COMPOSITIO MATHEMATICA

On the family of affine threefolds $x^m y = F(x, z, t)$

Neena Gupta

Compositio Math. **150** (2014), 979–998.

doi:10.1112/S0010437X13007793

FOUNDATION COMPOSITIO MATHEMATICA

LONDON MATHEMATICAL SOCIETY

Compositio Math. 150 (2014) 979–998
doi:10.1112/S0010437X13007793

On the family of affine threefolds $x^m y = F(x, z, t)$

Neena Gupta

Abstract

Let k be a field and $\mathbb{V}$ the affine threefold in $\mathbb{A}^4_k$ defined by $x^m y = F(x, z, t)$, $m \geqslant 2$. In this paper, we show that $\mathbb{V} \cong \mathbb{A}^3_k$ if and only if $f(z, t) := F(0, z, t)$ is a coordinate of $k[z, t]$. In particular, when k is a field of positive characteristic and f defines a non-trivial line in the affine plane $\mathbb{A}^2_k$ (we shall call such a $\mathbb{V}$ as an Asanuma threefold), then $\mathbb{V} \ncong \mathbb{A}^3_k$ although $\mathbb{V} \times \mathbb{A}^1_k \cong \mathbb{A}^4_k$, thereby providing a family of counter-examples to Zariski's cancellation conjecture for the affine 3-space in positive characteristic. Our main result also proves a special case of the embedding conjecture of Abhyankar–Sathaye in arbitrary characteristic.

1. Introduction

Let $S^{[n]}$ denote a polynomial ring in n variables over a ring S. Let k be a field of characteristic $p(>0)$ and $R = k[X, Y, Z, T]/(X^m Y + Z^{p^e} + T + T^{sp})$, where m, e, s are positive integers such that $p^e \nmid sp$ and $sp \nmid p^e$. In [Asa87], Asanuma constructed the above threefold and proved the following properties:

(i) $R^{[1]} \cong_{k[x]} k[x]^{[3]} \cong_k k^{[4]}$, where x is the image of X in R;

(ii) $R \ncong_{k[x]} k[x]^{[2]}$; and

(iii) $R \otimes_{k[x]} k(p) \cong k(p)^{[2]}\ \forall p \in \operatorname{Spec}(k[x])$.

Originally Asanuma constructed this example as an illustration of a non-trivial $\mathbb{A}^2$-fibration (defined in § 2) over a PID. The example showed that the theorem of Sathaye [Sat83] establishing that any $\mathbb{A}^2$-fibration over a PID S containing $\mathbb{Q}$ (the field of rational numbers) must be isomorphic to $S^{[2]}$ does not extend to the case $\mathbb{Q} \nsubseteq S$.

The example was soon to acquire a wider significance. In a subsequent paper [Asa94, Theorem 2.2], Asanuma used the above example to construct non-linearizable algebraic torus actions on $\mathbb{A}^n_k$ over any infinite field k of positive characteristic when $n \geqslant 4$. He then asked the following question:

Question. Is $R \cong_k k^{[3]}$?

Recall that Zariski's cancellation problem asks: if $\mathbb{V}$ is an affine variety over a field k such that $\mathbb{V} \times \mathbb{A}^1_k \cong \mathbb{A}^{n+1}_k$, does it follow that $\mathbb{V} \cong \mathbb{A}^n_k$? It was known that, for $n \leqslant 2$, Zariski's cancellation problem has an affirmative answer over any field k. This was shown by Abhyankar, Eakin and Heinzer [AEH72] for the case $n = 1$ (quoted in § 2 as Corollary 2.5), Fujita [Fuj79] and Miyanishi

Received 3 June 2013, accepted in final form 26 September 2013, published online 9 June 2014.
2010 Mathematics Subject Classification 14R10 (primary), 14R25, 13B25, 13A50 (secondary).
Keywords: polynomial algebra, graded ring, $\mathbb{G}_a$-action, Derksen invariant, Makar-Limanov invariant, cancellation problem, embedding problem, $\mathbb{A}^2$-fibration, localization theorem of K-theory.
The author is supported by the DST INSPIRE Faculty Award.

and Sugie [MS80] for the case $n = 2$ and ch $k = 0$, and Russell [Rus81] for the case $n = 2$ and ch $k > 0$.

Subsequently, Asanuma observed [Asa94, Remark 2.3] that if the above ring R is not isomorphic to $k^{[3]}$ then Zariski's cancellation problem for the affine space $\mathbb{A}^3_k$ has a negative solution for positive characteristic; and if R is isomorphic to $k^{[3]}$ then the linearisation problem for the affine space $\mathbb{A}^3_k$ has a negative solution for positive characteristic. This dichotomy has been popularized by Russell as *Asanuma's dilemma* (cf. [FR05, Problem 2, p. 9]). Recently, the author showed in [Gup14] that Asanuma's threefold R is not isomorphic to $k^{[3]}$ when $m \geqslant 2$, thus showing that Zariski's cancellation problem does not hold for the affine 3-space $\mathbb{A}^3_k$ in positive characteristic.

Now recall that over any field k of positive characteristic p, there exist non-trivial lines in $k^{[2]}$, i.e., there exists $f(Z,T) \in k[Z,T]$ satisfying $k[Z,T]/(f) \cong k^{[1]}$ but $k[Z,T] \neq k[f]^{[1]}$. Examples of non-trivial lines have been given by Segre [Seg56] and Nagata [Nag72]. An example of a Segre–Nagata non-trivial line is the polynomial $f(Z,T) = Z^{p^e} + T + T^{sp}$, where $p^e \nmid sp$ and $sp \nmid p^e$. A recent article of Ganong [Gan11] gives a nice overview on such non-trivial lines (called 'exotic lines' by Ganong).

Thus Asanuma's threefold is of the form $R = k[X,Y,Z,T]/(X^mY + f(Z,T))$, where f is the above-mentioned Segre–Nagata non-trivial line. After receiving a preprint of the paper [Gup14], Russell asked the author to consider the more general class of rings $A = k[X,Y,Z,T]/(X^mY - f)$, where $f \in k[Z,T]$ is *any* non-trivial line, and examine whether such rings are necessarily non-trivial (i.e., not isomorphic to $k^{[3]}$).

This paper, following on from [Gup14], answers Russell's question in the affirmative for $m \geqslant 2$. In fact, in § 3, we establish the following result which is independent of the characteristic of the field k (Theorem 3.11).

Theorem A. *Let k be a field of any characteristic and A an integral domain defined by*

$$A = k[X,Y,Z,T]/(X^mY - F(X,Z,T)) \quad \text{where } m > 1.$$

Set $f(Z,T) := F(0,Z,T)$ and $G := X^mY - F(X,Z,T)$. Then the following statements are equivalent.

(i) *$f(Z,T)$ is a variable in $k[Z,T]$.*

(ii) *$A \cong_{k[x]} k[x]^{[2]}$, where x denotes the image of X in A.*

(iii) *$A \cong_k k^{[3]}$.*

(iv) *G is a variable in $k[X,Y,Z,T]$.*

(v) *G is a variable in $k[X,Y,Z,T]$ along with X.*

Over a field k of positive characteristic, the implication (iii) $\Rightarrow$ (i) will answer Russell's question (Theorem 4.3). Criterion (i) also provides a general framework for understanding the non-triviality of the Russell–Koras threefold defined by $x^2y + x + z^2 + t^3 = 0$ over a field of any characteristic. The non-triviality of the Russell–Koras threefold was first proved by Makar-Limanov [Mak96] over a field of characteristic zero, and extended to arbitrary fields by Crachiola [Cra05]. When $k = \mathbb{C}$ (the field of complex numbers), the implications (iii) $\Rightarrow$ (i) and (iii) $\Longrightarrow$ (v) also follow from results of Kaliman [Kal02] and Kaliman *et al.* [KVZ04], respectively. Our proof is purely algebraic and valid for any characteristic. The implication (iii) $\Rightarrow$ (iv) establishes a special case of the Abhyankar–Sathaye embedding conjecture in arbitrary characteristic (see Remark 3.12). We mention here that interest in the rings defined by $x^my = F(x,z,t)$ goes back to the 1970s (cf. [VD74]).

It can be seen that the ring $A = k[X, Y, Z, T]/(X^mY - F(X, Z, T))$, where $m \geqslant 2$, is an $\mathbb{A}^2$-fibration over $k[x]$ if and only if $f(Z, T) := F(0, Z, T)$ is a line in $k[Z, T]$ (cf. Lemma 3.2). If $f(Z, T)$ is a trivial line (i.e., a variable) then, by Theorem A, $A \cong_k k^{[3]}$. In § 4, we examine a few properties of the ring A when $f(Z, T)$ is a non-trivial line in $k[Z, T]$. We shall see that $A^{[1]} \cong_k k^{[4]}$ although $A \ncong_k k^{[3]}$ (Theorems 4.2 and 4.3). We then compute the Derksen invariant and the Makar-Limanov invariant of the ring A (Lemma 4.6) and give a characterization of $\mathrm{Aut}_k(A)$ (Propositions 4.9 and 4.10). Finally, we show (Theorem 4.11) that if $n \neq m$, or if g and f are inequivalent non-trivial lines in $k[Z, T]$ (i.e., if there does not exist any $\theta \in \mathrm{Aut}_k(k[Z, T])$ such that $\theta(f) = g$), then $k[X, Y, Z, T]/(X^mY - f(Z, T)) \ncong k[X, Y, Z, T]/(X^nY - g(Z, T))$, although the two rings are stably isomorphic. Thus we have an infinite family of non-isomorphic rings which are counter-examples to Zariski's cancellation conjecture, each of them being a non-trivial $\mathbb{A}^2$-fibration over $k^{[1]}$.

2. Preliminaries

Throughout the paper, all rings will be assumed to be commutative.

Notation. Recall that $S^{[n]}$ denotes a polynomial ring in n variables over a ring S. Thus, for a subring S of a ring B, the notation $B = S^{[n]}$ will mean that $B = S[T_1, T_2, \ldots, T_n]$, where $T_1, \ldots, T_n$ are algebraically independent over S.

For a ring S, the notation S^* will denote the group of units of S.

For a prime ideal p of S, $k(p)$ denotes the field S_p/pS_p.

Definition. A subring S of an integral domain B is said to be *factorially closed* in B if for any non-zero $a, b \in B$, the condition $ab \in S$ implies both $a \in S$ and $b \in S$. A factorially closed subring of B is also known as an inert subring of B.

It is easy to see that if S is factorially closed in B and B is a UFD then S is also a UFD.

Definition. Let k be a field and A be a k-algebra. A is said to be *geometrically factorial* over k if $A \otimes_k \tilde{k}$ is a UFD for any algebraic extension $\tilde{k}$ of k.

We now state two applications of Russell–Sathaye criteria for a ring to be a polynomial algebra in one variable over a subring.

Theorem 2.1. *Let k be a field and $F \in k[X, Y]$ be such that $k[X, Y] \otimes_{k[F]} k(F) = k(F)^{[1]}$. Then $k[X, Y] = k[F]^{[1]}$.*

Proof. By [RS79, Theorem 2.4.2], it is enough to show that $k[X, Y] \cap k(F) = k[F]$. Set $D := k[X, Y] \cap k(F)$. Now $k[F] \subseteq D \subseteq k(F)$. Since $D^* = k^*$, we have $D = k[F]$. □

The following version of the Russell–Sathaye criterion [RS79, Theorem 2.3.1] is presented in [BD94, Theorem 2.8].

Theorem 2.2. *Let $R \subset D$ be integral domains such that D is a finitely generated R-algebra. Suppose that there exists a prime element π in R such that π remains prime in D, $D[\pi^{-1}] = R[\pi^{-1}]^{[1]}$, $\pi D \cap R = \pi R$ and $R/\pi R$ is algebraically closed in $D/\pi D$. Then $D = R^{[1]}$.*

We deduce a consequence of Theorems 2.1 and 2.2 for later use.

