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Existence of the zero-temperature limit of equilibrium states on topologically transitive countable Markov shifts

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Abstract. Consider a topologically transitive countable Markov shift Σ and a summable locally constant potential ϕ with finite Gurevich pressure and $\mathrm{Var}_1(\phi) < \infty$. We prove the existence of the limit $\lim_{t\to\infty} \mu_t$ in the weak* topology, where μ_t is the unique equilibrium state associated to the potential $t\phi$. In addition, we present examples where the limit at zero temperature exists for potentials satisfying more general conditions.

Key words: countable Markov shift, equilibrium state, maximizing measure, renewal shift, stationary Markov measure, topologically transitive
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1. *Introduction*

Consider X a metric space and let (X, T) be a dynamical system. The thermodynamic formalism studies the existence, uniqueness and properties of T-invariant probability measures that maximize the value $h(\mu) + \int \phi \, d\mu$, where $h(\mu)$ is the metric entropy, whose measures are widely known in the mathematical literature as equilibrium states. In this paper, we consider (X, T) as a countable Markov shift and $\phi : X \to \mathbb{R}$ as a continuous potential. Several properties about these observables have been studied using the so-called Ruelle operator and variational principles. In particular, in both the context of finite Markov shifts and countable Markov shifts, the notions of pressure, recurrence and transience were presented to guarantee the existence and uniqueness, see for instance [1, 13, 20, 24].

For any $t \geq 1$, we denote by μ_t the equilibrium state associated to the potential $t\phi$. An interesting problem in ergodic optimization is to study the weak* accumulation points at infinity of the family $(\mu_t)_{t\geq 1}$. The foregoing occurs because those accumulation points, also called in the mathematical literature as ground states, usually result in maximizing measures for the potential ϕ, that is, those measures are the ones giving greater mass to the potential ϕ on the set of all the T-invariant probability measures. In addition, the entropy of the ground states usually exhibits interesting properties among the entropies of all the maximizing measures for the potential ϕ. Actually, the fact that these accumulation points usually become maximizing measures for the system shows an interesting connection between thermodynamic formalism and ergodic optimization.

From the point of view of statistical mechanics, the equilibrium state μ_t describes the equilibrium of the system whose interactions are given by the potential ϕ at temperature $1/t$. Thus, the existence of accumulation points of the sequence $(\mu_t)_{t\geq 1}$ is associated with the freezing of the system. Because of that, the accumulation points when $t \to \infty$ are also known as zero-temperature limits. A first study about limits at zero temperature in the setting of countable Markov shifts was developed by Coelho in [10]. In fact, in that work, the properties of the pressure associated to the potential $t\phi$ were studied as well as a version of the central limit theorem in the context of aperiodic finite Markov shifts, also known as topologically mixing shifts of finite type.

When (X, T) is a finite Markov shift, the existence of ground states follows as an immediate consequence of the compactness of the set of Borel probability measures on X. However, the uniqueness of the accumulation point at zero temperature was studied in the setting of locally constant potentials in [6, 8, 18] assuming transitivity on the dynamics. It is important to mention that Chazottes and Hochman in [9] reported an example where the existence of the limit at zero temperature of the equilibrium state fails when the potential is not locally constant. Another interesting example of the existence of more than one accumulation point at zero temperature in the setting of the so-called XY models was presented in [28]. When the alphabet is countable infinite, seminal works about the existence of accumulation points at zero temperature were performed considering the well-known finitely irreducible condition, which is a strong assumption on the combinatorics of the shift X that allowed the generalization of several of the main results of the thermodynamic formalism in the countable Markov shifts context, see for instance [15, 17, 20]. Actually, in [15], the existence of ground states was proved in the setting of summable potentials. Moreover, the uniqueness was obtained in [17] assuming the so-called big images and preimages property (BIP) condition on X, in the setting of locally constant potentials. Later, in [12], the existence of accumulation points at zero temperature was shown under the hypothesis of transitivity. Also, in [27], the existence of weak* accumulation points at zero temperature was proved for a wider class of positive recurrent potentials defined on topologically transitive countable Markov shifts satisfying suitable conditions. However, in [19], conditions were presented to guarantee the existence of ground states in a wider class of linear dynamical systems defined on Banach spaces of infinite dimension.

In this paper, we prove the uniqueness of the accumulation point at infinity of the family of equilibrium states $(\mu_t)_{t\geq 1}$ assuming that the potential ϕ has bounded variations and finite Gurevich pressure. The above serve as a generalization of the results presented in [17] to the case of topologically transitive countable Markov shifts. To do that, we show that the equilibrium states can be expressed as stationary Markov measures, see [13] for details. In addition, we use an approximation of the topologically transitive countable Markov shift by finite Markov subshifts, in a similar way as in [17], with the aim of obtaining an approximation of the unique accumulation point at zero temperature of the family $(\mu_t)_{t\geq 1}$ by the ones obtained in the setting of finite Markov subshifts. It is important to mention that we assume uniqueness of the accumulation point at zero temperature in the compact context, which was actually proved in [6, 8, 18]. Additionally, we present some examples where the uniqueness of the accumulation point is guaranteed assuming weaker conditions on the potential ϕ.

The paper is organized as follows. In §2, we introduce some definitions on thermodynamic formalism in the setting of countable Markov shifts and recall some previously known results. In §3, we study the accumulation points of the family of equilibrium states $(\mu_t)_{t\geq 1}$ at infinity and we prove the existence of the zero-temperature limit of equilibrium states on topologically transitive countable Markov shifts. Finally, in §4, we present two examples of the zero-temperature limit of the equilibrium state on the renewal shift.

2. *Preliminaries*

Let S be a countable set of *states* (when $|S| = \infty$, let us consider $S = \mathbb{N}$). Assume that $A = (A(i, j))_{S\times S}$ is a square matrix of zeroes and ones with no columns or rows whose entries are all zeroes. Fix the set $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. The *countable Markov shift* is the set of all the sequences allowed by the matrix A, that is,

$$\Sigma := \{x = (x_0x_1x_2 \ldots) \in S^{\mathbb{N}_0} : A(x_i, x_{i+1}) = 1 \text{ for all } i \geq 0\},$$

equipped with the topology generated by the collection of *cylinders*

$$[x_0x_1 \ldots x_{n-1}] := \{y \in \Sigma : y_i = x_i, 0 \leq i \leq n-1\},$$

where $x_i \in S$, for every $0 \leq i \leq n-1$, and the shift map (to be defined below) acting on it. The sigma-algebra considered on Σ is the smallest one containing all the cylinders, that is, the Borel σ-algebra. In the case where the set of states S is finite, the shift Σ is known as a finite Markov shift. A *path* of length n, denoted by $\underline{\gamma} = x_0x_1 \ldots x_n$, is an element of S^{n+1} satisfying $[x_0x_1 \ldots x_n] \neq \emptyset$ and we say that the path $\underline{\gamma}$ passes from x_0 to x_n through the states $x_1, \ldots, x_{n-1}$. The set of paths of length n is denoted by $\mathcal{P}_n(\Sigma)$ and we denote the set of paths on Σ by $\mathcal{P}(\Sigma) := \bigcup_{n\geq 1} \mathcal{P}_n(\Sigma)$. As usual, the function $\sigma : \Sigma \to \Sigma$ defined by $(\sigma x)_i = x_{i+1}$, for every $i \in \mathbb{N}_0$, is called the *shift map*.

The countable Markov shift Σ is *topologically transitive* if for every $a, b \in S$, there is a path connecting a and b, and it is *topologically mixing* if there exists $N \in \mathbb{N}$ such that there is a path of length n connecting a and b, for all $n \geq N$. Also, we say that Σ satisfies

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the BIP if there are $b_1, b_2, \ldots, b_N \in S$ such that, for all $a \in S$, there exists $1 \leq i, j \leq N$ such that $A(a, b_i) = A(b_j, a) = 1$.

Throughout the paper, we call *potential* a continuous function $\phi : \Sigma \to \mathbb{R}$ determining the interactions on the lattice (that is, the one used to define the Ruelle operator). For each $n \geq 1$, we define $S_n\phi(x) := \sum_{i=0}^{n-1} \phi(\sigma^i(x))$ as the *nth ergodic sum* and the *nth variation* of ϕ as

$$\mathrm{Var}_n(\phi) := \sup\{|\phi(x) - \phi(y)| : x, y \in \Sigma, x_i = y_i, 0 \leq i \leq n-1\}.$$

We say that ϕ has *bounded variations* if

$$\overline{\mathrm{Var}}(\phi) := \sum_{n=1}^{\infty} \mathrm{Var}_n(\phi) < \infty,$$

and has *summable variations* if $\sum_{n=2}^{\infty} \mathrm{Var}_n(\phi) < \infty$. Also, ϕ is a *locally constant* if there exists an $n \in \mathbb{N}$ such that $\mathrm{Var}_n(\phi) = 0$. However, we say that ϕ is a *summable potential* if it satisfies

$$\sum_{i \in \mathbb{N}} \exp(\sup(\phi|_{[i]})) < \infty,$$

where $\sup(\phi|_{[i]}) := \sup\{\phi(x) : x \in [i]\}$, for every $i \in \mathbb{N}$. The so-called summability condition becomes important here, because it allows to guarantee a suitable behaviour of the Gurevich pressure, also allows to have a uniform control on the tails of the measures belonging to the family $(\mu_t)_{t\geq 1}$ and implies the existence of maximizing measures for the potential ϕ (see for instance [**3**, **15**]).

