*Probability in the Engineering and Informational Sciences* (2023), **37**:4 943–960
doi:10.1017/S0269964822000316

CAMBRIDGE
UNIVERSITY PRESS

RESEARCH ARTICLE

# Preservation properties of some reliability classes by lifetimes of coherent and mixed systems and their signatures

Salman Izadkhah[1], Ebrahim Amini-Seresht[2] and Narayanaswamy Balakrishnan[3]

[1] Department of Statistics, Campus of Bijar, University of Kurdistan, Bijar, Iran
[2] Department of Statistics, Bu-Ali Sina University, Hamedan, Iran. E-mail: e.amini64@yahoo.com
[3] Department of Mathematics and Statistics, McMaster University, Hamilton, ON, Canada.

**Keywords:** Aging class, Coherent system, Copula function, Exponential distribution, Residual lifetime, Signatures, Stochastic order

**Abstract**
This paper examines the preservation of several aging classes of lifetime distributions in the formation of coherent and mixed systems with independent and identically distributed (i.i.d.) or identically distributed (i.d.) component lifetimes. The increasing mean inactivity time class and the decreasing mean time to failure class are developed for the lifetime of systems with possibly dependent and i.d. component lifetimes. The increasing likelihood ratio property is also discussed for the lifetime of a coherent system with i.i.d. component lifetimes. We present sufficient conditions satisfied by the signature of a coherent system with i.i.d. components with exponential distribution, under which the decreasing mean remaining lifetime, the increasing mean inactivity time, and the decreasing mean time to failure are all satisfied by the lifetime of the system. Illustrative examples are presented to support the established results.

## 1. Introduction

Coherent systems are basic systems in reliability theory. A system is coherent if each component is relevant and its structure function increases in each component (see, e.g., [3]). The lifetime of a system is determined by its components and its structure. Samaniego [17] introduced the concept of "signature" of a system, which depends on the structure of the system, and proved that the lifetime distribution of a coherent system, whose components have continuous and independent and identically distributed (i.i.d.) lifetimes, can be obtained as a linear combination of distributions of order statistics obtained from the lifetimes of the components. The signature $\boldsymbol{p} = (p_1, \ldots, p_n)$ of a coherent system with $n$ i.i.d. lifetimes of the components is the $n$-dimensional probability vector whose $i$th element is $p_i = \mathbb{P}(\tau_{\boldsymbol{X}}(\boldsymbol{p}) = X_{i:n})$, where $\tau_{\boldsymbol{X}}(\boldsymbol{p})$ denotes the lifetime of the coherent system and $X_{1:n}, \ldots, X_{n:n}$ denotes the order statistics of $n$ i.i.d. component lifetimes $\boldsymbol{X} = (X_1, \ldots, X_n)$ with a common continuous distribution function. The reliability function of $\tau_{\boldsymbol{X}}(\boldsymbol{p})$ can be expressed as

$$\bar{F}_{\tau_{\boldsymbol{X}}(\boldsymbol{p})}(t) = \sum_{j=1}^{n} p_j \bar{F}_{j:n}(t). \tag{1}$$

Coherent systems with i.i.d. components have been discussed extensively in reliability theory. Here, we give some sufficient conditions on the system signature under which the distribution of the lifetime of the coherent system is preserved under some aging classes such as increasing likelihood ratio ($ILR$), or equivalently, log-concave density function, decreasing mean residual life ($DMRL$), increasing mean inactivity time ($IMIT$), and decreasing mean time to failure ($DMTTF$). We also present some sufficient conditions on the system signature which preserve some stochastic orderings between the coherent systems under the formation of signature representation of coherent systems.

Several authors have studied the preservation of aging classes and some stochastic orders under the formation of coherent systems in the general case of systems with dependent components or independent components. Nanda *et al.* [11] and Belzunce *et al.* [4] have established some preservation of stochastic orders for the case of i.i.d. components showing that some stochastic orders between the component lifetimes are translated into the same stochastic orders between the coherent systems. Navarro *et al.* [14] used a general approach (distortion functions) to study whether a coherent system with possibly dependent components preserves the ageing classes. Navarro *et al.* [15] considered the preservation of stochastic orders such as the usual stochastic, hazard rate, and reversed hazard rate orders, under the formation of coherent systems and then obtained some results in the case of identically distributed components. Navarro [12] presented sufficient conditions for the preservation of decreasing mean residual lifetime ($DMRL$) class and increasing mean residual lifetime ($IMRL$) class under the formation of a coherent system with possibly dependent components. Lindqvist and Samaniego [10] presented some sufficient conditions for preserving the new better than used in expectation ($NBUE$) class in a coherent system with i.i.d. component lifetimes. Recently, Rychlik and Szymkowiak [16] considered coherent systems composed of components with i.i.d. exponential lifetimes and presented conditions on the system signature which determine monotonicity, unimodality, and strong unimodality of density functions of system lifetimes. They also obtained conditions on the system signature which preserve the system under the increasing failure rate ($IFR$) and increasing mean residual life ($IMRL$) properties.

In what follows, we use "increasing" to mean "non-decreasing" and "decreasing" to mean "non-increasing".

## 2. Preliminaries

Let $X$ be a non-negative random variable with distribution function $F$, reliability function $\bar{F} = 1 - F$, and probability density function (PDF) $f$ whenever it exists. Then, the hazard rate (HR) of $X$ is defined by

$$h_X(t) = \frac{f(t)}{\bar{F}(t)}, \quad \text{for all } t \geq 0 \text{ for which } F(t) < 1, \tag{2}$$

and the $MRL$ function of $X$, which has a finite mean, is defined as (see, e.g., [5])

$$m_X(t) = E(X - t \mid X > t) = \frac{\int_t^\infty \bar{F}(x)\,dx}{\bar{F}(t)}, \quad \text{for all } t \geq 0 \text{ for which } F(t) < 1. \tag{3}$$

In contrast to the $HR$ and $MRL$ functions, which are descriptive measures for the instantaneous risk of failure and the entire residual lifetime of a life span after a specific time, respectively, the reversed hazard rate (RHR) function and the MIT function of $X$ is considered in the context of the lives lost before a certain time. The RHR function of $X$ is defined as

$$r_X(t) = \frac{f(t)}{F(t)}, \quad \text{for all } t \geq 0 \text{ for which } F(t) > 0, \tag{4}$$

and the $MIT$ function of $X$ is given by Kayid and Ahmad [6]

$$\tilde{m}_X(t) = E(t - X \mid X \leq t) = \frac{\int_0^t F(x)\,dx}{F(t)}, \quad \text{for all } t \text{ for which } F(t) > 0. \tag{5}$$

https://doi.org/10.1017/S0269964822000316 Published online by Cambridge University Press

In the age replacement model, the $MTTF$ is as (see, e.g., [8])

$$m^*_X(t) = \frac{\int_0^t \bar F(x)\,dx}{F(t)}, \quad \text{for all } t \text{ for which } F(t) > 0. \tag{6}$$