Lemma 2.3. *Let k be a field and $F \in k[Z, T]$ be such that $k[F]$ is algebraically closed in $k[Z, T]$. Suppose that $k[Y, Z, T] \otimes_{k[Y, F]} k(Y, F) = k(Y, F)^{[1]}$ for an indeterminate Y over $k[Z, T]$. Then $k[Z, T] = k[F]^{[1]}$.*

Proof. Let $h \in k^{[2]}$ be such that $k[Y,Z,T,1/h(Y,F)] = k[Y,F,1/h(Y,F)]^{[1]}$. Then $h(Y,F) = Y^n h_1(Y,F)$ for some $n \geqslant 0$ and $h_1 \in k^{[2]}$ such that $h_1(0,F) \neq 0$. Set $R := k[Y,F,1/h_1(Y,F)]$ and $D := k[Y,Z,T,1/h_1(Y,F)]$. Then Y is a prime element of both R and D, $YD \cap R = YR$ and $D[Y^{-1}] = R[Y^{-1}]^{[1]}$. Since $k[F]$ is algebraically closed in $k[Z,T]$, it follows that $R/YR(= k[F,1/h_1(0,F)])$ is algebraically closed in $D/YD = k[Z,T,1/h_1(0,F)]$. Hence $D = R^{[1]}$ by Theorem 2.2. Thus, there exists $G \in k[Y,Z,T]$ such that $k[Y,Z,T,1/h_1(Y,F)] = k[Y,F,G, 1/h_1(Y,F)]$. Let $\alpha, \beta \in k^{[3]}$ be such that

$$h_1(Y,F)^r Z = \alpha(Y,F,G) \quad \text{and} \quad h_1(Y,F)^s T = \beta(Y,F,G)$$

for some $r, s \geqslant 0$. Then we have

$$h_1(0,F)^r Z = \alpha(0,F,G(0,Z,T)) \quad \text{and} \quad h_1(0,F)^s T = \beta(0,F,G(0,Z,T)).$$

Therefore, $k[Z,T] \otimes_{k[F]} k(F) = k(F)[G(0,Z,T)] = k(F)^{[1]}$. Hence $k[Z,T] = k[F]^{[1]}$ by Theorem 2.1. □

We now quote three well-known results from [AEH72, 2.6, 2.8, 4.8].

Theorem 2.4. *Let k be a field and A be a one-dimensional normal k-subalgebra of $k[X_1,\ldots,X_n]$. Then $A = k^{[1]}$.*

Corollary 2.5. *Let k be a field and A an affine k-algebra. Suppose that $A^{[m]} \cong_k k^{[m+1]}$. Then $A = k^{[1]}$.*

Theorem 2.6. *Let R be a UFD and D be an R-algebra such that $R \subset D \subset R[X_1,\ldots,X_n]$,* $\operatorname{tr.deg}_R D = 1$ *and D is factorially closed in $R[X_1,\ldots,X_n]$. Then $D = R^{[1]}$.*

We shall use the following definition of $\mathbb{A}^n$-fibration that was given by Sathaye in [Sat83].

Definition. A finitely generated flat S-algebra R is said to be an $\mathbb{A}^n$*-fibration* over S if $R \otimes_S k(\mathfrak{p}) = k(\mathfrak{p})^{[n]}$ for every prime ideal $\mathfrak{p}$ of S.

We shall also use the following term from affine algebraic geometry.

Definition. An element $f \in k[Z,T]$ is called a *line* if $k[Z,T]/(f) = k^{[1]}$. A line f is called a *non-trivial line* if $k[Z,T] \neq k[f]^{[1]}$.

We now define the key ingredients in our proof of Theorem A: the exponential map (a formulation of the concept of $\mathbb{G}_a$-action) and associated invariants.

Definition. Let A be a k-algebra and let $\phi : A \longrightarrow A^{[1]}$ be a k-algebra homomorphism. For an indeterminate U over A, let the notation ϕ_U denote the map $\phi : A \longrightarrow A[U]$. ϕ is said to be an *exponential map on A* if ϕ satisfies the following two properties.

(i) $\varepsilon_0 \phi_U$ is identity on A, where $\varepsilon_0 : A[U] \longrightarrow A$ is the evaluation at $U = 0$.
(ii) $\phi_V \phi_U = \phi_{V+U}$, where $\phi_V : A \longrightarrow A[V]$ is extended to a homomorphism $\phi_V : A[U] \longrightarrow A[V, U]$ by setting $\phi_V(U) = U$.

The ring of ϕ-invariants of an exponential map ϕ on A is a subring of A given by

$$A^{\phi} = \{a \in A \mid \phi(a) = a\}.$$

https://doi.org/10.1112/S0010437X13007793 Published online by Cambridge University Press

An exponential map ϕ is said to be *non-trivial* if $A^\phi \neq A$. For an affine domain A over a field k, let $\mathrm{EXP}(A)$ denote the set of all exponential maps on A. The *Derksen invariant* of A is a subring of A defined by

$$\mathrm{DK}(A) = k[f \mid f \in A^\phi, \phi \text{ a non-trivial exponential map}],$$

and the *Makar-Limanov invariant* (also known as *AK-invariant*) of A is a subring of A defined by

$$\mathrm{ML}(A) = \bigcap_{\phi \in \mathrm{EXP}(A)} A^\phi.$$

We recall below a crucial observation (cf. [Gup14, Lemma 2.4] and [Cra05, Example 2.1]).

Lemma 2.7. *Let k be a field and $A = k^{[n]}$, where $n > 1$. Then $\mathrm{DK}(A) = A$ and $\mathrm{ML}(A) = k$.*

We summarise below some useful properties of an exponential map ϕ (cf. [Cra05, pp. 1291–1292] and [Gup14, Lemma 2.1]).

Lemma 2.8. *Let A be an affine domain over a field k. Suppose that there exists a non-trivial exponential map ϕ on A. Then the following statements hold.*

(i) *A^ϕ is factorially closed in A.*
(ii) *A^ϕ is algebraically closed in A.*
(iii) $\mathrm{tr.deg}_k(A^\phi) = \mathrm{tr.deg}_k(A) - 1$.
(iv) *There exists $c \in A^\phi$ such that $A[c^{-1}] = A^\phi[c^{-1}]^{[1]}$.*
(v) *If $\mathrm{tr.deg}_k(A) = 1$ and $\tilde{k}$ is the algebraic closure of k in A, then $A = \tilde{k}^{[1]}$ and $A^\phi = \tilde{k}$.*
(vi) *Let S be a multiplicative subset of $A^\phi \backslash \{0\}$. Then ϕ extends to a non-trivial exponential map $S^{-1}\phi$ on $S^{-1}A$ by setting $(S^{-1}\phi)(a/s) = \phi(a)/s$ for $a \in A$ and $s \in S$. Moreover, the ring of invariants of $S^{-1}\phi$ is $S^{-1}(A^\phi)$.*

We shall also use the following result proved in [Gup14, Lemma 3.3].

Lemma 2.9. *Let B be an affine domain over an infinite field k. Let $f \in B$ be such that $f - \lambda$ is a prime element of B for infinitely many $\lambda \in k$. Let ϕ be a non-trivial exponential map on B such that $f \in B^\phi$. Then there exist infinitely many $\beta \in k$ such that each $f - \beta$ is a prime element of B and ϕ induces a non-trivial exponential map on $B/(f - \beta)$.*

Finally, we define the concept of an admissible proper $\mathbb{Z}$-filtration on an affine domain.

Definition. Let A be an affine domain over a field k. A collection of k-linear subspaces $\{A_n\}_{n \in \mathbb{Z}}$ of A is said to be a *proper $\mathbb{Z}$-filtration* if it satisfies the following conditions:

(i) $A_n \subseteq A_{n+1}$ for all $n \in \mathbb{Z}$;
(ii) $A = \bigcup_{n \in \mathbb{Z}} A_n$;
(iii) $\bigcap_{n \in \mathbb{Z}} A_n = (0)$; and
(iv) $(A_n \backslash A_{n-1}) \cdot (A_m \backslash A_{m-1}) \subseteq A_{n+m} \backslash A_{n+m-1}$ for all $n, m \in \mathbb{Z}$.

We shall call a proper $\mathbb{Z}$-filtration $\{A_n\}_{n \in \mathbb{Z}}$ of A *admissible* if there exists a finite generating set Γ of A such that, for any $n \in \mathbb{Z}$ and $a \in A_n$, a can be written as a finite sum of monomials in elements of Γ and each of these monomials is an element of A_n.

Any proper $\mathbb{Z}$-filtration on A determines the $\mathbb{Z}$-graded integral domain

$$\mathrm{gr}(A) := \bigoplus_i A_i / A_{i-1},$$

and a map

$$\rho : A \longrightarrow \mathrm{gr}(A) \quad \text{defined by } \rho(a) = a + A_{n-1} \quad \text{if } a \in A_n \backslash A_{n-1}.$$

An exponential map ϕ on a graded ring A is said to be *homogeneous* if $\phi : A \longrightarrow A[U]$ becomes homogeneous when $A[U]$ is given a grading induced from A such that U is a homogeneous element.

Remark 2.10. Note that if ϕ is a homogeneous exponential map on a graded ring A, then A^{ϕ} is a graded subring of A.

We state below a result on homogenization of exponential maps due to Derksen *et al.* [DHM01]; the following version of the result is presented in [Cra05, Theorem 2.6] (cf. [Gup14, Theorem 2.3]).

Theorem 2.11. *Let A be an affine domain over a field k with an admissible proper $\mathbb{Z}$-filtration and $\mathrm{gr}(A)$ the induced $\mathbb{Z}$-graded domain. Let ϕ be a non-trivial exponential map on A. Then ϕ induces a non-trivial homogeneous exponential map $\bar{\phi}$ on $\mathrm{gr}(A)$ such that $\rho(A^{\phi}) \subseteq \mathrm{gr}(A)^{\bar{\phi}}$.*

3. Main theorem

In this section we shall prove Theorem A. We first record two observations about the coordinate ring of the threefold $x^m y = F(x, z, t)$.

Lemma 3.1. *Let k be a field and A be an integral domain defined by*

$$A = k[X, Y, Z, T]/(X^m Y - F(X, Z, T)) \quad \textit{where } m \geqslant 1.$$

Let $f(Z, T) = F(0, Z, T)$ and x denote the image of X in A. Then the following statements are equivalent.

(i) *A is a UFD.*
(ii) *x is prime in A or x is a unit in A.*
(iii) *$f(Z, T)$ is irreducible in $k[Z, T]$ or $f(Z, T) \in k^*$.*

Proof. (i) $\Rightarrow$ (ii): It is enough to show that either x is an irreducible element in A or x is a unit in A. Let z, t respectively denote the images of Z, T in A. Suppose that x is not irreducible in A. Then, since x is irreducible in $k[x, z, t]$, there exist $a, b \in A$ such that $x = ab$ and $a \notin k[x, z, t]$. Since $A \subseteq A[x^{-1}] = k[x, x^{-1}, z, t]$, we have $a = \alpha/x^i$ and $b = \beta/x^j$ for some $\alpha, \beta \in k[x, z, t]$ and some integers $i, j \geqslant 0$. Therefore, $x^{i+j+1} = \alpha\beta$ in $k[x, z, t]$. Since x is prime in $k[x, z, t]$, we have $\alpha = \lambda x^r$, for some $\lambda \in k^*$ and $r \geqslant 0$. Thus $a = \lambda x^{r-i}$. Since $a \notin k[x, z, t]$, we have $r - i < 0$ and hence $x^{-1} \in A$.

(ii) $\Rightarrow$ (i): $A[x^{-1}] = k[x, x^{-1}]^{[2]}$ is a UFD. Therefore, if x is prime in A then, by Nagata's well-known criterion, A is a UFD. If x is a unit in A, then clearly $A = A[x^{-1}]$ is a UFD.