Throughout the paper, $\mathcal{M}(\Sigma)$ denotes the set of Borel probability measures on Σ, $\mathcal{M}_\sigma(\Sigma)$ the set of σ-invariant Borel probability measures on Σ and $\mathcal{M}_{\mathrm{erg}}(\Sigma)$ the set of Borel ergodic probability measures on Σ. For any $\mu \in \mathcal{M}(\Sigma)$, we use the following notation:

$$\mu(\phi) := \int_\Sigma \phi \, \mathrm{d}\mu.$$

For every $\nu \in \mathcal{M}_\sigma(\Sigma)$, the *metric pressure* is defined by the following quantity:

$$P_\nu := h(\nu) + \nu(\phi), \tag{2.1}$$

where $h(\nu)$ is the metric entropy associated to measure ν (see [**24**, **23**]). The thermodynamic formalism studies the existence and properties of measures $\nu \in \mathcal{M}_\sigma(\Sigma)$ that maximize the value of the metric pressure defined in equation (2.1). Note that the sum at the right side of equation (2.1) is not always well defined, the foregoing occurs because the potential ϕ may not be ν-integrable or it could even happen that $h(\nu) = +\infty$ and $\nu(\phi) = -\infty$. By the above, the usual definition of *topological pressure* in the setting of countable Markov shifts is given by

$$P_{\mathrm{top}}(\phi) := \sup\{h(\nu) + \nu(\phi) : \nu \in \mathcal{M}_\sigma(\Sigma) \text{ such that } -\nu(\phi) < \infty\}. \tag{2.2}$$

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However, the *Gurevich pressure* of ϕ is defined by

$$P_G(\phi) := \limsup_{n\to\infty} \frac{1}{n} \log Z_n(\phi, a), \tag{2.3}$$

where $Z_n(\phi, a) := \sum_{\sigma^n x = x} \exp(S_n\phi(x))\mathbb{1}_{[a]}(x)$. It is well known that $P_G(\phi)$ is independent of the choice of $a \in \mathbb{N}$ when the countable Markov shift Σ is topologically transitive. Moreover, under the assumptions above, $-\infty < P_G(\phi) \le \infty$. Actually, Sarig in [24] showed that for a topologically mixing countable Markov shift Σ and a potential ϕ with summable variations and $\sup \phi < \infty$, the Gurevich pressure satisfies the *variational principle* and thus

$$P_G(\phi) = P_{\mathrm{top}}(\phi). \tag{2.4}$$

Furthermore, an analogous result in the setting of topologically transitive countable Markov shifts was presented in [7].

A measure $\mu \in \mathcal{M}_\sigma(\Sigma)$ is an *equilibrium state* associated to the potential ϕ if the supremum of equation (2.2) is attained for μ, that is, when it satisfies

$$P_\mu = h(\mu) + \mu(\phi) = P_{\mathrm{top}}(\phi). \tag{2.5}$$

We say that a potential ϕ is *recurrent* if

$$\sum_{n\ge 1} \exp(-nP_G(\phi))Z_n(\phi, a) = \infty.$$

For every $n \ge 1$ and $a \in S$, let

$$Z_n^*(\phi, a) := \sum_{\sigma^n x = x} \exp(S_n\phi(x))\mathbb{1}_{[\phi_a = n]}(x),$$

where $\phi_a(x) = \mathbb{1}_{[a]}(x) \inf\{n \ge 1 : \sigma^n x \in [a]\}$ and $\inf \emptyset := \infty$ (with $0 \cdot \infty := 0$). Fix some $a \in S$. We say that a recurrent potential ϕ is *positive recurrent* if

$$\sum_{n\ge 1} n \exp(-nP_G(\phi))Z_n^*(\phi, a) < \infty.$$

In the case where the above series diverges, the potential ϕ is called *null recurrent*. It is important to point out that when Σ is topologically transitive, all modes of recurrence defined above are independent of the choice of $a \in S$. We refer the reader to [23] for details. Also, when $|S| < \infty$, we have that any potential ϕ is positive recurrent. The positive recurrent potentials have an important role in the setting in which we are interested in, because they are the ones with an equilibrium state associated to them, as we describe below.

A well-known tool in thermodynamic formalism useful to find equilibrium states is the so-called *Ruelle operator*, which is defined in the setting of countable Markov shifts by the following equation:

$$L_\phi f(x) := \sum_{\sigma(y)=x} \exp(\phi(y)) f(y). \tag{2.6}$$

When $|S| < \infty$, the Markov shift Σ is a compact metric space, the Ruelle operator is well defined on the space of functions $C(\Sigma)$ and we have the famous Ruelle's

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Perron–Frobenius theorem, which guarantees the existence of a main eigenvalue with the associated eigenfunctions to the Ruelle operator and eigenmeasures for the dual of the Ruelle operator, see [5, 22] for complete details. In general, for countable Markov shifts with $|S| = \mathbb{N}$, the series in equation (2.6) can be infinite. However, in [11, 20, 24, 26], one can find different types of regularity that can be considered, both on the countable Markov shift Σ and on the potential $\phi : \Sigma \to \mathbb{R}$, to have the Ruelle operator in equation (2.6) well defined and obtain an analogous result to the so-called Ruelle's Perron–Frobenius theorem.

For positive recurrent potentials ϕ with $P_G(\phi) < \infty$, O. Sarig showed that there exists a ϕ-conformal sigma-finite measure ν, that is, a finite Borel measure satisfying

$$\nu(L_\phi f) = \exp(P_G(\phi))\nu(f) \quad \text{for each } f \in L^1(\nu). \tag{2.7}$$

Here the identity in equation (2.7) is denoted by $L_\phi^* \nu = \lambda \nu$ (see Theorem 4.9 in [23]).

In the context of topologically transitive countable Markov shifts, for any potential ϕ bounded from above, with summable variations and finite Gurevich pressure, there is at most one equilibrium state and, in the case where the existence is guaranteed, the equilibrium state is given by $\mathrm{d}\,\mu = h\,\mathrm{d}\,\nu$, where h is the main eigenfunction of L_ϕ, that is, $L_\phi h = \exp(P_G(\phi))h$, and ν is a sigma-finite measure such that $L_\phi^* \nu = \exp(P_G(\phi))\nu$ (for more details, see Theorems 1.1 and 1.2 in [7]).

Remark 2.1. The hypotheses that we assume throughout the paper to prove the existence of the zero-temperature limit are the following: we consider Σ as a topologically transitive countable Markov shift and $\phi : \Sigma \to \mathbb{R}$ as a summable potential such that $\overline{\mathrm{Var}}(\phi) < \infty$ and $P_G(\phi) < \infty$. Under these hypotheses, Theorem 1.1 from [7] assures that the Gurevich pressure satisfies the variational principle in equation (2.4). Moreover, by Theorem 1 from [12], we have that

$$P_G(t\phi) = P_{\mathrm{top}}(t\phi) = h(\mu_t) + t\mu_t(\phi),$$

for every $t \geq 1$, where μ_t is the unique equilibrium state associated to the potential $t\phi$. In addition, the existence of accumulation points at infinity for the family $(\mu_t)_{t\geq 1}$ is also guaranteed. It is important to point out that we will additionally assume in Theorem 3.1 that ϕ is a locally constant potential.

Let us define

$$\alpha(\phi) := \sup\{\nu(\phi) : \nu \in \mathcal{M}_\sigma(\Sigma)\}. \tag{2.8}$$

A measure $\mu \in \mathcal{M}_\sigma(\Sigma)$ is called *ϕ-maximizing* if $\alpha(\phi) = \mu(\phi)$. We denote by $\mathcal{M}_{\max}(\phi)$ the set of ϕ-maximizing measures.

In the setting of finite Markov shifts, it is widely known that the existence of maximizing measures is a direct consequence of the compactness of the subshift. Nevertheless, in the non-compact approach, that is, when the alphabet S is countable infinite, one requires additional conditions on the regularity of the potential. Indeed, in Theorem 1 of [3], the authors proposed conditions on the potential ϕ that guarantee the existence of such a kind of measure. To be specific, they proved that any coercive potential with bounded variations has a maximizing measure supported on a finite Markov shift Σ_I, where $I \subset \mathbb{N}$ is a finite set such that (Σ_I, σ) is a topologically transitive countable Markov shift. Actually, the class of potentials satisfying the so-called coercive property considered in [3] strictly contains the class of summable potentials. For instance, the potential $\phi(x) := -\log(x_0)$ is coercive, but it is not a summable one.

Let $\mathrm{Per}_p(\Sigma)$ be the set of points $x \in \Sigma$ such that $\sigma^p(x) = x$ and consider $\mathrm{Per}(\Sigma) := \bigcup_{p\geq 1} \mathrm{Per}_p(\Sigma)$. For every $x \in \Sigma$, define $\alpha(\phi, x) := \limsup_{n\to\infty}(1/n)S_n\phi(x)$. So, when $x \in \mathrm{Per}(\Sigma)$, it follows that $\alpha(\phi, x) = (1/p)S_p\phi(x)$, where p is the period of x (that is, the minimum $p \in \mathbb{N}$ such that $\sigma^p(x) = x$). Denote by $\mathcal{M}_{\mathrm{per}}(\Sigma)$ the set of periodic probability measures on Σ, that is, those supported on periodic orbits. Since

$$\mathcal{M}_{\mathrm{erg}}(\Sigma) = \overline{\mathcal{M}_{\mathrm{per}}(\Sigma)}\ ,$$

by the ergodic decomposition theorem, we have $\alpha(\phi) = \sup\{\alpha(\phi, x) : x \in Per(\Sigma)\}$ (see for instance [3]). This last identity will be used later in Example 4.2.

3. *Zero-temperature limits on topologically transitive countable Markov shifts*

As we already said in the previous section, for any $t \geq 1$, there is a unique equilibrium state $\mu_t \in \mathcal{M}_\sigma(\Sigma)$ associated to the potential $t\phi$. Moreover, by [12], the family of equilibrium states $(\mu_t)_{t\geq 1}$ has weak* accumulation points at $t \to \infty$. In addition, Theorem 3.1, which is the main result of this paper, states that there is at most one of those accumulation points for $(\mu_t)_{t\geq 1}$ when ϕ is a locally constant potential, that is, there is $n \in \mathbb{N}$ such that $\phi = \phi(x_0 \ldots x_{n-1})$. The statement of the result is the following.