To obtain the expression of the $MRL$, $MIT$, and $MTTF$ of the lifetime of a coherent system with independent component lifetimes $X_1, \ldots, X_n$ with common distribution function $F$ we first get the distribution and the reliability functions of $\tau_{\boldsymbol{X}}(\boldsymbol{p})$. From (1), the distribution function of $\tau_{\boldsymbol{X}}(\boldsymbol{p})$ is given by

$$F_{\tau_{\boldsymbol{X}}(\boldsymbol{p})}(t) = \sum_{j=1}^{n} P_j \binom{n}{j} F^j(t)\bar F^{n-j}(t) \tag{7}$$

and the associated reliability function is given by

$$\bar F_{\tau_{\boldsymbol{X}}(\boldsymbol{p})}(t) = \sum_{j=0}^{n-1} \bar P_j \binom{n}{j} F^j(t)\bar F^{n-j}(t), \tag{8}$$

where $P_j = \sum_{k=1}^{j} p_k$, $j = 1, 2, \ldots, n$, and $\bar P_j = \sum_{k=j+1}^{n} p_k$, $j = 0, 1, \ldots, n-1$. The $MRL$ function of $\tau_{\boldsymbol{X}}(\boldsymbol{p})$ is obtained by replacing (8) in place of $\bar F$ in (3) as

$$m_{\tau_{\boldsymbol{X}}(\boldsymbol{p})}(t) = \frac{\int_t^\infty \bar F_{\tau_{\boldsymbol{X}}(\boldsymbol{p})}(u)\,du}{\bar F_{\tau_{\boldsymbol{X}}(\boldsymbol{p})}(t)} = \frac{\sum_{j=0}^{n-1}\binom{n}{j}\bar P_j \int_t^\infty F^j(u)\bar F^{n-j}(u)\,du}{\sum_{j=0}^{n-1}\binom{n}{j}\bar P_j F^j(t)\bar F^{n-j}(t)}. \tag{9}$$

The $MIT$ function of $\tau_{\boldsymbol{X}}(\boldsymbol{p})$ is obtained by replacing (7) in place of $F$ in (5) as follows:

$$\tilde m_{\tau_{\boldsymbol{X}}(\boldsymbol{p})}(t) = \frac{\int_0^t F_{\tau_{\boldsymbol{X}}(\boldsymbol{p})}(u)\,du}{F_{\tau_{\boldsymbol{X}}(\boldsymbol{p})}(t)} = \frac{\sum_{j=1}^{n} P_j\binom{n}{j} \int_0^t F^j(u)\bar F^{n-j}(u)\,du}{\sum_{j=1}^{n} P_j\binom{n}{j} F^j(t)\bar F^{n-j}(t)}. \tag{10}$$

The $MTTF$ function of $\tau_{\boldsymbol{X}}(\boldsymbol{p})$ is derived by replacing (7) in place of $F$ and also by replacing (8) in place of $\bar F$ in (6) as

$$m^*_{\tau_{\boldsymbol{X}}(\boldsymbol{p})}(t) = \frac{\int_0^t \bar F_{\tau_{\boldsymbol{X}}(\boldsymbol{p})}(u)\,du}{F_{\tau_{\boldsymbol{X}}(\boldsymbol{p})}(t)} = \frac{\sum_{j=0}^{n-1} \bar P_j\binom{n}{j} \int_0^t F^j(u)\bar F^{n-j}(u)\,du}{\sum_{j=1}^{n} P_j\binom{n}{j} F^j(t)\bar F^{n-j}(t)}. \tag{11}$$

Some regular classes of lifetime distributions have been proposed in the literature using the foregoing measures, and they are as given in the following definition (see, e.g., [7,8,18]).

**Definition 2.1.** *The non-negative random variable $X$ with distribution, reliability, density, failure rate, and reversed failure rate functions, $F$, $\bar F$, $f$, $h_X(t)$, and $r_X(t)$, respectively, is said to have*

(i) *$ILR$ property if $f(x+t)/f(t)$ is decreasing in $t > 0$ for all $x \geq 0$, or equivalently, if $f(t)$ is log-concave in $t \geq 0$;*
(ii) *$IFR$ property if $\bar F(x+t)/\bar F(t)$ is decreasing in $t > 0$, or equivalently, if $h_X(t) \leq h_X(t+x)$ for every $x, t \geq 0$;*
(iii) *$DRFR$ property if $F(x+t)/F(t)$ is decreasing in $t > 0$, or equivalently, if $r_X(t) \geq r_X(t+x)$ for every $x, t \geq 0$;*
(iv) *$DMRL$ property if $m_X(t)$ is non-increasing in $t \geq 0$, or equivalently, if $\int_t^{+\infty} \bar F(x)\,dx$ is log-concave in $t \geq 0$;*
(v) *$IMIT$ property if $\tilde m_X(t)$ is non-decreasing in $t \geq 0$, or equivalently, if $\int_0^t F(x)\,dx$ is log-concave in $t \geq 0$;*
(vi) *$DMTTF$ property of $m^*_X(t)$ is non-increasing in $t \geq 0$.*

https://doi.org/10.1017/S0269964822000316 Published online by Cambridge University Press

The following stochastic orderings have also been discussed extensively in the literature (see, e.g., [6,8,18]).

**Definition 2.2.** *The non-negative random variable $X$ with distribution and reliability functions, $F$ and $\bar{F}$, respectively, is said to be less than the non-negative random variable $Y$ with distribution and reliability functions, $G$ and $\bar{G}$, respectively, in*

(i) *mean residual life order (denoted as $X \leq_{MRL} Y$) whenever*

$$m_X(t) = \frac{\int_t^\infty \bar{F}(x)\,dx}{\bar{F}(t)} \leq m_Y(t) = \frac{\int_t^\infty \bar{G}(x)\,dx}{\bar{G}(t)} = m_Y(t), \quad \textit{for all } t \geq 0;$$

(ii) *mean inactivity time order (denoted as $X \leq_{MIT} Y$) whenever*

$$\tilde{m}_X(t) = \frac{\int_0^t F(x)\,dx}{F(t)} \geq \frac{\int_0^t G(x)\,dx}{G(t)} = \tilde{m}_Y(t), \quad \textit{for all } t \geq 0;$$

(iii) *mean time to failure order (denoted as $X \leq_{MTTF} Y$) whenever*

$$*m_X^*(t) = \frac{\int_0^t \bar{F}(x)\,dx}{F(t)} \leq \frac{\int_0^t \bar{G}(x)\,dx}{G(t)} = m_Y^*(t), \quad \textit{for all } t \geq 0.$$

We must remark that the $MTTF$ order given in Definition 2.2(iii) which can be found in Kayid *et al.* [8], is related to comparison of lifetimes of two units in an age replacement policy. Let, for any $0 < x \leq 1$ and for $a > 0$ and $b > 0$,

$$B_x(a,b) = \int_0^x t^{a-1}(1-t)^{b-1}\,dt, \quad I_x(a,b) = \frac{B_x(a,b)}{B(a,b)},$$

where $B(a,b) = \Gamma(a)\Gamma(b)/\Gamma(a+b)$ is the complete beta function. For positive integer values of $j$ and $n$, such that $j < n$ we have

$$\begin{aligned} B_x(n-j, j+1) &= \frac{\Gamma(n-j)\Gamma(j+1)}{\Gamma(n+1)} I_x(n-j, j+1) \\ &= \frac{(n-j-1)!j!}{n!} \sum_{i=n-j}^{n} \binom{n}{i} x^i (1-x)^{n-i} \\ &= \frac{1}{(j+1)\binom{n}{n-j-1}} \sum_{i=n-j}^{n} \binom{n}{i} x^i (1-x)^{n-i}. \end{aligned} \tag{12}$$

