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SOME CRITERIA FOR SOLVABILITY AND NILPOTENCY OF FINITE GROUPS BY STRONGLY MONOLITHIC CHARACTERS

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Abstract

Gagola and Lewis [‘A character theoretic condition characterizing nilpotent groups’, *Comm. Algebra* **27** (1999), 1053–1056] proved that a finite group G is nilpotent if and only if $\chi(1)^2$ divides $|G : \ker\chi|$ for every irreducible character χ of G. The theorem was later generalised by using monolithic characters. We generalise the theorem further considering only strongly monolithic characters. We also give some criteria for solvability and nilpotency of finite groups by their strongly monolithic characters.

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1. Introduction

In this paper, all groups under consideration are finite and all characters are complex. Notation is standard and taken from [5]. Let G be a finite group and $\chi \in \mathrm{Irr}(G)$, where $\mathrm{Irr}(G)$ denotes the set of all irreducible complex characters of G. We say that χ is a monolithic character if $G/\ker\chi$ has a unique minimal normal subgroup. These characters play an important role in understanding the structure of finite groups (for more details, see [2]).

A monolithic character χ of G is said to be a strongly monolithic character if either $Z(\chi) = \ker\chi$ or $G/\ker\chi$ is a p-group whose commutator subgroup is its unique minimal normal subgroup. The strongly monolithic characters of a group were first studied in [3]. It is proven in [3] that linear characters of a group are not strongly monolithic and every nonabelian group has at least one strongly monolithic character. Moreover, by [5, Lemma 12.3], the nonlinear irreducible characters whose kernels are maximal among the kernels of all nonlinear irreducible characters of a nonabelian solvable group are monomial and strongly monolithic. Thus, every nonabelian solvable

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group has at least one monomial strongly monolithic character. We use the notation $\mathrm{cd}_{\mathrm{sm}}(G)$ and $\ker_{\mathrm{msm}}(G)$ to denote the set of all strongly monolithic character degrees of G and the set of all monomial strongly monolithic character kernels of G, respectively. Given $N \lhd G$, we write $\mathrm{cd}(G|N)$ to denote the set of all character degrees of $\mathrm{Irr}(G|N)$, where the set $\mathrm{Irr}(G|N) = \mathrm{Irr}(G) - \mathrm{Irr}(G/N)$.

Berkovich proved in [1] that a finite group G is solvable whenever $|\mathrm{cd}_{\mathrm{m}}(G)| \le 3$, where $\mathrm{cd}_{\mathrm{m}}(G)$ is the set of all monolithic character degrees of G. Gagola and Lewis proved in [4] that a group G is nilpotent if and only if $\chi(1)^2$ divides $|G : \ker\chi|$ for every irreducible character χ of G. Later, Lu *et al.* [10] generalised this theorem for monolithic characters. Motivated by these papers, we give some criteria for solvability and nilpotency of finite groups by their strongly monolithic characters.

THEOREM 1.1. *Let G be a nonabelian group. Then G is solvable if one of the following conditions holds:*

(i) $|\mathrm{cd}_{\mathrm{sm}}(G)| \le 2$*;*
(ii) $\chi(1)$ *is a prime for every strongly monolithic character* χ *of G;*
(iii) *every strongly monolithic character of G is monomial.*

The next result generalises the theorem of Gagola and Lewis in [4] by considering only strongly monolithic characters.

THEOREM 1.2. *Let G be a nonabelian group.*

(i) *G is nilpotent if and only if* $\chi(1)^2$ *divides* $|G : \ker\chi|$ *for every strongly monolithic character* χ *of G.*
(ii) *If G is solvable, then G is nilpotent if and only if* $\chi(1)^2$ *divides* $|G : \ker\chi|$ *for every monomial strongly monolithic character* χ *of G.*

THEOREM 1.3. *Let G be a solvable group.*

(i) *If* $|\ker_{\mathrm{msm}}(G)| = 1$*, then* G' *is nilpotent.*
(ii) *If* $|\ker_{\mathrm{msm}}(G)| = 2$*, then* G' *is meta-nilpotent.*

2. Proofs of the main theorems

Let G be a group, $N \lhd G$ and $\chi \in \mathrm{Irr}(G)$ such that $N \subseteq \ker\chi$. Since there exists a one-to-one correspondence between irreducible characters of G/N and irreducible characters of G with kernel containing N, it follows that χ is a strongly monolithic character of G if and only if it is a strongly monolithic character of G/N. We use this fact in the proofs.

PROOF OF THEOREM 1.1. Let G be a minimal counterexample for all the situations. Suppose that G has two distinct minimal normal subgroups N_1 and N_2. Then, both G/N_1 and G/N_2 are solvable groups by induction. However, G is isomorphic to a subgroup of $G/N_1 \times G/N_2$, in contrast to the assumption that G is not solvable. Thus, G has a unique minimal normal subgroup in every situation of Theorem 1.1 and so

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it has at least one faithful irreducible character. Let N be the unique minimal normal subgroup of G. It follows that N is not solvable and $N' = N$ because G/N is a solvable group by induction. Moreover, all faithful irreducible characters of G are strongly monolithic since $Z(G) = 1$.