THEOREM 3.1. *Let Σ be a topologically transitive countable Markov shift and let $\phi : \Sigma \to \mathbb{R}$ be a summable locally constant potential such that $\mathrm{Var}_1(\phi) < \infty$ and $P_G(\phi) < \infty$. Then, the limit $\lim_{t\to\infty} \mu_t$ exists in the weak* topology, where $\mu_t \in \mathcal{M}_\sigma(\Sigma)$ is the unique equilibrium state associated to the potential $t\phi$, for $t \geq 1$. Furthermore, the limit measure μ_∞ is ϕ-maximizing.*

The main idea to prove the above theorem is to obtain an approximation of the weak* accumulation points at ∞ of the family $(\mu_t)_{t\geq 1}$ by the unique accumulation point at ∞ of a family of equilibrium states $(\vartheta_t)_{t\geq 1}$ defined on a suitable finite Markov shift Σ', which guarantees uniqueness in the non-compact context. Here we use a similar technique to that in [17] guaranteeing that in our setting, any equilibrium state can be expressed as a stationary Markov measure and that expression can be used to obtain the desired approximation.

Our first goal here is to characterize the behaviour of those accumulation points and the asymptotic behaviour of the map $t \mapsto P_G(t\phi)$. To do that, next we present a lemma that will be useful to state and prove Proposition 3.1, which assures that every weak* accumulation point at ∞ of the family $(\mu_t)_{t\geq 1}$ is a maximizing measure for the potential ϕ. The following lemma is already a well-known result in the matter of countable Markov shifts satisfying the BIP condition, for details see [4], and its validity basically depends

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on the existence of equilibrium states μ_t, for each $t \geq 1$. Since, the existence of each one of the μ_t states is guaranteed in the topologically transitive context (see [12]), here, we expect to obtain something similar to the result in [4]. The statement of the lemma is the following.

LEMMA 3.2. *Given $t \geq 1$, assume that $\mu_t \in \mathcal{M}_\sigma(\Sigma)$ is an equilibrium state associated to the potential $t\phi$. Then:*

(i) *the family $\{h(\mu_t)\}_{t\geq 1}$ is decreasing;*

(ii) $\lim_{t\to\infty}(P_G(t\phi)/t) = \alpha(\phi)$.

Proof. A similar procedure to that in Lemma 9 from [4] shows that the family $\{\mu_t(\phi)\}_{t\geq 1}$ is increasing. Fix $1 \leq t_1 < t_2$. Since $\mu_1 \in \mathcal{M}_\sigma(\Sigma)$ is an equilibrium state for the potential $t_1\phi$, we have

$$h(\mu_{t_1}) = P_G(t_1\phi) - t_1\mu_{t_1}(\phi). \tag{3.1}$$

Note that by the variational principle in equation (2.4), we have $P_G(t_1\phi) > h(\mu_{t_2}) + t_1\mu_{t_2}(\phi)$. Hence, replacing in equation (3.1), we obtain

$$h(\mu_{t_1}) > h(\mu_{t_2}) + t_1(\mu_{t_2}(\phi) - \mu_{t_1}(\phi)). \tag{3.2}$$

As $\mu_{t_2}(\phi) - \mu_{t_1}(\phi) > 0$, so $h(\mu_{t_1}) > h(\mu_{t_2})$ and thus the family $\{h(\mu_t)\}_{t\geq 1}$ is decreasing.

Now we will prove item (ii). Note that $|\mathcal{M}_{\max}(\phi)| \neq \emptyset$, see Theorem 1 from [3]. Let $\nu \in \mathcal{M}_{\max}(\phi)$ and $t \geq 1$, then by the variational principle of equation (2.4), we have

$$h(\nu) + t\nu(\phi) \leq \sup\{h(\mu) + t\mu(\phi) : \mu \in \mathcal{M}_\sigma(\Sigma) \text{ and } -\mu(\phi) < \infty\} = P_G(t\phi).$$

Later,

$$t\alpha(\phi) \leq P_G(t\phi). \tag{3.3}$$

However, the map $t \mapsto P_G(t\phi) - t\alpha(\phi)$ is decreasing in $[1,\infty)$, see [4]. Therefore, $0 \leq P_G(t\phi) - t\alpha(\phi) \leq P_G(\phi) - \alpha(\phi) < \infty$ for every $t \geq 1$, so

$$\lim_{t\to\infty} \frac{P_G(t\phi)}{t} = \lim_{t\to\infty} \frac{1}{t}(P_G(t\phi) - t\alpha(\phi)) + \alpha(\phi) = \alpha(\phi). \qquad \square$$

PROPOSITION 3.1. *Every weak* accumulation point at ∞ of the family of equilibrium states $(\mu_t)_{t\geq 1}$ belongs to the set $\mathcal{M}_{\max}(\phi)$.*

Proof. Consider some arbitrary accumulation point $\mu_\infty \in \mathcal{M}_\sigma(\Sigma)$ of the family $(\mu_t)_{t\geq 1}$. It is enough to verify that

$$\alpha(\phi) \leq \mu_\infty(\phi).$$

Indeed, since that map $\mu \mapsto \mu(\phi)$, from $\mathcal{M}_\sigma(\Sigma)$ into $[0,\infty)$, is upper semi-continuous in the weak* topology, see Lemma 1 in [15], we have

$$\limsup_{t\to\infty} \mu_t(\phi) \leq \mu_\infty(\phi). \tag{3.4}$$

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However, by variational principle, for each $t \geq 1$, we have

$$\frac{P_G(t\phi)}{t} = \frac{h(\mu_t)}{t} + \mu_t(\phi).$$

Now, taking the lim sup in the last equality, since $(h(\mu_t))_{t\geq 1}$ is bounded from above (see Lemma 3.2), we obtain that

$$\alpha(\phi) = \limsup_{t\to\infty} \frac{P_G(t\phi)}{t} \leq \limsup_{t\to\infty} \mu_t(\phi) \leq \mu_\infty(\phi). \qquad \square$$

An important condition necessary to prove the convergence in the weak* topology of the family of equilibrium states $(\mu_t)_{t\geq 1}$ at ∞ is to show that the potential $t\phi$ is positively recurrent for each $t \geq 1$. That statement is verified in the following proposition.

PROPOSITION 3.2. *For every $t \geq 1$, the potential $t\phi$ is positive recurrent.*

Proof. Note that the potential ϕ is positive recurrent. Theorem 1 in [**12**] guarantees that for every $t > 1$, there is a unique equilibrium state μ_t associated to potential $t\phi$, so by Theorem 1.2 in [**7**], the potential $t\phi$ is positive recurrent, for every $t > 1$. $\square$

Remark 3.3. Under the same hypotheses of the previous lemma, but assuming that Σ is a finitely primitive countable Markov shift, Morris showed in [**21**] that

$$h(\mu_\infty) = \lim_{t\to\infty} h(\mu_t) = \sup_{\nu\in\mathcal{M}_{\max}(\phi)} h(\nu), \tag{3.5}$$

where $\mu_\infty \in \mathcal{M}_\sigma(\Sigma)$ is some accumulation point of the family of equilibrium states $(\mu_t)_{t\geq 1}$. Actually, Freire and Vargas in [**12**] obtained an extension of equation (3.5) for the setting of topologically transitive countable Markov shifts.

We say the two potentials ϕ and ψ are cohomologous when there is η such that $\phi = \psi + \eta - \eta \circ \sigma$. It is not difficult to check that cohomology between potentials is an equivalence relation preserving equilibrium states and maximizing measures.

Since there is an equilibrium measure associated with the potential ϕ, we have that there is a potential $\widetilde{\phi}$ cohomologous to ϕ, such that $\widetilde{\phi} \leq \alpha(\phi)$, see [**16**]. The potential $\widetilde{\phi} - \alpha(\phi)$ is called *normalized potential*. So, for ease of computation, from now on, we consider the normalization $\widetilde{\phi} - \alpha(\phi)$ of the potential ϕ and, to not overload the notation, we will denote the normalized one simply by ϕ.

PROPOSITION 3.3. *The following properties are satisfied:*

(i) $P_G(t\phi) \geq 0$, *for every* $t \geq 1$;
(ii) *the function* $t \mapsto P_G(t\phi)$ *is decreasing;*
(iii) $\lim_{t\to\infty} P_G(t\phi) = h(\mu_\infty)$, *where* $\mu_\infty \in \mathcal{M}_{\max}(\Sigma)$ *is a weak** *accumulation point at* ∞ *for the family of equilibrium states* $(\mu_t)_{t\geq 1}$.

Proof. The proofs of items (i) and (ii) are obtained directly from Lemma 3.2. Now we proceed to check item (iii). Let $\{t_k\}_{k\in\mathbb{N}}$ be an increasing sequence of numbers greater than one converging to infinity such that $\mu_{t_k} \to \mu_\infty$ as $k \to \infty$ in the weak* topology. Note

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that $\lim_{k\to\infty}\mu_{t_k}(\phi)=0$, see Proposition 3.1. So, by Theorem 2 in [12], we obtain

$$\begin{aligned} h(\mu_\infty) &= \limsup_{k\to\infty} h(\mu_{t_k}) \\ &= \limsup_{k\to\infty}(P_G(t_k\phi)-t_k\cdot\mu_{t_k}(\phi)) \\ &\geq \limsup_{k\to\infty}(P_G(t_k\phi)-\mu_{t_k}(\phi)) \\ &= \lim_{k\to\infty} P_G(t_k\phi)-\liminf_{k\to\infty}\mu_{t_k}(\phi) \\ &= \lim_{t\to\infty} P_G(t\phi). \end{aligned}$$

However, $P_G(t\phi)\geq h(\mu_\infty)+t\mu_\infty(\phi)=h(\mu_\infty)$. Therefore, $\lim_{t\to\infty}P_G(t\phi)=h(\mu_\infty)$. □

3.1. *Existence of the zero-temperature limit for locally constant potentials.* From now on, we consider locally constant potentials $\phi:\Sigma\to\mathbb{R}$, that is, we assume that there exists $n\in\mathbb{N}_0$ such that $\phi(x)=\phi(x_0\ldots x_{n-1})$ for any $x\in\Sigma$. We also consider the normalization $\widetilde{\phi}-\alpha(\phi)$ of the potential ϕ such that $\widetilde{\phi}\leq\alpha(\phi)$. As stated above, the normalized potential will be denoted also by ϕ (see [16] for details). In fact, without loss of generality, we can assume ϕ as a Markov potential as well, that is, $\phi(x)=\phi(x_0x_1)$. This is true because we can recode the Markov shift Σ to guarantee that any word of length $r+1$ can be written as a word of length 2 in the new codification (we refer the reader to [8] for details).