(ii) $\Leftrightarrow$ (iii) holds since $A/xA = k[Y, Z, T]/(f) = (k[Z, T]/(f))^{[1]}$. □

Lemma 3.2. *Let k, A, f and x be as in Lemma 3.1. Then the following statements are equivalent.*

(i) *A is an $\mathbb{A}^2$-fibration over $k[x]$.*
(ii) *$A/xA = k^{[2]}$.*
(iii) *$f(Z, T)$ is a line in $k[Z, T]$.*

Proof. (i) $\Rightarrow$ (ii) follows from the definition of $\mathbb{A}^2$-fibration.

(ii) $\Rightarrow$ (iii): Since $A/xA = k[Y, Z, T]/(f) = (k[Z, T]/(f))^{[1]}$ and $A/xA = k^{[2]}$, by Corollary 2.5, we have $k[Z, T]/(f) = k^{[1]}$.

(iii) $\Rightarrow$ (i): We have $A/xA = (k[Z, T]/(f))^{[1]} = k^{[2]}$. Let $\mathfrak{p}$ be a prime ideal of $k[x]$ other than $xk[x]$. Since $x \notin \mathfrak{p}$, $\mathfrak{p}k[x, x^{-1}]$ is a prime ideal of $k[x, x^{-1}]$. Since $A[x^{-1}] = k[x, x^{-1}]^{[2]}$, we have $A \otimes_{k[x]} k(\mathfrak{p}) = k(\mathfrak{p})^{[2]}$. Hence, for any prime ideal P of $k[x]$, $A \otimes_{k[x]} k(P) = k(P)^{[2]}$. Since $k[x]$ is a PID and A is an integral domain containing $k[x]$ (in particular, A is a torsion-free $k[x]$-module), it follows that A is flat over $k[x]$. Thus A is an $\mathbb{A}^2$-fibration over $k[x]$. $\square$

We shall see (Theorem 3.11) that, when $m > 1$, the above ring A is $k[x]^{[2]}$ if and only if $f(Z, T)$ is a variable in $k[Z, T]$. We now prove a few technical results needed to establish this.

Lemma 3.3. *Let k, A, f and x be as in Lemma 3.1. Let x, y, z and t respectively denote the images of X, Y, Z and T in A. Also let $B = k[X, Y, Z, T]/(X^m Y - f(Z, T))$. Then there exists a proper $\mathbb{Z}$-filtration $\{A_n\}_{n \in \mathbb{Z}}$ on A with $x \in A_{-1} \backslash A_{-2}$ and $z, t \in A_0 \backslash A_{-1}$ such that the induced graded ring $\operatorname{gr}(A) \cong B$.*

Proof. We note that $A \hookrightarrow k[x, x^{-1}, z, t]$ and that $k[x, x^{-1}, z, t]$ is a $\mathbb{Z}$-graded ring $k[x, x^{-1}, z, t] = \bigoplus_{i \in \mathbb{Z}} \mathcal{F}_i$, where $\mathcal{F}_i = k[z, t]x^i$. Consider the proper $\mathbb{Z}$-filtration $\{A_n\}_{n \in \mathbb{Z}}$ on A defined by $A_n := A \cap \bigoplus_{i \geqslant -n} \mathcal{F}_i$. Then $x \in A_{-1} \backslash A_{-2}$, $z, t \in A_0 \backslash A_{-1}$ and since A is an integral domain, $f(z, t) \neq 0$ and hence $y \in A_m \backslash A_{m-1}$. Using the relation $x^m y = F(x, z, t)$, we see that each element $g \in A$ can be written uniquely as

$$g = \sum_{n \geqslant 0} g_n(z, t) x^n + \sum_{j > 0} g_{ij}(z, t) x^i y^j \quad \text{where } 0 \leqslant i < m \tag{1}$$

and $g_n(z, t), g_{ij}(z, t) \in k[z, t]$. Let $\widetilde{A}$ denote the graded ring $\operatorname{gr}(A)(:= \bigoplus_{n \in \mathbb{Z}} A_n/A_{n-1})$ with respect to the above filtration. For $g \in A$, let $\bar{g}$ denote the image of g in $\widetilde{A}$. From the filtration on A and (1), it can be seen that

$$\bar{g} = g_i(\bar{z}, \bar{t}) \bar{x}^i, \quad \text{for some } i \geqslant 0 \text{ if } g \in k[x, z, t],$$

and

$$\bar{g} = g_{ij}(\bar{z}, \bar{t}) \bar{x}^i \bar{y}^j, \quad \text{for some } j > 0, 0 \leqslant i < m \text{ if } g \notin k[x, z, t]. \tag{2}$$

It also follows from (1) that the filtration defined on A is admissible with the generating set $\Gamma := \{x, y, z, t\}$. Hence $\widetilde{A}$ is generated by $\bar{x}$, $\bar{y}$, $\bar{z}$ and $\bar{t}$ (cf. [Gup14, Remark 2.2(2)]).

We now show that $\widetilde{A} \cong B$. Let $F(X, Z, T) := f(Z, T) + X f_1(Z, T) + \cdots + X^n f_n(Z, T)$. Since $x^m y$ and $f(z, t) \in A_0$, and $x^m y - f$ $(= x f_1 + \cdots + x^n f_n) \in A_{-1}$, we see that $\bar{x}^m \bar{y} - \bar{f} = 0$ in $\widetilde{A}$ (cf. [Gup14, Remark 2.2(1)]). As $\widetilde{A}$ can be identified with a subring of $\operatorname{gr}(k[x, x^{-1}, z, t]) \cong k[x, x^{-1}, z, t]$, we see that the elements $\bar{x}$, $\bar{z}$ and $\bar{t}$ of $\widetilde{A}$ are algebraically independent over k. Now as $k[X, Y, Z, T]/(X^m Y - f(Z, T))$ is an integral domain, we have $\widetilde{A} \cong B$. $\square$

Proposition 3.4. *Let k be a field and B the integral domain defined by*

$$B = k[X, Y, Z, T]/(X^m Y - f(Z, T)) \quad \textit{where } m \geqslant 1.$$

Let x, y, z and t respectively denote the images of X, Y, Z and T in B. Consider $B = \bigoplus_{i \in \mathbb{Z}} B_i$ as a graded subring of $k[x, x^{-1}, z, t]$ with $B_i = B \cap k[z, t]x^i$ for each $i \in \mathbb{Z}$. Suppose that there exists a non-trivial homogeneous exponential map ϕ on the graded ring B such that $k[y] \subseteq B^{\phi}$. Then there exists $w \in B^{\phi}$ such that $k[z, t] = k[w]^{[1]}$.

https://doi.org/10.1112/S0010437X13007793 Published online by Cambridge University Press

Proof. *Case* 1. Suppose that $B^\phi \subseteq k[y,z,t]$. Set $D := B^\phi \cap k[z,t]$. Since $y \in B^\phi$ and $\operatorname{tr.deg}_k B^\phi = 2$, it follows that $D \subsetneqq k[z,t]$. By Lemma 2.8(i), B^ϕ is a factorially closed subring of B and hence D is a factorially closed subring of $k[z,t]$. As $D \subsetneqq k[z,t]$, it then follows that $\operatorname{tr.deg}_k D \leqslant 1$. Since B^ϕ is a graded subring of $k[y,z,t] = \bigoplus_{n\in\mathbb{Z}} k[z,t]y^n$ (cf. Remark 2.10) and $\operatorname{tr.deg}_k B^\phi = 2$, we have $k \subsetneqq D$. Thus $\operatorname{tr.deg}_k D = 1$. Therefore, by Theorem 2.6, $D = k[w]$ for some $w \in k[z,t]$.

We now show that $B^\phi = k[y,w]$. Since B^ϕ is a graded subring of B, it is enough to show that if $u \in B^\phi$ is a homogeneous element then $u \in k[y,w]$. Since u is homogeneous, we have $u = h(z,t)y^i$ for some polynomial $h(z,t) \in k[z,t]$ and $i \in \mathbb{Z}_{\geqslant 0}$. By Lemma 2.8(i), $h(z,t) \in D$ and hence $u \in k[y,w]$.

We now show that $k[z,t] = k[w]^{[1]}$. Let $S = k[y,w](=B^\phi)$ and $L = k(y,w)$ be the quotient field of B^ϕ. By Lemma 2.8(iv), we have $B \otimes_S L = L^{[1]}$. Hence, since

$$L \subseteq k[y,z,t] \otimes_S L \subseteq B \otimes_S L = L^{[1]}$$

and $k[y,z,t] \otimes_S L$ is a normal domain, we have $k[y,z,t] \otimes_S L = L^{[1]}$ by Theorem 2.4. Since $k[y,z,t] \otimes_S k(y,w) = k(y,w)^{[1]}$ and $k[w]$ is algebraically closed in $k[z,t]$, we have $k[z,t] = k[w]^{[1]}$ by Lemma 2.3.

Case 2. Now suppose that $B^\phi \nsubseteq k[y,z,t]$. Since B^ϕ is a graded subring of B, it follows from Lemma 2.8(i) that $x \in B^\phi$. By Lemma 2.8(vi), ϕ induces a non-trivial exponential map ϕ_1 on

$$\tilde{B} := B \otimes_{k[x]} k(x) = k(X)[Y,Z,T]/(X^mY - f(Z,T)) = k(x)[z,t]$$

such that $\tilde{B}^{\phi_1} = B^\phi \otimes_{k[x]} k(x)$. Since $\operatorname{tr.deg}_{k(x)} \tilde{B}^{\phi_1} = 1$ and $\tilde{B}^{\phi_1}$ is a factorially closed subring of $\tilde{B} = k(x)[z,t]$, we have $\tilde{B}^{\phi_1} = k(x)[w_1]$ for some $w_1 \in k(x)[z,t]$ by Theorem 2.6. Again by Lemma 2.8(iv), $k(x)[z,t] \otimes_{k(x)[w_1]} k(x,w_1) = k(x,w_1)^{[1]}$. Hence, by Theorem 2.1, we have $k(x)[z,t] = k(x)[w_1]^{[1]}$. Now $w_1 = \alpha(x,z,t)/\beta(x)$ for some $\alpha(x,z,t) \in k[x,z,t]$ and $\beta(x) \in k[x]$. Set $w_2 := \beta(x)w_1$. Then $k(x)[w_1] = k(x)[w_2]$. Now $w_2 \in \tilde{B}^{\phi_1} \cap k[x,z,t] \subseteq B^\phi$. Let $w_2 = h_0(z,t) + h_1(z,t)x + \cdots + h_r(z,t)x^r$ for some $h_i(z,t) \in k[z,t]$, $0 \leqslant i \leqslant r$. Then $h_i(z,t) \in B^\phi$ for each i (since B^ϕ is a graded subring of B). Set $E := B^\phi \cap k[z,t]$. Then, since B^ϕ is a factorially closed subring of B, we have that E is a factorially closed subring of $k[z,t]$. Since $\operatorname{tr.deg}_k B^\phi = 2$ and $x \in B^\phi$, it follows that $E \subsetneqq k[z,t]$ and since $h_i(z,t) \in E$ for each i, $0 \leqslant i \leqslant r$, we have $k \subsetneqq E$. Hence, $E = k[w]$ for some $w \in k[z,t]$ by Theorem 2.6. Now, $E = k[w] \subseteq B^\phi \subseteq \tilde{B}^{\phi_1} = k(x)[w_2]$ and $k(x)[w_2] \subseteq k(x)[w]$. Hence $k(x)[w] = k(x)[w_2] = k(x)[w_1]$. Therefore, since $k(x)[z,t] = k(x)[w_1]^{[1]} = k(x)[w]^{[1]}$ and $k[w]$ $(=E)$ is algebraically closed in $k[z,t]$, we have $k[z,t] = k[w]^{[1]}$ by Lemma 2.3. □

The following result was proved by Makar-Limanov [Mak01] when the characteristic of the field k is zero. We modify his arguments to give a characteristic-free proof.