Suppose the underlying distribution of component lifetimes is exponential with distribution function $F_\lambda(t) = 1 - e^{-\lambda t}$ and reliability function $\bar{F}_\lambda(t) = e^{-\lambda t}$, where $t, \lambda > 0$. By (12), we then obtain

$$\begin{aligned} \int_t^\infty F^j(u)\bar{F}^{n-j}(u)\,du &= \frac{1}{\lambda}\int_0^{\bar{F}_\lambda(t)} t^{n-j-1}(1-t)^j\,dt \\ &= \frac{1}{\lambda} B_{\bar{F}_\lambda(t)}(n-j, j+1) \\ &= \frac{1}{\lambda(j+1)\binom{n}{n-j-1}} \sum_{i=n-j}^{n} \binom{n}{i} \bar{F}_\lambda^i(t) F_\lambda^{n-i}(t) \\ &= \frac{1}{\lambda(n-j)\binom{n}{j}} \sum_{i=0}^{j} \binom{n}{i} F_\lambda^i(t) \bar{F}_\lambda^{n-i}(t). \end{aligned}$$

In a similar manner as in (12), we can derive

$$B_x(j+1,n-j) = \frac{1}{(n-j)\binom{n}{j}} \sum_{i=j+1}^{n} \binom{n}{i} x^i (1-x)^{n-i}. \tag{13}$$

By (13), one gets

$$\begin{aligned}\int_0^t F^j(u)\bar{F}^{n-j}(u)\,du &= \frac{1}{\lambda}\int_0^{F_\lambda(t)} t^j(1-t)^{n-j-1}\,dt \\ &= \frac{1}{\lambda} B_{F_\lambda(t)}(j+1,n-j) \\ &= \frac{1}{(n-j)\binom{n}{j}} \sum_{i=j+1}^{n} \binom{n}{i} \bar{F}_\lambda^i(t) F_\lambda^{n-i}(t).\end{aligned}$$

We use the convention that $\sum_{k=k_1}^{k_2} a_k = 0$ if $k_1 > k_2$, and so $P_0 = 0, P_n = 1, \bar{P}_0 = 1$, and $\bar{P}_n = 0$.

## 3. Preservation under an arbitrary component lifetime distribution

In this section, the preservation of several reliability properties is established for lifetimes of coherent systems under some conditions on the distortion functions and/or the signatures vector. We first present sufficient conditions for the $IMIT$ property and the $DMTTF$ property. Then, sufficient conditions for the $ILR$ aging property are presented. In Theorems 3.1 and 3.4, we consider a coherent system with lifetime $\tau_{\boldsymbol{X}}(\boldsymbol{p})$ based on possibly dependent components with lifetimes $X_1, \ldots, X_n$. We further assume that $X_1, \ldots, X_n$ are identically distributed (i.d.) with common distribution and reliability functions $F$ and $\bar{F}$, respectively. In this situation, as pointed out by Navarro *et al.* [14], the reliability function of the lifetime $\tau_{\boldsymbol{X}}(\boldsymbol{p})$ of the coherent system satisfies

$$\bar{F}_{\tau_{\boldsymbol{X}}(\boldsymbol{p})}(t) = H(\bar{F}(t)) \tag{14}$$

and the corresponding distribution function is such that

$$F_{\tau_{\boldsymbol{X}}(\boldsymbol{p})}(t) = \bar{H}(F(t)) \tag{15}$$

where $H$ is called domination function and $\bar{H}$ given by $\bar{H}(u) = 1 - H(1-u)$ for $u \in [0,1]$ is the distortion function. We assume further that $H$ is a differentiable function. The function $\bar{H}$ is an increasing continuous function in $[0,1]$ such that $\bar{H}(0) = 0$ and $\bar{H}(1) = 1$. In the case when the components are i.i.d., we have

$$\bar{H}(u) = \sum_{j=1}^{n} P_j \binom{n}{j} u^j (1-u)^{n-j} \tag{16}$$

and

$$H(u) = \sum_{j=0}^{n-1} \bar{P}_j \binom{n}{j} u^{n-j} (1-u)^j. \tag{17}$$

**Theorem 3.1.** *If $X_1$ is $IMIT$ and, further,*

$$\inf_{u\in(0,v]} \frac{u}{\bar{H}(u)} \geq \frac{v^2 \bar{H}'(v)}{\bar{H}^2(v)}, \quad \text{for all } v \in (0,1), \tag{18}$$

*then $\tau_{\boldsymbol{X}}(\boldsymbol{p})$ is IMIT.*

*Proof.* From inequality (7) in Kayid and Izadkhah [7], $X_1$ is $IMIT$ if, and only if,

$$\frac{F^2(t)}{f(t)\int_0^t F(x)\,dx} \geq 1, \quad \text{for all } t \geq 0. \tag{19}$$

In addition, $\tau_{\boldsymbol{X}}(\boldsymbol{p})$ is IMIT if, and only if,

$$\frac{\bar{H}^2(F(t))}{f(t)\bar{H}'(F(t))\int_0^t \bar{H}(F(x))\,dx} \geq 1, \quad \text{for all } t \geq 0.$$

By assumption, for all $0 < x \leq t$, we have

$$\begin{aligned} \frac{F(x)}{\bar{H}(F(x))} &\geq \inf_{u\in(0,F(t)]} \frac{u}{\bar{H}(u)} \\ &\geq \frac{F^2(t)\bar{H}'(F(t))}{\bar{H}^2(F(t))}. \end{aligned} \tag{20}$$

Then multiplying both sides of (20) by $\bar{H}(F(x))$ and then integrating both sides with respect to $x$ over $(0,t]$ concludes that

$$\frac{\bar{H}^2(F(t))}{f(t)\bar{H}'(F(t))\int_0^t \bar{H}(F(x))\,dx} \geq \frac{F^2(t)}{f(t)\int_0^t F(x)\,dx}, \quad \text{for all } t \geq 0,$$

which by (19) concludes the proof. □

In the following example, the result of Theorem 3.5 given in Ahmad *et al.* [1] where it is established that the $IMIT$ class is preserved under the formation of series systems with i.i.d. components is considered to fulfill sufficient condition (18) in Theorem 3.1.

**Example 3.2.** Suppose $X_1, X_2, \ldots, X_n$ are i.i.d. lifetimes of $n$ components of a series system with common distribution function $F$ which is IMIT. Then, $\tau_{\boldsymbol{X}}(\boldsymbol{p}) = \min\{X_1, X_2, \ldots, X_n\}$ has distribution function $F_{\tau_{\boldsymbol{X}}(\boldsymbol{p})}(t) = 1-(1-F(t))^n$. Thus, from 15, $\bar{H}(u) = 1-(1-u)^n$. It is readily seen that $\bar{H}$ is concave on [0,1] and so $u/\bar{H}(u)$ is increasing in $u \in [0,1]$. Therefore,

$$\inf_{u\in(0,v]} \frac{u}{\bar{H}(u)} = \lim_{u\to 0^+} \frac{u}{\bar{H}(u)} = \lim_{u\to 0^+} \frac{u}{1-(1-u)^n} = \frac{1}{n}.$$

Now, in view of the above identities, (18) holds true if for all $v \in [0,1]$,

$$\frac{v^2\bar{H}'(v)}{\bar{H}^2(v)} = \frac{n(1-\bar{v})^2\bar{v}^{n-1}}{(1-\bar{v}^n)^2} \leq \frac{1}{n}, \tag{21}$$

where $\bar{v} = 1-v$. Since $(1-\bar{v}^n)^2 = (1-\bar{v})^2(1+\bar{v}+\cdots+\bar{v}^{n-1})^2$, we note that (21) is equivalent to

$$\frac{1+\bar{v}+\ldots+\bar{v}^{n-1}}{n} \geq \bar{v}^{(n-1)/2}, \quad \text{for all } v \in [0,1]. \tag{22}$$

Let $Y$ be a discrete random variable uniformly distributed over $\{0,1,\ldots,n-1\}$. By standard algebraic calculations, we see that $g(y) = \bar{v}^y$ is convex in $y \geq 0$. By Jensen's inequality, we then has $E[g(Y)] \geq g(E[Y])$. We observe that $E[g(Y)] = (1+\bar{v}+\cdots+\bar{v}^{n-1})/n$ and $E[Y] = (n-1)/2$ so that have $g(E[Y]) = \bar{v}^{E[Y]} = \bar{v}^{(n-1)/2}$. The inequality in (22) then follows readily. Hence, the sufficient condition in (18) holds and the $IMIT$ property is preserved under the considered coherent system.