By [24], we have that the potential $\phi(x)=\phi(x_0x_1)$ has an associated equilibrium state $\mu\in\mathcal{M}_\sigma(\Sigma)$ because it is positive recurrent, see Lemma 3.9. So the Ruelle operator L_ϕ is well defined on the space of bounded continuous functions. In particular, for any function of the form $\psi(x)=\psi(x_0)$, we have

$$L_\phi(\psi)(x)=L_\phi(\psi)(x_0)=\sum_{\substack{a\in S\\ A(a,x_0)=1}}\exp(\phi(ax_0))\psi(a). \tag{3.6}$$

From Theorem 1.2 in [7], it follows that the operator L_ϕ has a strictly positive eigenfunction h, $L_\phi h=\lambda h$, where $\lambda=\exp(P_G(\phi))$ and by equation (3.6), we have $h(x)=h(x_0)$. In this case, we also can define the transpose of the Ruelle operator, L_ϕ^{T}, calculated in a function $\psi(x)=\psi(x_0)$ as

$$L_\phi^{\mathsf{T}}(\psi)(x)=L_\phi^{\mathsf{T}}(\psi)(x_0)=\sum_{\substack{a\in S\\ A(x_0a)=1}}\exp(\phi(x_0a))\psi(a).$$

It is not difficult to check that the operator L_ϕ^{T} has a strictly positive eigenfunction h^{T}, satisfying $L_\phi^{\mathsf{T}}(h^{\mathsf{T}})=\lambda h^{\mathsf{T}}$ and $h^{\mathsf{T}}(x)=h^{\mathsf{T}}(x_0)$, where λ is the main eigenvalue of the operator L_ϕ and $\sum_{a\in S}h(a)h^{\mathsf{T}}(a)=1$. A detailed proof about this claim can be found in [13].

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Remark 3.4. Theorem C in [**13**] states that the equilibrium state μ is unique and it is a stationary Markov measure given by the formula:

$$\mu([x_0x_1 \ldots x_n]) = \pi(x_0)p(x_0x_1)p(x_1x_2)\ldots p(x_{n-1}x_n), \tag{3.7}$$

where $\pi(a) = h(a)h^{\mathsf{T}}(a) > 0$ for all $a \in S$ is the stationary probability measure. Moreover, here h, h^{T} are the main eigenfunctions of the previously indicated operators L_ϕ and L_ϕ^{T}, respectively. The explicit form for the transition probabilities is given by

$$p(a, b) = \frac{h(b)}{h(a)} \exp(\phi(ab) - P_G(\phi)). \tag{3.8}$$

Here, it is convenient to define a measure $\widehat{\mu}_t \in \mathcal{M}_\sigma(\widehat{\Sigma})$ on the bilateral countable Markov shift $\widehat{\Sigma} := \{x \in S^{\mathbb{Z}} : A(x_i, x_{i+1}) = 1, \text{ for all } i \in \mathbb{Z}\}$, associated to the potential $t\widehat{\phi}((x_i)_{i\in\mathbb{Z}}) = t\phi(x_0x_1)$, given by $\widehat{\mu}_t([x_mx_{m-1}\ldots x_n]) = \mu_t([x'_0x'_1\ldots x'_{n-m}])$, $m, n \in \mathbb{Z}$ and $m \le n$, where $x'_0 = x_m, \ldots, x'_{n-m} = x_n$. The measure $\widehat{\mu}_t$ is invariant under the bilateral shift map and convergence of the family $(\widehat{\mu}_t)_{t\ge1}$ implies the convergence of the family $(\mu_t)_{t\ge1}$.

To facilitate the computations that appear below, we will use the measures $\widehat{\mu}_t$ defined on the bilateral countable Markov shift $\widehat{\Sigma}$ instead of the measures μ_t defined on the unilateral one Σ. Trying to not overload the notation, hereafter, we will denote the bilateral countable Markov shift by Σ, we will use the notation μ_t for its corresponding equilibrium states and we will denote by σ the map given by $(\sigma x)_i = x_{i+1}$ for any $i \in \mathbb{Z}$.

Also, to simplify our notation, for each path $\underline{\gamma} = x_0x_1\ldots x_n$, we use $l(\underline{\gamma})$ to denote its length and $\phi(\underline{\gamma}) := S_n\phi(x)$, $x \in [\underline{\gamma}]$. Since ϕ is a Markov potential, it follows that $\phi(\underline{\gamma})$ is constant on each $x \in [\underline{\gamma}]$, so this notation is not ambiguous. Similarly, for any probability measure $\mu \in \mathcal{M}_\sigma(\Sigma)$, we write $\mu(\underline{\gamma}) := \mu([\underline{\gamma}])$. Now, for a typical path $\underline{\gamma} = x_0x_1\ldots x_n \in \mathcal{P}(\Sigma)$, from equations (3.7) and (3.8) of Remark 3.4, we have

$$\begin{aligned}\mu_t(\underline{\gamma}) &= \pi(x_0)\prod_{k=0}^{n-1}\frac{h(x_{k+1})}{h(x_k)}\exp(t\phi(x_kx_{k+1}) - P_G(t\phi))\\ &= \pi(x_0)\frac{h(x_n)}{h(x_0)}\exp(t\phi(\underline{\gamma}) - nP_G(t\phi)).\end{aligned} \tag{3.9}$$

Obviously $\pi(x_0) = \mu_t(x_0) > 0$, when $\underline{\gamma}$ is a loop, that is, $x_0 = x_n$, we have $h(x_0) = h(x_n)$ and consequently

$$\mu_t(\underline{\gamma}) = \mu_t(x_0)\exp(t\phi(\underline{\gamma}) - nP_G(t\phi)). \tag{3.10}$$

This identity was deduced by Kempton in a more restrictive case (see [**17**]). In fact, the positive recurrence of the potential $t\phi$ and the topologically transitive condition of the Markov shift Σ are necessary and sufficient conditions to get equation (3.10) (see Theorems C and D from [**13**] for more details).

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Notice that by the notation introduced earlier, we have that

$$(t\phi - P_G(t\phi))(\underline{\gamma}) = t\phi(\underline{\gamma}) - nP_G(t\phi),$$

for every $t \geq 1$. Therefore, for any loop $\underline{\gamma} = x_0 x_1 \ldots x_n$ satisfying $x_0 = x_n$, equation (3.10) can be re-written into the form

$$\frac{\mu_t(\underline{\gamma})}{\mu_t([x_0])} = \exp((t\phi - P_G(t\phi))(\underline{\gamma})). \tag{3.11}$$

However, by Theorem 1 from [3], there exists a finite set $I \subset \mathbb{N}$ such that any ϕ-maximizing measure $\mu \in \mathcal{M}_\sigma(\Sigma)$ satisfies $\sup(\mu) \subset \Sigma_I$. From now on, the finite set I will denote the set given for this theorem.

Remark 3.5. Since ϕ is a coercive potential, by Lemma 2 in [3], there exists $d > 0$ such that $\sup \phi|_{[i]} < -d$, for every $i \notin I$, where $\sup \phi|_{[i]} := \sup\{\phi(x) : x \in [i]\}$.

By Proposition 3.1, we have that any weak* accumulation point μ_∞ of the family of equilibrium states $(\mu_t)_{t\geq 1}$ is a maximizing measure supported on $\bigcup_{i\in I}[i]$. As a consequence, the existence of the limit $\lim_{t\to\infty} \mu_t$ in the weak* topology is equivalent to showing existence of the limit $\lim_{t\to\infty} \mu_t[a]$ for all $a \in I$ (see for instance [17]). Because of that, it is enough to check the convergence of the ratios $\lim_{t\to\infty}(\mu_t([b])/\mu_t([a]))$, for all $a, b \in I$, to show the convergence of $(\mu_t)_{t\geq 1}$ in the weak* topology. In fact, the limit of the ratios can even be infinite.

From now on, let us fix $a, b \in I$. We define

$$\Sigma(a) := \{x \in \Sigma : x_i = a \text{ for infinitely many } i \in \mathbb{N}_0\}.$$

Clearly, $\Sigma(a)$ is σ-invariant. For every $t \geq 1$, the potential $t\phi$ is positive recurrent, see Lemma 3.2, so that $\mu_t \in \mathcal{M}_\sigma(\Sigma)$ is an ergodic measure and moreover $\mu_t(\Sigma(a)) = 1$, for all $t \geq 1$. Then we have that $\mu_t([b]) = \mu_t(\Sigma(a) \cap [b])$, for every $b \in I$.

Let $\Gamma(a)$ denote the set of paths $\underline{\gamma} = x_0 x_1 \ldots x_n$, $n \geq 1$, such that $x_j = a$ if and only if $j \in \{0, n\}$. Since $\Gamma(a)$ is countable (because it is a countable union of countable sets), this allows to write them as $\Gamma(a) = \{\underline{\gamma_i}\}_{i=1}^{\infty}$, where every $\underline{\gamma}_i \in \Gamma(a)$ for $i \in \mathbb{N}$. Notice that $\Sigma(a)$ can be split as

$$\Sigma(a) = \bigcup_{i=1}^{\infty} \bigcup_{k=1}^{l(\underline{\gamma}_i)} \sigma^k[\underline{\gamma}_i],$$

and so,

$$\Sigma(a) \cap [b] = \bigcup_{i=1}^{\infty} \bigcup_{k=1}^{l(\underline{\gamma}_i)} \sigma^k[\underline{\gamma}_i] \cap [b]. \tag{3.12}$$

For any loop $\underline{\gamma}_i \in \Gamma(a)$, let $N(b, \underline{\gamma}_i)$ be the number of occurrences of the symbol b within the loop $\underline{\gamma}_i$, note that $N(b, \underline{\gamma}_i) = \sum_{k=1}^{l(\underline{\gamma}_i)} \mathbb{1}_{[b]}(\sigma^k[\underline{\gamma}_i])$.