Suppose that $|\mathrm{cd}_{\mathrm{sm}}(G)| \leq 2$. Then $|\mathrm{cd}(G|N)| \leq 2$ since $\mathrm{cd}(G|N) \subseteq \mathrm{cd}_{\mathrm{sm}}(G)$. It follows from [8, Theorem B] that N is solvable. This contradiction completes the proof of (i).

Now, let $\nu \in \mathrm{Irr}(N)$ and $\nu(1) > 1$. Then, there exists an irreducible character ψ of G such that $[\psi, \nu^G] \neq 0$. Thus, $\ker \psi_N = N \cap \ker \psi \leq \ker \nu < N$ because $[\psi_N, \nu] \neq 0$ by Frobenius reciprocity. Hence, $\ker \psi = 1$ since N is the unique minimal normal subgroup of G and $\ker \psi_N < N$. Therefore, ψ is a strongly monolithic character of G. Also, from Clifford's theorem, $\nu(1) \mid \psi(1)$. In the case (ii), $\nu(1) = \psi(1)$ and the degrees of all nonlinear irreducible characters of N are prime numbers. It follows from [9] that N is solvable, which is a contradiction. This completes the proof of (ii).

In the case (iii), N must be a proper subgroup of G. Otherwise, all nonlinear irreducible characters of G would be faithful and so strongly monolithic. However, this is a contradiction since all M-groups are solvable. Let χ be a faithful irreducible character of G of least degree. Then χ is a strongly monolithic character and so monomial by hypothesis. So there exists a linear character λ of a subgroup H of G such that $\lambda^G = \chi$. Since $[G : H] = \chi(1) > 1$, it follows that $H < G$ and $(1_H)^G$ is reducible. Let θ be an irreducible constituent of $(1_H)^G$, so that $\theta(1) < \chi(1)$. By the choice of χ, we deduce that $N \leq \ker \theta$. Moreover, $N \leq \ker (1_H)^G \leq H$ because $N \leq \ker \theta$ for every irreducible constituent θ of $(1_H)^G$. It follows that $N \leq \ker \lambda$ since $N = N'$ and λ is a linear character of H. Furthermore, $N \leq (\ker \lambda)^g$ for every $g \in G$, since $N \lhd G$. This gives the contradiction that $N \leq \ker(\lambda^G) = \ker \chi = 1$. This contradiction completes the proof of (iii). □

A group G may not be solvable when $|\mathrm{cd}_{\mathrm{sm}}(G)| > 2$. For instance, $|\mathrm{cd}_{\mathrm{sm}}(S_5)| = 3$, where S_5 is the symmetric group on five letters. Additionally, a strongly monolithic character of a solvable group may not be monomial. For example, the semidirect product $\mathrm{He}_3 \rtimes \mathrm{SD}_{16}$ (SmallGroup(432,520) in GAP), where the semidihedral group SD_{16} acts faithfully on the Heisenberg group He_3, has some strongly monolithic characters which are not monomial.

Proof of Theorem 1.2. If G is a nilpotent group, then by [4, Theorem A], $\chi(1)^2$ divides $|G : \ker \chi|$ for every $\chi \in \mathrm{Irr}(G)$. Thus, it is clear that the necessary conditions in both cases of Theorem 1.2 hold.

Let G be a minimal counterexample for the sufficient conditions in both cases of the theorem. Since the hypotheses of Theorem 1.2 are inherited by G/N for all $N \lhd G$, it follows that G has a unique minimal normal subgroup M and therefore G has at least one faithful irreducible character. Moreover, $Z(G) = 1 = \Phi(G)$ since G is a minimal counterexample. Thus all faithful irreducible characters of G are strongly monolithic. By the hypothesis of case (i) in the theorem, $\theta(1)^2 \mid |G : \ker \theta|$ for every faithful irreducible character θ of G. Moreover, $\chi(1)^2 \mid |G : \ker \chi|$ for every $\chi \in \mathrm{Irr}(G)$,

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because G/M is nilpotent by induction. Thus, by [4, Theorem A], G is nilpotent, which is a contradiction. This contradiction completes the proof of (i).

