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ENRIQUES INVOLUTIONS AND BRAUER CLASSES

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Abstract. We prove that every element of order 2 in the Brauer group of a complex Kummer surface X descends to an Enriques quotient of X. In *generic* cases, this gives a bijection between the set $\mathcal{E}nr(X)$ of Enriques quotients of X up to isomorphism and the set of Brauer classes of X of order 2. For some K3 surfaces of Picard rank 20, we prove that the fibers of $\mathcal{E}nr(X) \to \mathrm{Br}(X)[2]$ above the nonzero points have the same cardinality.

§1. Introduction

Let S be a complex Enriques surface, and let $\pi\colon X \to S$ be its K3 étale double cover. J.-L. Colliot-Thélène asked whether the induced map of Brauer groups $\pi^*\colon \mathrm{Br}(S) \simeq \mathbb{Z}/2 \to \mathrm{Br}(X)$ is injective or zero[1]. Beauville has given a necessary and sufficient condition for the injectivity of π^* [B, Cor. 5.7] and showed that the Enriques surfaces S for which this map is zero form a countable union of hypersurfaces in the moduli space of Enriques surfaces [B, Cor. 6.5]. Enriques surfaces with injective π^* are used in explicit constructions of Enriques surfaces over $\mathbb{Q}$ for which the Brauer–Manin obstruction fails to control weak approximation [HS1] and the Hasse principle [VV]). Enriques surfaces over $\mathbb{Q}$ such that the map π^* is zero have been constructed in [HS2, GS].

From a different perspective, one can start with a K3 surface X and consider the set $\mathcal{F}(X) \subset \mathrm{Aut}(X)$ of fixed point free involutions $\sigma\colon X \to X$, which are precisely the involutions such that the quotient X/σ is an Enriques surface.

In this paper, we are interested in the map

$$\Phi_X\colon \mathcal{F}(X) \longrightarrow \mathrm{Br}(X)[2],$$

which sends $\sigma \in \mathcal{F}(X)$ to $\pi^*(b_S)$, where $\pi\colon X \to X/\sigma = S$ is the quotient morphism, and b_S is the unique nonzero element of $\mathrm{Br}(S)$. A combination of results of Beauville and of Keum and Ohashi show that $\mathrm{Im}(\Phi_X)$ depends only on the isomorphism class of the transcendental lattice $T(X)$ of X (see Corollary 2.6). A description of all lattices $T(X)$ such that $\mathcal{F}(X) \neq \emptyset$ can be found in [BSV, Th. 1.6].

Let $\mathcal{E}nr(X)$ be the set of Enriques quotients of X, considered up to isomorphism of varieties. Equivalently, $\mathcal{E}nr(X)$ is the set of conjugacy classes of $\mathrm{Aut}(X)$ contained in $\mathcal{F}(X)$ (see [O1, Prop. 2.1]). Ohashi proved that the set $\mathcal{E}nr(X)$ is always finite [O1, Cor. 0.4] although its size is not bounded [O1, Th. 0.1]. The map Φ_X is $\mathrm{Aut}(X)$-equivariant, where $\mathrm{Aut}(X)$ acts on $\mathcal{F}(X)$ by conjugation, so Φ_X descends to a map

$$\varphi_X\colon \mathcal{E}nr(X) \longrightarrow \mathrm{Br}(X)[2]/\mathrm{Aut}(X).$$

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[1] Private communication to the first named author in the early 2000s.

The action of $\mathrm{Aut}(X)$ on $\mathrm{Br}(X)[2]$ factors through the action of the group of Hodge isometries of the integral Hodge structure on $T(X)$, so when $\mathrm{Aut}_{\mathrm{Hdg}}(T(X)) = \{\pm 1\}$ the action of $\mathrm{Aut}(X)$ on $\mathrm{Br}(X)[2]$ is trivial. In such a *generic* situation, φ_X is a map $\mathcal{E}nr(X) \to \mathrm{Br}(X)[2]$. In this case, the set $\mathcal{E}nr(X)$ depends only on the isomorphism class of the lattice $T(X)$ (see the discussion after Theorem 2.5).

Examples show that the set $\mathcal{E}nr(X)$ can be empty or very large, so in general φ_X is neither surjective nor injective. A very general Enriques surface S (corresponding to the points of the moduli space outside a countable union of hypersurfaces) is the unique Enriques quotient of its K3 cover X; by Beauville, in this case, $\varphi_X(\mathcal{E}nr(X))$ is a certain nonzero element of $\mathrm{Br}(X)[2]$.

The aim of this paper is to clarify the structure of Φ_X and φ_X in some favourable situations. Keum [K, Th. 2] proved that every Kummer surface is a double cover of some Enriques surface. His method can be used to prove the following.

THEOREM A. *Let X be a Kummer surface. Then, for every $\alpha \in \mathrm{Br}(X)$ of order 2, there is an Enriques quotient $\pi_S\colon X \to S$ such that $\alpha = \pi_S^*(b_S)$.*

In other words, for Kummer surfaces, the set $\mathrm{Br}(X)[2] \setminus \{0\}$ is contained in the image of Φ_X. As a kind of partial converse, in Corollary 2.7, we show that if X is a K3 surface such that the abelian group $\mathrm{Br}(X)[2]$ is generated by the image of Φ_X, then the transcendental lattice of X is divisible by 2 as an even lattice. We do not know if there exist Kummer surfaces such that $\Phi_X^{-1}(0)$ is non-empty. At the end of §2, we give examples of non-Kummer K3 surfaces such that $\mathrm{Im}(\Phi_X) = \{0\}$.

In two *generic* cases, Ohashi classified all Enriques quotients of a given K3 surface. Combining Theorem A with his results [O1, Th. 4.1], [O2, Th. 1.1] we obtain the following corollary.

COROLLARY B. *Let X be the Kummer surface attached to any of the following abelian surfaces:*

(i) *a product of two non-isogenous elliptic curves;*
(ii) *the Jacobian J of a curve of genus 2 such that $\mathrm{NS}(J) \cong \mathbb{Z}$.*

Then φ_X is a bijection between $\mathcal{E}nr(X)$ and $\mathrm{Br}(X)[2] \setminus \{0\}$.

For some K3 surfaces of maximal Picard rank, the following result gives information about the fibers of φ_X. Its proof uses a certain Galois action on $\mathrm{Br}(X)[2]$ constructed by the second named author in [V].

THEOREM C. *Let X be a K3 surface of Picard rank 20. Let $E = \mathbb{Q}(\sqrt{-d})$, where d is the discriminant of the transcendental lattice $T(X)$. Assume that $\mathrm{End}_{\mathrm{Hdg}}(T(X))$ is the ring of integers $\mathcal{O}_E \subset E$ and, moreover, 2 is inert in E and $E \neq \mathbb{Q}(\sqrt{-3})$. Then $\mathrm{Aut}_{\mathrm{Hdg}}(T(X)) = \{\pm 1\}$ and the fibers of $\varphi_X\colon \mathcal{E}nr(X) \to \mathrm{Br}(X)[2]$ above the nonzero points have the same cardinality.*

The conditions in Theorem C are easy to check. Let

$$\begin{pmatrix} 2a & b \\ b & 2c \end{pmatrix}$$

be the Gram matrix of $T(X)$, where $a,b,c \in \mathbb{Z}$, so that $-d = b^2 - 4ac < 0$. Write $-d = f^2 D$, where $f \in \mathbb{Z}$ and D is the discriminant of E. By [V, Th. 3.2] we have $\mathrm{End}_{\mathrm{Hdg}}(T(X)) = \mathcal{O}_E$

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if and only if $f = \gcd(a,b,c)$. Next, 2 is inert in E if and only if $D \equiv 5 \bmod 8$. If f is odd, so that $-d \equiv 5 \bmod 8$, we have $\mathcal{E}nr(X) = \emptyset$ by [S], so in this case, the fibers of φ_X are empty. Using Theorem A, it is easy to see that for each $D \equiv 5 \bmod 8$, $D \neq -3$, there are infinitely many pairwise non-isomorphic K3 surfaces of Picard rank 20 with complex multiplication by $\mathcal{O}_{\mathbb{Q}(\sqrt{D})}$ such that the fibers of φ_X above the nonzero points of $\mathrm{Br}(X)[2]$ have the same *positive* number of elements.