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Fix $t \geq 1$, recalling that $\mu_t([b]) = \mu_t(\Sigma(a) \cap [b])$ and μ_t is invariant by the action of the bilateral shift σ. From equation (3.12), we see that

$$\mu_t([b]) = \sum_{i=1}^{\infty} \sum_{k=1}^{l(\underline{\gamma}_i)} \mu_t(\sigma^k[\underline{\gamma}_i])\mathbb{1}_{[b]}(\sigma^k[\underline{\gamma}_i]) \tag{3.13}$$

$$= \sum_{i=1}^{\infty} \mu_t(\underline{\gamma}_i)N(b, \underline{\gamma}_i). \tag{3.14}$$

From equation (3.11), we know that for any closed loop $\underline{\gamma}_i \in \Gamma(a)$,

$$\mu_t(\underline{\gamma}_i) = \mu_t([a]) \exp((t\phi - P_G(t\phi))(\underline{\gamma}_i)), \tag{3.15}$$

so

$$\mu_t([b]) = \sum_{i=1}^{\infty} \mu_t([a]) \exp((t\phi - P_G(t\phi))(\underline{\gamma}_i))N(b, \underline{\gamma}_i) \tag{3.16}$$

and hence

$$\frac{\mu_t([b])}{\mu_t([a])} = \sum_{i=1}^{\infty} \exp((t\phi - P_G(t\phi))(\underline{\gamma}_i))N(b, \underline{\gamma}_i). \tag{3.17}$$

Actually, the finiteness of $\mu_t([b])/\mu_t([a])$, for all $t \geq 1$, is guaranteed by the positive recurrence of the potential $t\phi$ (see for instance [**24**]).

Those closed loops $\underline{\gamma}_i \in \Gamma(a)$ which do not pass through of the symbol b have no relevance at the right-hand of equation (3.17), because $N(b, \underline{\gamma}_i) = 0$. In this case, equation (3.17) is equivalent to

$$\frac{\mu_t([b])}{\mu_t([a])} = \sum_{m=1}^{\infty} m \sum_{\underline{\gamma}_i \in \Gamma(a)} \exp((t\phi - P_G(t\phi))(\underline{\gamma}_i))\mathbb{1}_{[N(b,\underline{\gamma}_i)=m]}, \tag{3.18}$$

where $\mathbb{1}_{[N(b,\underline{\gamma}_i)=m]} = 1$ if $N(b, \underline{\gamma}_i) = m$ and otherwise $\mathbb{1}_{[N(b,\underline{\gamma}_i)=m]} = 0$.

Definition 3.1. Let $a, b \in I$. We say that $\underline{\gamma} = x_0 \ldots x_n$, $n \geq 1$, is a main path in Σ that starts at i and ends at j, where $i, j \in \{a, b\}$ if and only if $x_0 = i$, $x_n = j$ and $x_m \notin \{a, b\}$ for $m \in \{1, 2, \ldots, n-1\}$. This means that the symbols a, b do not appear in the middle of the path $\underline{\gamma}$. We will denote by $\{\underline{\gamma} : i \to j\}$ the set of all the main paths that start at i and end at j.

Note that $\{\underline{\gamma} : a \to a\}$ and $\Gamma(a)$ do not represent the same set. The foregoing is true because $\Gamma(a)$ contains paths with the symbol b while $\{\underline{\gamma} : a \to a\}$ does not. For $\widetilde{\Sigma}$, a topologically transitive $\sigma-$invariant subset of the Markov shift Σ, we use the notation $\{\underline{\gamma} : i \to j \in \mathcal{P}(\widetilde{\Sigma})\}$ to indicate that each main path that starts at i and ends at j is a path of $\widetilde{\Sigma}$ (remember that $\mathcal{P}(\widetilde{\Sigma})$ denotes the set of paths in $\widetilde{\Sigma}$).

For every $i, j \in \{a, b\}$, we define

$$p_{ij}^t := \sum_{\underline{\gamma} \in \{\underline{\gamma}: i \to j\}} \exp((t\phi - P_G(t\phi))(\underline{\gamma})). \tag{3.19}$$

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Note from equation (3.11) that $p^t_{ii} < 1$, for every $i \in I$. Indeed, the probability, with respect to μ_t, that a path from i returns to i eventually is one, so from equations (3.11) and (3.19), we observe that it can be split into p^t_{ii}, the probability that a path from i returns to i without passing through j, where $j \neq i$, and $p^t_{ij}(\sum_{n\geq 1}(p^t_{jj})^n)p^t_{ji}$, the probability that a path from i returns to i passing through j at least once. Therefore,

$$p^t_{ii} + p^t_{ij}p^t_{ji}\sum_{n\geq 1}(p^t_{jj})^n = 1. \tag{3.20}$$

Recalling that $t\phi$ is a Markov potential, for all $m \in \mathbb{N}$, we get from equation (3.19),

$$\sum_{\underline{\gamma}\in\Gamma(a)} \exp((t\phi - P_G(t\phi))(\underline{\gamma}))\mathbb{1}_{[N(b,\underline{\gamma})=m]} = p^t_{ab}(p^t_{bb})^{m-1}p^t_{ba}. \tag{3.21}$$

Therefore, from equation (3.18), we obtain

$$\frac{\mu_t([b])}{\mu_t([a])} = \sum_{m=1}^{\infty} mp^t_{ab}(p^t_{bb})^{m-1}p^t_{ba} = \frac{p^t_{ab}p^t_{ba}}{(1-p^t_{bb})^2}. \tag{3.22}$$

By inverting the roles of a and b, we get $\mu_t([a])/\mu_t([b]) = p^t_{ba}p^t_{ab}/(1-p^t_{aa})^2$ and combining both expressions, we conclude

$$\frac{\mu_t([b])}{\mu_t([a])} = \frac{1-p^t_{aa}}{1-p^t_{bb}}. \tag{3.23}$$

Therefore, proving the convergence of the equilibrium states $(\mu_t)_{t\geq 1}$, $\lim_{t\to\infty}\mu_t$ reduces to showing the existence of $\lim_{t\to\infty}(1-p^t_{aa})/(1-p^t_{bb})$ for all $a, b \in I$.

It is well known that in the setting of finite Markov shifts, the existence of the limit at zero temperature is satisfied for families of equilibrium states associated to Markov potentials. Below, we present a definition of a suitable collection of finite Markov shifts contained in Σ which are useful to approximate the unique accumulation point of the family $(\mu_t)_{t\geq 1}$ in the weak* topology.

Definition 3.2. Let Σ be a topologically transitive countable Markov shift, $\phi : \Sigma \to \mathbb{R}$ be a Markov potential $\phi(x) = \phi(x_0x_1)$ and let $c \in (0, \infty)$. We denote by Σ_{-c} a minimal topologically transitive subshift of Σ that contains all the symbols $i \in \mathbb{N}$ such that $\sup \phi|_{[i]} \geq -c$.

Remark 3.6. In Definition 3.2, minimal is understood in the sense that any other topologically transitive subshift strictly contained in Σ_{-c} does not exist that includes all the symbols $i \in \mathbb{N}$ satisfying $\sup \phi|_{[i]} \geq -c$.

By summability, Σ_{-c} is a finite Markov shift when c is large enough. Furthermore, for any $c \leq c'$, it is possible to find $\Sigma_{-c'}$ satisfying Definition 3.2 such that $\Sigma_{-c} \subset \Sigma_{-c'}$. In the following, the key argument to prove the existence of the zero-temperature limit of equilibrium states on topologically transitive countable Markov shifts is the construction of an appropriate finite Markov subshift (which remains fixed for all $t \geq 1$), whose equilibrium states approximate those defined on the countable Markov shift Σ.

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For every $a, b \in I$, we will denote by $\underline{\gamma_a^b}$ the shortest path connecting a to b. From the construction, we immediately get that $\underline{\gamma_a^b}$ contains the symbols a and b only at the ends, so that $\underline{\gamma_a^b} \in \{\underline{\gamma} : a \to b\}$. This path always exists because Σ is topologically transitive which allows to guarantee that any pair of symbols can be linked by a finite path. We denote $\underline{\gamma_a^b}\underline{\gamma_b^a}$ the concatenation of the paths $\underline{\gamma_a^b}$ with $\underline{\gamma_b^a}$. With this, we have that there is always a loop that passes through a and b. We now consider the set

$$\mathcal{C} := \bigcup_{a \in I, b \in I} \{\underline{\gamma_a^b}\underline{\gamma_b^a} \in \mathcal{P}(\Sigma)\},$$

which is a non-empty and finite set. In addition, for each pair $a, b \in I$ and each non-empty finite set of symbols $J \subset I$, consider $\underline{\gamma_a^b}(J)$ as the shortest path connecting a to b and avoiding in the middle any symbol belonging to J, the foregoing, in the case where there is at least a path satisfying those conditions. We define the set of all the paths between a and b avoiding some subset of I as

$$\widetilde{\mathcal{C}} := \bigcup_{a,b \in I; J \subset I} \{\underline{\gamma_a^b}(J) \in \mathcal{P}(\Sigma)\}.$$

Finally, we fix

$$N := \max\{l(\underline{\gamma}) : \underline{\gamma} \in \mathcal{C} \cup \widetilde{\mathcal{C}}\} < \infty, \tag{3.24}$$

$$C := \max\{-\phi(\underline{\gamma}) : \underline{\gamma} \in \mathcal{C} \cup \widetilde{\mathcal{C}}\} < \infty. \tag{3.25}$$

$$c := C + \frac{2d}{7} > 0, \tag{3.26}$$

where $d > 0$ was given in Remark 3.5. The above construction leads us to the following proposition.