Lemma 3.5. *Let k be a field, $P(Z) \in k[Z]$ a polynomial of $\deg_Z P(Z) > 1$ and*

$$D = k[X,Y,Z]/(X^mY - P(Z)) \quad \text{where } m > 1.$$

Let x, y, z respectively denote the images of X, Y, Z in D. Then there does not exist any non-trivial exponential map ϕ on D such that $y \in D^\phi$.

Proof. Let $r = \deg_Z P(Z)$ and λ be the coefficient of Z^r in $P(Z)$. We note that $D \hookrightarrow k[x,x^{-1},z]$ and that $k[x,x^{-1},z]$ is a $\mathbb{Z}$-graded ring $k[x,x^{-1},z] = \bigoplus_{i\in\mathbb{Z}} C_i$, where $C_i = k[x,x^{-1}]z^i$. Consider the proper $\mathbb{Z}$-filtration $\{D_n\}_{n\in\mathbb{Z}}$ on D defined by $D_n := D \cap \bigoplus_{i\leqslant n} C_i$. Let E denote the graded ring $\operatorname{gr}(D)(:= \bigoplus_{n\in\mathbb{Z}} D_n/D_{n-1})$ with respect to the above filtration. For $g \in D$, let $\bar{g}$ denote the

https://doi.org/10.1112/S0010437X13007793 Published online by Cambridge University Press

image of g in E. Note that $\bar{y} = \lambda \bar{z}^r / \bar{x}^m$. Thus, in the graded ring E, $\deg(\bar{x}) = 0$, $\deg(\bar{z}) = 1$ and $\deg(\bar{y}) = r$. We now show that

$$E \cong k[X,Y,Z]/(X^m Y - \lambda Z^r). \tag{3}$$

Each element $g \in D$ can be written uniquely as

$$g = \sum_{n \geqslant 0} g_n(z) x^n + \sum_{j>0} g_{ij}(z) x^i y^j \quad \text{where } 0 \leqslant i < m.$$

From this expression, it can be easily seen that the filtration defined on D is admissible with the generating set $\Gamma := \{x, y, z\}$. Hence E is generated by $\bar{x}$, $\bar{y}$ and $\bar{z}$. We also note that $\bar{x}^m \bar{y} - \lambda \bar{z}^r = 0$ in E. As E can be identified with a subring of $\mathrm{gr}(k[x, x^{-1}, z]) \cong k[x, x^{-1}, z]$, we see that the elements $\bar{x}$ and $\bar{z}$ of E are algebraically independent over k. Now as $k[X,Y,Z]/(X^m Y - \lambda Z^r)$ is an integral domain, the isomorphism in (3) holds.

Suppose that there exists a non-trivial exponential map ϕ on D such that $y \in D^\phi$. By Theorem 2.11, ϕ induces a non-trivial exponential map $\bar{\phi}$ on E such that $\bar{y} \in E^{\bar{\phi}}$, i.e., $k[\bar{y}] \subseteq E^{\bar{\phi}}$. By Lemma 2.8(vi), $\bar{\phi}$ induces a non-trivial exponential map on

$$E \otimes_{k[\bar{y}]} k(\bar{y}) = k(\bar{y})[\bar{x}, \bar{z}] \cong k(\bar{y})[X, Z]/(\bar{y} X^m - \lambda Z^r),$$

which contradicts Lemma 2.8(v), as $E \otimes_{k[\bar{y}]} k(\bar{y})$ is not a normal domain. □

We record an observation on the Derksen invariant and the Makar-Limanov invariant of the affine threefold $x^m y = F(x,z,t)$.

Lemma 3.6. *Let k be a field and A be an integral domain defined by*

$$A = k[X,Y,Z,T]/(X^m Y - F(X,Z,T)) \quad \textit{where } m \geqslant 1.$$

Let x, y, z and t respectively denote the images of X, Y, Z and T in A. Then $k[x,z,t] \subseteq \mathrm{DK}(A)$ and $\mathrm{ML}(A) \subseteq k[x]$.

Proof. Define ϕ_1 by

$$\phi_1(x) = x, \quad \phi_1(z) = z, \quad \phi_1(t) = t + x^m U \quad \text{and} \quad \phi_1(y) = \frac{F(x,z,t+x^m U)}{x^m} = y + U\alpha(x,z,t,U) \tag{4}$$

and define ϕ_2 by

$$\phi_2(x) = x, \quad \phi_2(t) = t, \quad \phi_2(z) = z + x^m U \quad \text{and} \quad \phi_2(y) = \frac{F(x, z+x^m U, t)}{x^m} = y + U\beta(x,z,t,U). \tag{5}$$

Note that $\alpha(x,z,t,U), \beta(x,z,t,U) \in k[x,z,t,U]$ and that $k[x,z]$ and $k[x,t]$ are algebraically closed in A of transcendence degree two over k. It then follows that ϕ_1 and ϕ_2 are non-trivial exponential maps on A with $A^{\phi_1} = k[x,z]$ and $A^{\phi_2} = k[x,t]$. Hence $k[x,z,t] \subseteq \mathrm{DK}(A)$ and $\mathrm{ML}(A) \subseteq k[x,z] \cap k[x,t] = k[x]$. □

We now show that, for $m > 1$, a necessary condition for the above ring A to be polynomial ring (whence $\mathrm{DK}(A) = A$ by Lemma 2.7) is that $F(0,Z,T)$ can be expressed as a linear polynomial.

Proposition 3.7. *Let k be a field and A be an integral domain defined by*

$$A = k[X,Y,Z,T]/(X^m Y - F(X,Z,T)) \quad \textit{where } m > 1.$$

Set $f(Z,T) := F(0,Z,T)$. Let x, y, z and t respectively denote the images of X, Y, Z and T in A. Suppose that $\mathrm{DK}(A) \neq k[x,z,t]$. Then the following statements hold.

(i) *There exist* $Z_1, T_1 \in k[Z,T]$ *and* $a_0, a_1 \in k^{[1]}$ *such that* $k[Z,T] = k[Z_1, T_1]$ *and* $f(Z,T) = a_0(Z_1) + a_1(Z_1)T_1$.

(ii) *If* $k[Z,T]/(f) = k^{[1]}$, *then* $k[Z,T] = k[f]^{[1]}$.

Proof. (i) By Lemma 3.6, $k[x,z,t] \subseteq \mathrm{DK}(A)$ and now since $\mathrm{DK}(A) \neq k[x,z,t]$, there exists a non-trivial exponential map ϕ on A such that $A^{\phi} \nsubseteq k[x,z,t]$. Choose an element $g \in A^{\phi} \backslash k[x,z,t]$. Consider the proper $\mathbb{Z}$-filtration $\{A_n\}_{n\in\mathbb{Z}}$ on A and the induced graded ring $B = \mathrm{gr}(A)$ of Lemma 3.3. For $h \in A$, let $\bar{h}$ denote the image of h in B. By Theorem 2.11, ϕ induces a non-trivial homogeneous exponential map $\bar{\phi}$ on B such that $\bar{g} \in B^{\bar{\phi}}$. By the relation (2) in the proof of Lemma 3.3, $\bar{g} = g_{ab}(\bar{z}, \bar{t})\bar{x}^a\bar{y}^b (\in B^{\bar{\phi}})$ for some $0 \leqslant a < m$, $b > 0$ and $g_{ab}(\bar{z}, \bar{t}) \in k[\bar{z}, \bar{t}]$. Since $B^{\bar{\phi}}$ is factorially closed in B (cf. Lemma 2.8(i)), it follows that $\bar{y} \in B^{\bar{\phi}}$. Therefore, by Proposition 3.4, there exists $\bar{z_1} \in k[\bar{z}, \bar{t}]$ such that $k[\bar{z}, \bar{t}] = k[\bar{z_1}]^{[1]}$ and $\bar{z_1} \in B^{\bar{\phi}}$. Then $k[Z,T] = k[Z_1, T_1]$ where Z_1 is the pre-image of $\bar{z_1}$ in $k[Z,T]$. Let $h \in k^{[2]}$ be such that

$$h(Z_1, T_1) = f(Z,T) = a_0(Z_1) + a_1(Z_1)T_1 + \cdots + a_n(Z_1){T_1}^n.$$

Let $\tilde{k}$ be an algebraic closure of the field k. Then $\bar{\phi}$ induces a non-trivial exponential map $\tilde{\phi}$ on

$$\tilde{B} := B \otimes_k \tilde{k} = \tilde{k}[X, Y, Z_1, T_1]/(X^mY - h(Z_1, T_1)) = \tilde{k}[\bar{x}, \bar{y}, \bar{z_1}, \bar{t_1}]$$

such that $\tilde{k}[\bar{y}, \bar{z_1}] \subseteq \tilde{B}^{\tilde{\phi}}$. Since there exist infinitely many $\beta \in \tilde{k}$ such that $\bar{z_1} - \beta$ is a prime element of $\tilde{B}$, by Lemma 2.9, we may choose β such that $\tilde{\phi}$ induces a non-trivial exponential map on the ring $\tilde{B}/(\bar{z_1} - \beta)$ and which also satisfies $a_n(\beta) \neq 0$. Thus, there exists a non-trivial exponential map on the ring

$$\frac{\tilde{B}}{(\bar{z_1} - \beta)} \cong \frac{\tilde{k}[X, Y, T_1]}{(X^mY - h(\beta, T_1))} = \frac{\tilde{k}[X, Y, T_1]}{(X^mY - (a_0(\beta) + a_1(\beta)T_1 + \cdots + a_n(\beta){T_1}^n))}$$

with the image of $\bar{y}$ in $\tilde{B}/(\bar{z_1} - \beta)$ lying in the ring of invariants. Hence, by Lemma 3.5, $n = 1$. Thus, $f(Z,T) = a_0(Z_1) + a_1(Z_1)T_1$ for some $Z_1, T_1 \in k[Z,T]$ satisfying $k[Z,T] = k[Z_1, T_1]$.

(ii) Since $f(Z,T)$ is a line, we have $A/xA = k^{[2]}$ and hence $(A/xA)^* = k^*$. By (i) above, there exist $Z_1, T_1 \in k[Z,T]$ and $a_0, a_1 \in k^{[1]}$ such that $k[Z,T] = k[Z_1, T_1]$ and $f(Z,T) = a_0(Z_1) + a_1(Z_1)T_1$. If $a_1(Z_1) = 0$, then $f(Z,T) = a_0(Z_1)$ is clearly a linear polynomial in Z_1 (since $f(Z,T)$ is a line) and hence a variable in $k[Z,T]$. Now suppose that $a_1(Z_1) \neq 0$. As $f(Z,T)$ is irreducible in $k[Z,T]$, $a_0(Z_1)$ and $a_1(Z_1)$ are coprime in $k[Z_1]$. Hence $A/xA \cong k[Z, 1/a_1(Z_1)]^{[1]}$ and, since $(A/xA)^* = k^*$, we have $a_1(Z_1) \in k^*$. This again implies that $f(Z,T)$ is a variable in $k[Z,T]$. □

We shall now prove Theorem A (Theorem 3.11); we shall also prove the equivalence of five more conditions each involving the Derksen invariant $\mathrm{DK}(A)$. In the proof of the implications (v) ⇒ (vii) ⇒ (viii) of Theorem 3.11, we shall use a few results on the groups $\mathbf{K}_i$ of algebraic **K**-theory. However, when $F(0,Z,T)$ is a line, one has a simpler proof of Theorem A which does not need the language of **K**-theory (see Remark 3.13(1)).

We first quote a result from **K**-theory due to Quillen [Sri08, Corollary 5.5].

Theorem 3.8. *Let* R *be a regular ring and* U *an indeterminate over* R. *Then the following statements hold.*

(i) *The inclusion map* $R \hookrightarrow R[U]$ *induces an isomorphism from* $\mathbf{K}_i(R)$ *to* $\mathbf{K}_i(R[U])$ *for each* $i \geqslant 0$.