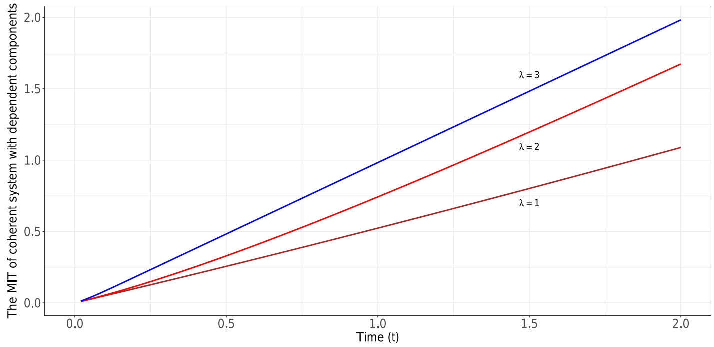


***Figure 1.*** *The MIT function in Example 3.3 when* $\lambda = 1, 2, 3$

Next, we show that the $IMIT$ property is preserved under a series system with two dependent components in the special case where the components lifetimes follow Clayton-Oakes copula at a certain level of its parameter.

**Example 3.3.** Suppose $X_1$ and $X_2$ are dependent but identically distributed components' lifetimes having the $IMIT$ property with joint reliability function $\bar{F}(x_1, x_2) = \hat{C}(\bar{F}(x_1), \bar{F}(x_2))$ in which $\hat{C}$ is the Clayton-Oakes survival copula

$$\hat{C}(u, v) = \frac{uv}{u + v - uv}.$$

Then, the lifetime $\tau_{\boldsymbol{X}}(\boldsymbol{p}) = \min\{X_1, X_2\}$ has $CDF$ as

$$\begin{aligned} F_{\tau_{\boldsymbol{X}}(\boldsymbol{p})}(t) &= 1 - \bar{F}(t, t) \\ &= 1 - \frac{\bar{F}^2(t)}{2\bar{F}(t) - \bar{F}^2(t)} \\ &= \frac{2F(t)}{1 + F(t)}. \end{aligned}$$

So, from (15), $\bar{H}(u) = 2u/(u+1)$. It is evident that $u/\bar{H}(u) = (u+1)/2$ is increasing in $u \in [0, 1]$. Thus,

$$\inf_{u \in (0, v]} \frac{u}{\bar{H}(u)} = \lim_{u \to 0^+} \frac{u}{\bar{H}(u)} = \lim_{u \to 0^+} \frac{u+1}{2} = \frac{1}{2}.$$

We also observe that $v^2 \bar{H}'(v)/\bar{H}^2(v) = \frac{1}{2}$ and thus the equality in (18) is satisfied and the $IMIT$ property is preserved. Suppose now that components have lifetimes with exponential distribution with CDF $F(t) = 1 - \exp(-\lambda t)$, $t, \lambda > 0$ which is $IMIT$. Then, the $MIT$ of $\tau_{\boldsymbol{X}}(\boldsymbol{p}) = \min\{X_1, X_2\}$ is given by

$$\tilde{m}_{\tau_{\boldsymbol{X}}(\boldsymbol{p})}(t) = \frac{2t + (1/\lambda)\ln(D(t))}{1 - D(t)}, \quad t \geq 0,$$

where $D(t) = 1/(2e^{\lambda t} - 1)$. The curves of the $MIT$ function of the series system with two dependent components with a common exponential distribution are plotted in Figure 1.

**Theorem 3.4.** *If* $X_1$ *is* $DMTTF$ *and, in addition,*

$$\sup_{u \in [v, 1)} \frac{u}{H(u)} \leq \frac{v(1-v)H'(v)}{H(v)(1 - H(v))}, \quad \text{for all } v \in (0, 1), \tag{23}$$

*then* $\tau_{\boldsymbol{X}}(\boldsymbol{p})$ *is also DMTTF.*

https://doi.org/10.1017/S0269964822000316 Published online by Cambridge University Press

*Proof.* Note that $\bar{F}_{\tau_{\boldsymbol{X}}(\boldsymbol{p})}(t) = H(\bar{F}(t))$, where $H$ is an increasing differentiable function. It can be readily seen that $X_1$ is $DMTTF$ if, and only if,

$$\frac{\bar{F}(t)(1-\bar{F}(t))}{f(t)\int_0^t \bar{F}(x)\,dx} \leq 1, \quad \text{for all } t \geq 0. \tag{24}$$

Furthermore, $\tau_{\boldsymbol{X}}(\boldsymbol{p})$ is $DMTTF$ if, and only if,

$$\frac{H(\bar{F}(t))(1-H(\bar{F}(t)))}{f(t)H'(\bar{F}(t))\int_0^t H(\bar{F}(x))\,dx} \leq 1, \quad \text{for all } t \geq 0.$$

From the assumption, for all $0 < x \leq t$, we have

$$\begin{aligned}\frac{\bar{F}(x)}{H(\bar{F}(x))} &\leq \sup_{u\in[\bar{F}(t),1)} \frac{u}{H(u)} \\ &\leq \frac{\bar{F}(t)(1-\bar{F}(t))H'(\bar{F}(t))}{H(\bar{F}(t))(1-H(\bar{F}(t)))}.\end{aligned} \tag{25}$$

From (25), we deduce that

$$\frac{H(\bar{F}(t))(1-H(\bar{F}(t)))}{f(t)H'(\bar{F}(t))\int_0^t H(\bar{F}(x))\,dx} \leq \frac{\bar{F}(t)(1-\bar{F}(t))}{f(t)\int_0^t \bar{F}(x)\,dx}, \quad \text{for all } t \geq 0,$$

which together with (24) concludes the proof. □

In the following example, we prove that the $DMTTF$ class is preserved under the formation of parallel systems with i.i.d. components (i.e., the result of Theorem 3.1 in [9]).