It would be interesting to describe the K3 surfaces X such that Φ_X is surjective onto $\mathrm{Br}(X)[2]$ or onto $\mathrm{Br}(X)[2] \setminus \{0\}$. In this direction, we have the following result, whose proof uses Nikulin's theory of lattices [N] and surjectivity of the period map for K3 surfaces.

THEOREM D. *Let X be a K3 surface such that* $\mathrm{rk}(\mathrm{NS}(X)) \geq 12$. *Then there exist infinitely many K3 surfaces Y such that:*

(1) $T(X)_{\mathbb{Q}} \cong T(Y)_{\mathbb{Q}}$ *as polarized Hodge structures.*
(2) *The discriminants of $T(Y)$ are pairwise different.*
(3) *There is a natural isomorphism* $\mathrm{Br}(X)[2] \cong \mathrm{Br}(Y)[2]$ *under which*

$$\mathrm{Im}(\Phi_X) \setminus \{0\} = \mathrm{Im}(\Phi_Y) \setminus \{0\}.$$

We recall results of Beauville, Keum, and Ohashi, and then prove some useful lemmas in §2. Theorem A and Corollary B are proved in §3, Theorem C is proved in §4, and Theorem D in §5.

§2. Lattices and the topology of Enriques quotients

A *lattice* L is a free finitely generated abelian group with a non-degenerate integral symmetric bilinear form. Write $L(2)$ for the same group with the form $2(x.y)$.

For a lattice L, we denote by $A_L = L^*/L$ the discriminant group of L. If L is even, then $q_L \colon A_L \to \mathbb{Q}/2\mathbb{Z}$ is the associated quadratic form.

If $L \subset M$ are lattices, we denote by $L^{\perp}_M$ the orthogonal complement to L in M. It is clear that $L^{\perp}_M$ is a primitive sublattice of M.

Let U be the hyperbolic plane. Write $\mathrm{U} = \mathbb{Z}e \oplus \mathbb{Z}f$, where $(e^2) = (f^2) = 0$, $(e.f) = 1$. We denote by E_8 the negative-definite, even, unimodular lattice of the root system E_8. Write

$$\Lambda = \mathrm{E}_8^{\oplus 2} \oplus \mathrm{U}^{\oplus 3}, \qquad M = \mathrm{U}(2) \oplus \mathrm{E}_8(2), \qquad N = \mathrm{U} \oplus \mathrm{U}(2) \oplus \mathrm{E}_8(2).$$

Here, Λ is the K3 lattice. Let $\iota \colon \Lambda \to \Lambda$ be the involution permuting two copies of $\mathrm{E}_8 \oplus \mathrm{U}$, and acting as -1 on the third copy of U. Then $\Lambda^+ \cong M$ and $\Lambda^- \cong N$, where $\Lambda^{\pm}$ is the ± 1-eigenspace of ι. By [H2, (vii) on p. 305], for any Enriques quotient $\pi_S \colon X \to S = X/\sigma$, the induced map

$$\pi_S \colon \mathrm{H}^2(S,\mathbb{Z})/_{\mathrm{tors}} \longrightarrow \mathrm{H}^2(X,\mathbb{Z})$$

can be identified with the composition

$$\mathrm{H}^2(S,\mathbb{Z})/_{\mathrm{tors}} \simeq \mathrm{U}\oplus \mathrm{E}_8 \xrightarrow{\mathrm{diag}} (\mathrm{U}\oplus \mathrm{E}_8)^{\oplus 2} \subset \Lambda \simeq \mathrm{H}^2(X,\mathbb{Z}).$$

Here, the fixed point free involution $\sigma\colon X\to X$ induces the involution ι on Λ.

The lattice N has a canonical character $N\to \mathbb{Z}/2$ which will play a crucial role in what follows.

Lemma 2.1. *The homomorphism* $\varepsilon\colon N\to\mathbb{Z}/2$ *given by* $\varepsilon(x) := \big(x.(e+f)\big) \bmod 2$, *where* e *and* f *are standard generators of* $\mathrm{U}\subset N$, *does not depend on the embedding of lattices* $\mathrm{U}\hookrightarrow N$. *Hence,* $\alpha^*(\varepsilon)=\varepsilon$ *for any* $\alpha\in\mathrm{Aut}(N)$.

Proof. Let e',f' be standard generators of U embedded in N. Write $e' = ae+bf+u$, $f' = ce+df+w$, where $a,b,c,d\in\mathbb{Z}$ and $u,w\in \mathrm{U}(2)\oplus\mathrm{E}_8(2)$. We have $2ab+(u^2) = 2cd+(w^2)=0$ and $ad+bc+(u.w)=1$. Since (u^2) and (w^2) are divisible by 4, and $(u.w)$ is even, we see that ab is even, cd is even, and $ad+bc$ is odd. It follows that either a,d are odd and b,c are even, or a,d are even and b,c are odd. In both cases, $e'+f'$ equals $e+f$ modulo $2\mathrm{U}\oplus\mathrm{U}(2)\oplus\mathrm{E}_8(2)$, hence the result. ∎

Lemma 2.2. *If* $x\in N$ *is such that* $(x^2)\equiv 2 \bmod 4$, *then* $\varepsilon(x)=0$.

Proof. Write $x = ae+bf+u$, where $a,b\in\mathbb{Z}$ and $u\in\mathrm{U}(2)\oplus\mathrm{E}_8(2)$. Then a and b are both odd, hence $\varepsilon(x)\equiv a+b\equiv 0 \bmod 2$. ∎

Lemma 2.3. *Let* L *be a sublattice of* N. *If the restriction of* $\varepsilon\colon N\to\mathbb{Z}/2$ *to* L *is nonzero, then* $L^{\perp}_N = L'(2)$ *for some even lattice* L'.

Proof. Suppose $\varepsilon(x)\neq 0$ for some $x\in L$. Writing $x=ae+bf+u$, where $a,b\in\mathbb{Z}$ and $u\in\mathrm{U}(2)\oplus\mathrm{E}_8(2)$, we see that a and b have opposite pairity. If $y = ce+df+w\in L^{\perp}_N$, where $c,d\in\mathbb{Z}$ and $w\in\mathrm{U}(2)\oplus\mathrm{E}_8(2)$, then $ad+bc$ is even, which implies that either c or d is even. Then $(y^2)=2cd+(w^2)$ is divisible by 4, hence $L^{\perp}_N = L'(2)$ for some even lattice L'. ∎

The importance of the character $\varepsilon\colon N\to\mathbb{Z}/2$ has been revealed by Beauville. Namely, let $\pi_S\colon X\to S = X/\sigma$ be an Enriques quotient of a K3 surface X. Let $T(X)\subset\Lambda$ be the transcendental lattice of X. Recall the canonical isomorphism

$$\mathrm{Br}(X)\cong \mathrm{Hom}(T(X),\mathbb{Q}/\mathbb{Z})$$

(see [CS, (5.5) on p. 130, p. 142]). It is well known that the involution σ is not symplectic [H2, Cor. 15.1.5 and (ii) on p. 356], so it acts on $\mathrm{H}^0(X,\Omega^2_X)$ as -1. Therefore, $\sigma^*=\iota$ acts on $T(X)$ as -1, so $T(X)\subset N$.

Theorem 2.4 (Beauville). *Let* $\pi_S\colon X\to S$ *be an Enriques quotient of a K3 surface* X. *Then* $\pi_S^*(b_S)\in\mathrm{Br}(X)[2]$ *is the restriction of* $\varepsilon\colon N\to\mathbb{Z}/2$ *to* $T(X)$.

Proof. See [B, Prop s. 3.4 and 5.3]. ∎

An embedding $T(X)\subset N$ coming from an Enriques quotient of X is clearly primitive. The orthogonal complement $T(X)^{\perp}_N\subset N$ contains no (-2)-elements x, because by Riemann–Roch either x or $-x$ is effective, but σ^* preserves effectivity. In fact, these are the only conditions. Horikawa's theorem on the surjectivity of the period map for Enriques surfaces [H1] leads to the following result. See [K, Th. 1], which was extended in [O2, Prop. 2.1].

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Theorem 2.5 (Keum, Ohashi). *Let X be a K3 surface. Associating to an Enriques quotient of X a primitive embedding $T(X)\subset N$ defines a bijection between $\mathcal{E}nr(X)$ and the set of equivalence classes of primitive embeddings of $T(X)$ into N without (-2)-elements in the orthogonal complement. Here the embeddings i_1 and i_2 are equivalent if there is an automorphism $\tilde{\phi}$ of the lattice N and a $\phi\in \mathrm{Aut}_{\mathrm{Hdg}}(T(X))$ such that $i_2\circ\phi=\tilde{\phi}\circ i_1$.*

If $\mathrm{Aut}_{\mathrm{Hdg}}(T(X))=\{\pm 1\}$ (which holds, e.g., when the Picard number of X is odd), the set $\mathcal{E}nr(X)$ depends only on the lattice $T(X)$.