PROPOSITION 3.4. *Let $N \in \mathbb{N}$, $C \geq 0$ and $c > 0$ as in equations (3.24), (3.25) and (3.26), respectively. Then, for every $a, b \in I$, we have:*

(i) *there exists a loop $\underline{\gamma} \in \mathcal{P}(\Sigma)$ connecting a to b such that $\phi(\underline{\gamma}) \geq -C$ and $l(\underline{\gamma}) \leq N$;*

(ii) *if there exists a loop $\underline{\gamma}$ connecting a to a and avoiding the set symbols of $J \subset I$, then $\phi(\underline{\gamma}) \geq -C$ and $l(\underline{\gamma}) \leq N$.*

Also, each symbol of I belongs to the symbols of the topologically transitive finite Markov shift Σ_{-c}, that is, $\Sigma_I \subset \Sigma_{-c}$.

Proof. Note that items (i) and (ii) are a direct consequence of equations (3.24), (3.25) and (3.26). Furthermore, fixing $i, j \in I$, it follows that

$$-c + \frac{2d}{7} \leq \phi(\underline{\gamma_i^j}\underline{\gamma_j^i}) \leq \sup \phi|_{[i]},$$

so, by Definition 3.2, we have $\Sigma_I \subset \Sigma_{-c}$. □

The main difference here between the BIP case and the topologically transitive case is the following: in the BIP case, we are able to link a and b using only symbols of the finite set $\{b_1, b_2, \ldots, b_N\}$ and, thus, the length of the path is at most N. However, in the

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topologically transitive setting, we are only able to guarantee the existence of a finite path linking a and b, but we do not have any control on the length of the paths. Despite this, the finiteness of the set I makes it possible to build a compact subshift as in Proposition 3.4.

Fix the topologically transitive finite Markov shift $\Sigma' := \Sigma_{-7c}$, note that $\Sigma_I \subset \Sigma_{-c} \subset \Sigma' \subset \Sigma$. For the subshift Σ', we will denote by ϑ_t the equilibrium state associated with the potential $t\phi$ restricted to Σ' (this will be denoted by $t\phi|_{\Sigma'}$), and $Q(t\phi) \leq P_G(t\phi)$ is the topological pressure of the potential $t\phi|_{\Sigma'}$.

For every $i, j \in \{a, b\}$ and $t \geq 1$, we define

$$q_{ij}^t := \sum_{\{\underline{\gamma}: i \to j \in \mathcal{P}(\Sigma')\}} \exp((t\phi - Q(t\phi))(\underline{\gamma})), \tag{3.27}$$

where $\{\underline{\gamma} : i \to j \in \mathcal{P}(\Sigma')\}$ is the set of paths in Σ' that connect i to j and such that do not have an intermediate occurrence of neither a nor b. Similarly to equation (3.23), we see that

$$\frac{\vartheta_t([b])}{\vartheta_t([a])} = \frac{1 - q_{aa}^t}{1 - q_{bb}^t}, \tag{3.28}$$

for each $a, b \in I$. Since Σ' is a finite Markov shift, we have an existence of the limit $\lim_{t\to\infty}(\vartheta_t([b])/\vartheta_t([a]))$ for any $a, b \in I$, see [**6**, **8**, **18**] for details.

Next, we will show the following equality:

$$\lim_{t\to\infty} \frac{\mu_t([b])}{\mu_t([a])} = \lim_{t\to\infty} \frac{\vartheta_t([b])}{\vartheta_t([a])}, \tag{3.29}$$

for each $a, b \in I$. Since, we have an existence of the limit in the right side of the equation above (see for instance [**6**, **8**, **18**]), by Proposition 3.1, it follows that equation (3.29) guarantees the main theorem of this work (that is, Theorem 3.1).

So, by equations (3.23) and (3.28), it is only necessary to prove the following:

$$\lim_{t\to\infty} \frac{1 - p_{aa}^t}{1 - p_{bb}^t} = \lim_{t\to\infty} \frac{1 - q_{aa}^t}{1 - q_{bb}^t}. \tag{3.30}$$

To study the asymptotic behaviour of $1 - p_{aa}^t$ and $1 - q_{aa}^t$, we will use item (i) of Proposition 3.4 to find their lower bounds. Now we will find a lower bound for $1 - p_{aa}^t$. Fix $a, b \in I$ and $t \geq 1$. By Proposition 3.4, the concatenation $\underline{\gamma_a^b \gamma_b^a}$ is a path from a to a passing though b with length at most N satisfying $\phi(\underline{\gamma_a^b \gamma_b^a}) \geq -C$, thus

$$\exp((t\phi - P_G(t\phi))(\underline{\gamma_a^b \gamma_b^a})) \geq \exp(-tC - NP_G(t\phi)). \tag{3.31}$$

Therefore, from equations (3.20) and (3.31), we obtain the lower bound

$$1 - p_{aa}^t = p_{ab}^t p_{ba}^t \sum_{n\geq 1} (p_{bb}^t)^n \geq \exp(-tC - NP_G(t\phi)). \tag{3.32}$$

Obviously, the same argument gives us lower bounds for $1 - p_{bb}^t$, $1 - q_{aa}^t$ and $1 - q_{bb}^t$.

To continue analysing the asymptotic behaviour of $(1 - p^t_{aa})/(1 - p^t_{bb})$ and $(1 - q^t_{aa})/(1 - q^t_{bb})$, we introduce the terms:

$$r^t_{ij} = \sum_{\{\underline{\gamma}: i \to j \in \mathcal{P}(\Sigma')\}} \exp(t\phi - P_G(t\phi))(\underline{\gamma}), \tag{3.33}$$

for each $i, j \in \{a, b\}$. It can be observed that $r^t_{ij} \le p^t_{ij}$ and $r^t_{ij} \le q^t_{ij}$. Note that r^t_{ij} is obtained by taking the sum only on the main paths in Σ', while p^t_{ij} considers all the main paths of Σ. Later, in Lemma 3.9, we will show that for each $a \in I$ fixed, the value of r^t_{aa} is close to p^t_{aa} and q^t_{aa} for t large enough.

The main tool to prove equation (3.30) is Lemma 3.9; in fact, Lemmas 3.7 and 3.8 allow us to prove this lemma. The aforementioned results are the same as those obtained by T. Kempton for countable Markov shifts satisfying the BIP condition and the proofs are similar, which can be found in [**17**]. This is due to Proposition 3.4 and because the Gurevich pressure on topologically transitive countable Markov shifts has a similar behaviour to the one observed in the case of countable Markov shifts satisfying the BIP condition. This was proved in Proposition 3.3.

For each main path $\underline{\gamma} \in \{\underline{\gamma} : a \to a\}$, we write $n(\underline{\gamma})$ to denote the number of times that a symbol of I appears in $\underline{\gamma}$ without taking into account the symbols that appear at the end. So, when $n(\underline{\gamma}) = n$, these symbols of I can be labelled as $i_0, i_1, i_2, \ldots, i_{n+1}$ with the convention $i_0 = i_{n+1} = a$. We denote by X^n_{aa} all those main paths satisfying $n(\underline{\gamma}) = n$. From the definition of a main path, we necessarily have $i_k \ne \{a, b\}$, $1 \le k \le n$. By calling X^n_{aa} the collection of main paths such that $n(\underline{\gamma}) = n$, we get

$$p^t_{aa} = \sum_{n=0}^{\infty} p^t_{aa}(n), \tag{3.34}$$

where

$$p^t_{aa}(n) = \sum_{i_0,\ldots,i_{n+1} \in X^n_{aa}} \prod_{k=0}^{n} \Bigg(\sum_{\{\underline{\gamma}: i_k \hookrightarrow i_{k+1} \in \mathcal{P}(\Sigma)\}} \exp(t\phi - P_G(t\phi))(\underline{\gamma}) \Bigg). \tag{3.35}$$

Note that for every $n \ge 0$, the terms $p^t_{aa}(n)$ are of the form $\exp(t\phi - P_G(t\phi))(\underline{\gamma})$, where $\underline{\gamma} \in \mathcal{P}(\Sigma)$ such that $n(\underline{\gamma}) = n$. The following lemma gives a lower bound for $p^t_{aa}(n)$, for every $a \in I$.

LEMMA 3.7. *For every $r \in \mathbb{N}_0$ and $r|I| \le n < (r+1)|I|$, we have that*

$$p^t_{aa}(n) \le (1 - \exp(-Ct - N P_G(t\phi)))^r.$$

Similarly to equations (3.34), (3.35), we now define $r^t_{aa} = \sum_{n=0}^{\infty} r^t_{aa}(n)$, where

$$r^t_{aa}(n) := \sum_{i_0,\ldots,i_{n+1} \in X^n_{aa}} \prod_{k=0}^{n} \Bigg(\sum_{\{\underline{\gamma}: i_k \hookrightarrow i_{k+1} \in \mathcal{P}(\Sigma')\}} \exp(t\phi - P_G(t\phi))(\underline{\gamma}) \Bigg).$$

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By calling

$$\epsilon(n) := \frac{p^t_{aa}(n)}{r^t_{aa}(n)},$$

it can be checked that

$$0 \leq p^t_{aa} - r^t_{aa} = \sum_{n=0}^{\infty} p^t_{aa}(n)\bigg(1 - \frac{1}{\epsilon(n)}\bigg).$$

LEMMA 3.8. *There exists $T > 0$ and K_1 such that for all $t > T$, the following hold.*

