10, it is easily verified that there exists an $(\omega - 2)$-colouring of $B_{2,3}$ with colours in $\{3, 4, \ldots, \omega\}$.

Since $B_{2,3}$ and A_2 are anti-complete by M6, the above colouring gives a proper ω-colouring of $\mathcal{P} \cup A_2 \cup B_{2,3} \cup B_{4,5}$.

Subcase 3.2. A_2 is stable but not empty. Suppose first that A_3 contains an edge. By M8, $A_5 = B_{4,5} = \emptyset$. By M8, A_4 contains no edges since $A_2 \neq \emptyset$.

If A_4 is empty, one can easily verify that the following is a proper ω-colouring of $\mathcal{P} \cup A \cup B_{2,3} \cup B_{4,5}$.

- Colour $\mathcal{P} := u_1, u_2, u_3, u_4, u_5$ with colours $1, 2, 1, 3, 2$ in order.
- Colour A_2 with 1 and colour each component of A_3 with colours in $\{2, 3, \ldots, \omega\}$.
- Colour vertices in $B_{2,3}$ with colours in $\{3, 4, \ldots, \omega\}$.

If A_4 is stable but not empty, then $B_{2,3} = \emptyset$ by M9. That is, $V(G) = \mathcal{P} \cup A$. One can obtain a proper colouring of $\mathcal{P} \cup A$ as follows.

- Colour $\mathcal{P} := u_1, u_2, u_3, u_4, u_5$ with colours $1, 2, 1, 3, 2$ in order.
- Colour A_2 and A_4 with colours 3 and 2, respectively, and colour each component of A_3 with colours in $\{2, 3, \ldots, \omega\}$.

Now suppose that A_3 is stable but not empty. Then, by M9, $B_{4,5} = \emptyset$, and by M8, both A_4 and A_5 are stable since $A_2 \neq \emptyset$. So, each A_i is stable for $2 \leq i \leq 5$. We can obtain a proper colouring of $\mathcal{P} \cup A \cup B$ as follows.

- Colour $\mathcal{P} := u_1, u_2, u_3, u_4, u_5$ with colours $1, 2, 1, 3, 2$ in order.
- Colour A_2, A_3, A_4 and A_5 with colours 3, 3, 2 and 1, respectively, and colour each component of $B_{2,3}$ with colours in $\{3, 4, \ldots, \omega\}$.

Therefore, we may suppose that $A_3 = \emptyset$. Then, by M8, both A_4 and A_5 are stable since $A_2 \neq \emptyset$ and, by M9, either $A_4 = \emptyset$ or $B_{2,3} = \emptyset$. Now we consider the following two colourings.

(i) $A_4 = \emptyset$.

  - Colour $\mathcal{P} := u_1, u_2, u_3, u_4, u_5$ with colours $3, 2, 1, 2, 1$ in order.
  - Colour A_2 and A_5 with colours 1 and 2, respectively.
  - By M10, there exists an $(\omega - 2)$-colouring of $B_{2,3} \cup B_{4,5}$ with colours in $\{3, 4, \ldots, \omega\}$.

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(ii) $A_4 \neq \emptyset$, that is, $B_{2,3} = \emptyset$.

- Colour $\mathcal{P} := u_1, u_2, u_3, u_4, u_5$ with colours $3, 2, 1, 2, 1$ in order.
- Colour A_2, A_4 and A_5 with colours 1, 3 and 2, respectively.
- By M4, there exists an $(\omega - 2)$-colouring of $B_{4,5}$ with colours in $\{3, 4, \ldots, \omega\}$.

By M4 and M10, one can easily verify that the above is a proper ω-colouring of $\mathcal{P} \cup A \cup B_{2,3} \cup B_{4,5}$.

Subcase 3.3. A_2 is empty. Suppose first that A_3 contains an edge. By M8, $A_5 = B_{4,5} = \emptyset$. By M9, either $A_4 = \emptyset$ or $B_{2,3} = \emptyset$. We consider the following two colourings.

(i) $A_4 = \emptyset$.

- Colour $\mathcal{P} := u_1, u_2, u_3, u_4, u_5$ with colours $3, 2, 1, 2, 1$ in order.
- Colour each component of A_3 with colours in $\{2, 3, \ldots, \omega\}$.
- Colour vertices in $B_{2,3}$ with colours in $\{3, 4, \ldots, \omega\}$.

(ii) $A_4 \neq \emptyset$, that is, $B_{2,3} = \emptyset$.