(ii) *For each $i \geqslant 1$, the sequence*

$$0 \longrightarrow \mathbf{K}_i(R[U]) \longrightarrow \mathbf{K}_i(R[U, U^{-1}]) \longrightarrow \mathbf{K}_{i-1}(R) \longrightarrow 0$$

is a split short exact sequence, where the map $\mathbf{K}_i(R[U]) \longrightarrow \mathbf{K}_i(R[U, U^{-1}])$ is induced by the inclusion map $R[U] \hookrightarrow R[U, U^{-1}]$.

The following result follows from [Sri08, Proposition 5.15, §5.6 p. 52 and §5.16 p. 61].

THEOREM 3.9. *Let R be a regular ring and x be a non-zero-divisor of R such that R/xR is a regular ring. Let $j : R \longrightarrow R[x^{-1}]$ be the inclusion map. Then we have the following long exact sequence of K-groups:*

$$\longrightarrow \mathbf{K}_i(R/xR) \longrightarrow \mathbf{K}_i(R) \xrightarrow{j_*} \mathbf{K}_i(R[x^{-1}]) \xrightarrow{\partial} \mathbf{K}_{i-1}(R/xR) \longrightarrow$$

Moreover, if $\phi : R \longrightarrow S$ is a flat ring homomorphism with $u = \phi(x)$ such that S and S/uS are regular rings, then we have the following natural commutative diagram:

$$\begin{tikzcd}
{} \arrow[r] & \mathbf{K}_i(R/xR) \arrow[r] \arrow[d] & \mathbf{K}_i(R) \arrow[r] \arrow[d] & \mathbf{K}_i(R[x^{-1}]) \arrow[r, "\partial"] \arrow[d] & \mathbf{K}_{i-1}(R/xR) \arrow[r] \arrow[d] & {} \\
{} \arrow[r] & \mathbf{K}_i(S/uS) \arrow[r] & \mathbf{K}_i(S) \arrow[r] & \mathbf{K}_i(S[u^{-1}]) \arrow[r, "\partial"] & \mathbf{K}_{i-1}(S/uS) \arrow[r] & {}
\end{tikzcd}$$

where the vertical maps are induced by ϕ.

We also observe an elementary result.

LEMMA 3.10. *Let $\phi : R \longrightarrow B$ be an injective ring homomorphism. Then the map $\phi_* : \mathbf{K}_1(R) \longrightarrow \mathbf{K}_1(B)$, induced by ϕ, maps the subgroup R^* of $\mathbf{K}_1(R)$ injectively into the subgroup B^* of $\mathbf{K}_1(B)$.*

We now prove our main theorem.

THEOREM 3.11. *Let k be a field and*

$$A = k[X, Y, Z, T]/(X^m Y - F(X, Z, T)) \quad \text{where } m > 1.$$

Let x, y, z and t respectively denote the images of X, Y, Z and T in A. Set $f(Z, T) := F(0, Z, T)$ and $G := X^m Y - F(X, Z, T)$. Then the following statements are equivalent.

(i) $k[X, Y, Z, T] = k[X, G]^{[2]}$.
(ii) $k[X, Y, Z, T] = k[G]^{[3]}$.
(iii) $A = k[x]^{[2]}$.
(iv) $A = k^{[3]}$.
(v) $A^{[\ell]} \cong_k k^{[\ell+3]}$ *for some integer $\ell \geqslant 0$ and* $\mathrm{DK}(A) \neq k[x, z, t]$.
(vi) A *is an $\mathbb{A}^2$-fibration over $k[x]$ and* $\mathrm{DK}(A) \neq k[x, z, t]$.
(vii) A *is geometrically factorial over k,* $\mathrm{DK}(A) \neq k[x, z, t]$ *and the canonical map $k^* \to \mathbf{K}_1(A)$ (induced by the inclusion $k \hookrightarrow A$) is an isomorphism.*
(viii) A *is geometrically factorial over k,* $\mathrm{DK}(A) \neq k[x, z, t]$ *and* $(A/xA)^* = k^*$.
(ix) $k[Z, T] = k[f]^{[1]}$.
(x) $k[Z, T]/(f) = k^{[1]}$ *and* $\mathrm{DK}(A) \neq k[x, z, t]$.

https://doi.org/10.1112/S0010437X13007793 Published online by Cambridge University Press

Proof. (i) $\Rightarrow$ (ii) $\Rightarrow$ (iv), (i) $\Rightarrow$ (iii) are trivial. It suffices to prove (iv) $\Rightarrow$ (v) $\Rightarrow$ (vii) $\Rightarrow$ (viii) $\Rightarrow$ (ix) $\Rightarrow$ (i) and (iii) $\Rightarrow$ (vi) $\Leftrightarrow$ (x) $\Rightarrow$ (ix).

(iv) $\Rightarrow$ (v) follows from Lemma 2.7.

(v) $\Rightarrow$ (vii) follows from Theorem 3.8 and the fact that $\mathbf{K}_1(k) = k^*$ for any field k.

(vii) $\Rightarrow$ (viii): Since $\mathrm{DK}(A) \neq k[x, z, t]$, by Proposition 3.7(i), there exist $Z_1, T_1 \in k[Z, T]$ and $a_0, a_1 \in k^{[1]}$ such that $k[Z, T] = k[Z_1, T_1]$ and $f(Z, T) = a_0(Z_1) + a_1(Z_1)T_1$. Without loss of generality, we may assume that $Z_1 = Z$, $T_1 = T$ and $f(Z, T) = a_0(Z) + a_1(Z)T$. We now consider two cases.

Case 1: $a_1(Z) = 0$. Let $\tilde{k}$ be an algebraic closure of k. Since $A \otimes_k \tilde{k}$ is a UFD, it follows from Lemma 3.1 that $a_0(Z)$ is either irreducible or a non-zero constant in $\tilde{k}[Z, T]$. But if $a_0(Z)$ is a non-zero constant, then $A = k[x, x^{-1}, z, t]$ and hence $\mathbf{K}_1(A) \neq k^*$, contradicting the hypothesis. Thus, $a_0(Z)$ is irreducible in $\tilde{k}[Z, T]$ and hence a linear polynomial in Z. Therefore, $f(Z, T)$ is a variable in $k[Z, T]$. Hence $A/xA = k^{[2]}$, which implies that $(A/xA)^* = k^*$.

Case 2: $a_1(Z) \neq 0$. Since $a_0(Z) + a_1(Z)T$ is irreducible in $k[Z, T]$ (cf. Lemma 3.1), we have $(a_0(Z), a_1(Z)) = 1$ and hence it follows that $A/xA = k[Z, T, Y]/(a_0(Z) + a_1(Z)T) \cong k[Z, 1/a_1(Z)][Y]$. Also $A[x^{-1}] = k[x, x^{-1}]^{[2]}$. Since $A[x^{-1}]$ and A/xA are regular rings, we have A is a regular ring. Hence, by Theorem 3.9, we have an exact sequence:

$$\longrightarrow \mathbf{K}_2(A[x^{-1}]) \xrightarrow{\partial} \mathbf{K}_1(A/xA) \longrightarrow \mathbf{K}_1(A) \xrightarrow{j_*} \mathbf{K}_1(A[x^{-1}]) \longrightarrow \tag{6}$$

where j_* is induced by the inclusion $j : A \longrightarrow A[x^{-1}]$ and ∂ is the connecting morphism. Let n and j denote the inclusion maps $n : k \longrightarrow A$ and $j : A \longrightarrow A[x^{-1}]$. By Lemma 3.10, $j_* \circ n_*$ maps k^* injectively into $(A[x^{-1}])^*$. Since $\mathbf{K}_1(A) = n_*(\mathbf{K}_1(k))$ by hypothesis, it follows that j_* maps $\mathbf{K}_1(A)$ injectively into $\mathbf{K}_1(A[x^{-1}])$. Thus, from the exact sequence (6), we have the exact sequence

$$\longrightarrow \mathbf{K}_2(A[x^{-1}]) \xrightarrow{\partial} \mathbf{K}_1(A/xA) \longrightarrow 0. \tag{7}$$

Since $A[x^{-1}] = k[x, x^{-1}]^{[2]}$, by Theorem 3.8(i), the inclusion map $k[x, x^{-1}] \hookrightarrow A[x^{-1}]$ induces an isomorphism from $\mathbf{K}_2(k[x, x^{-1}])$ to $\mathbf{K}_2(A[x^{-1}])$. Again, by Theorem 3.8(ii), the sequence

$$0 \longrightarrow \mathbf{K}_2(k[x]) \longrightarrow \mathbf{K}_2(k[x, x^{-1}]) \longrightarrow \mathbf{K}_1(k) \longrightarrow 0 \tag{8}$$

is a split short exact sequence, where the map $\mathbf{K}_2(k[x]) \longrightarrow \mathbf{K}_2(k[x, x^{-1}])$ is induced by the inclusion map $k[x] \hookrightarrow k[x, x^{-1}]$. Now consider the inclusion $k[x] \hookrightarrow A$. Since A is flat over $k[x]$, by Theorem 3.9, we have the following commutative diagram between the exact sequences (8) and (7):

$$\begin{tikzcd}
\mathbf{K}_2(k[x, x^{-1}]) \arrow[r, "\partial"] \arrow[d, "\cong"'] & \mathbf{K}_1(k) \arrow[r] \arrow[d, "\phi_*"'] & 0 \\
\mathbf{K}_2(A[x^{-1}]) \arrow[r, "\partial"] & \mathbf{K}_1(A/xA) \arrow[r] & 0
\end{tikzcd}$$

where the vertical maps are induced by the canonical inclusion maps. From this commutative diagram it follows that the canonical map $\phi_* : \mathbf{K}_1(k)(= k^*) \to \mathbf{K}_1(A/xA)$, induced by the inclusion $\phi : k \hookrightarrow A/xA$, is surjective. By Lemma 3.10, ϕ_* maps k^* injectively into the subgroup $(A/xA)^*$ of $\mathbf{K}_1(A/xA)$. Hence $(A/xA)^* = k^*(= \mathbf{K}_1(A/xA))$.

(viii) $\Rightarrow$ (ix): As before, we may assume that $f(Z, T) = a_0(Z) + a_1(Z)T$. Suppose that $a_1(Z) = 0$, i.e., $f(Z, T) = a_0(Z)$. Then $A/xA = k[Y, Z, T]/(a_0(Z))$. Since $(A/xA)^* = k^*$, it follows that $a_0(Z) \notin k^*$. Then, as before, we have that $a_0(Z)$ is irreducible in $\tilde{k}[Z, T]$ and hence a linear polynomial in Z. Therefore, $f(Z, T)$ is a variable in $k[Z, T]$.

Now suppose that $a_1(Z) \neq 0$. Then, since $a_0(Z) + a_1(Z)T$ is irreducible in $k[Z, T]$ (cf. Lemma 3.1), we have $(a_0(Z), a_1(Z)) = 1$ and hence $A/xA = k[Z, T, Y]/(a_0(Z) + a_1(Z)T) \cong k[Z, 1/a_1(Z)][Y]$. Therefore, since $(A/xA)^* = k^*$, we have $a_1(Z) \in k^*$. Thus $f(Z, T)$ is a variable in $k[Z, T]$.

(ix) $\Rightarrow$ (i): Without loss of generality, we may assume that $f(Z, T) = Z$. Set $D := k[X, Y, Z, T]$ and $R := k[X, G, T]$. Then $D[X^{-1}] = R[X^{-1}][Z]$ and $D/XD = (R/XR)^{[1]}$. Hence, by Theorem 2.2, $D = R^{[1]}$, i.e., $k[X, Y, Z, T] = k[X, G, T]^{[1]} = k[X, G]^{[2]}$.

