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ON THE LOWEST EIGENVALUE OF THE FRACTIONAL LAPLACIAN FOR THE INTERSECTION OF TWO DOMAINS

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Abstract

We extend a result of Lieb [‘On the lowest eigenvalue of the Laplacian for the intersection of two domains’, *Invent. Math.* **74**(3) (1983), 441–448] to the fractional setting. We prove that if A and B are two bounded domains in $\mathbb{R}^N$ and $\lambda_s(A)$, $\lambda_s(B)$ are the lowest eigenvalues of $(-\Delta)^s$, $0 < s < 1$, with Dirichlet boundary conditions, there exists some translation B_x of B such that $\lambda_s(A \cap B_x) < \lambda_s(A) + \lambda_s(B)$. Moreover, without the boundedness assumption on A and B, we show that for any $\varepsilon > 0$, there exists some translation B_x of B such that $\lambda_s(A \cap B_x) < \lambda_s(A) + \lambda_s(B) + \varepsilon$.

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1. Introduction

Let Ω be an open subset of $\mathbb{R}^N$. Denote by $\lambda(\Omega)$ the lowest eigenvalue of the Laplace operator $-\Delta$ in Ω with Dirichlet boundary conditions. It is well known that if $\lambda(\Omega)$ is small, then Ω must be ‘large’. For example, when Ω is empty, then $\lambda(\Omega) = +\infty$. The Faber–Krahn inequality for the Laplace operator states that

$$\lambda(\Omega) \geq C_N |\Omega|^{-2/N},$$

where C_N is the lowest eigenvalue of a ball with unit volume and $|\Omega|$ is the volume of Ω [3, 5]. In other words, among all domains with fixed volume, the ball has the smallest λ.

Geometrically, when $\lambda(\Omega)$ is small, Ω is not only large, but also ‘fat’ in some sense. As shown in [6], the inequality $\lambda(\Omega) \geq \alpha_N R^{-2}$, where R is the radius of the largest ball contained in Ω, is not true when $N > 1$. This implies that Ω need not contain any ball of fixed radius R, even if $\lambda(\Omega)$ is sufficiently small. Nevertheless, if $\lambda(\Omega)$ is small, then Ω contains ‘most of’ a ball of radius $R \sim \lambda^{-1/2}(\Omega)$. This assertion is derived from the following inequality: for any $\varepsilon \in (0, 1)$, there exists $\alpha_N(\varepsilon)$, with $\alpha_N(\varepsilon) \to 0$ as $\varepsilon \to 1$, such that

$$\lambda(\Omega) \geq \alpha_N(\varepsilon) R^{-2}, \tag{1.1}$$

where R is the largest radius such that $|\Omega \cap B_R| \geq \varepsilon |B_R|$ for some ball B_R. More precisely, fix $\varepsilon \in (0,1)$ (ε may be close to 1) and let $R > 0$ be the largest radius such that $|\Omega \cap B_R| \geq \varepsilon |B_R|$ for some ball B_R. By (1.1),

$$R > \sqrt{\alpha_N(\varepsilon)} \lambda^{-1/2}(\Omega).$$

This means that Ω contains 'most of' the ball $B_{\sqrt{\alpha_N(\varepsilon)}\lambda^{-1/2}(\Omega)}$ of large radius when $\lambda(\Omega)$ is small.

The inequality (1.1) follows from [6, Corollary 2] which is a consequence of the following theorem.

THEOREM 1.1. *Let A and B be nonempty open sets in $\mathbb{R}^N$, $N \geq 1$, and let $\lambda(A)$, $\lambda(B)$ be the corresponding lowest eigenvalues of the Laplacian $-\Delta$ with Dirichlet boundary conditions. Let $B_x = x + B$ denote B translated by $x \in \mathbb{R}^N$. Let $\varepsilon > 0$. Then, there exists an x such that*

$$\lambda(A \cap B_x) < \lambda(A) + \lambda(B) + \varepsilon. \tag{1.2}$$

If A and B are both bounded, then there is an $x \in \mathbb{R}^N$ such that

$$\lambda(A \cap B_x) < \lambda(A) + \lambda(B). \tag{1.3}$$

Moreover, the mapping $x \mapsto \lambda(A \cap B_x)$ is upper-semicontinuous, so that the set of x for which (1.2) or (1.3) holds is open.