- Colour $\mathcal{P} := u_1, u_2, u_3, u_4, u_5$ with colours $2, 3, 1, 2, 1$ in order.
- Colour each component of A_3 with colours in $\{2, 3, \ldots, \omega\}$.
- Colour each component of A_4 with colours in $\{1, 3, 4, \ldots, \omega\}$.

One can easily verify that the above is a proper ω-colouring of $\mathcal{P} \cup A \cup B_{2,3} \cup B_{4,5}$.

Now suppose that A_3 is stable but not empty. Then, by M9, $B_{4,5}$ is empty and, by M8, A_5 is stable. We consider the following two colourings.

(i) $A_4 = \emptyset$.

- Colour $\mathcal{P} := u_1, u_2, u_3, u_4, u_5$ with colours $3, 1, 2, 1, 2$ in order.
- Colour A_3 and A_5 with colours 1 and 3, respectively.
- Colour vertices in $B_{2,3}$ with colours in $\{3, 4, \ldots, \omega\}$.

(ii) $A_4 \neq \emptyset$, that is, $B_{2,3} = \emptyset$.

- Colour $\mathcal{P} := u_1, u_2, u_3, u_4, u_5$ with colours $1, 3, 2, 1, 2$ in order.
- Colour A_3 and A_5 with colours 1 and 3, respectively, and colour each component of A_4 with colours in $\{2, 3, \ldots, \omega\}$.

By M2 and M3, one can easily verify that the above is a proper ω-colouring of $\mathcal{P} \cup A \cup B_{2,3} \cup B_{4,5}$.

Finally, we suppose that A_3 is empty. That is, $V(G) = \mathcal{P} \cup A_4 \cup A_5 \cup B_{2,3} \cup B_{4,5}$. By M9, either $A_4 = \emptyset$ or $B_{2,3} = \emptyset$. Since G is diamond-free, the edges (if there are any) between $B_{2,3}$ and each component of A_5 form a matching. Consider the following two colourings of $\mathcal{P} \cup A_4 \cup A_5 \cup B_{2,3} \cup B_{4,5}$.

(i) $A_4 = \emptyset$.

- Colour $\mathcal{P} := u_1, u_2, u_3, u_4, u_5$ with colours $3, 2, 1, 2, 1$ in order.

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- Colour each component of A_5 with colours in $\{2, 3, \ldots, \omega\}$.
- By Lemma 3.1, there exists an $(\omega - 2)$-colouring of $B_{2,3}$ with colours in $\{3, 4, \ldots, \omega\}$ by permuting colours in A_5 (if necessary).
- Colour vertices in $B_{4,5}$ with colours in $\{3, 4, \ldots, \omega\}$.

(ii) $A_4 \neq \emptyset$, that is, $B_{2,3} = \emptyset$.

- Colour $\mathcal{P} := u_1, u_2, u_3, u_4, u_5$ with colours $3, 1, 2, 1, 2$ in order.
- Colour each component of A_4 with colours in $\{2, 3, \ldots, \omega\}$.
- Colour each component of A_5 with colours in $\{1, 3, 4, \ldots, \omega\}$.
- Colour $B_{4,5}$ with colours in $\{3, 4, \ldots, \omega\}$.

Since $B_{2,3}$ and A_2 are anti-complete, the above colouring gives a proper ω-colouring of $\mathcal{P} \cup A_4 \cup A_5 \cup B_{2,3} \cup B_{4,5}$. This concludes the proof of Theorem 3.2. □

Now we can easily deduce Theorem 1.1.

Proof of Theorem 1.1. If $\omega \leq 2$, then the theorem follows from Lemma 2.2. Therefore, we can assume that $\omega \geq 3$ and we prove the theorem by induction on $|V|$. We may assume that G is connected. For otherwise, the theorem holds by applying the inductive hypothesis to each connected component of G. If G contains a clique cutset S, that is, $G[V - S]$ is the disjoint union of two subgraphs X_1 and X_2, then $\chi(G) = \max\{\chi(G[V(X_1) \cup S]), \chi(G[V(X_2) \cup S])\}$ directly from the inductive hypothesis. If G contains two nonadjacent vertices x and y such that $N(y) \subseteq N(x)$, then $\chi(G) = \chi(G[V - \{y\}])$ and $\omega(G) = \omega(G[V - \{y\}])$, and the theorem holds by applying the inductive hypothesis to $G[V - \{y\}]$. Therefore, we can assume that G is a connected graph with no pair of comparable vertices and no clique cutsets. Thus, the theorem follows directly from Theorem 3.2. □

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