(i) *For each $0 \leq n < |I| - 1$, the following inequality holds:*

$$\epsilon(n) \leq 1 + K_1 \exp(-5Ct).$$

(ii) *For each $r \geq 1$ and $r|I| \leq n < (r+1)|I|$, the following statement is satisfied:*

$$\epsilon(n) \leq (1 + K_1 \exp(-5Ct))^r.$$

Proof. From Proposition 3.3(ii) and Theorem 2 from [12], the Gurevich pressure $P_G(t\phi)$ decreases to $h_{\max}$. So, there exists $T > 0$ such that

$$P_G(t\phi) \leq h_{\max} + d \quad \text{for all } t \geq T + 1,$$

where d was given in Remark 3.5, also

$$h_{\max} \leq P_G(t\phi) \quad \text{for all } t \geq 1.$$

Therefore,

$$-d \leq P_G(t\phi) - P_G(T\phi) < 0 \quad \text{for all } t \geq T + 1, \tag{3.36}$$

and thus the difference between the pressure $P_G(t\phi)$ and $P_G(T\phi)$ can be controlled, for $t \gg 0$. Let $i_k, i_{k+1} \in I$ be arbitrary. Consider a path $\underline{\gamma} = i_k x_1 \ldots x_{m-1} i_{k+1} : i_k \hookrightarrow i_{k+1} \in \mathcal{P}(\Sigma \setminus \Sigma')$, that is, $x_n \notin I$ for all $1 \leq n \leq m-1$ and at least one symbol x_n, $n = 1, 2, \ldots, m-1$ does not belongs to the alphabet associated to the finite Markov shift Σ'. From Remark 3.5, we have $\phi(i_k x_1) \leq 0$, $\phi(x_n x_{n+1}) < -d$ for $n = 1, \ldots, m-2$ and $\phi(x_{m-1} i_{k+1}) < -d$, because $x_n \notin I$ for $n = 1, \ldots, m-1$. Moreover, since $\underline{\gamma} \in \mathcal{P}(\Sigma \setminus \Sigma')$, there exists some $j \in \{1, 2, \ldots, m-1\}$ such that $\phi(x_j x_{j+1}) < -7c$ and consequently

$$\phi(\underline{\gamma}) = \phi(i_k x_1) + \phi(x_1 x_2) + \cdots + \phi(x_{m-1} i_{k+1}) \leq -d(m-2) - 7c.$$

In addition, from equation (3.36), we have $P_G(T\phi) - P_G(t\phi) \leq d \leq (t - T)d$ for $t \geq T + 1$, so that

$$\begin{aligned}
(t\phi - P_G(t\phi))(\underline{\gamma}) - (T\phi - P_G(T\phi))(\underline{\gamma}) &= ((t-T)\phi - P_G(t\phi) + P_G(T\phi))(\underline{\gamma}) \\
&= m(P_G(T\phi) - P_G(t\phi)) + (t-T)\phi(\underline{\gamma}) \\
&\leq (t-T)md + (t-T)(-d(m-2) - 7c) \\
&= (t-T)(2d - 7c) \\
&\leq -7C(t-T).
\end{aligned} \tag{3.37}$$

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We now fix the constant:

$$K := \exp(7CT) \max_{i_k, i_{k+1} \in I} \sum_{\underline{\gamma}: i_k \hookrightarrow i_{k+1} \in \mathcal{P}(\Sigma \setminus \Sigma')} \exp((T\phi - P_G(T\phi))(\underline{\gamma})), \tag{3.38}$$

note that $K < \infty$, because $\sum_{\underline{\gamma}: i_k \hookrightarrow i_{k+1} \in \mathcal{P}(\Sigma \setminus \Sigma')} \exp((T\phi - P_G(T\phi))(\underline{\gamma})) \leq p^T_{i_k i_{k+1}} < \infty$. From equation (3.37), we have for $t \geq T + 1$,

$$(t\phi - P_G(t\phi))(\underline{\gamma}) \leq (T\phi - P_G(T\phi))(\underline{\gamma}) - 7C(t - T),$$

so that

$$\begin{aligned} &\sum_{\underline{\gamma}: i_k \hookrightarrow i_{k+1} \in \mathcal{P}(\Sigma \setminus \Sigma')} \exp((t\phi - P_G(t\phi))(\underline{\gamma})) \\ &\quad \leq \exp(-7Ct + 7CT) \sum_{\underline{\gamma}: i_k \hookrightarrow i_{k+1} \in \mathcal{P}(\Sigma \setminus \Sigma')} \exp(((T\phi - P_G(T\phi))(\underline{\gamma})) \\ &\quad \leq \exp(-7Ct)K. \end{aligned}$$

The proof continues following the same steps as [17] to obtain items (i) and (ii). □

Due to Lemmas 3.7 and 3.8, the following lemma is obtained.

LEMMA 3.9. *There exists $T > 0$ and $0 < M < \infty$ such that for each pair $a, b \in I$ and for all $t > T$:*

(i) $p^t_{aa} \leq r^t_{aa} + M \exp(-3Ct)$;

(ii) $p^t_{ab} p^t_{ba} \leq r^t_{ab} r^t_{ba} + M \exp(-3Ct)$.

We write $a(t) \sim b(t)$ to express that

$$\lim_{t \to \infty} \frac{a(t)}{b(t)} = 1.$$

As a consequence of the previous lemma, one can obtain that

$$1 - p^t_{aa} \sim 1 - r^t_{aa}, \quad 1 - r^t_{aa} \sim 1 - q^t_{aa}, \tag{3.39}$$

for every $a \in I$, see [17] for complete details.

Finally, from equation (3.39), we have that

$$\frac{\mu_t([b])}{\mu_t([a])} = \frac{1 - p^t_{aa}}{1 - p^t_{bb}} \sim \frac{1 - r^t_{aa}}{1 - r^t_{bb}} \sim \frac{1 - q^t_{aa}}{1 - q^t_{bb}} = \frac{\vartheta_t([b])}{\vartheta_t([a])}.$$

Therefore,

$$\lim_{t \to \infty} \frac{\mu_t([b])}{\mu_t([a])} = \lim_{t \to \infty} \frac{\vartheta_t([b])}{\vartheta_t([a])},$$

since that $\lim_{t \to \infty} (\vartheta_t([b])/\vartheta_t([a]))$ exists for all $a, b \in I$, we finally have that $\lim_{t \to \infty} \mu_t$ exists in the weak* topology.

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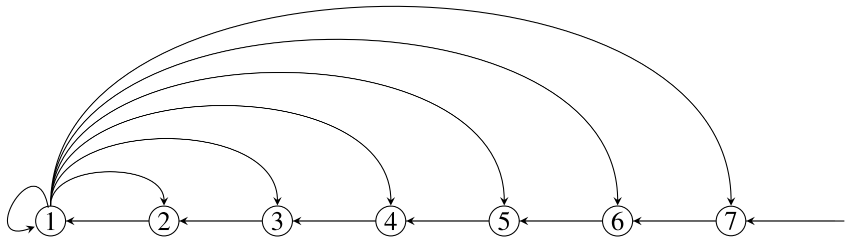

FIGURE 1. Renewal shift.

4. *Examples on the renewal shift*

Throughout this section, we present some examples where there is selection at zero temperature for the family of equilibrium states $(\mu_t)_{t\geq}$, that is, where the limit $\lim_{t\to\infty}\mu_t$ exists in the weak* topology. In fact, those examples are given in the context of the so-called renewal shifts (to be defined below), when the potentials do not necessarily satisfy the conditions stated in Theorem 3.1.

The *renewal shift* is the countable Markov shift whose transition matrix $(A(i,j))_{\mathbb{N}\times\mathbb{N}}$ has entries $A(1,1)$, $A(1,i)$, and $A(i,i-1)$ are equal to 1 for every $i>1$, and the other entries are equal to 0. Note that the renewal shift is topologically mixing and does not satisfy the BIP property (see Figure 1).

In this subsection, we will present two examples of the zero-temperature limit of equilibrium states on the renewal shift. Example 4.1 is for the potential $\phi(x)=-x_0$. Note that this potential satisfies the hypotheses of Theorem 3.1. However, Example 4.2 is for the potential $\phi(x)=x_0-x_1$, which is not a summable potential; however, it has zero temperature limits for its associated equilibrium states.

In a renewal shift with $\phi:\Sigma\to\mathbb{R}$ a weakly Hölder continuous function such that $\sup\phi<\infty$, Sarig [**25**] showed that there exists $t_c>0$ and a unique equilibrium state $\mu_t\in\mathcal{M}_\sigma(\Sigma)$ associated to the potential $t\phi$, for $t\in(0,t_c)$. In that same context, Iommi [**14**] showed that if $t_c=\infty$, then the set of ϕ-maximizing $\mathcal{M}_{\max}(\phi)\neq\emptyset$, otherwise there is no maximizing measure associated to potential ϕ.

Example 4.1. Consider the renewal shift Σ and a potential $\phi:\Sigma\to\mathbb{R}$ given by

$$\phi(x)=-x_0.$$

We will show that $\lim_{t\to\infty}\mu_t$ exists and is a maximizing measure, where μ_t is the equilibrium state associated to potential $t\phi$. Also, $\lim_{t\to\infty}\mu_t([a])=0$, for all $a\geq 2$.

Note that ϕ is a summable potential with $\overline{\mathrm{Var}}(\phi)=0$. Also,

$$P_G(t\phi)\leq\log(2)-t\quad\text{for all } t\geq 1.$$

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So, by Theorem 1 in [12], the family $(\mu_t)_{t\geq 1}$ of equilibrium states associated to potential $t\phi$ have accumulation points. To verify the statement of the previous example, we need the following affirmations to hold.