(iii) $\Rightarrow$ (vi) follows from Lemma 2.7.

(vi) $\Leftrightarrow$ (x) follows from Lemma 3.2.

(x) $\Rightarrow$ (ix) follows from Proposition 3.7(ii). □

Remark 3.12. A version of the 'epimorphism problem' or 'embedding problem' for hypersurfaces over a field k asks the following: if $k[X_1, X_2, \ldots, X_n]/(G) \cong k^{[n-1]}$, then is $k[X_1, X_2, \ldots, X_n] = k[G]^{[n-1]}$? The Segre–Nagata non-trivial lines mentioned in the introduction show that the problem has a negative solution when characteristic $k > 0$ (unless there are some additional conditions on G). When characteristic $k = 0$, an affirmative answer for $n = 2$ has been given independently by Abhyankar and Moh and by Suzuki (see [Abh77, Corollary 9.21]), and the Abhyankar–Sathaye conjecture envisages an affirmative solution for $n \geqslant 3$. While the conjecture remains open for $n \geqslant 3$, affirmative solutions to the epimorphism problem are known for a few special cases of G even in arbitrary characteristic. When $n = 3$, ch $k = 0$ and G is a 'linear plane in $\mathbb{A}^3_k$' defined by $aX_3 - b$, where $a, b \in k[X_1, X_2]$, it was shown by Sathaye [Sat76] that there exist coordinates X, Y for which $a \in k[X]$ and $k[X, Y] = k[X_1, X_2]$ and that G is a variable in $k[X, Y, X_3](= k[X_1, X_2, X_3])$ along with X. This result was extended by Russell [Rus76] to fields k of arbitrary characteristic.

Thus the implication (iv) $\Rightarrow$ (i) of Theorem 3.11 may be thought of as a partial extension of the Sathaye–Russell theorem on linear planes in $\mathbb{A}^3_k$ to the linear hypersurfaces in $\mathbb{A}^4_k$ of the form $x^m y = F(x, z, t)$ in arbitrary characteristic. When $k = \mathbb{C}$, more general cases of linear hypersurfaces have been proved in [KVZ04].

Remark 3.13. (1) In the case where $f(Z, T) := F(0, Z, T)$ is a line in $k[Z, T]$, Proposition 3.7(ii), along with Lemma 2.7, gives an alternative proof of (iv) $\Rightarrow$ (ix) in Theorem 3.11 (and hence a proof of Theorem A) without using any machinery of **K**-theory. Note that this case will already address the question of Asanuma–Russell mentioned in the introduction, for $m > 1$.

(2) In Theorem 3.11, for the case $m = 1$, the implications (ix) $\Rightarrow$ (i) $\Rightarrow$ (ii) $\Rightarrow$ (iv) $\Rightarrow$ (v) $\Rightarrow$ (vii) and (i) $\Rightarrow$ (iii) $\Rightarrow$ (vi) $\Leftrightarrow$ (x) $\Rightarrow$ (viii) hold; however, (vii) $\nRightarrow$ (ix) and (viii) $\nRightarrow$ (ix) (cf. Remark 4.7(2)).

We end this section with a partial converse to Proposition 3.7(i).

Proposition 3.14. *Let k be a field and A be an integral domain defined by*

$$A = k[X, Y, Z, T]/(X^m Y - f(Z, T)) \quad \textit{where } m \geqslant 1$$

and $f(Z, T) = a_0(Z) + a_1(Z)T$ for some $a_0(Z), a_1(Z) \in k[Z]$. Then $\mathrm{DK}(A) = A$.

Proof. Let x, y, z and t respectively denote the images of X, Y, Z and T in A. By Lemma 3.6, we have $k[x, z, t] \subseteq \mathrm{DK}(A)$. We define an exponential map ϕ on A. If $a_1(Z) = 0$, then ϕ is defined by

$$\phi(x) = x, \quad \phi(z) = z, \quad \phi(y) = y \quad \text{and} \quad \phi(t) = t + U; \tag{9}$$

if $a_1(Z) \neq 0$, then ϕ is defined by

$$\phi(x) = x + a_1(z)U, \quad \phi(z) = z, \quad \phi(y) = y \quad \text{and} \quad \phi(t) = \frac{(x + a_1(z)U)^m y - a_0(z)}{a_1(z)}. \tag{10}$$

In either case, ϕ is an exponential map of A such that $k[y, z] \subseteq A^{\phi}$. Hence $\mathrm{DK}(A) = A$. $\square$

4. Asanuma threefolds

Let k be a field of characteristic $p > 0$ and

$$A = k[X, Y, Z, T]/(X^m Y - F(X, Z, T)) \quad \text{where } m \geqslant 1. \tag{11}$$

Let $f(Z, T) := F(0, Z, T)$. Let x denote the image of X in A. We have seen that:

(i) A is a UFD $\Longleftrightarrow$ $f(Z, T)$ is either a constant or an irreducible in $k[Z, T]$;

(ii) A is an $\mathbb{A}^2$-fibration over $k[x]$ $\Longleftrightarrow$ $f(Z, T)$ is a line in $k[Z, T]$.

Moreover, when $m \geqslant 2$, we also have

(iii) $A = k[x]^{[2]} \Longleftrightarrow A = k^{[3]} \Longleftrightarrow f(Z, T)$ is a variable in $k[Z, T]$.

We shall call a ring A defined in (11) an *Asanuma threefold* if $f(Z, T)$ is a non-trivial line in $k[Z, T]$. Thus, for $m \geqslant 2$, an Asanuma threefold is a non-trivial $\mathbb{A}^2$-fibration over $k[x]$ which is not isomorphic to $k^{[3]}$. Recall that Asanuma made pioneering investigations on such a ring A; for instance he considered the ring R mentioned in the introduction which was obtained from Segre and Nagata's non-trivial line $f(Z, T) = Z^{p^e} + T + T^{sp}$, where $p^e \nmid sp$ and $sp \nmid p^e$. In this section, we shall see that, when $m > 1$, any Asanuma threefold is a counter-example to Zariski's cancellation problem for the affine 3-space $\mathbb{A}^3_k$ in positive characteristic. Finally, we shall describe the isomorphism classes of certain Asanuma threefolds.

We first prove an elementary lemma.

Lemma 4.1. *Let k be a field and D an affine k-domain. Let $F(X) \in D[X]$ and $f := F(0)$. Suppose that $D/(f) = k^{[1]}$. Then $D[X]/(X^m, F) = (k[X]/(X^m))^{[1]}$ for every $m \geqslant 1$.*

Proof. Fix m. Let $F(X) = f + XF_1(X)$ for some $F_1(X) \in D[X]$. Let $h \in D$ be such that $D = k[h] + fD$. Then $D[X] = k[h][X] + fD[X]$, i.e.,

$$D[X] = k[h][X] + (F(X) - XF_1(X))D[X] \subseteq k[h, X] + XD[X] + F(X)D[X] \subseteq D[X].$$

Thus,

$$\begin{aligned} D[X] &= k[h] + XD[X] + F(X)D[X] \\ &= k[h] + X(k[h] + XD[X] + F(X)D[X]) + F(X)D[X] \\ &= k[h] + Xk[h] + X^2 D[X] + F(X)D[X] \\ &\quad \cdots \\ &= k[h] + Xk[h] + \cdots + X^{m-1}k[h] + X^m D[X] + F(X)D[X]. \end{aligned}$$

Hence, $D[X]/(X^m, F(X)) = (k[X]/(X^m))^{[1]}$. $\square$

https://doi.org/10.1112/S0010437X13007793 Published online by Cambridge University Press

Since any Asanuma threefold A is an $\mathbb{A}^2$-fibration over its subfield $k[x]$ (Lemma 3.2), by a theorem of Asanuma [Asa87, Proposition 2.5], we know that $A^{[\ell]} = k[x]^{[2+\ell]}$ for some $\ell \geqslant 0$. We now present a generalised version of Asanuma's stability theorem [Asa87, Theorem 5.1] showing that we actually have $A^{[1]} = k[x]^{[3]}$.

Theorem 4.2. *Let k be any field of characteristic p ($\geqslant 0$) and*

$$A = k[X, Y, Z, T]/(X^m Y - F(X, Z, T)) \quad \textit{where } m \geqslant 1.$$

Let $f(Z, T) := F(0, Z, T)$ be such that $k[Z, T]/(f) = k^{[1]}$. Then

$$A^{[1]} \cong_{k[x]} k[x]^{[3]} \cong_k k^{[4]},$$

where x denotes the image of X in A.

Proof. Let y be the image of Y in A. Since $k[X, Z, T] \hookrightarrow A$, identifying X, Z, and T with their images in A, we have $A = k[X, Z, T, y]$. Let U be an indeterminate over $k[X]$ and $\Psi : k[X, U] \longrightarrow k[X, U]/(X^m)$ be the natural surjective map. Since $k[Z, T]/(f) = k^{[1]}$, by Lemma 4.1, we have a surjective k-algebra homomorphism $\Phi : k[X, Z, T] \longrightarrow k[X, U]/(X^m)$ with kernel $(X^m, F(X, Z, T))$ satisfying $\Phi(X) = \Psi(X)$. Let $h(X, Z, T) \in k[X, Z, T]$ and $P(X, U), Q(X, U) \in k[X, U]$ be such that

$$\Phi(h) = \Psi(U), \quad \Phi(Z) = \Psi(P(X, U)) \quad \text{and} \quad \Phi(T) = \Psi(Q(X, U)).$$

Let W be an indeterminate over A. Set

$$\begin{aligned} W_1 &:= X^m W + h(X, Z, T), \\ Z_1 &:= \frac{Z - P(X, W_1)}{X^m}, \\ T_1 &:= \frac{T - Q(X, W_1)}{X^m}. \end{aligned}$$

We show that $A[W] = k[X, Z_1, T_1, W_1]$. Set $B := k[X, Z_1, T_1, W_1]$. We have

$$\begin{aligned} Z &= P(X, W_1) + X^m Z_1, \\ T &= Q(X, W_1) + X^m T_1, \\ y &= \frac{F(X, Z, T)}{X^m} = \frac{F(X, X^m Z_1 + P(X, W_1), X^m T_1 + Q(X, W_1))}{X^m} \\ &= \frac{F(X, P(X, W_1), Q(X, W_1))}{X^m} + \alpha(X, Z_1, T_1, W_1), \\ W &= \frac{W_1 - h(X, Z, T)}{X^m} = \frac{W_1 - h(X, X^m Z_1 + P(X, W_1), X^m T_1 + Q(X, W_1))}{X^m} \\ &= \frac{W_1 - h(X, P(X, W_1), Q(X, W_1))}{X^m} + \beta(X, Z_1, T_1, W_1) \end{aligned}$$

for some $\alpha, \beta \in B$. Since $\Psi(F(X, P(X, U), Q(X, U))) = \Phi(F(X, Z, T))$, we see that

$$F(X, P(X, W_1), Q(X, W_1)) \in X^m k[X, W_1] \subseteq X^m B.$$

Thus $y \in B$. Also, since $\Psi(h(X, P(X, U), Q(X, U))) = \Phi(h(X, Z, T)) = \Psi(U)$, we see that

$$h(X, P(X, W_1), Q(X, W_1)) - W_1 \in X^m k[X, W_1] \subseteq X^m B.$$

https://doi.org/10.1112/S0010437X13007793 Published online by Cambridge University Press

Thus $W \in B$. Hence, $A[W] \subseteq B$. We now show that $B \subseteq A[W]$. Now

$$Z_1 = \frac{Z - P(X, W_1)}{X^m} = \frac{Z - P(X, X^m W + h(X, Z, T))}{X^m} = \frac{Z - P(X, h(X, Z, T))}{X^m} + \gamma(X, Z, T, W)$$

and

$$T_1 = \frac{T - Q(X, W_1)}{X^m} = \frac{T - Q(X, X^m W + h(X, Z, T))}{X^m} = \frac{T - Q(X, h(X, Z, T))}{X^m} + \delta(X, Z, T, W)$$

for some $\gamma, \delta \in A[W]$. Since $\Phi(Z - P(X, h)) = \Psi(P(X, U)) - \Psi(P(X, U)) = 0$ and $\Phi(T - Q(X, h)) = \Psi(Q(X, U)) - \Psi(Q(X, U)) = 0$, we have

$$Z - P(X, h) = a(X, Z, T)X^m + b(X, Z, T)F(X, Z, T)$$

and

$$T - Q(X, h) = c(X, Z, T)X^m + d(X, Z, T)F(X, Z, T),$$

for some $a, b, c, d \in k[X, Z, T]$. Hence,

$$\frac{Z - P(X, h)}{X^m} = a(X, Z, T) + b(X, Z, T)y$$

and

$$\frac{T - Q(X, h)}{X^m} = c(X, Z, T) + d(X, Z, T)y.$$

Thus, $Z_1, T_1 \in A[W]$. Hence $B \subseteq A[W]$. Since $B = k[X]^{[3]}$, the result follows. $\square$

The next theorem highlights the non-triviality of Asanuma threefolds for $m > 1$.