Since $\Phi(G) = 1$, there exists a subgroup H of G such that $G = MH$ and $M \cap H = 1$. Then H is nilpotent by induction. Using Gaschütz's theorem, $(|M|, |H|) = 1$. It follows from [6, Theorem B] that there exists a $\lambda \in \mathrm{Irr}(M)$ with $\lambda \neq 1$ such that $|I_H(\lambda)| \leq (|H|/q)^{1/q}$, where $I_H(\lambda)$ is the inertia group of λ in H and q is the smallest prime divisor of $|H|$. This gives the inequality:

$$|I_H(\lambda)| \leq \left(\frac{|H|}{q}\right)^{1/q} \leq \left(\frac{|H|}{2}\right)^{1/2} < |H|^{1/2}.$$

From [5, Problem 6.18], there exists a linear character $\psi \in \mathrm{Irr}(I_G(\lambda))$ such that $\psi_M = \lambda$, where $I_G(\lambda)$ is the inertia group of λ in G. This implies that ψ^G is a monomial irreducible character of G. Let $\chi = \psi^G$. Then χ is a faithful irreducible character of G. Otherwise, $M \leq \ker\chi = \bigcap_{g\in G}(\ker\psi)^g \leq \ker\psi$. However, this is a contradiction since $\psi_M = \lambda \neq 1$. Therefore, χ is a monomial strongly monolithic character of G. By the hypothesis of (ii), $\chi(1)^2 \mid |G|$. Also $\chi(1) \mid |H|$ by Ito's theorem (see [5]). Then, $|G| = |M||I_H(\lambda)|\chi(1) < |M||H| = |G|$, which is a contradiction. This completes the proof of (ii). □

We note that the commutator subgroup G' of a solvable group G may not be nilpotent when G has only one maximal kernel among the kernels of the strongly monolithic characters of G. For example, all nonlinear irreducible characters of the symmetric group S_4 are strongly monolithic and there is only one maximal kernel among the kernels of these characters. However, the alternating group A_4 is not nilpotent. Also the commutator subgroup G' of a solvable group G may not be abelian when $|\ker_{\mathrm{msm}}(G)| = 1$. For instance, $\mathrm{SL}(2,3)$ has only one (monomial) strongly monolithic character and its commutator subgroup is not abelian.

PROOF OF THEOREM 1.3. Let G be a minimal counterexample to the case (i). Assume that G has two distinct minimal normal subgroups, say N_1 and N_2. It follows that $|\ker_{\mathrm{msm}}(G/N_i)| \leq 1$ for $i = 1, 2$. Since a solvable group does not have any monomial strongly monolithic characters if and only if it is abelian, the commutator subgroups of G/N_1 and G/N_2 are nilpotent by induction. Thus, G' is nilpotent since G is isomorphic to a subgroup of $G/N_1 \times G/N_2$. This contradiction shows that G has a unique minimal normal subgroup and it has at least one faithful irreducible character.

Let N be the unique minimal normal subgroup of G. Assume that $Z(G) > 1$. Then $G'Z(G)/Z(G)$, the commutator subgroup of $G/Z(G)$, is nilpotent by induction. Therefore, G' is also nilpotent, which is a contradiction. Thus, $Z(G)$ must be trivial and so all faithful irreducible characters of G must be strongly monolithic. However, $\Phi(G) = 1$. Otherwise, $G'\Phi(G)/\Phi(G)$ would be nilpotent by induction. Thus, $G'\Phi(G)$ would be nilpotent by [7, Theorem 8.24], which is a contradiction. Since $\Phi(G) = 1$, there is a subgroup H of G such that $G = NH$ and $N \cap H = 1$. Let $1_N \neq \lambda \in \mathrm{Irr}(N)$. From [5, Problem 6.18], there exists a linear character $\theta \in \mathrm{Irr}(I_G(\lambda))$ such that $\theta_N = \lambda$, where $I_G(\lambda)$ is the inertia group of λ in G. This implies that $\theta^G \in \mathrm{Irr}(G)$ is a faithful

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monomial strongly monolithic character of G since $\ker\theta^G = 1$. Since $|\ker_{\mathrm{msm}}(G)| = 1$, the maximal kernel among the kernels of all nonlinear irreducible characters of G must be trivial. It follows that all nonlinear irreducible characters of G are faithful. Thus, G' is the unique minimal normal subgroup of G, which is a contradiction. This completes the proof of (i).

To prove (ii), let G again be a minimal counterexample. Then G has a unique minimal normal subgroup N. Assume that $Z(G) > 1$. Then $G'Z(G)/Z(G)$ is meta-nilpotent by induction. Thus, there is a normal subgroup A of $G'Z(G)$ such that both $G'Z(G)/A$ and $A/Z(G)$ are nilpotent. Since $A/Z(G)$ is nilpotent, A is nilpotent and so $A \cap G'$ is nilpotent. Moreover, $G'/(A \cap G')$ is nilpotent because $G'/(A \cap G') \cong G'Z(G)/A$. Then G' is also meta-nilpotent, which is a contradiction. Hence, $Z(G) = 1$. Similarly, it is clear that $\Phi(G) = 1$. Therefore, G has a faithful monomial strongly monolithic character χ as in the proof of (i). Since $|\ker_{\mathrm{msm}}(G)| = 2$, we see that $|\ker_{\mathrm{msm}}(G/N)| = 1$. Thus, G'/N is nilpotent by (i) of the theorem, that is, G' is meta-nilpotent. This contradiction completes the proof of (ii). □

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