Corollary 2.6. *For any K3 surface X, the following statements hold.*

(i) $\mathrm{Im}(\Phi_X)\setminus\{0\}$ *is the set of nonzero $\alpha\in \mathrm{Br}(X)[2]\cong \mathrm{Hom}(T(X),\mathbb{Z}/2)$, for which there exists a primitive embedding $i\colon T(X)\hookrightarrow N$ such that $\alpha=i^*(\varepsilon)$.*
(ii) $0\in \mathrm{Im}(\Phi_X)$ *if and only if there exists a primitive embedding $i\colon T(X)\hookrightarrow N$ without (-2)-elements in the orthogonal complement such that $i^*(\varepsilon)=0$.*
(iii) *If $x\in T(X)$ is such that $(x^2)\equiv 2 \bmod 4$, then $\alpha(x)=0$ for any $\alpha\in \mathrm{Im}(\Phi_X)$.*

Proof. Parts (i) and (ii) formally follow from Theorems 2.4 and 2.5 and Lemma 2.3. In particular, Lemma 2.3 implies that $i(T(X))^{\perp}_N$ does not contain (-2)-classes. Part (iii) follows from Lemma 2.2. □

Corollary 2.7. *If X is a K3 surface such that the abelian group $\mathrm{Br}(X)[2]$ is generated by the image of Φ_X, then there is an even lattice T' such that $T(X)\cong T'(2)$.*

Proof. It is enough to show that for every $x\in T(X)$ we have $(x^2)\equiv 0 \bmod 4$. Suppose that there is an element $y\in T(X)$ such that $(y^2)\equiv 2 \bmod 4$. Then y is not divisible by 2 in $T(X)$. By Corollary 2.6(iii), the nonzero class of y in $T(X)/2T(X)$ is in the kernel of every $\alpha\in \mathrm{Im}(\Phi_X)$. Thus $\mathrm{Im}(\Phi_X)$ is contained in a proper subgroup of $\mathrm{Br}(X)[2]$. □

Corollary 2.8. *Let X be a K3 surface such that $T(X)$ has a basis $e_1,\ldots,e_n$ with $(e_i^2)\equiv 2 \bmod 4$ for $i=1,\ldots,n$. Then either $\mathcal{E}nr(X)=\emptyset$ or $\mathrm{Im}(\Phi_X)=\{0\}$.*

Proof. Suppose that a nonzero $\alpha\in \mathrm{Hom}(T(X),\mathbb{Z}/2)$ is in the image of Φ_X. By Theorem 2.4, there is a primitive embedding $i\colon T(X)\to N$ such that $i^*(\varepsilon)=\alpha$. By Lemma 2.2, we have $\alpha(e_i)=0$ for $i=1,\ldots,n$, hence $\alpha(T(X))=0$ which is a contradiction. □

This can be used to give examples of K3 surfaces X such that $\mathrm{Im}(\Phi_X)=\{0\}$. For example, one can take the K3 surface X of Picard rank 20 with transcendental lattice

$$\begin{pmatrix} 2 & 0 \\ 0 & 2c \end{pmatrix}$$

with $c=3,5,7$. Indeed, by [SV, Table 3.1] in these cases, we have $|\mathcal{E}nr(X)|=1$.

§3. Kummer surfaces

Proof of Theorem A

By Corollary 2.6(i), it is enough to construct, for any nonzero $\alpha\in \mathrm{Hom}(T(X),\mathbb{Z}/2)$, a primitive embedding $i\colon T(X)\hookrightarrow N=\mathrm{U}\oplus \mathrm{U}(2)\oplus \mathrm{E}_8(2)$ such that $\varepsilon(x)=\alpha(x)$ for any $x\in T(X)$. We use Morrison's classification of transcendental lattices of Kummer surfaces (see [H2, Cor. 14.3.20]). For each of them, Keum [K, pp. 106–108] constructed a primitive embedding into N; we follow the same strategy to construct all 2^n-1 embeddings, where

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$n=\mathrm{rk}(T(X))$. We keep the notation of [K], in particular, e,f is a standard basis of U and h,k is a standard basis of U(2). We denote by ρ the Picard rank of X.

In the proof below, we shall use the following particular case of a result of Nikulin.

LEMMA 3.1. *Any even negative-definite lattice of rank at most 4 has a primitive embedding in* E_8.

Proof. This follows from [N, Th. 1.12.4] using the fact that E_8 is a unique even unimodular negative-definite lattice of rank 8. ∎

$\rho=20$

In this case, the lattice $T=\mathbb{Z}x\oplus\mathbb{Z}y$ is positive-definite with Gram matrix

$$\begin{pmatrix} 4a & 2b \\ 2b & 4c \end{pmatrix},$$

where $a,b,c\in\mathbb{Z}$. The three primitive embeddings can be given by sending x,y to the following two elements of N:

$$(e+2af,2bf+h+ck),\quad (2bf+h+ak,e+2cf),\quad (e+2af,e+(2b-2a)f+h+(c-b+a)k).$$

$\rho=19$

Now T has signature $(2,1)$. We can choose an integral basis x,y,t of T so that the Gram matrix is

$$\begin{pmatrix} 4a & 2d & 2l \\ 2d & 4b & 2m \\ 2l & 2m & 4c \end{pmatrix},$$

where $a,b,c,d,l,m\in\mathbb{Z}$ and $a,b,c<0$. The embeddings we need to construct are numbered by the nonzero vectors $(v_1,v_2,v_3)\in(\mathbb{F}_2)^3$ given by evaluating ε on the images of x,y,t in this order. By symmetry it is enough to construct embeddings labeled $(1,0,0)$, $(1,1,0)$, and $(1,1,1)$. The first two can be given by sending x,y,t to the following three elements of N, where w is a primitive element of $\mathrm{E}_8(2)$ such that $(w^2)=4c$:

$$(e+2af,2df+h+bk,2lf+mk+w);$$

$$(e+2af,e+(2d-2a)f+h+(b-d+a)k,2lf+(m-l)k+w).$$

Next, we deal with $(1,1,1)$. Without loss of generality, we can assume $m>0$. Take

$$(e+k+ah,e+2mf+(d-m)h+w',e+lh+w),$$

where $\mathbb{Z}w'\oplus\mathbb{Z}w$ is a primitive sublattice of $\mathrm{E}_8(2)$ such that $(w'^2)=4b-4m<0$, $(w^2)=4c<0$, $(w'.w)=0$.

$\rho=18$

Here, the lattice T is the orthogonal direct sum of $\mathbb{Z}x\oplus\mathbb{Z}y$ with signature $(1,1)$ and Gram matrix

$$\begin{pmatrix} 4a & 2b \\ 2b & 4c \end{pmatrix}$$

and $\mathrm{U}(2)=\mathbb{Z}r\oplus\mathbb{Z}s$. Without loss of generality, we assume that $a,c<0$ and $b>0$. Let w and u be primitive vectors of $\mathrm{E}_8(2)$ such that $(w^2)=4c<0$ and $(u^2)=4(a-b+c)<0$. We

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label the embeddings in the same way as above. Up to exchanging the roles of x and y, and of r and s, it is enough to construct embeddings with the following labels:

$$(1,0,0,0),(1,1,0,0),(1,0,1,0),(0,0,1,0),(0,0,1,1),(1,1,1,0),(1,0,1,1),(1,1,1,1).$$

Let us first construct primitive embeddings with labels $(1,0,0,0)$ and $(1,1,0,0)$ by taking the direct sum of a primitive embedding $\mathbb{Z}x\oplus\mathbb{Z}y$ into $\mathrm{U}\oplus\mathrm{E}_8(2)$ and the identity embedding $\mathrm{U}(2)\tilde{\longrightarrow}\mathrm{U}(2)$. We send x,y to

$$(e+2af,2bf+w),\quad (e+2af,e+(2b-2a)f+u).$$

The embedding with label $(1,0,1,0)$ can be obtained by sending x,y,r,s to

$$(e+2af-ak,2bf-bk+w,e+h,k).$$

For $(0,0,1,0)$, we take $(h+w_1,bk+w_2,e,2e+2f+w_3)$, where $\mathbb{Z}w_1\oplus\mathbb{Z}w_2\oplus\mathbb{Z}w_3$ is a primitive sublattice of $\mathrm{E}_8(2)$ with diagonal Gram matrix such that $(w_1^2)=4a<0$, $(w_2^2)=4c<0$, $(w_3^2)=-8$.