Lieb [6] used Theorem 1.1 to prove two important corollaries: a lower bound for $\sup_x\{\text{Volume}(A \cap B_x)\}$ in terms of $\lambda(A)$ when B is a ball (see [6, Corollary 2]) and a compactness lemma for certain sequences in $W^{1,p}(\mathbb{R}^N)$ that under some conditions, a bounded sequence in $W^{1,p}(\mathbb{R}^N)$ has a nonzero weak limit (see [6, Lemma 6]).

Inspired by [6], our purpose in this paper is to generalise the main result in [6] for the fractional Laplacian $(-\Delta)^s$, $0 < s < 1$. The fractional Laplacian $(-\Delta)^s$ is defined as a nonlocal pseudodifferential operator on the space of rapidly decreasing functions by

$$(-\Delta)^s u(x) = c_{N,s} P.V. \int_{\mathbb{R}^N} \frac{u(x) - u(\xi)}{|x - \xi|^{N+2s}}\, d\xi = c_{N,s} \lim_{\varepsilon \to 0} \int_{\mathbb{R}^N \setminus B(x,\varepsilon)} \frac{u(x) - u(\xi)}{|x - \xi|^{N+2s}}\, d\xi,$$

where $c_{N,s}$ is a normalisation constant. By using the Fourier transform, the fractional Laplacian can also be defined as

$$\mathcal{F}((-\Delta)^s u)(\xi) = |\xi|^{2s} \mathcal{F} u(\xi).$$

In the distributional sense, the fractional Laplacian can be extended on $\mathcal{L}_s(\mathbb{R}^N)$, where

$$\mathcal{L}_s(\mathbb{R}^N) = \Big\{ u \in L^1_{\text{loc}}(\mathbb{R}^N) : \int_{\mathbb{R}^N} \frac{|u(x)|}{(|x| + 1)^{N+2s}}\, dx < \infty \Big\},$$

by

$$((-\Delta)^s u, \varphi) = (u, (-\Delta)^s \varphi), \quad \varphi \in C_0^\infty(\mathbb{R}^N).$$

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Some elementary properties of the fractional Laplacian can be found in [2]. Recent progress on the eigenvalue bounds for the fractional Laplacian and the fractional Schrödinger operator can be found in the survey [4] and the references given therein.

Denote by $\dot{H}^s(\mathbb{R}^N)$ the space of functions $f \in L^2(\mathbb{R}^N)$ such that

$$\|f\|^2_{\dot{H}^s(\mathbb{R}^N)} = \iint_{\mathbb{R}^{2N}} \frac{(f(x) - f(y))^2}{|x - y|^{N+2s}}\, dx\, dy < \infty. \tag{1.4}$$

For $A \subset \mathbb{R}^N$, we define the space $H^s_0(A)$ as the completion of $C^\infty_0(A)$ with norm $\|\cdot\|_{\dot{H}^s(\mathbb{R}^N)}$ and

$$\lambda_s(A) = \inf\{\|f\|^2_{\dot{H}^s(\mathbb{R}^N)} : f \in H^s_0(A), \|f\|_{L^2(A)} = 1 \text{ and } f \not\equiv 0\}.$$

We can now state our main result.