**Example 3.5.** Suppose $X_1, X_2, \ldots, X_n$ are i.i.d. lifetimes of $n$ components of a parallel system with common reliability function $\bar{F}$. Then, $\tau_{\boldsymbol{X}}(\boldsymbol{p}) = \max\{X_1, X_2, \ldots, X_n\}$ has reliability function $\bar{F}_{\tau_{\boldsymbol{X}}(\boldsymbol{p})}(t) = 1-(1-\bar{F}(t))^n$ which together with (14) gives $H(u) = 1-(1-u)^n$. It is seen that $u/H(u)$ is increasing in $u$ and so,

$$\sup_{u\in[v,1)} \frac{u}{H(u)} = \frac{1}{H(1)} = 1.$$

Therefore, (23) holds if, and only if,

$$H'(v) \geq \frac{H(v)}{v}\frac{1-H(v)}{1-v}, \quad \text{for all } v \in [0,1], \tag{26}$$

which holds if, and only if,

$$f_1(v) = (1-v)^n \geq 1-nv = f_2(v), \quad \text{for all } v \in [0,1]. \tag{27}$$

As $(1-u)^{n-1} \leq 1$ for all $u \in [0,1]$, we have $f_1'(u) - f_2'(u) = n - n(1-u)^{n-1} \geq 0$ for all $u \in [0,1]$. Now, since $f_1(0) = f_2(0) = 0$, we deduce that $f_1(v) - f_2(v) = \int_0^v (f_1'(u) - f_2'(u))\,du \geq 0$, for all $v \in [0,1]$. Hence, (27) holds true and the sufficient condition (23) in Theorem 3.4 is satisfied and the $DMTTF$ property is preserved under the formation of a parallel system with i.i.d. components.

Next, it is shown that the $DMTTF$ property is preserved under the formation of a parallel system consisting two dependent components in the particular case where the components lifetimes follow Clayton-Oakes copula at a certain level of its parameter.

https://doi.org/10.1017/S0269964822000316 Published online by Cambridge University Press

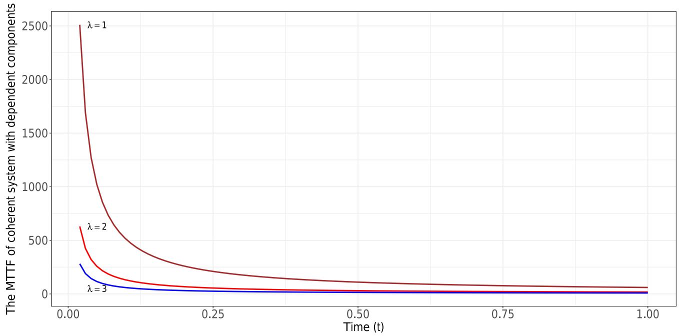


***Figure 2.*** *The MTTF function in Example 3.6 when* $\lambda = 1, 2, 3$.

**Example 3.6.** Suppose $X_1$ and $X_2$ are dependent but identically distributed components' lifetimes with the $DMTTF$ property with joint reliability function $\bar{F}(x_1, x_2) = \hat{C}(\bar{F}(x_1), \bar{F}(x_2))$ in which $\hat{C}$ is the Clayton-Oakes survival copula. Then, $\bar{F}(t,t) = \hat{C}(\bar{F}(t), \bar{F}(t)) = \bar{F}^2(t)/(2\bar{F}(t) - \bar{F}^2(t))$. The $SF$ of $\tau_{\boldsymbol{X}}(\boldsymbol{p}) = \max\{X_1, X_2\}$ is given by

$$\begin{aligned} \bar{F}_{\tau_{\boldsymbol{X}}(\boldsymbol{p})}(t) &= P(X_1 > t) + P(X_2 > t) - P(X_1 > t, X_2 > t) \\ &= 2\bar{F}(t) - \bar{F}(t,t) \\ &= \frac{3\bar{F}(t) - 2\bar{F}^2(t)}{2 - \bar{F}(t)}, \end{aligned}$$

and so, from (14) we get $H(u) = (3u - 2u^2)/(2-u)$. It is easily seen that $u/H(u) = (2-u)/(3-2u)$ is increasing in $u \in [0,1]$. As a result,

$$\sup_{u \in [v,1)} \frac{u}{H(u)} = \frac{1}{H(1)} = 1.$$

Moreover, we have $H'(v) = 2(1-v)(3-v)/(2-v)$, $H(v)/v = (3-2v)/(2-v)$ and $(1-H(v))/(1-v) = 2(1-u)/(2-u)$. Thus, it follows that (26) holds and, consequently, (23) in Theorem 3.4 holds true. By Theorem 3.4, it follows that $\tau_{\boldsymbol{X}}(\boldsymbol{p})$ is $DMTTF$. Suppose that lifetimes of components have exponential distribution with reliability function $\bar{F}(t) = \exp(-\lambda t)$, $t, \lambda > 0$, which is $DMTTF$. The $MTTF$ of $\tau_{\boldsymbol{X}}(\boldsymbol{p}) = \max\{X_1, X_2\}$ is obtained as

$$m^*_{\tau_{\boldsymbol{X}}(\boldsymbol{p})}(t) = \frac{(1+U(t))(2U(t) - \ln(1+U(t)))}{\lambda((1+U(t)) - (1-U(t))(2U(t)+1))}, \quad t > 0,$$

where $U(t) = 1 - e^{-\lambda t}$. The curves of the $MTTF$ function of the parallel system with two dependent components with a common exponential distribution are plotted in Figure 2.

Navarro *et al.* [14] gave some conditions involving the domination function $H$, used in (14), under which the $ILR$ class is preserved under the formation of a coherent system with possibly dependent identically distributed components lifetimes. To be more specific, in Proposition 2.2(i) of their work, by considering $\tau_{\boldsymbol{X}}(\boldsymbol{p}) = \psi(X_1, \ldots, X_n)$ as the lifetime of the coherent system, letting the domination function $H$ is twice differentiable in $[0,1]$ and taking

$$\beta(u) = \frac{uH''(u)}{H'(u)} \quad \text{and} \quad \bar{\beta}(u) = \frac{(1-u)H''(u)}{H'(u)},$$

they proved that if $X_1$ is $ILR$ and there exists $a \in [0,1]$ such that $\beta$ is non-negative and decreasing in $(0,a)$ and $\bar{\beta}$ is non-positive and decreasing in $(a,1)$, then $\tau_{\boldsymbol{X}}(\boldsymbol{p})$ is also $ILR$. We apply this result to

https://doi.org/10.1017/S0269964822000316 Published online by Cambridge University Press

coherent systems with i.i.d. components lifetimes, that is, the case when $\tau_{\boldsymbol{X}}(\boldsymbol{p})$ is the lifetime of the coherent system with reliability function $\bar{F}_{\tau_{\boldsymbol{p}}(\boldsymbol{X})}(t) = H(\bar{F}(t))$ where $H$ is as obtained in (14). By taking derivatives of $H$, one gets

$$H'(u) = \sum_{i=1}^{n} ip_i \binom{n}{i} u^{n-i}(1-u)^{i-1}$$

and

$$H''(u) = \sum_{i=1}^{n} ip_i \binom{n}{i} (n-i-(n-1)u)u^{n-i-1}(1-u)^{i-2}.$$

Therefore, from Proposition 2.2(i) of Navarro *et al.* [14] if there exists $a \in [0, 1]$ such that

$$\frac{\sum_{i=1}^{n} ip_i \binom{n}{i}(n-i-(n-1)u)u^{n-i}(1-u)^{i-2}}{\sum_{i=1}^{n} ip_i \binom{n}{i} u^{n-i}(1-u)^{i-1}}$$

is non-negative and decreasing in $u \in (0, a)$, also

$$\frac{\sum_{i=1}^{n} ip_i \binom{n}{i}(n-i-(n-1)u)u^{n-i-1}(1-u)^{i-1}}{\sum_{i=1}^{n} ip_i \binom{n}{i} u^{n-i}(1-u)^{i-1}}$$

is non-positive and decreasing in $u \in (a, 1)$, and further if $X_1$ is $ILR$, then $\tau_{\boldsymbol{p}}(\boldsymbol{X})$ is also $ILR$. However, this is a sufficient (not necessary) condition for preservation of $ILR$ class in a coherent system as our observation in one example will affirm it in the sequel. Next result, presents different sufficient conditions for the preservation of the $ILR$ class under the formation of a coherent system with i.i.d. components lifetimes by imposing conditions on its signatures and also conditions on the common lifetime distribution that the components follow it. In the following theorem, it is shown that many conditions may be needed to get the preservation of the $ILR$ class under the formation of a coherent or mixed system composed of i.i.d. component lifetimes.