Theorem 4.3. *Let k be any field of characteristic $p(>0)$ and $f(Z, T) \in k[Z, T]$ be such that*

$$k[Z, T]/(f) = k^{[1]} \quad \text{but } k[Z, T] \neq k[f]^{[1]}.$$

Let

$$A = k[X, Y, Z, T]/(X^m Y - F(X, Z, T)) \quad \text{where } m > 1 \text{ and } F(0, Z, T) = f(Z, T).$$

Then $A \ncong_k k^{[3]}$.

Proof. Follows from Lemma 2.7 and Proposition 3.7(ii). $\square$

Corollary 4.4. *Zariski's cancellation conjecture does not hold for any Asanuma threefold A defined by $A = k[X, Y, Z, T]/(X^m Y - F(X, Z, T))$, where $m > 1$ and $F(0, Z, T)$ is any non-trivial line in $k[Z, T]$.*

Proof. Follows from Theorems 4.2 and 4.3. $\square$

Remark 4.5. Theorem 4.3 gives us a better understanding of the main theorem in [Gup14]; the *arguments* are now independent of the characteristic of the field k. However, the *hypotheses* of Theorem 4.3 are fulfilled only for characteristic $p > 0$ since, by a famous theorem of Abhyankar and Moh and of Suzuki [Abh77, Corollary 9.21], there does not exist any non-trivial line in $k^{[2]}$ when $\operatorname{ch} k = 0$. As mentioned earlier, when $\operatorname{ch} k = p > 0$, we do have non-trivial lines (e.g., the Segre–Nagata lines $f(Z, T) = Z^{p^e} + T + T^{sp}$, where $p^e \nmid sp$ and $sp \nmid p^e$).

In the rest of this section (except for Remark 4.7) we shall consider an affine k-domain A satisfying the hypotheses of Theorem 4.3 and use the notation x, y, z and t to denote the images in A of X, Y, Z and T, respectively. We first compute the Derksen invariant and the Makar-Limanov invariant of A.

Lemma 4.6. *Let A be as in Theorem 4.3. Then $\mathrm{DK}(A) = k[x, z, t]$ and $\mathrm{ML}(A) = k[x]$.*

https://doi.org/10.1112/S0010437X13007793 Published online by Cambridge University Press

Proof. Since $f(Z, T)$ is a non-trivial line, $\mathrm{DK}(A) = k[x, z, t]$ by Proposition 3.7(ii). By Lemma 3.6, $\mathrm{ML}(A) \subseteq k[x]$. We now show that $k[x] \subseteq \mathrm{ML}(A)$. Let ϕ be any non-trivial exponential map on A. We show that $x \in A^\phi$. Since $\operatorname{tr.deg}_k A^\phi = 2$, there exist two algebraically independent elements $\alpha, \beta \in A^\phi \subset \mathrm{DK}(A) = k[x, z, t]$. Let

$$\alpha = x\alpha_1(x, z, t) + \alpha_0(z, t) \quad \text{and} \quad \beta = x\beta_1(x, z, t) + \beta_0(z, t) \quad \text{for some } \alpha_0, \alpha_1, \beta_0, \beta_1 \in k^{[3]}.$$

Suppose, if possible, that $\alpha_0(z, t)$ and $\beta_0(z, t)$ are algebraically independent over k. Consider the proper $\mathbb{Z}$-filtration $\{A_n\}_{n\in\mathbb{Z}}$ on A and the induced graded ring $B = \mathrm{gr}(A)$ of Lemma 3.3. For $h \in A$, let $\bar{h}$ denote the image of h in B. Then $\bar{\alpha} = \alpha_0(\bar{z}, \bar{t})$ and $\bar{\beta} = \beta_0(\bar{z}, \bar{t})$. Now, by Theorem 2.11, ϕ induces a non-trivial homogeneous exponential map $\bar{\phi}$ on B such that $k[\alpha_0(\bar{z}, \bar{t}), \beta_0(\bar{z}, \bar{t})] \subseteq B^{\bar{\phi}}$. By the structure of B, $k[\bar{z}, \bar{t}] \cong k^{[2]}$ and hence $\alpha_0(\bar{z}, \bar{t})$ and $\beta_0(\bar{z}, \bar{t})$ are algebraically independent over k. By Lemma 2.8(ii), it follows that $k[\bar{z}, \bar{t}] \subseteq B^{\bar{\phi}}$. Since $\bar{x}^m\bar{y} = f(\bar{z}, \bar{t}) \in B^{\bar{\phi}}$, we have $\bar{x}, \bar{y} \in B^{\bar{\phi}}$ by Lemma 2.8(i). But this contradicts the fact that $\bar{\phi}$ is non-trivial.

Hence, $\alpha_0(z, t)$ and $\beta_0(z, t)$ are algebraically dependent. Thus, there exists a polynomial $H \in k^{[2]}$ such that $H(\alpha_0, \beta_0) = 0$. Therefore, $H(\alpha, \beta) \in xk[x, z, t] \subset xA$. Since $H(\alpha, \beta) \in A^\phi$, we have $x \in A^\phi$ by Lemma 2.8(i). Thus $k[x] \subseteq \mathrm{ML}(A)$. □

Remark 4.7. (1) Let A be the coordinate ring of the affine threefold $x^m y = F(x, z, t)$, where $m > 1$ and $F(0, z, t)$ is a line in $k[z, t]$. Then it follows from Lemma 2.7, Theorem 3.11 and Lemma 4.6 that either $\mathrm{DK}(A) = A$ (respectively, $\mathrm{ML}(A) = k$) or $\mathrm{DK}(A) = k[x, z, t]$ (respectively, $\mathrm{ML}(A) = k[x]$), according as $A = k[x]^{[2]}$ or $A \neq k[x]^{[2]}$.

(2) Some of our results on Asanuma threefolds (stated for $m \geqslant 2$) are not true for $m = 1$. For instance, in contrast to Lemma 4.6, the Derksen and Makar-Limanov invariants of an Asanuma threefold $A = k[X, Y, Z, T]/(XY - F(X, Z, T))$ are always trivial, i.e., $\mathrm{DK}(A) = A$ and $\mathrm{ML}(A) = k$. To see this, recall that $k[x, z, t] \subseteq \mathrm{DK}(A)$ and $\mathrm{ML}(A) \subseteq k[x]$ by Lemma 3.6. Interchanging the role of x with y (which we can do only for $m = 1$), we also have $k[y, z, t] \subseteq \mathrm{DK}(A)$ and $\mathrm{ML}(A) \subseteq k[y]$. Therefore, $\mathrm{DK}(A) = k[x, z, t, y] = A$ and $\mathrm{ML}(A) = k[x] \cap k[y] = k$. We still do not know whether A is isomorphic to $k^{[3]}$.

The following result shows that the ring of invariants of any non-trivial $\mathbb{G}_a$-action on an Asanuma threefold A ($m > 1$) is a polynomial ring in two variables.

COROLLARY 4.8. *Let A be as in Theorem 4.3 and ϕ be a non-trivial exponential map on A. Then $A^\phi = k[x]^{[1]}$.*

Proof. By Lemma 4.6, $x \in A^\phi$. Thus ϕ extends to a non-trivial exponential map ϕ_1 on $A_1 := A[x^{-1}] = k[x, x^{-1}, z, t]$. Since $k[x, x^{-1}]$ is a UFD and ${A_1}^{\phi_1}$ is factorially closed in $A_1 (= k[x, x^{-1}]^{[2]})$, we have ${A_1}^{\phi_1} = k[x, x^{-1}]^{[1]}$ by Theorem 2.6. By Lemma 2.8(vi), we have $A^\phi[x^{-1}] = {A_1}^{\phi_1} = k[x, x^{-1}]^{[1]}$. By Lemma 3.2, x is prime in A and $A/xA = k^{[2]}$. Since A^ϕ is factorially closed in A, we have that x is prime in A^ϕ and $xA \cap A^\phi = xA^\phi$ and hence

$$k \hookrightarrow A^\phi/xA^\phi \hookrightarrow A/xA = k^{[2]}.$$

Thus, $k[x]/(x)$ is algebraically closed in A^ϕ/xA^ϕ. Hence, by Theorem 2.2, $A^\phi = k[x]^{[1]}$. □

We shall now describe a necessary and sufficient condition for an endomorphism of an Asanuma threefold A (for $m > 1$) to be an automorphism of A.

PROPOSITION 4.9. *Let A be as in Theorem 4.3 and $\psi \in \mathrm{Aut}_k(A)$. Then:*

(i) *$\psi(k[x]) = k[x]$ and $\psi(k[x, z, t]) = k[x, z, t]$.*

(ii) *$\psi(I) = I$, where I is the ideal $(x^m, F(x, z, t))$.*

Proof. (i) Let ψ' be the extension of ψ to $A[U]$ defined by $\psi'(U) = U$. Then, for any $\phi \in \mathrm{EXP}(A)$, $\psi'\phi\psi^{-1}$ is also an exponential map on A and hence

$$\psi(\mathrm{DK}(A)) \subseteq \mathrm{DK}(A) \quad \text{and} \quad \psi(\mathrm{ML}(A)) \subseteq \mathrm{ML}(A).$$

Thus, $\psi(k[x,z,t]) \subseteq k[x,z,t]$ and $\psi(k[x]) \subseteq k[x]$ by Lemma 4.6. Since ψ is an automorphism we have $\psi(k[x]) = k[x]$ and $\psi(k[x,z,t]) = k[x,z,t]$.

(ii) Since ψ restricts to an automorphism of $k[x]$, we see that $\psi(x) = \lambda x + \mu$ for some $\lambda \in k^*$ and $\mu \in k$. Now $\psi(y) = F(\psi(x), \psi(z), \psi(t))/(\psi(x))^m$. Since $\psi(y) \in A \subseteq k[x, x^{-1}, z, t]$, there exists an integer $i > 0$ such that $x^i\psi(y) \in k[x,z,t]$. Hence, $x^iF(\psi(x),\psi(z),\psi(t))/(\lambda x+\mu)^m \in k[x,z,t]$. If $\mu \neq 0$, then $(\lambda x + \mu)^m \mid F(\psi(x), \psi(z), \psi(t))$ in $k[x,z,t]$, which would imply that $\psi(y) \in k[x, z, t]$ and hence $\psi(A) \subseteq k[x,z,t]$, a contradiction. Thus, $\psi(x) = \lambda x$ and $\psi(y) = F(\psi(x), \psi(z), \psi(t))/\lambda^m x^m$.

Note that $x^mA \cap k[x,z,t] = I$. Thus, $\lambda^m x^m \psi(y) = F(\psi(x), \psi(z), \psi(t)) \in I$ and hence, $\psi(I) \subseteq I$. Since ψ is an automorphism, we have $\psi(I) = I$. $\square$

We now prove the converse of Proposition 4.9.