For $(0,0,1,1)$, we take $(h+w_1,bk+w_2,e,e+2f+w_3)$, where $\mathbb{Z}w_1\oplus\mathbb{Z}w_2\oplus\mathbb{Z}w_3$ is a primitive sublattice of $\mathrm{E}_8(2)$ with diagonal Gram matrix such that $(w_1^2)=4a<0$, $(w_2^2)=4c<0$, $(w_3^2)=-4$.

For $(1,1,1,0)$, we take $(e+2af-ak,e+(2b-2a)f+(a-b)k+u,e+h,k)$.

For $(1,0,1,1)$, we take $(e+2af-ak,2bf-bk+w,e+h,e+k+h+w')$, where $\mathbb{Z}w\oplus\mathbb{Z}w'$ is a primitive sublattice of $\mathrm{E}_8(2)$ such that $(w^2)=4c<0$, $(w'^2)=-4$, $(w.w')=0$.

For $(1,1,1,1)$, we take $(e+2af-ak,e+(2b-2a)f+(a-b)k+u,e+h,e+k+h+w')$, where $\mathbb{Z}u\oplus\mathbb{Z}w'$ is a primitive sublattice of $\mathrm{E}_8(2)$ such that $(u^2)=4(a-b+c)<0$, $(w'^2)=-4$, $(u.w')=0$.

$\rho=17$

Here, we have $T=\mathrm{U}(2)\oplus\mathrm{U}(2)\oplus(-4m)$, where $m\geq 1$. A standard basis is $\{x,y,x',y',t\}$. Up to swapping the two copies of $\mathrm{U}(2)$ and swapping the elements of a standard basis of each $\mathrm{U}(2)$ it is enough to construct embeddings with the following labels:

$$(1,0,0,0,0),(1,1,0,0,0),(1,0,0,0,1),(1,1,0,0,1),(0,0,0,0,1),$$

$$(1,1,1,1,0),(1,1,1,1,1),(1,0,1,0,0),(1,1,1,0,0),(1,1,1,0,1),(1,0,1,0,1).$$

The first five embeddings are obtained as direct sums of a primitive embedding of $\mathrm{U}(2)\oplus(-4m)$ into $\mathrm{U}\oplus\mathrm{E}_8(2)$ and the identity embedding $\mathrm{U}(2)\tilde{\longrightarrow}\mathrm{U}(2)$. The respective primitive embeddings of $\mathrm{U}(2)\oplus(-4m)$ into $\mathrm{U}\oplus\mathrm{E}_8(2)$ are given by sending x,y,t to the following triples:

$$(e,2e+2f+u_1,v_1),(e,e+2f+u_2,v_2),(e,2e+2f+u_3,e+v_3),(e,e+2f+u_4,e+v_4).$$

Here, $\mathbb{Z}u_i\oplus\mathbb{Z}v_i$ is a primitive sublattice of $\mathrm{E}_8(2)$ such that:
$(u_1^2)=-8$, $(v_1^2)=-4m$, $(u_1.v_1)=0$;
$(u_2^2)=-4$, $(v_2^2)=-4m$, $(u_2.v_2)=0$;
$(u_3^2)=-8$, $(v_3^2)=-4m$, $(u_3.v_3)=-2$;
$(u_4^2)=-4$, $(v_4^2)=-4m$, $(u_4.v_4)=-2$.

The embedding labeled $(0,0,0,0,1)$ can be obtained by sending x,y,t to

$$(2e+2f+w_0, 2e+2f+w_1, e+w_2),$$

where w_0, w_1, w_2 generate a primitive sublattice of $\mathrm{E}_8(2)$ with Gram matrix

$$\begin{pmatrix} -8 & -6 & -2 \\ -6 & -8 & -2 \\ -2 & -2 & -4m \end{pmatrix}.$$

Indeed, this matrix is negative-definite.

To construct the last six embeddings, we exhibit the images of x,y,x',y',t. In the case of $(1,1,1,1,0)$, we consider

$$(e, e+2f+k+w_0, e-h, e-h-k+w_1, w_2),$$

where w_0, w_1, w_2 generate a primitive sublattice of $\mathrm{E}_8(2)$ with diagonal Gram matrix such that $(w_0^2) = (w_1^2) = -4$ and $(w_2^2) = -4m$.

In the case of $(1,1,1,1,1)$, we take

$$(e, e+2f+k+w_0, e-h, e-h-k+w_1, e+w_2),$$

where w_0, w_1, w_2 generate a primitive sublattice of $\mathrm{E}_8(2)$ with the negative-definite Gram matrix

$$\begin{pmatrix} -4 & 0 & -2 \\ 0 & -4 & 0 \\ -2 & 0 & -4m \end{pmatrix}.$$

In the case of $(1,0,1,0,0)$, we take $(e, 2f+k, e-h, -k, w)$, where w is a primitive element of $\mathrm{E}_8(2)$ with $(w^2) = -4m$.

For $(1,1,1,0,0)$, we take $(e, e+2f+k+u_2, e-h, -k, v_2)$.

For $(1,1,1,0,1)$, we take $(e, e+2f+k+u_4, e-h, -k, e+v_4)$.

For $(1,0,1,0,1)$, we take $(e, 2e+2f+k+u_3, e-h, -k, e+v_3)$.

Proof of Corollary B

(i) Let E_1 and E_2 be non-isogenous elliptic curves, and let $X = \mathrm{Kum}(E_1 \times E_2)$. By [O1, §4], we have $\mathrm{Aut}_{\mathrm{Hdg}}(T(X)) = \{\pm 1\}$ and $|\mathcal{E}nr(X)| = 15$. (The 15 Enriques involutions can be described geometrically as the Lieberman involutions and the Kondo–Mukai involutions.) We have $\mathrm{rk}(T(X)) = 4$, hence $|\mathrm{Br}(X)[2] \setminus \{0\}| = 15$.

(ii) Let C be a smooth projective curve of genus 2 such that $\mathrm{NS}(\mathrm{Jac}(C)) \cong \mathbb{Z}$. Let $X = \mathrm{Kum}(\mathrm{Jac}(C))$. Condition $\mathrm{Aut}_{\mathrm{Hdg}}(T(X)) = \{\pm 1\}$ is satisfied since the Picard rank of X is odd. Ohashi [O2] shows that $|\mathcal{E}nr(X)| = 31$ and describes these 31 involutions geometrically. In this case $\mathrm{rk}(T(X)) = 5$, so $|\mathrm{Br}(X)[2] \setminus \{0\}| = 31$.

Taking into account (i) and (ii), Corollary B follows from Theorem A since a surjective map of finite sets of the same cardinality is a bijection. ∎

§4. Singular K3 surfaces

K3 surfaces over $\overline{\mathbb{Q}}$

For a variety X over $\overline{\mathbb{Q}}$ and an element $g \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, we define $X^g = X \times_{\overline{\mathbb{Q}},g} \overline{\mathbb{Q}}$. Then, we have a morphism $g\colon X \to X^g$ making the following diagram commutative:

$$\begin{array}{ccc} X & \xrightarrow{\quad g \quad} & X^g \\ \downarrow & & \downarrow \\ \mathrm{Spec}(\overline{\mathbb{Q}}) & \xrightarrow{(g^{-1})^*} & \mathrm{Spec}(\overline{\mathbb{Q}}) \end{array} .$$

Here, the vertical arrows are structure morphisms. A morphism of $\overline{\mathbb{Q}}$-varieties $\phi\colon X \to Y$ gives rise to a morphism of $\overline{\mathbb{Q}}$-varieties $\phi^g = g\phi g^{-1}\colon X^g \to Y^g$.

Let $K \subset \overline{\mathbb{Q}}$ be a subfield, and let $G_K = \mathrm{Gal}(\overline{\mathbb{Q}}/K)$.

DEFINITION 4.1. Let X be a variety over $\overline{\mathbb{Q}}$.

(i) The field of moduli of X over K is the subfield $K(X) \subset \overline{\mathbb{Q}}$ fixed by the group $\{g \in G_K | X \cong X^g\}$.