**Theorem 3.7.** *Let $\tau_{\boldsymbol{X}}(\boldsymbol{p})$ be the lifetime of a coherent system with $X_i$, $i = 1, \ldots, n$, having common distribution and density functions $F$ and $f$, respectively, and corresponding signature vector $\boldsymbol{p} = (p_1, \ldots, p_n)$. Let $\Phi$ be a polynomial function of degree n defined by*

$$\Phi(u) = \sum_{j=0}^{n-1} s_j u^{n-j} \tag{28}$$

*in which $s_j = \binom{n}{j}(n-j)\sum_{i=j}^{n-1} p_{i+1}(-1)^{i-j}\binom{n-j-1}{n-i-1}$, $j = 0, \ldots, n-1$. If*

(i) *$X_1$ is ILR and it has a log-concave hazard rate,*
(ii) *for some point $\alpha \in (0, 1]$, $u\Phi'(u)/\Phi(u)$ is decreasing and non-negative for all $u \in (0, \alpha]$, and*
(iii) *$(1-u)\Phi'(u)/\Phi(u)$ is decreasing and non-positive for all $u \in (\alpha, 1]$,*

*then $\tau_{\boldsymbol{X}}(\boldsymbol{p})$ is ILR.*

*Proof.* The density function of $\tau_{\boldsymbol{X}}(\boldsymbol{p})$ can be written as (see [2])

$$f_{\tau_{\boldsymbol{X}}(\boldsymbol{p})}(t) = h_X(t)\sum_{j=0}^{n} s_j \bar{F}^{n-j}(t) = h_X(t)\Phi(\bar{F}(t)),$$

https://doi.org/10.1017/S0269964822000316 Published online by Cambridge University Press

where $\Phi$ is the function given in (28). To reach the desired result, it suffices to show that $f_{\tau_X(\boldsymbol{p})}(t+x)/f_{\tau_X(\boldsymbol{p})}(t)$ is also decreasing in $t$. We have

$$\frac{f_{\tau_X(\boldsymbol{p})}(t+x)}{f_{\tau_X(\boldsymbol{p})}(t)} = \frac{h_X(t+x)}{h_X(t)} \frac{\Phi(\bar{F}(t+x))}{\Phi(\bar{F}(t))}. \tag{29}$$

From the second part of the assumption (i), the first ratio in the right-hand side of (29) is decreasing in $t$ for all $x > 0$. The proof obtains if we show that the second ratio in (29) is also decreasing in $t$ for all $x > 0$. Let us denote $\Lambda(t) = \Phi(\bar{F}(t+x))/\Phi(\bar{F}(t))$. Let $t_\alpha$ be the solution of $\bar{F}(t) = \alpha$. Taking the derivative of $\Lambda(t)$ with respect $t$, we observe the following:

Case (A): For all $t \geq t_\alpha$, we have

$$\begin{aligned}
\Lambda'(t) &\overset{\text{sgn}}{=} h_X(t)\frac{\bar{F}(t)\Phi'(\bar{F}(t))}{\Phi(\bar{F}(t))} - h_X(t+x)\frac{\bar{F}(t+x)\Phi'(\bar{F}(t+x))}{\Phi(\bar{F}(t+x))} \\
&\leq h_X(t+x)\left[\frac{\bar{F}(t)\Phi'(\bar{F}(t))}{\Phi(\bar{F}(t))} - \frac{\bar{F}(t+x)\Phi'(\bar{F}(t+x))}{\Phi(\bar{F}(t+x))}\right] \\
&\overset{\text{sgn}}{=} \frac{u\Phi'(u)}{\Phi(u)} - \frac{u^*\Phi'(u^*)}{\Phi(u^*)} \leq 0,
\end{aligned}$$

where $h_X(t)$ is the common HR function of components defined in (2), $u = \bar{F}(t)$ and $u^* = \bar{F}(t+x)$. As $\bar{F}$ is a decreasing function, we have $u \geq u^*$. The first inequality follows from the fact that $ILR$ class implies $IFR$ class , that is, $h_X(t) \leq h_X(t+x)$ for all $t, x \geq 0$, while the second inequality follows from assumption (ii).

Case (B): For all $t < t_\alpha$, we have

$$\begin{aligned}
\Lambda'(t) &\overset{\text{sgn}}{=} r_X(t)\frac{(1-\bar{F}(t))\Phi'(\bar{F}(t))}{\Phi(\bar{F}(t))} - r_X(t+x)\frac{(1-\bar{F}(t+x))\Phi'(\bar{F}(t+x))}{\Phi(\bar{F}(t+x))} \\
&\leq r_X(t+x)\left[\frac{(1-\bar{F}(t))\Phi'(\bar{F}(t))}{\Phi(\bar{F}(t))} - \frac{(1-\bar{F}(t+x))\Phi'(\bar{F}(t+x))}{\Phi(\bar{F}(t+x))}\right] \\
&\overset{\text{sgn}}{=} \frac{u\Phi'(u)}{\Phi(u)} - \frac{u^*\Phi'(u^*)}{\Phi(u^*)} \leq 0,
\end{aligned}$$

where $r_X(t)$ is the common RHR function of components defined in (4), $u = \bar{F}(t)$ and $u^* = \bar{F}(t+x)$. The first inequality follows from the fact that $ILR$ class implies $DRFR$ class, that is, $r_X(t) \geq r_X(t+x)$ for all $t, x \geq 0$, while the second inequality follows from assumption (iii).

□

**Example 3.8.** Let us consider a system, with 4-independent component lifetimes, with lifetime

$$\tau_X(\boldsymbol{p}) = \max\{\min\{X_1, X_2\}, \min\{X_3, X_4\}\},$$

where $\boldsymbol{p} = (0, \frac{2}{3}, \frac{1}{3}, 0)$. It is easy to see that the function $\Phi$ in (28) is acquired as $\Phi(u) = 4u^2(u^2 - 1)$. We further observe that

$$\Upsilon_1(u) = \frac{u\Phi'(u)}{\Phi(u)} = \frac{4u^2 - 2}{u^2 - 1}$$

and

$$\Upsilon_2(u) = \frac{(1-u)\Phi'(u)}{\Phi(u)} = \frac{-4u^3 + 4u^2 + 2u - 2}{u^3 - u}.$$

https://doi.org/10.1017/S0269964822000316 Published online by Cambridge University Press