Proposition 4.10. *Let A be as in Theorem 4.3 and ψ be an endomorphism of the ring A satisfying* (i) *and* (ii) *of Proposition 4.9. Then ψ is an automorphism of the ring A.*

Proof. Since $k[\psi(x)] = k[x]$ and $\psi(I) = I$, we must have $\psi(x) = \lambda x$ for some $\lambda \in k^*$. Since $\psi(k[x,z,t]) = k[x,z,t]$, we have that ψ is injective. Therefore, it is enough to show that $y = F(x, z,t)/x^m \in \psi(k[x,z,t,y])$, i.e., $y \in k[x,z,t,\psi(y)]$. Since $F(x,z,t) \in \psi(I)$, we have $F(x,z,t) = \alpha' x^m + \beta' F(\psi(x), \psi(z), \psi(t))$ for some $\alpha', \beta' \in k[x,z,t]$. Hence $y = F(x,z,t)/x^m = \alpha' + \beta'\lambda^m\psi(y) \in k[x,z,t,\psi(y)]$. $\square$

Finally, we investigate the isomorphism classes of Asanuma threefolds of the form

$$A(m,f) := k[X,Y,Z,T]/(X^mY - f(Z,T)) \quad \text{where } m \geqslant 2,$$

k is a field of positive characteristic and f is a non-trivial line in $k[Z,T]$. By Theorem 4.2, we have $A(m,f)^{[1]} \cong k^{[4]}$. The next result describes the condition when two such rings are isomorphic. (In fact the proof will also show that two affine threefolds $x^my = F(x,z,t)$ and $x^ny = G(x,z,t)$ will be isomorphic only if $m = n$ and there exists an automorphism θ of $k[z,t]$ satisfying $\theta(F(0,z,t)) = \epsilon G(0,z,t)$ for some $\epsilon \in k^*$.)

Theorem 4.11. *$A(m,f)$ is isomorphic to $A(n,g)$ if and only if $m = n$ and there exists a k-algebra automorphism θ of $k[Z,T]$ such that $\theta(g) = \delta f$ for some $\delta \in k^*$.*

Proof. Clearly, if $m = n$ and $\theta(g) = \epsilon f$ for some $\theta \in \mathrm{Aut}_k(k[Z,T])$ and $\epsilon \in k^*$, then $A(m,f) \cong A(n,g)$.

Set $A := A(m,f) = k[x,y,z,t]$ and $B := A(n,g) = k[x_1,y_1,z_1,t_1]$, where x,y,z,t (respectively, x_1,y_1,z_1,t_1) denote the images of X,Y,Z,T in A (respectively, B).

Suppose that there exists a k-algebra isomorphism $\psi : B \longrightarrow A$. Replacing B by $\psi(B)$, we may assume that $B = A$. By Lemma 4.6, we have $\mathrm{ML}(A) = k[x] = k[x_1]$ and $\mathrm{DK}(A) = k[x, z,t] = k[x_1,z_1,t_1]$. Hence, $x_1 = \lambda x + \mu$ for some $\lambda \in k^*$ and $\mu \in k$. Now $y_1 = g(z_1,t_1)/{x_1}^n$ and $y_1 \in k[x,z,t,x^{-1}]$. Hence, there exists an integer $i \geqslant 0$ such that $x^iy_1 \in k[x,z,t]$, i.e., $x^ig(z_1,t_1)/(\lambda x+\mu)^n \in k[x,z,t]$. If $\mu \neq 0$, then $(\lambda x+\mu)^n \mid g(z_1,t_1)$ in $k[x,z,t]$, which implies that $y_1 \in k[x, z, t]$. Thus $k[x_1, z_1, t_1, y_1] \subseteq k[x, z, t]$, a contradiction. Therefore, $x_1 = \lambda x$ for some $\lambda \in k^*$. Now, since $xA \cap k[x, z, t] = x_1 A \cap k[x_1, z_1, t_1]$, we have

$$(x, f(z, t))k[x, z, t] = (x_1, g(z_1, t_1))k[x, z, t]. \tag{12}$$

Therefore, since $x_1 = \lambda x$, by (12), we have $f(z, t) = \epsilon g(z_1, t_1) + x_1 g_1(x_1, z_1, t_1)$ for some $\epsilon \in k^*$ and $g_1(x_1, z_1, t_1) \in k[x_1, z_1, t_1]$.

Suppose, if possible, that $m > n$. Set $m := nq - r$, where $q, r \in \mathbb{Z}_{\geqslant 0}$ and $0 \leqslant r < n$. Note that $q > 1$. Since $y \in A = k[x_1, y_1, z_1, t_1]$, we have

$$y = \frac{f(z, t)}{x^m} = h_1(x_1, z_1, t_1) + \sum_{0 \leqslant i < n, 0 < j} h_{ij}(z_1, t_1) {x_1}^i {y_1}^j,$$

where $h_1(x_1, z_1, t_1) \in k[x_1, z_1, t_1]$ and $h_{ij}(z_1, t_1) \in k[z_1, t_1]$. Thus,

$$\frac{f(z, t)}{x^m} = \epsilon \lambda^m \frac{g(z_1, t_1)}{{x_1}^m} + \lambda^m \frac{g_1(x_1, z_1, t_1)}{{x_1}^{m-1}} = h_1(x_1, z_1, t_1) + \sum_{0 \leqslant i < n, 0 < j} h_{ij}(z_1, t_1) {x_1}^i \left(\frac{g(z_1, t_1)}{{x_1}^n} \right)^j.$$

Comparing the coefficient of ${x_1}^{-m}$ from both sides, we get $\epsilon \lambda^m g(z_1, t_1) = h_{rq}(z_1, t_1)(g(z_1, t_1))^q$. Since $q > 1$ and $g(z_1, t_1) \notin k^*$, we have a contradiction. Hence $m \leqslant n$. Similarly, we have $n \leqslant m$. Therefore, $m = n$.

Let $z_1 = \alpha(x, z, t)$ and $t_1 = \beta(x, z, t)$. Then, since $k[x, z, t] = k[x_1, z_1, t_1]$ and $x_1 = \lambda x$, we have $k[z, t] = k[\alpha(0, z, t), \beta(0, z, t)]$. Now, by (12), we have $f(z, t) = \epsilon g(\alpha(0, z, t), \beta(0, z, t))$. Consider the automorphism θ of the the ring $k[z, t]$ such that $\theta(z) = \alpha(0, z, t)$ and $\theta(t) = \beta(0, z, t)$. Then $\theta(g) = \epsilon^{-1} f$. Hence the result. ◻

Thus the Asanuma threefolds provide an infinite family of non-isomorphic affine rings which are stably isomorphic to $k^{[3]}$.

Acknowledgements

The author thanks Professors Shrikant M. Bhatwadekar and Amartya K. Dutta for carefully going through the earlier draft and suggesting simplifications and refinements. She is grateful to Professor T. Asanuma for his lectures on affine fibrations during his visits to ISI Kolkata in February 2011 and March 2012. The author thanks Professor Russell for his question which led to Theorem A and for drawing her attention to the reference [KVZ04]. The author also thanks the referee for drawing her attention to the papers [Gan11, Seg56, VD74] and for useful suggestions.

References

Abh77 S. Abhyankar, *Lectures on expansion techniques in algebraic geometry* (Notes by Balwant Singh), Tata Institute of Fundamental Research Lectures on Mathematics and Physics, vol. 57 (Tata Institute of Fundamental Research, Bombay, 1977).

AEH72 S. Abhyankar, P. Eakin and W. Heinzer, *On the uniqueness of the coefficient ring in a polynomial ring*, J. Algebra **23** (1972), 310–342.

Asa87 T. Asanuma, *Polynomial fibre rings of algebras over Noetherian rings*, Invent. Math. **87** (1987), 101–127.

Asa94 T. Asanuma, *Non-linearizable algebraic group actions on* $\mathbb{A}^n$, J. Algebra **166** (1994), 72–79.

BD94 S. M. Bhatwadekar and A. K. Dutta, *Linear planes over a discrete valuation ring*, J. Algebra **166** (1994), 393–405.

Cra05 A. J. Crachiola, *The hypersurface $x + x^2y + z^2 + t^3 = 0$ over a field of arbitrary characteristic*, Proc. Amer. Math. Soc. **134** (2005), 1289–1298.

DHM01 H. Derksen, O. Hadas and L. Makar-Limanov, *Newton polytopes of invariants of additive group actions*, J. Pure Appl. Algebra **156** (2001), 187–197.

FR05 G. Freudenburg and P. Russell, *Open problems in affine algebraic geometry*, in *Affine algebraic geometry*, Contemporary Mathematics, vol. 369 (American Mathematical Society, Providence, RI, 2005), 1–30.

Fuj79 T. Fujita, *On Zariski problem*, Proc. Japan Acad. **55** (1979), 106–110.

Gan11 R. Ganong, *The pencil of translates of a line in the plane*, in *Affine algebraic geometry*, CRM Proceedings Lecture Notes, vol. 54 (American Mathematical Society, Providence, RI, 2011), 57–71.

Gup14 N. Gupta, *On the cancellation problem for the affine space $\mathbb{A}^3$ in characteristic p*, Invent. Math. **195** (2014), 279–288, doi:10.1007/s00222-013-0455-2.

Kal02 S. Kaliman, *Polynomials with general $\mathbb{C}^2$-fibers are variables*, Pacific J. Math. **203** (2002), 161–190.

KVZ04 S. Kaliman, S. Vénéreau and M. Zaidenberg, *Simple birational extensions of the polynomial algebra $\mathbb{C}^{[3]}$*, Trans. Amer. Math. Soc. **356** (2004), 509–555.

Mak96 L. Makar-Limanov, *On the hypersurface $x + x^2y + z^2 + t^3 = 0$ in $\mathbb{C}^4$ or a $\mathbb{C}^3$-like threefold which is not $\mathbb{C}^3$*, Israel J. Math. B **96** (1996), 419–429.

Mak01 L. Makar-Limanov, *On the group of automorphism of a surface $x^ny = P(z)$*, Israel J. Math. **121** (2001), 113–123.

MS80 M. Miyanishi and T. Sugie, *Affine surfaces containing cylinderlike open sets*, J. Math. Kyoto Univ. **20** (1980), 11–42.

Nag72 M. Nagata, *On automorphism group of $k[x, y]$*, Lectures in Mathematics, vol. 5 (Kyoto University, Tokyo, 1972).

Rus76 P. Russell, *Simple birational extensions of two dimensional affine rational domains*, Compositio Math. **33** (1976), 197–208.

Rus81 P. Russell, *On affine-ruled rational surfaces*, Math. Ann. **255** (1981), 287–302.

RS79 P. Russell and A. Sathaye, *On finding and cancelling variables in $k[X, Y, Z]$*, J. Algebra **57** (1979), 151–166.

Sat76 A. Sathaye, *On linear planes*, Proc. Amer. Math. Soc. **56** (1976), 1–7.

Sat83 A. Sathaye, *Polynomial ring in two variables over a D.V.R.: A criterion*, Invent. Math. **74** (1983), 159–168.

Seg56 B. Segre, *Corrispondenze di Möbius e trasformazioni cremoniane intere*, Atti Accad. Sci. Torino. Cl. Sci. Fis. Mat. Natur. **91** (1956/1957), 3–19 (in Italian).

Sri08 V. Srinivas, *Algebraic K-theory* (Birkhäuser, Boston, 2008).

VD74 B. Veĭsfeĭler and I. V. Dolgačev, *Unipotent group schemes over integral rings*, Izv. Akad. Nauk SSSR Ser. Mat. **38** (1974), 757–799 (in Russian).

Neena Gupta neenag@isical.ac.in, rnanina@gmail.com
Stat-Math Unit, Indian Statistical Institute, 203 B.T. Road,
Kolkata 700108, India