THEOREM 1.2. *Let* $0 < s < 1$. *Suppose that* A *and* B *are two nonempty open sets in* $\mathbb{R}^N$, $N \geq 1$. *Let* $\lambda_s(A)$ *and* $\lambda_s(B)$ *be the corresponding lowest eigenvalues of the fractional Laplacian* $(-\Delta)^s$ *with Dirichlet boundary conditions. Then, the following assertions hold.*

(i) *For any* $\varepsilon > 0$, *there exists an* $x \in \mathbb{R}^N$ *such that*

$$\lambda_s(A \cap B_x) < \lambda_s(A) + \lambda_s(B) + \varepsilon,$$

where B_x *is the translate of* B *by* x.

(ii) *In addition, if* A *and* B *are bounded, then there is an* $x \in \mathbb{R}^N$ *such that*

$$\lambda_s(A \cap B_x) < \lambda_s(A) + \lambda_s(B).$$

(iii) *The mapping* $x \mapsto \lambda_s(A \cap B_x)$ *is upper-semicontinuous.*

As applications of Theorem 1.2, by the same arguments as [6, Corollary 2], we can first prove a lower bound for $\sup_x\{\text{Volume}(A \cap B_x)\}$ in terms of $\lambda_s(A)$ when B is a ball.

COROLLARY 1.3. *Let* A *be a nonempty open set in* $\mathbb{R}^N$, $N \geq 1$. *Let* B_r *be a ball of radius* r. *Given* δ *with* $0 < \delta < 1$, *put*

$$\sigma_{N,s}(\delta) = \alpha_{N,s} w_N^{-2s/N}(\delta^{-2s/N} - 1) > 0,$$

where w_N *is the volume of the unit ball and* $\alpha_{N,s}$ *is the lowest eigenvalue of* $(-\Delta)^s$ *on a ball of unit volume. Suppose that for* $R > 0$,

$$\lambda_s(A) \leq \sigma_{N,s}(\delta) R^{-2s}.$$

Then, for every r *with* $0 < r < R$, *there is an* $x \in \mathbb{R}^N$ *and a ball* $B(x, r)$ *of radius* r *centred at* x *such that*

$$|A \cap B(x, r)| > \delta |B_r| = \delta w_N r^N.$$

Second, we can use Theorem 1.2 to give a different proof of the compactness lemma (see [1, Lemma 2.1]) in the space $H^s(\mathbb{R}^N)$ with $0 < s < 1$. The proof is similar to the one of [6, Lemma 6] and we omit the details. The proof of Theorem 1.2 is inspired

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by the approach in [6]. However, because of the presence of the fractional Laplacian, some difficulties arise. For instance, the product rule for derivatives cannot be applied and one needs to use delicate integral estimates (see the proof below).

The rest of this paper is devoted to the proof of Theorem 1.2.

2. Proof of Theorem 1.2

In this section, we give the proof of our main result. We begin with the first assertion in Theorem 1.2.

2.1. Proof of assertion (i). By the definition,

$$\begin{aligned}\lambda_s(A) &= \inf\{\|f\|^2_{\dot{H}^s(\mathbb{R}^N)} : f \in H^s_0(A), \|f\|_{L^2(A)} = 1 \text{ and } f \not\equiv 0\}\\ &= \inf\{\|f\|^2_{\dot{H}^s(\mathbb{R}^N)} : f \in C^\infty_0(A), \|f\|_{L^2(A)} = 1 \text{ and } f \not\equiv 0\},\end{aligned}$$

where we have used the density of $C^\infty_0(A)$ in $H^s_0(A)$.

For $\varepsilon > 0$, there are real-valued functions $f \in C^\infty_0(A)$ and $g \in C^\infty_0(B)$ such that

$$\|f\|_{L^2(A)} = 1, \quad \|g\|_{L^2(B)} = 1,$$

$$\|f\|^2_{\dot{H}^s(\mathbb{R}^N)} < \lambda_s(A) + \frac{\varepsilon}{2} \quad \text{and} \quad \|g\|^2_{\dot{H}^s(\mathbb{R}^N)} < \lambda_s(B) + \frac{\varepsilon}{2}.$$