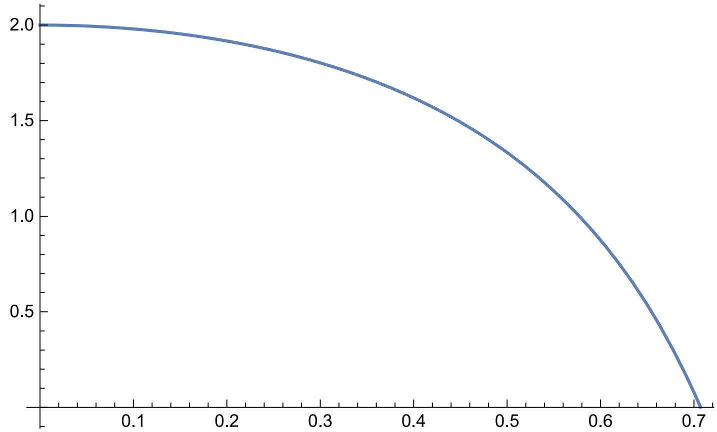


***Figure 3.*** *Plot of* $\Upsilon_1(u)$ *for all* $u \in (0, 0.70701)$ *in Example 3.8.*

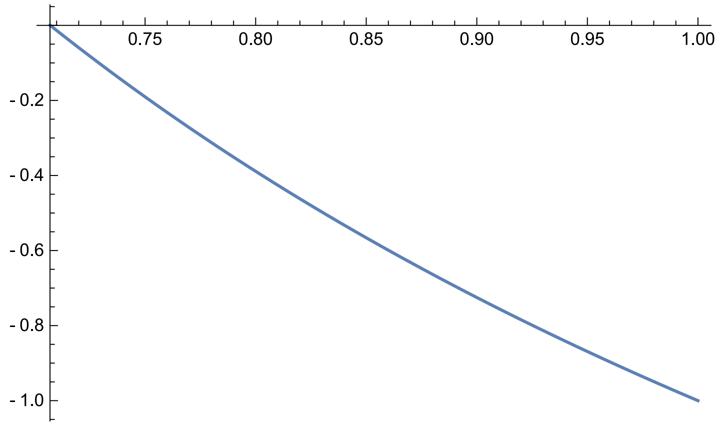


***Figure 4.*** *Plot of* $\Upsilon_2(u)$ *for all* $u \in (0.70701, 1)$ *in Example 3.8.*

In Figures 3 and 4, $\Upsilon_1(u)$ and $\Upsilon_2(u)$ have been plotted.

It can be seen that $\Upsilon_1(u)$ is decreasing and positive in $u \in (0, 0.70701)$ and $\Upsilon_2(u)$ is decreasing and non-positive in $u \in (0.70701, 1)$, which means that the conditions in Theorem 3.7 are satisfied. Suppose the lifetimes of components have exponential distribution with PDF $f(t) = \lambda \exp(-\lambda t),\ t, \lambda > 0$, which is log-concave and so $f(t)$ is ILR. Also, $h_X(t+x)/h_X(t) = 1$ for all $t, x \geq 0$, which fulfills condition (v) in Theorem 3.7. The PDF of $\tau_{\boldsymbol{X}}(\boldsymbol{p}) = \max\{\min\{X_1, X_2\}, \min\{X_3, X_4\}\}$ is given by

$$f_{\tau_{\boldsymbol{X}}(\boldsymbol{p})}(t) = 4f(t)\bar{F}(t)(1-\bar{F}^2(t)) = 4\lambda e^{-2\lambda t}(1-e^{-2\lambda t}),\ t > 0.$$

The curve of $l(t) = \ln\{f_{\tau_{\boldsymbol{X}}(\boldsymbol{p})}(t)\}$, which has to exhibit concavity, has been plotted in Figure 5.

**Remark 3.9.** In the context of Theorem 3.7 and also using notations given in Proposition 2.2(i) in Navarro *et al.* [14], it is plainly seen that $u\Phi'(u)/\Phi(u) = 1 + \beta(u)$ and also $(1-u)\Phi'(u)/\Phi(u) = (1-u)/u + \bar{\beta}(u)$, where $\Phi$ is as defined in (28). These identities are useful to make a comparison between conditions of Theorem 3.7 and the condition imposed in Proposition 2.2(i) in Navarro *et al.* [14].

The next example acknowledges that there is a situation where Theorem 3.7 is applicable for preservation of the *ILR* class by a particular mixed system for which the sufficient condition in Proposition 2.2(i) of Navarro *et al.* [14] does not hold and, therefore, their result is not applicable.

https://doi.org/10.1017/S0269964822000316 Published online by Cambridge University Press

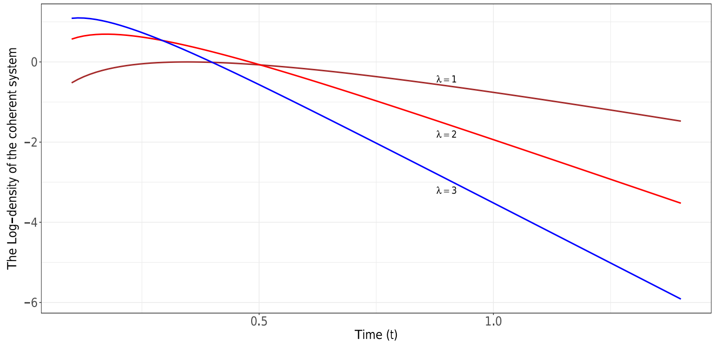


***Figure 5.*** *Plot of $l(t) = \ln\{f_{\tau_{\boldsymbol{X}}(\boldsymbol{p})}(t)\}$ in Example 3.8 when $\lambda = 1, 2, 3$.*

**Example 3.10.** Consider a mixed system with two i.i.d. component lifetimes and signature $\boldsymbol{p} = (p_1, p_2)$ where it is assumed that $p_2 \in (\frac{1}{2}, \frac{2}{3}]$. From (14), we get $H(u) = (1 - 2p_2)u^2 + 2p_2 u$ and as a result, $H'(u) = 2(1 - 2p_2)u + 2p_2$ and $H''(u) = 2(1 - 2p_2)$ from which we get

$$\beta(u) = \frac{uH''(u)}{H'(u)} = \frac{2u(1 - 2p_2)}{2(1 - 2p_2)u + 2p_2}$$

and

$$\bar{\beta}(u) = \frac{(1 - u)H''(u)}{H'(u)} = \frac{2(1 - u)(1 - 2p_2)}{2(1 - 2p_2)u + 2p_2}.$$

Since $p_2 > \frac{1}{2}$, thus for all $u \in (0, 1)$ it holds that $\beta(u) \leq 0$ and also that $\beta'(u) < 0$. Furthermore, since $p_2 > \frac{1}{2}$, thus for all $u \in (0, 1)$, we have $\bar{\beta}(u) \leq 0$ and also $\bar{\beta}'(u) \geq 0$. Therefore, $\beta$ is a non-positive and decreasing function while $\bar{\beta}$ is a non-positive and increasing function. Thus, the sufficient condition in Proposition 2.2(i) does not hold true. However, in view of Remark 3.9, we have

$$\frac{u\Phi'(u)}{\Phi(u)} = 1 + \beta(u) = \frac{p_2 + 2(1 - 2p_2)u}{p_2 + u(1 - 2p_2)}$$

which is non-negative for all $u \in (0, 1)$ since $\frac{1}{2} < p_2 \leq \frac{2}{3}$ and also it is decreasing in $u \in (0, 1)$, since $(d/du)(u\Phi'(u)/\Phi(u)) = \beta'(u) \leq 0$, where $\Phi$ is as given in (28). Therefore, if $\alpha = 1$ in Theorem 3.7, the the condition (ii) is satisfied, the condition (iii) is relaxed and if $X_1$ is $ILR$ with a log-concave hazard rate, then $\tau_{\boldsymbol{p}}(\boldsymbol{X})$ is $ILR$.