(ii) Let $B \subset \mathrm{Br}(X)$ be a finite subgroup. The field of moduli of the pair (X,B) over K is the subfield $K(X,B) \subset \overline{\mathbb{Q}}$ fixed by the group

$$\{g \in G_K | \exists \text{an isomorphism}\, f\colon X^g \to X \,\text{such that}\, (g^* \circ f^*)|_B = \mathrm{id}_B\}.$$

Let us fix an embedding $\overline{\mathbb{Q}} \subset \mathbb{C}$. For a K3 surface X over $\overline{\mathbb{Q}}$ we write $T(X)$ for the transcendental lattice of $X_{\mathbb{C}}$. One has natural isomorphisms ([CS, Prop. 5.2.3 and p. 142])

$$\mathrm{Br}(X) \cong \mathrm{Br}(X_{\mathbb{C}}) \cong \mathrm{Hom}(T(X), \mathbb{Q}/\mathbb{Z}).$$

REMARK 4.2. Let X be a K3 surface over $\overline{\mathbb{Q}}$ of Picard rank at least 12. According to [V, Rem. 6.1(2), p. 32] a Hodge isometry $h\colon T(X^g) \tilde{\to} T(X)$ exists if and only if $X \cong X^g$. It follows that in this case $K(X,B)$ is the fixed field of the group

$$\{g \in G_K | \exists \text{a Hodge isometry}\, h\colon T(X^g) \to T(X) \,\text{such that}\, (g^* \circ h^*)|_B = \mathrm{id}_B\}.$$

For a K3 surface over $\overline{\mathbb{Q}}$, we have $\mathrm{Aut}(X) = \mathrm{Aut}(X_{\mathbb{C}})$, since $Aut_{X/\overline{\mathbb{Q}}}$ is a discrete group scheme. Hence, the set of conjugacy classes of fixed point free involutions $\mathcal{E}nr(X) \subset \mathrm{Aut}(X)$ coincides with $\mathcal{E}nr(X_{\mathbb{C}})$.

PROPOSITION 4.3. *Let X be a K3 surface over $\overline{\mathbb{Q}}$ such that $\mathrm{Aut}_{\mathrm{Hdg}}(T(X)) = \{\pm 1\}$. The Galois group $G_{K(X)}$ acts naturally on $\mathcal{E}nr(X)$ and on $\mathrm{Br}(X)[2]$ so that the map $\varphi_X\colon \mathcal{E}nr(X) \to \mathrm{Br}(X)[2]$ is $G_{K(X)}$-equivariant.*

Proof. Write $K := K(X)$. We use σ and τ to denote arbitrary elements of G_K. By Definition 4.1(i), we can find an isomorphism $f_{\sigma,\tau}\colon X^\sigma \tilde{\longrightarrow} X^\tau$.

Let us denote the conjugacy class of $\psi \in \mathrm{Aut}(X)$ by $[\psi]$.

A fixed point free involution $\iota\colon X \to X$ gives rise to a fixed point free involution $\iota^\sigma = \sigma\iota\sigma^{-1}\colon X^\sigma \to X^\sigma$, and one has $(\iota^\sigma)^\tau = \iota^{\tau\sigma}$. We define an action of G_K on $\mathcal{E}nr(X)$ by making σ send $[\iota]$ to $[f_{1,\sigma}^{-1}\iota^\sigma f_{1,\sigma}]$. This class depends neither on the choice of ι in its conjugacy class, nor on the choice of $f_{1,\sigma}$. We have

$$[f_{1,\tau}^{-1}(f_{1,\sigma}^{-1}\iota^\sigma f_{1,\sigma})^\tau f_{1,\tau}] = [(f_{1,\sigma}^\tau f_{1,\tau})^{-1}\iota^{\tau\sigma}(f_{1,\sigma}^\tau f_{1,\tau})] = [f_{1,\tau\sigma}^{-1}\iota^{\tau\sigma} f_{1,\tau\sigma}],$$

because $f_{1,\tau\sigma}$ and $f_{1,\sigma}^{\tau} f_{1,\tau}$ are both isomorphisms $X \xrightarrow{\sim} X^{\tau\sigma}$, so replacing one of them by the other does not change the conjugacy class.

Let us now define an action of G_K on $\mathrm{Br}(X)[2]$ by making $\sigma \in G_K$ act as $f_{1,\sigma}^*(\sigma^{-1})^*$ which is induced by $\sigma^{-1} f_{1,\sigma}\colon X \to X^\sigma \to X$. This action on $\mathrm{Br}(X)[2]$ does not depend on the choice of $f_{1,\sigma}$. Indeed, $f_{1,\sigma}$ is well defined up to an automorphism of X, but the action of $\mathrm{Aut}(X)$ on $\mathrm{Br}(X)[2]$ factors through the action of $\mathrm{Aut}_{\mathrm{Hdg}}(T(X))$. The latter group is $\{\pm 1\}$ by assumption, so $\mathrm{Aut}(X)$ acts on $\mathrm{Br}(X)[2]$ trivially. The map $(f_{1,\sigma})^\tau = \tau f_{1,\sigma} \tau^{-1}$ is an isomorphism $X^\tau \xrightarrow{\sim} X^{\tau\sigma}$, hence $(f_{1,\sigma})^\tau f_{1,\tau}$ is an isomorphism $X \to X^{\tau\sigma}$, so for the purpose of calculating the induced action of $\mathrm{Br}(X)[2]$, we can replace it with $f_{1,\tau\sigma}$. This shows that sending $\sigma \in G_K$ to the map induced on $\mathrm{Br}(X)[2]$ by $\sigma^{-1} f_{1,\sigma}$ is indeed an action.

We have a commutative diagram

$$\begin{array}{ccccc} X & \xrightarrow{f_{1,\sigma}} & X^\sigma & \xrightarrow{\sigma^{-1}} & X \\ \downarrow & & \downarrow & & \downarrow \\ X/(f_{1,\sigma}^{-1}\iota^\sigma f_{1,\sigma}) & \longrightarrow & X^\sigma/\iota^\sigma & \longrightarrow & X/\iota \end{array},$$

where the vertical maps are quotients by the respective fixed point free involutions. Thus the image of the nonzero element of $\mathrm{Br}(X/\iota)$ in $\mathrm{Br}(X)[2]$ followed by the action of σ on $\mathrm{Br}(X)[2]$ is the same as the image of the nonzero element of $\mathrm{Br}(X/(f_{1,\sigma}^{-1}\iota^\sigma f_{1,\sigma}))$ in $\mathrm{Br}(X)[2]$. This proves that φ_X is G_K-equivariant. ∎

Moduli fields of singular K3 surfaces

Let X be a singular K3 surface, that is, a K3 surface of maximal Picard rank 20. It is well known that every singular K3 surface is defined over $\overline{\mathbb{Q}}$ and has complex multiplication by the imaginary quadratic field $E = \mathrm{End}_{\mathrm{Hdg}}(T(X)_{\mathbb{Q}})$. Assume that $\mathrm{End}_{\mathrm{Hdg}}(T(X))$ is the ring of integers $\mathcal{O}_E \subset E$. In this situation, the results of [V] give explicit descriptions of the moduli fields $E(X)$ and $E(X, \mathrm{Br}(X)[n])$ which we now recall.

The group $\mathrm{Br}(X) \cong \mathrm{Hom}(T(X), \mathbb{Q}/\mathbb{Z})$ is naturally an $\mathcal{O}_E$-module. Let K_n/E be the ray class field of E with modulus $n\mathcal{O}_E$, and let $\mathrm{Cl}_n(E) \cong \mathrm{Gal}(K_n/E)$. The complex conjugation c acts on $\mathrm{Cl}_n(E)$. Let $\mathrm{Cl}_n(E)^c$ be the c-invariant subgroup of $\mathrm{Cl}_n(E)$. Define $\widetilde{K}_n \subset K_n$ as the fixed field of $\mathrm{Cl}_n(E)^c$, so that $\mathrm{Gal}(\widetilde{K}_n/E) \cong \mathrm{Cl}_n(E)/\mathrm{Cl}_n(E)^c$. Note that K_1 is the Hilbert class field of E and $\mathrm{Cl}_1(E) = \mathrm{Cl}(E)$ is the usual class group. The complex conjugation c acts on $\mathrm{Cl}(E)$ as -1.

THEOREM 4.4. *Let X be a singular K3 surface. Then $\widetilde{K}_n = E(X, \mathrm{Br}(X)[n])$.*

Proof. See [V, Th. 11.2 and Rem. 9.2 on p. 41]. ∎

In particular, we have $\widetilde{K}_1 = E(X)$. If n divides m, then $\widetilde{K}_n \subset \widetilde{K}_m$.