Following [6], given $x \in \mathbb{R}^N$, we define $h_x(y) = f(y)g(y - x)$, for $y \in \mathbb{R}^N$, and note that $h_x(y)$ belongs to $C^\infty_0(A \cap B_x)$. Put

$$D(x) := \|h_x\|^2_{L^2(\mathbb{R}^N)}.$$

By Fubini's theorem,

$$\int_{\mathbb{R}^N} D(x)\,dx = \iint_{\mathbb{R}^{2N}} f^2(y)g^2(y - x)\,dx\,dy = 1.$$

We next estimate $\|h_x\|_{\dot{H}^s(\mathbb{R}^N)}$. In the local case $s = 1$, by using the product rule $\nabla h_x(y) = \nabla f(y)g(y - x) + f(y)\nabla g(y - x)$, Lieb proved that

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} |\nabla h_x|^2\,dy\,dx = \int_{\mathbb{R}^N} |\nabla f|^2\,dx + \int_{\mathbb{R}^N} |\nabla g|^2\,dx.$$

This is the key estimate in the proof of Lieb [6]. However, in the case of the fractional Laplacian, one cannot use the product rule as above. Instead, we use the integral representation of the fractional norm and some integral estimates. We shall establish the following inequality:

$$\int_{\mathbb{R}^N} \|h_x\|^2_{\dot{H}^s(\mathbb{R}^N)}\,dx \le \|f\|^2_{\dot{H}^s(\mathbb{R}^N)} + \|g\|^2_{\dot{H}^s(\mathbb{R}^N)}, \tag{2.1}$$

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which is also the key in our proof. From the definition (1.4),

$$\begin{aligned}\|h_x\|^2_{\dot{H}^s(\mathbb{R}^N)} &= \iint_{\mathbb{R}^{2N}} \frac{(f(y)g(y-x)-f(z)g(z-x))^2}{|y-z|^{N+2s}}\,dz\,dy \\ &= \iint_{\mathbb{R}^{2N}} \frac{(f(y)-f(z))g(y-x)+f(z)(g(y-x)-g(z-x))^2}{|y-z|^{N+2s}}\,dz\,dy \\ &= I_1(x)+I_2(x)+2I_3(x),\end{aligned} \tag{2.2}$$

where

$$\begin{aligned} I_1(x) &:= \iint_{\mathbb{R}^{2N}} \frac{(f(y)-f(z))^2 g^2(y-x)}{|y-z|^{N+2s}}\,dz\,dy,\\ I_2(x) &:= \iint_{\mathbb{R}^{2N}} \frac{f^2(z)(g(y-x)-g(z-x))^2}{|y-z|^{N+2s}}\,dz\,dy,\\ I_3(x) &:= \iint_{\mathbb{R}^{2N}} \frac{(f(y)-f(z))f(z)g(y-x)(g(y-x)-g(z-x))}{|y-z|^{N+2s}}\,dz\,dy.\end{aligned}$$

By Fubini's theorem,

$$\int_{\mathbb{R}^N} I_1(x)\,dx = \iint_{\mathbb{R}^{2N}} \frac{(f(y)-f(z))^2}{|y-z|^{N+2s}}\,dz\,dy \int_{\mathbb{R}^N} g^2(y-x)\,dx = \|f\|^2_{\dot{H}^s(\mathbb{R}^N)}.$$

However, by a change of variable,

$$\begin{aligned} I_2(x) &= \iint_{\mathbb{R}^{2N}} \frac{f^2(z)(g(y-x)-g(z-x))^2}{|y-z|^{N+2s}}\,dz\,dy\\ &= \iint_{\mathbb{R}^{2N}} \frac{f^2(z+x)(g(y)-g(z))^2}{|y-z|^{N+2s}}\,dz\,dy.\end{aligned}$$