AFFIRMATION 1. *For $a, n \in \mathbb{N}$, we have*

$$Z_n(t\phi, a) \leq \exp(nP_G(t\phi)).$$

Proof. Let $a, n \in \mathbb{N}$. It can be verified that $Z_n(t\phi, a) = (L^n_{t\phi}\mathbb{1}_{[a]})(x)$ for all $x \in [a]$. Let us integrate that expression with respect to ν_t, which is the eigenmeasure of the dual of the Ruelle operator, that is, $L^*_{t\phi}\nu_t = \exp(P_G(t\phi))\nu_t$. So, we obtain

$$\begin{aligned} Z_n(t\phi, a) &= \frac{1}{\nu_t[a]}\int_{[a]} L^n_{t\phi}\mathbb{1}_{[a]}(x)\,\mathrm{d}\,\nu_t \\ &\leq \frac{1}{\nu_t[a]}\int L^n_{t\phi}\mathbb{1}_{[a]}(x)\,\mathrm{d}\,\nu_t \\ &= \frac{\exp(nP_G(t\phi))}{\nu_t[a]}\int \mathbb{1}_{[a]}(x)\,\mathrm{d}\,\nu_t = \exp(nP_G(t\phi)). \end{aligned}$$

□

AFFIRMATION 2. *For any $a \in \mathbb{N}$, there is a constant $C = \exp(-(a-1)(a+2)/2)$ such that*

$$(L^n_{t\phi}\mathbb{1}_{[1]})(x) \leq C \cdot Z_{n+a-1}(t\phi, 1),$$

where $x \in [a]$.

Proof. Fix $x \in [a]$. We define the application bijective:

$$\begin{aligned} \theta\ :\ \{y \in [1] :\ \sigma^n(y) = x\} &\longrightarrow \{z \in [1] : \sigma^{n+a-1}(z) = z, z_n = a\} \\ y = 1, y_1, \ldots, y_{n-1}, x_0^\infty &\longmapsto z = \overline{1, y_1, \ldots, y_{n-1}, a, (a-1), \ldots, 2}_{\text{per}}. \end{aligned}$$

Note that for every $y \in \mathrm{Dom}(\theta)$, $S_n\phi(y) - S_{n+a-1}\phi(\theta(y)) = -(a-1)(a+2)/2$. So,

$$\begin{aligned} (L^n_{t\phi}\mathbb{1}_{[1]})(x) &= \exp\bigg(-\frac{(a-1)(a+2)}{2}t\bigg)\sum_{z\in \mathrm{Im}(\theta)} \exp(tS_{n+a-1}\phi(z)) \\ &\leq \exp\bigg(-\frac{(a-1)(a+2)}{2}t\bigg)\sum_{\sigma^{n+a-1}z=z} \exp(tS_{n+a-1}\phi(z))\mathbb{1}_{[1]}(z) \\ &= \exp\bigg(-\frac{(a-1)(a+2)}{2}t\bigg)\cdot Z_{n+a-1}(t\phi, 1). \end{aligned}$$

□

AFFIRMATION 3. *For $a \in \mathbb{N}$, we have*

$$\nu_t([a]) = \exp\bigg(-\frac{(a+2)(a-1)}{2}t - (a-1)P_G(t\phi)\bigg)\nu_t([1]).$$

Proof. Fix $t \geq 1$. By the generalized Ruelle's Perron–Frobenius theorem (see Theorem 4.9 in [23]), there is an eigenmeasure ν_t such that

$$\nu_t(L_\phi f) = \lambda\nu_t(f), \quad \text{para } f \in L^1(\nu_t). \tag{4.1}$$

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Let $a \geq 2$. Consider $f := \mathbb{1}_{[a]}$. Substituting in equation (4.1), we have

$$\begin{aligned}\exp(P_G(t\phi))\nu_t([a]) &= \int \sum_{\sigma y=x} \exp(t\phi(y))\mathbb{1}_{[a]}(y) \,\mathrm{d}\, \nu_t(x)\\ &= \int_{[a-1]} \sum_{\sigma y=x} \exp(t\phi(y))\mathbb{1}_{[a]}(y) \,\mathrm{d}\, \nu_t(x)\\ &= \exp(-at)\nu_t([a-1]).\end{aligned}$$

So,

$$\nu_t([a]) = \exp(-at - P_G(t\phi))\nu_t([a-1]), \tag{4.2}$$

using equation (4.2) recursively follows the statement. □

Note that the potential $t\phi$ is positive recurrent, for every $t \geq 1$. Consider $a \geq 1$, then by the generalized Ruelle's Perron–Frobenius theorem, we have

$$\begin{aligned}\frac{\mu_t([a])}{\nu_t([a])} &= h(x) \quad \text{for all } x \in [a]\\ &= \frac{1}{\nu_t([1])} \lim_{n\to\infty} \exp(-nP_G(t\phi))(L^n_{t\phi}\mathbb{1}_{[1]})(x)\\ &\leq \frac{\exp(-(((a-1)(a+2))/2)t)}{\nu_t([1])} \exp((a-1)P_G(t\phi)),\end{aligned}$$

where in the third line, the Affirmations 1 and 2 were used. Later, by Affirmation 3, we have

$$\mu_t([a]) \leq \exp(-(a+2)(a-1)t).$$

Therefore,

$$\lim_{t\to\infty} \mu_t([a]) = 0 \quad \text{for all } a \geq 2. \tag{4.3}$$

We will show that $(\mu_t)_{t\geq 1}$ has only one accumulation point and this is $\delta_{\overline{1}} \in \mathcal{M}_\sigma(\Sigma)$, where this measure is the one supported at the point $\overline{1} = 111\ldots1\ldots \in \Sigma$. Consider $\mu_\infty \in \mathcal{M}_\sigma(\Sigma)$ an arbitrary accumulation point of $(\mu_t)_{t\geq 1}$. Note that it is enough to show that $\mu_\infty = \delta_{\overline{1}}$. From equation (4.3), we have that $\mu_\infty([a]) = 0$, for all $a \geq 2$, and hence $\mu_\infty([1]) = 1$. Since the measure μ_∞ is σ-invariant, then

$$\mu_\infty([1]) = \mu_\infty([11]) + \mu_\infty([21]),$$

so, from equation (4.3) and the fact that $[1] = \bigcup_{i\geq 1}[1i]$, we have that $\mu_\infty([1a]) = 0$, for all $a \geq 2$. Similarly, we obtain that

$$\mu_\infty([11\ldots 1a]) = 0 \quad \text{for all } a \geq 2. \tag{4.4}$$

So, since $\mu_\infty([1]) = 1$, we have

$$\mu_\infty([1\ldots 1]) = 1, \tag{4.5}$$

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where $1 \dots 1$ is a word of arbitrary size composed only of the symbol one. To show that $\mu_\infty = \delta_{\overline{1}}$, it suffices to show that

$$\mu_\infty([\underline{\omega}]) = \delta_{\overline{1}}([\underline{\omega}]) \quad \text{for all } \underline{\omega} \in \mathcal{W}. \tag{4.6}$$

Note that for every $a \geq 2$ and $\underline{\omega} \in \mathcal{W}$ such that $[\underline{\omega}] \cap [a] \neq \emptyset$, we have that equation (4.6) is satisfied. Also, from equations (4.4) and (4.5), for every $[\underline{\omega}] \cap [1] \neq \emptyset$, we have that equation (4.6) is satisfied. Therefore, $\lim_{t\to\infty} \mu_t = \delta_{\overline{1}}$. Also, Proposition 3.1 ensures that $\delta_{\overline{1}} \in \mathcal{M}_{\max}(\phi)$.

Example 4.2. Consider the renewal shift Σ and the potential $\phi : \Sigma \to \mathbb{R}$ given by

$$\phi(x) = x_0 - x_1.$$

We will show that $\lim_{t\to\infty} \mu_t$ exists and it is a ϕ-maximizing measure, where $\mu_t \in \mathcal{M}_\sigma(\Sigma)$ is the equilibrium state associated to potential $t\phi$, for every $t \geq 1$. Also, $\lim_{t\to\infty} \mu_t([a]) > 0$, for every $a \in \mathbb{N}$.

Note that ϕ is a weakly Hölder continuous potential with $\mathrm{Var}_1(\phi) = +\infty$ and $\sup \phi < \infty$. Also $t_c = \infty$. Next, by Theorem 5 in [**25**], we know that the equilibrium state $\mu_t \in \mathcal{M}_\sigma(\Sigma)$ associated with the potential $t\phi$ exists, for every $t \geq 1$, and hence $\mathcal{M}_{\max}(\phi) \neq \emptyset$, see Theorem 1.1 in [**14**]. Fix $t \geq 1$, and notice that the Gurevich pressure of $P_G(t\phi)$ is constant, $P_G(t\phi) = \log 2$ (see [**2**]). Also, $\alpha(\phi) = 0$, and $|\mathcal{M}_{\max}(\phi)| = \infty$, because of the fact that for every $x \in Per(\Sigma)$ such that $\sigma^n x = x$, we have $S_n\phi(x) = 0$.

However, let $\underline{\omega} \in \mathcal{W}_m$, such that $\underline{\omega} \subset [a]$, then

$$\begin{aligned}
\mu_t([\underline{\omega}]) &= h_t(x)\nu_t([\underline{\omega}]), \quad x \in [a] \\
&= \lim_{n\to\infty} \frac{1}{2^n} L^n_{t\phi} \mathbb{1}_{[\underline{\omega}]}(x) \\
&= \lim_{n\to\infty} \frac{1}{2^n} \{y \in [\underline{\omega}] : \sigma^n y = x\}.
\end{aligned} \tag{4.7}$$

So, for any cylinder $\underline{\omega}$, we have that $\mu_t([\underline{\omega}])$ does not depend on t. Since $\underline{\omega}$ was arbitrary, we have that $(\mu_t)_{t\geq 1}$ is a singleton which we will denote by $\mu_\infty \in \mathcal{M}_\sigma(\Sigma)$. By Proposition 3.3(iii), $h(\mu_\infty) = \log 2$. So, by the variational principle, we have

$$P_G(t\phi) = h(\mu_\infty) + \mu_\infty(t\phi),$$

and thus $\mu_\infty(\phi) = 0$. Therefore, μ_∞ is a ϕ-maximizing measure. If we consider $\underline{\omega} = a$ in equation (4.7), we have $\mu_\infty([a]) = 1/2^a$.

Note that from Example 4.2, we have the existence of the zero temperature limit of equilibrium states in more general conditions than Theorem 3.1.

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