Proof of Theorem C

The assumptions of Theorem C imply that $\mathrm{Aut}_{\mathrm{Hdg}}(T(X)) = \mathcal{O}_E^\times = \{\pm 1\}$, so we can apply Proposition 4.3. Let ρ be the representation of $G_{\widetilde{K}_1}$ in $\mathrm{Br}(X)[2] \cong (\mathbb{Z}/2)^2$ constructed in the proof of Proposition 4.3. It is enough to show that under our assumptions one has $|\rho(G_{\widetilde{K}_1})| = 3$. Then $G_{\widetilde{K}_1}$ acts transitively on $\mathrm{Br}(X)[2] \setminus \{0\}$, so in view of the $G_{\widetilde{K}_1}$-equivariance established in Proposition 4.3 this will imply Theorem C. By Theorem 4.4, we need to prove that $[\widetilde{K}_2 : \widetilde{K}_1] = 3$.

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The following exact sequence describes the ray class group $\mathrm{Cl}_2(E)$:

$$0 \to \frac{\mathcal{O}_E^\times}{\{x \in \mathcal{O}_E^\times | x \equiv 1 \bmod 2\}} \to (\mathcal{O}_E/2)^\times \to \mathrm{Cl}_2(E) \to \mathrm{Cl}(E) \to 0.$$

Under our assumptions, we have $\mathcal{O}_E^\times = \{x \in \mathcal{O}_E^\times | x \equiv 1 \bmod 2\} = \{\pm 1\}$. Since 2 is inert in E, we have $\mathcal{O}_E/2 \cong \mathbb{F}_4$, and thus the sequence above becomes

$$0 \to \mathbb{F}_4^\times \to \mathrm{Cl}_2(E) \to \mathrm{Cl}(E) \to 0.$$

This is a sequence of G-modules, where $G = \{1, c\}$. We have $(\mathbb{F}_4^\times)^c = \{1\}$ and $\mathrm{H}^1(G, \mathbb{F}_4^\times) = 0$, and hence $\mathrm{Cl}_2(E)^c = \mathrm{Cl}(E)^c$. From this, we obtain the exact sequence

$$0 \to \mathbb{F}_4^\times \to \mathrm{Gal}(\widetilde{K}_2/E) \to \mathrm{Gal}(\widetilde{K}_1/E) \to 0.$$

Thus, $[\widetilde{K}_2 : \widetilde{K}_1] = 3$, as required. ∎

REMARK 4.5. When 2 is split, a similar argument shows that the $G_{\widetilde{K}_1}$-action on $\mathrm{Br}(X)[2]$ is trivial.

§5. Constructing Enriques involutions

For a finite abelian group G, we write $\ell(G)$ for the minimal number of generators of G. For a prime p we denote by G_p the p-primary subgroup of G. Recall that for a lattice L we write $A_L = L^*/L$ for the discriminant group of L. When L is even, we denote by $q_L\colon A_L \to \mathbb{Q}/2\mathbb{Z}$ the finite quadratic form of L.

We need to recall fundamental results of Nikulin about the existence of lattices and their primitive embeddings.

Let $q\colon A \to \mathbb{Q}/2\mathbb{Z}$ be a finite quadratic form. The signature $\mathrm{sign}(q) \in \mathbb{Z}/8\mathbb{Z}$ of q is defined as $(t_+ - t_-) \bmod 8$, where (t_+, t_-) is the signature of any even lattice whose discriminant form is isomorphic to (A, q) (such a lattice always exists and, moreover, this notion is well-defined). One also has

$$\mathrm{sign}(q \oplus q') = \mathrm{sign}(q) + \mathrm{sign}(q'). \tag{1}$$

Write $A = \bigoplus_p A_p$, where p ranges over the prime numbers. Then one has quadratic forms $q_p\colon A_p \to \mathbb{Q}_p/\mathbb{Z}_p$ when p is odd and $q_2\colon A_2 \to \mathbb{Q}_2/2\mathbb{Z}_2$ when $p = 2$. It is clear that q is the orthogonal direct sum of the forms q_p.

For an odd prime p, a finite abelian p-group A_p, and a quadratic form $q_p\colon A_p \to \mathbb{Q}_p/\mathbb{Z}_p$, Nikulin [N, Th. 1.9.1] showed that there is a unique $\mathbb{Z}_p$-lattice $K(q_p)$ of rank $\ell(A_p)$ whose quadratic form is isomorphic to q_p.

When $p = 2$, the same result of Nikulin says the following. Let $q_\theta^{(2)}(2)$ be the discriminant quadratic form of the rank one $\mathbb{Z}_2$-lattice (2θ), where $\theta \in \mathbb{Z}_2^\times$. For a finite abelian 2-group A_2 and a quadratic form $q_2\colon A_2 \to \mathbb{Q}_2/2\mathbb{Z}_2$ we have the following alternative. If q_2 splits as an orthogonal direct sum $q_2 = q_\theta^{(2)}(2) \oplus q_2'$, then there are precisely two even $\mathbb{Z}_2$-lattices of rank $\ell(A_2)$ whose quadratic form is isomorphic to q_2. If such a splitting of q_2 does not exist, there is a unique $\mathbb{Z}_2$-lattice $K(q_2)$ of rank $\ell(A_2)$ whose quadratic form is isomorphic to q_2. The following result is [N, Th. 1.10.1].

THEOREM 5.1 (Nikulin). *An even lattice with signature (t_+, t_-) and quadratic form $q\colon A \to \mathbb{Q}/2\mathbb{Z}$ exists if and only if the following conditions are satisfied:*

(1) $t_+ - t_- \equiv \operatorname{sign}(q) \bmod 8$;
(2) $t_+, t_- \geq 0$ *and* $t_+ + t_- \geq \ell(A)$;
(3) $(-1)^{t_-}|A_p| \equiv \operatorname{discr} K(q_p) \bmod \mathbb{Z}_p^{\times 2}$ *for the odd primes* p *such that* $t_+ + t_- = \ell(A_p)$;
(4) $|A_2| \equiv \pm \operatorname{discr} K(q_2) \bmod \mathbb{Z}_2^{\times 2}$ *if* $t_+ + t_- = \ell(A_2)$ *and* $q_2 \neq q_\theta^{(2)}(2) \oplus q_2'$ *for any* θ *and* q_2'.

The following result is a consequence of [N, Prop. 1.15.1] where we took into account that N is the unique lattice of signature $(2,10)$ whose quadratic form is isomorphic to q_N (see [N, Cor. 1.13.4].

THEOREM 5.2 (Nikulin). *Let* L *be an even lattice with signature* $(2_+, k_-)$ *and quadratic form* $q_L\colon A_L \to \mathbb{Q}/2\mathbb{Z}$. *The existence of a primitive embedding* $L \hookrightarrow N$ *is equivalent to the existence of the following data:*

- *subgroups* $H_L \subset A_L$ *and* $H_N \subset A_N$;
- *an isomorphism of finite quadratic forms* $\gamma\colon (H_L, q_L|_{H_L}) \xrightarrow{\sim} (H_N, q_N|_{H_N})$;
- *an even negative-definite lattice* K *of rank* $10-k$;
- *an isomorphism of finite quadratic forms* δ *from* $(A_K, -q_K)$ *to the restriction of* $q_L \oplus -q_N$ *to* $\Gamma_\gamma^\perp/\Gamma_\gamma$, *where the isotropic subgroup* $\Gamma_\gamma \subset A_L \oplus A_N$ *is the graph of* γ *in* $H_L \oplus H_N \subset A_L \oplus A_N$.

Moreover, if $i\colon L \hookrightarrow N$ *is a primitive embedding associated to* $(H_L, H_N, \gamma, K, \delta)$, *then* $K \cong i(L)^\perp$.

REMARK 5.3.

(1) If $f\colon \widetilde{K} \to K$ is an isomorphism of lattices and $\bar{f}\colon A_{\widetilde{K}} \to A_K$ is the induced isomorphism, then the primitive embeddings $L \hookrightarrow N$ associated to $(H_L, H_N, \gamma, K, \delta)$ and to $(H_L, H_N, \gamma, \widetilde{K}, \delta \circ \bar{f})$ are isomorphic.
(2) An analog of Theorem 5.2 gives the conditions for the existence of a primitive embedding of $L \otimes \mathbb{Z}_p$ into $N \otimes \mathbb{Z}_p$, for any prime p. The analog of (1) also holds in this context.