Using Fubini's theorem again,

$$\int_{\mathbb{R}^N} I_2(x)\,dx = \|g\|^2_{\dot{H}^s(\mathbb{R}^N)}.$$

For I_3, after a change of variable $y = z + h$, we obtain

$$I_3(x) = \iint_{\mathbb{R}^{2N}} \frac{(f(z+h)-f(z))f(z)}{|h|^{N+2s}} g(z-x+h)(g(z-x+h)-g(z-x))\,dz\,dh.$$

It follows from Fubini's theorem that

$$\int_{\mathbb{R}^N} I_3(x)\,dx = \int_{\mathbb{R}^N} \frac{K(h)}{|h|^{N+2s}} L(h)\,dh,$$

where

$$K(h) = \int_{\mathbb{R}^N} g(z-x+h)(g(z-x+h)-g(z-x))\,dx = \int_{\mathbb{R}^N} g(x)(g(x)-g(x-h))\,dx$$

and

$$L(h) = \int_{\mathbb{R}^N} (f(z+h)-f(z))f(z)\,dz.$$

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Applying Hölder's inequality,

$$\int_{\mathbb{R}^N} g(x)g(x-h)\,dx \le \Big(\int_{\mathbb{R}^N} g^2(x)\,dx \int_{\mathbb{R}^N} g^2(x-h)\,dx\Big)^{1/2} = \int_{\mathbb{R}^N} g^2(x)\,dx.$$

This implies that $K(h) \ge 0$. In the same way, we also obtain $L(h) \le 0$. Consequently,

$$\int_{\mathbb{R}^N} I_3(x)\,dx \le 0.$$

Combining the integral estimates of I_1, I_2, I_3 and (2.2), we arrive at the estimate (2.1). Using (2.1) and the choice of f and g, we obtain

$$\int_{\mathbb{R}^N} \|h_x\|^2_{\dot{H}^s(\mathbb{R}^N)}\,dx < \lambda_s(A) + \lambda_s(B) + \varepsilon.$$

Hence,

$$\int_{\mathbb{R}^N} [\|h_x\|^2_{\dot{H}^s(\mathbb{R}^N)} - (\lambda_s(A) + \lambda_s(B) + \varepsilon)D(x)]\,dx < 0,$$

which yields $0 \le \|h_x\|^2_{\dot{H}^s(\mathbb{R}^N)} < (\lambda_s(A) + \lambda_s(B) + \varepsilon)D(x)$ on a set of positive measure. By the definition, we get

$$\lambda_s(A \cap B_x) < \lambda_s(A) + \lambda_s(B) + \varepsilon,$$

for x in a set of positive measure. The existence of x is proved.

2.2. Proof of assertion (ii). Suppose that A and B are bounded. Then the embedding $L^2(A) \to H_0^s(A)$ is compact (see, for example, [2]). It follows that there is a function $\tilde{f} \in H_0^s(A)$ such that $\|\tilde{f}\|_{L^2(A)} = 1$ and

$$\lambda_s(A) = \|\tilde{f}\|^2_{\dot{H}^s(\mathbb{R}^N)}.$$

Similarly, there exists $\tilde{g} \in H_0^s(B)$ satisfying $\|\tilde{g}\|_{L^2(B)} = 1$ and

$$\lambda_s(B) = \|\tilde{g}\|^2_{\dot{H}^s(\mathbb{R}^N)}.$$

As above, we also define

$$\tilde{h}_x(y) = \tilde{f}(y)\tilde{g}(y-x) \quad \text{and} \quad \tilde{D}(x) = \int_{\mathbb{R}^N} \tilde{h}_x^2\,dy.$$

Applying Fubini's theorem,

$$\int_{\mathbb{R}^N} \tilde{D}(x)\,dx = 1. \tag{2.3}$$

This implies that $\tilde{D}(x) < \infty$ for almost all $x \in \mathbb{R}^N$ or $\tilde{h}_x \in L^2(\mathbb{R}^N)$ for almost all $x \in \mathbb{R}^N$.