DEFINITION 5.4. Let L be a lattice such that $0 < \operatorname{rk}(L) \leq 10$. We say that a sublattice $L' \subset L$ of finite index satisfies condition $(*)$ if

$$\gcd(2\operatorname{discr}(L), [L\colon L']) = 1,$$

and for each prime p not dividing $2\operatorname{discr}(L)$, we have $\ell(A_{L',p}) < 12 - \operatorname{rk}(L')$.

PROPOSITION 5.5. *Any lattice* L *such that* $0 < \operatorname{rk}(L) \leq 10$ *contains infinitely many distinct sublattices* $L' \subset L$ *satisfying condition* $(*)$.

Proof. Let p be any odd prime not dividing $\operatorname{discr}(L)$. As is well known (see, e.g., [N, Cor. 1.9.3]), the unimodular p-adic lattice $L \otimes \mathbb{Z}_p$ has an orthogonal $\mathbb{Z}_p$-basis $v_1, \ldots, v_n$ such that $(v_i^2) \in \mathbb{Z}_p^\times$ for $i = 1, \ldots, n$. The images of $v_1, \ldots, v_n$ in $(L \otimes \mathbb{Z}_p)/p \cong L/p$ form a basis of this $\mathbb{F}_p$-vector space. Let $L' \subset L$ be the inverse image of the hyperplane spanned by the images of $v_2, \ldots, v_n$. Thus $[L : L'] = p$, so that $\operatorname{discr}(L') = p^2 \operatorname{discr}(L)$. Since p does not divide $\operatorname{discr}(L)$, we have a canonical isomorphism $A_{L'} \cong A_L \oplus A_{L',p}$. It is enough to check that $\ell(A_{L',p}) = 1$, which says that $A_{L',p}$ is cyclic. It is clear that $A_{L',p} \cong A_{L' \otimes \mathbb{Z}_p}$, so it is enough to prove that $\operatorname{Hom}_{\mathbb{Z}_p}(L' \otimes \mathbb{Z}_p, \mathbb{Z}_p)/(L' \otimes \mathbb{Z}_p) \cong \mathbb{Z}/p^2$. The $\mathbb{Z}_p$-module $L' \otimes \mathbb{Z}_p$ is freely generated by $pv_1, v_2, \ldots, v_n$, hence the $\mathbb{Z}_p$-module $\operatorname{Hom}_{\mathbb{Z}_p}(L' \otimes \mathbb{Z}_p, \mathbb{Z}_p) \subset L' \otimes \mathbb{Q}_p$ is freely generated by $p^{-1}v_1, v_2, \ldots, v_n$, which implies the result. ∎

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Condition $(*)$ implies that $[L\colon L']$ is odd, and hence the inclusion $L' \subset L$ induces a natural isomorphism

$$\mathrm{Hom}(L', \mathbb{Z}/2\mathbb{Z}) \cong \mathrm{Hom}(L, \mathbb{Z}/2\mathbb{Z}). \tag{2}$$

Recall that for a primitive embedding $i\colon L \hookrightarrow N$ we denote by $i^*(\varepsilon)$ the precomposition of the character $\varepsilon\colon N \to \mathbb{Z}/2$ with i.

THEOREM 5.6. *Let $L' \subset L$ be an inclusion of even lattices of signature $(2_+, k_-)$, where $0 \le k \le 8$. Then we have the following statements.*

(a) *If $L' \subset L$ satisfies condition $(*)$, then for any primitive embedding $i\colon L \hookrightarrow N$ with $i^*(\varepsilon) \neq 0$ there exists a primitive embedding $i'\colon L' \hookrightarrow N$ such that $i'^*(\varepsilon) = i^*(\varepsilon)$ under the identification (2).*
(b) *If $[L\colon L']$ is odd, then for any primitive embedding $i'\colon L' \hookrightarrow N$ with $i'^*(\varepsilon) \neq 0$ there exists a primitive embedding $i\colon L \hookrightarrow N$ such that $i'^*(\varepsilon) = i^*(\varepsilon)$ under the identification (2).*

Proof. (a) Let $i\colon L \hookrightarrow N$ be a primitive embedding such that $i^*(\varepsilon) \neq 0$. Then $K := i(L)^{\perp}_N$ is an even negative-definite lattice of rank $10-k$. By Theorem 5.2, the embedding i corresponds to some datum $(H_L, H_N, \gamma, K, \delta)$.

Since $L' \subset L$ satisfies condition $(*)$, the index $[L : L']$ is coprime to $|A_L|$, hence $A_{L'}$ canonically isomorphic to $A_L \oplus A_{\mathrm{new}}$, where $|A_{\mathrm{new}}| = [L : L']^2$. Then $q_{L'}$ is an orthogonal direct sum $q_{L'} \cong q_L \oplus q_{\mathrm{new}}$, where q_{new} is a quadratic form on A_{new}.

We claim that there is a negative-definite lattice K' of rank $10-k$ such that $A_{K'} \cong A_K \oplus A_{\mathrm{new}}$ and $q_{K'} \cong q_K \oplus -q_{\mathrm{new}}$. Since L' is a sublattice of L of finite index and $\mathrm{rk}(K) = 10-k$, we have

$$\mathrm{sign}(q_L) \equiv \mathrm{sign}(q_{L'}) \bmod 8, \qquad k-10 \equiv \mathrm{sign}(q_K) \bmod 8.$$

Since $q_{L'} \cong q_L \oplus q_{\mathrm{new}}$, we have that $\mathrm{sign}(q_{L'}) = \mathrm{sign}(q_L) + \mathrm{sign}(q_{\mathrm{new}})$ by (1). Thus $\mathrm{sign}(q_{\mathrm{new}}) \equiv 0 \bmod 8$, which implies property (1) of Theorem 5.1.

By condition $(*)$, we know that $|A_{\mathrm{new}}|$ is odd and coprime to $|A_L|$. For any odd prime p, the $\mathbb{Z}_p$-lattices $L \otimes \mathbb{Z}_p$ and $K \otimes \mathbb{Z}_p$ are orthogonal complements of each other in the unimodular $\mathbb{Z}_p$-lattice $N \otimes \mathbb{Z}_p$, hence $|A_{L,p}| = |A_{K,p}|$. Thus, $|A_K|$ and $|A_{\mathrm{new}}|$ are coprime. This implies

$$\ell(A_K \oplus A_{\mathrm{new}}) = \max\{\ell(A_K), \ell(A_{\mathrm{new}})\} \le 10-k,$$

since $\ell(A_K) \le \mathrm{rk}(K) = 10-k$ and $\ell(A_{\mathrm{new}}) \le \ell(A_{L'}) < 12 - \mathrm{rk}(L)$ by condition $(*)$. Thus, property (2) of Theorem 5.1 also holds.

We now check properties (3) and (4) taking into account the coprimality of $|A_K|$ and $|A_{\mathrm{new}}|$. If p divides $|A_K|$, then (3) and (4) hold because they hold for A_K. If p divides $|A_{\mathrm{new}}|$, then $\ell(A_{\mathrm{new}}) < \mathrm{rk}(K')$ by condition $(*)$, so there is nothing to check.

Theorem 5.1 now implies the existence of K' with required properties.

Let us construct a datum defining the desired primitive embedding $L' \hookrightarrow N$. Since $2A_N = 0$, we have $2H_N = 0$ and thus $2H_L = 0$, so that $H_L \subset A_{L,2}$. In view of the canonical isomorphism $A_{L,2} \cong A_{L',2}$, we can keep the same $H_{L'} = H_L$, H_N and $\gamma' = \gamma$ as the first three entries of our datum.

Recall that $A_{L'} \cong A_L \oplus A_{\mathrm{new}}$. We have

$$\Gamma_{\gamma'} = \Gamma_\gamma \oplus 0 \subset \Gamma_{\gamma'}^{\perp} = \Gamma_\gamma^{\perp} \oplus A_{\mathrm{new}} \subset (A_L \oplus A_N) \oplus A_{\mathrm{new}},$$

hence $\Gamma_{\gamma'}^{\perp}/\Gamma_{\gamma'} = \Gamma_{\gamma}^{\perp}/\Gamma_{\gamma} \oplus A_{\text{new}} \cong A_K \oplus A_{\text{new}}$. The restriction of

$$q_{L'} \oplus -q_N \cong (q_L \oplus -q_N) \oplus q_{\text{new}}$$

to $\Gamma_{\gamma'}^{\perp}/\Gamma_{\gamma'}$ is isomorphic to $-q_K \oplus q_{\text{new}}$ via the isomorphism $\delta' := (\delta, \text{id})$.

Take a negative-definite lattice K' of rank $10-k$ as above, that is, such that $A_{K'} \cong A_K \oplus A_{\text{new}}$ and $q_{K'} \cong q_K \oplus -q_{\text{new}}$. Let $i' \colon L' \hookrightarrow N$ be a primitive embedding associated to the datum $(H_{L'}, H_N, \gamma', K', \delta')$.

To prove that $i'^*(\varepsilon) = i(\varepsilon)$ under the natural identification (2), it is enough to show that the induced embeddings of $\mathbb{Z}_2$-lattices $i_2 \colon L \otimes \mathbb{Z}_2 \hookrightarrow N \otimes \mathbb{Z}_2$ and $i_2' \colon L' \otimes \mathbb{Z}_2 \hookrightarrow N \otimes \mathbb{Z}_2$ are isomorphic.

First, we claim that $K \otimes \mathbb{Z}_2$ and $K' \otimes \mathbb{Z}_2$ are isomorphic $\mathbb{Z}_2$-lattices. Since K and K' are negative-definite of the same rank, and $|A_{K'}| = |A_K| \cdot |A_{\text{new}}|$, we have $\text{discr}(K') = \text{discr}(K) \cdot |A_{\text{new}}|$. Since $|A_{\text{new}}|$ is a square of an odd integer, the even 2-adic lattices $K \otimes \mathbb{Z}_2$ and $K' \otimes \mathbb{Z}_2$ have the same rank, the same discriminant form, and the same discriminant modulo $\mathbb{Z}_2^{\times 2}$. This implies that the $\mathbb{Z}_2$-lattices $K \otimes \mathbb{Z}_2$ and $K' \otimes \mathbb{Z}_2$ are isomorphic (see [Nik79, Cor. 1.9.3]).

It remains to show that after tensoring with $\mathbb{Z}_2$ the data $(H_L, H_N, \gamma, K, \delta)$ and $(H_{L'}, H_N, \gamma', K', \delta')$ give rise to isomorphic embeddings of $L' \otimes \mathbb{Z}_2 \cong L \otimes \mathbb{Z}_2$ into $N \otimes \mathbb{Z}_2$. The first three entries of each datum are the same. By Remark 5.3, it is enough to find an isomorphism of $\mathbb{Z}_2$-lattices $f \colon K' \otimes \mathbb{Z}_2 \to K \otimes \mathbb{Z}_2$ such that $\delta_2' = \delta_2 \circ \bar{f}$. The existence of such an f follows from [N, Th. 1.9.5]. This concludes the proof of (a).

(b) Write $A := A_L = A_2 \oplus A_{\text{odd}}$, where A_2 is the 2-primary subgroup of A. Similarly, write $A' := A_{L'} = A_2' \oplus A_{\text{odd}}'$. It is clear that $A_2 \cong A_2'$. Then $q_{L'}$ is an orthogonal direct sum of quadratic forms $q_{L,2}$ on A_2 and q_{odd} on A_{odd}'.

The overlattice L of L' defines an isotropic subgroup $I \subset A'$, where $|I| = [L : L']$, so that q_L is the quadratic form induced by $q_{L'}$ on $A = I^{\perp}/I$. Since $[L \colon L']$ is odd by assumption, we have $I \subset A_{\text{odd}}'$. Thus $I^{\perp} = A_2 \oplus I_{\text{odd}}^{\perp}$, where $I_{\text{odd}}^{\perp} = I^{\perp} \cap A_{\text{odd}}'$. This shows that $A = A_2 \oplus (I_{\text{odd}}^{\perp}/I)$.

Let $i' \colon L' \hookrightarrow N$ be a primitive embedding such that $i'^*(\varepsilon) \neq 0$. Then $K' := i(L')_N^{\perp}$ is an even negative-definite lattice of rank $10-k$. Let $(H_{L'}, H_N, \gamma', K', \delta')$ be a datum associated to $i' \colon L' \hookrightarrow N$ as in Theorem 5.2. In particular, δ' is an isomorphism of $-q_{K'}$ with the restriction of $q_{L'} \oplus -q_N$ to $\Gamma_{\gamma'}^{\perp}/\Gamma_{\gamma'}$. Since $2A_N = 0$, we have $2H_{L'} = 0$, so that $H_{L'} \subset A_2' = A_2$. Hence $\Gamma_{\gamma'} \subset A_2 \oplus A_N \subset A' \oplus A_N$ and thus $\Gamma_{\gamma'}^{\perp} = (\Gamma_{\gamma'}^{\perp})_2 \oplus A_{\text{odd}}'$, where $(\Gamma_{\gamma'}^{\perp})_2 = \Gamma_{\gamma'}^{\perp} \cap (A_2 \oplus A_N)$. This shows that δ' identifies the finite quadratic form $-q_{K'}$ on $A_{K'}$ with the restriction of $(q_{L,2} \oplus -q_N) \oplus q_{\text{odd}}$ to $\big((\Gamma_{\gamma'}^{\perp})_2/\Gamma_{\gamma'}\big) \oplus A_{\text{odd}}'$.

The isotropic subgroup $I \subset A_{\text{odd}}'$ gives rise, via δ', to an isotropic subgroup in $A_{K'}$. The latter defines an overlattice $K' \subset K$ with $[K : K'] = [L : L']$, so that δ' induces an isomorphism δ of the quadratic form $-q_K$ on A_K with the restriction of $(q_{L,2} \oplus -q_N) \oplus q_{\text{odd}}$ to $\big((\Gamma_{\gamma'}^{\perp})_2/\Gamma_{\gamma'}\big) \oplus (I_{\text{odd}}^{\perp}/I)$. Let $i \colon L \hookrightarrow N$ be a primitive embedding associated to the datum $(H_L, H_N, \gamma, K, \delta)$, where $H_L = H_{L'}$ and $\gamma = \gamma'$.

To complete the proof of (b), it remains to show that i and i' induce isomorphic embeddings of $\mathbb{Z}_2$-lattices. This is proved by the same arguments as in (a). ∎

Corollary 5.7. *Let L be an even lattice of signature $(2_+, k_-)$, where $0 \leq k \leq 8$. Write $\mathcal{S}(L)$ for the set of nonzero homomorphisms $\alpha \colon L \to \mathbb{Z}/2$ such that there is a*

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primitive embedding $i\colon L \hookrightarrow N$ *with* $\alpha = i^*(\varepsilon)$*. Let* L' *be a sublattice of* L *that satisfies condition* $(*)$*. Then, under the natural identification* $\mathrm{Hom}(L,\mathbb{Z}/2) \cong \mathrm{Hom}(L',\mathbb{Z}/2)$*, we have* $\mathcal{S}(L) = \mathcal{S}(L')$*.*

Proof. Part (a) of Theorem 5.6 implies $\mathcal{S}(L) \subset \mathcal{S}(L')$, whereas part (b) implies $\mathcal{S}(L') \subset \mathcal{S}(L)$ since $[L : L']$ is odd. ∎

Proof of Theorem D

By Proposition 5.5, there are infinitely many sublattices $T \subset T(X)$ with pairwise different discriminants that satisfy condition $(*)$. Endow T with the Hodge structure coming from $T(X)$. Since $\mathrm{rk}(T) \leq 10$, by [N, Th. 1.14.4], there exists a unique primitive embedding of the lattice T into the K3 lattice Λ. We equip Λ with the Hodge structure induced by the Hodge structure on T so that $T_{\Lambda}^{\perp} \subset \Lambda^{(1,1)}$. By the surjectivity of the period map, there is a K3 surface Y together with a Hodge isometry between Λ and $\mathrm{H}^2(Y,\mathbb{Z})$. The transcendental lattice $T(Y)$ is the orthogonal complement to $\mathrm{H}^2(Y,\mathbb{Z})^{(1,1)}$, hence $T(Y) \cong T$.

Applying Corollary 5.7 with $L = T(X)$, we obtain $\mathcal{S}(T(X)) = \mathcal{S}(T(Y))$. Now Corollary 2.6(i) (whose proof uses Lemma 2.3) gives $\mathrm{Im}(\Phi_X) \setminus \{0\} = \mathrm{Im}(\Phi_Y) \setminus \{0\}$. ∎

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