Define again $\|\tilde{h}_x\|^2_{\dot{H}^s(\mathbb{R}^N)}$ as in (2.2). Using the same argument as for $\|h_x\|^2_{\dot{H}^s(\mathbb{R}^N)}$ above, we also obtain

$$\int_{\mathbb{R}^N} \|\tilde{h}_x\|^2_{\dot{H}^s(\mathbb{R}^N)}\,dx \le \|\tilde{f}\|^2_{\dot{H}^s(\mathbb{R}^N)} + \|\tilde{g}\|^2_{\dot{H}^s(\mathbb{R}^N)} < \infty. \tag{2.4}$$

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This implies that $\|\tilde{h}_x\|_{\dot{H}^s(\mathbb{R}^N)} < \infty$ for almost all $x \in \mathbb{R}^N$ or $\tilde{h}_x \in \dot{H}^s(\mathbb{R}^N)$ for almost all $x \in \mathbb{R}^N$.

Combining (2.3) and (2.4), we arrive at

$$\int_{\mathbb{R}^N} (\|\tilde{h}_x\|^2_{\dot{H}^s(\mathbb{R}^N)} - (\lambda_s(A) + \lambda_s(B))\tilde{D}(x))\, dx \le 0,$$

which gives

$$0 \le \|\tilde{h}_x\|^2_{\dot{H}^s(\mathbb{R}^N)} \le (\lambda_s(A) + \lambda_s(B))\tilde{D}(x)$$

on a set of positive measure. Consequently,

$$\lambda_s(A \cap B_x) \le \lambda_s(A) + \lambda_s(B) \quad \text{with } x \text{ in a set of positive measure.}$$

It remains to show the strict inequality. It is sufficient to prove that

$$\|\tilde{h}_x\|^2_{\dot{H}^s(\mathbb{R}^N)} = (\lambda_s(A) + \lambda_s(B))\tilde{D}(x)$$

is not true for almost all x. Indeed, denote by χ_A the characteristic function of the set A. Put

$$\chi(x) = \chi_A * \chi_B(x) = |A \cap B_x|.$$

Then, $\chi \in C_0(\mathbb{R}^N)$. Given any $\varepsilon > 0$, there exists an open set C such that $0 < \chi(x) < \varepsilon$ and both $\tilde{f}(y)$ and $\tilde{g}(y - x)$ are positive in $A \cap B_x$ for all $x \in C$ (we can choose connected components of A and B if necessary). Hence, $\tilde{D}(x) > 0$ on C and if $\|\tilde{h}_x\|^2_{\dot{H}^s(\mathbb{R}^N)} = (\lambda_s(A) + \lambda_s(B))\tilde{D}(x)$ on C, then

$$\lambda_s(A \cap B_x) \le \lambda_s(A) + \lambda_s(B). \tag{2.5}$$

However, by the fractional Faber–Krahn inequality (see, for example, [4]),

$$C_{N,s}\varepsilon^{-2s/N} < \lambda_s(A \cap B_x). \tag{2.6}$$

Letting $\varepsilon \downarrow 0$, we obtain a contradiction from (2.5) and (2.6).

2.3. Proof of assertion (iii). It is not difficult to see that

$$\lambda_s(A \cap B_x) = \inf\left\{\frac{\|h_x\|^2_{\dot{H}^s(\mathbb{R}^N)}}{D(x)} : h_x = f(\cdot)g(\cdot - x), f \in C_0^\infty(A), g \in C_0^\infty(B)\right\},$$

where the function $\|h_x\|^2_{\dot{H}^s(\mathbb{R}^N)}/D(x)$ is the quotient of two continuous functions and then is extended to an upper-semicontinuous function by putting $\|h_x\|^2_{\dot{H}^s(\mathbb{R}^N)}/D(x) = \infty$ if $D(x) = 0$. Hence, $x \mapsto \lambda_s(A \cap B_x)$ is upper-semicontinuous.

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