## 4. Preservation under exponentially distributed component lifetimes

In the previous section, we considered coherent systems with general lifetime distributions with some positive aging properties ($IMIT$, $DMTTF$ and $ILR$) for the component lifetimes and presented conditions under which such properties are preserved by the lifetime of a coherent system. However, when a positive aging behavior of lifetimes of components is inherited by the lifetime of the coherent system, the speed of aging and that whether the system ages faster than the components are passed over. The exponential distribution is a standard lifetime distribution which is the only continuous lifetime distribution with no-aging property. Moreover, the exponential distribution fulfills the $DMRL$, $IMIT$, and $DMTTF$ properties. The preservation of these classes under the formation of coherent systems consisting of i.i.d. components with exponential distribution shows that the coherent system is aging faster than its components and that provides a new insight.

https://doi.org/10.1017/S0269964822000316 Published online by Cambridge University Press

The preservation properties of the ILR, IFR, increasing failure rate in average (IFRA), new better than used (NBU), and NBUE classes of lifetime distributions have been derived in the literature considering independent or dependent components with a general lifetime distribution. However, the results can also be applied to the case where component lifetimes are exponentially distributed (e.g., see Theorem 4.1 in Navarro [13]). For three classes that we consider in the sequel, sufficient conditions imposed are specific ones and the method given to prove the preservation properties are different. Furthermore, the approaches used to prove the results are affected by the choice of exponential distribution as the lifetime of components of the system.

We now consider the $DMRL$, $IMIT$, and $DMTTF$ classes and provide sufficient conditions on the signature vector for the preservation of the lifetime of coherent systems having exponentially distributed component lifetimes.

**Theorem 4.1.** *Let $\tau_{\mathbf{X}}(\boldsymbol{p})$ be the lifetime of a coherent system with signature vector* $\mathbf{p}$ *comprising independent components with lifetimes $X_i (i = 1, \ldots, n)$, following a common exponential distribution with mean $1/\lambda$. Then, $\tau_{\mathbf{X}}(\boldsymbol{p})$ is*

(i) *DMRL if $\bar{P}_j / \sum_{i=j}^{n-1} (n-i)^{-1} \bar{P}_i$ is non-decreasing in $j = 0, 1, \ldots, n-1$ for which $\bar{P}_j > 0$;*
(ii) *IMIT if $\sum_{i=0}^{j-1} (n-i)^{-1} P_i / P_j$ is non-decreasing in $j = 1, \ldots, n$ for which $P_j > 0$;*
(iii) *DMTTF if $P_j / \sum_{i=0}^{j-1} (n-i)^{-1} \bar{P}_i$ is non-decreasing in $j = 1, \ldots, n$ for which $P_j > 0$.*

*Proof.* (i) To prove the required monotonicity property, we first observe by using (9) that

$$\begin{aligned}\frac{1}{m_{\tau_{\mathbf{X}}(\boldsymbol{p})}(t)} &= \frac{\sum_{j=0}^{n-1} \binom{n}{j} \bar{P}_j F_\lambda^j(t) \bar{F}_\lambda^{n-j}(t)}{\sum_{j=0}^{n-1} \binom{n}{j} \bar{P}_j \int_t^\infty F_\lambda^j(u) \bar{F}_\lambda^{n-j}(u)\, du} \\ &= \lambda \frac{\sum_{j=0}^{n-1} \binom{n}{j} \bar{P}_j F_\lambda^j(t) \bar{F}_\lambda^{n-j}(t)}{\sum_{j=0}^{n-1} \sum_{i=0}^{j} (n-j)^{-1} \bar{P}_j \binom{n}{i} F_\lambda^i(t) \bar{F}_\lambda^{n-i}(t)} \\ &= \lambda \frac{\sum_{j=0}^{n-1} \binom{n}{j} \bar{P}_j [\psi(F_\lambda(t))]^j}{\sum_{j=0}^{n-1} \binom{n}{j} (\sum_{i=j}^{n-1} (n-i)^{-1} \bar{P}_i) [\psi(F_\lambda(t))]^j},\end{aligned}$$

where $\psi(x) = x/(1-x)$ for $0 \le x < 1$. Then, since $\psi(F_\lambda(t))$ is increasing in $t$, applying Lemma 2.1 of Amini-Seresht *et al.* [2], we can conclude that $1/m_{\tau_{\mathbf{X}}(\boldsymbol{p})}(t)$ is non-decreasing in $t \ge 0$, i.e., $\tau_{\mathbf{X}}(\boldsymbol{p})$ is $DMRL$.

(ii) To prove this assertion, by applying (10) we can write

$$\begin{aligned}\tilde{m}_{\tau_{\mathbf{X}}(\boldsymbol{p})}(t) &= \frac{\sum_{j=0}^{n} \binom{n}{j} P_j \int_0^t F_\lambda^j(u) \bar{F}_\lambda^{n-j}(u)\, du}{\sum_{j=0}^{n} \binom{n}{j} P_j F_\lambda^j(t) \bar{F}_\lambda^{n-j}(t)} \\ &= \frac{\sum_{j=0}^{n-1} \sum_{i=j+1}^{n} (n-j)^{-1} P_j \binom{n}{i} F_\lambda^i(t) \bar{F}_\lambda^{n-i}(t)}{\lambda \sum_{j=0}^{n} \binom{n}{j} P_j F_\lambda^j(t) \bar{F}_\lambda^{n-j}(t)} \\ &= \frac{\sum_{j=0}^{n} \binom{n}{j} \left( \sum_{i=0}^{j-1} (n-i)^{-1} P_i \right) F_\lambda^j(t) \bar{F}_\lambda^{n-j}(t)}{\lambda \sum_{j=0}^{n} \binom{n}{j} P_j F_\lambda^j(t) \bar{F}_\lambda^{n-j}(t)}, \\ &= \frac{\sum_{j=0}^{n} \binom{n}{j} \left( \sum_{i=0}^{j-1} (n-i)^{-1} P_i \right) [\psi(F_\lambda(t))]^j}{\lambda \sum_{j=0}^{n} \binom{n}{j} P_j [\psi(F_\lambda(t))]^j}.\end{aligned}$$

Once more, applying Lemma 2.1 of Amini-Seresht *et al.* [2], we see that $\tilde{m}_{\tau_{\mathbf{X}}(\boldsymbol{p})}(t)$ is non-decreasing in $t \ge 0$, i.e., $\tau_{\mathbf{X}}(\boldsymbol{p})$ is $IMIT$.

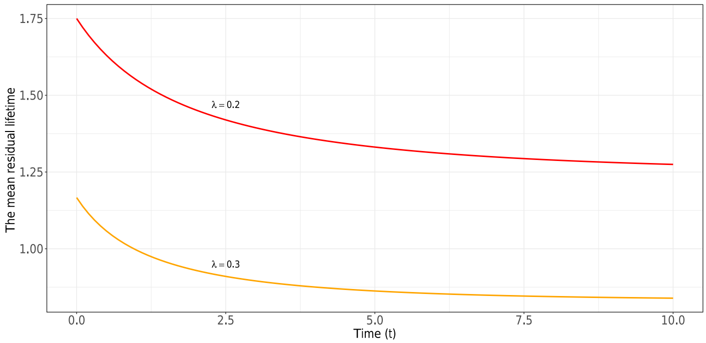


***Figure 6.*** *The mean residual lifetime in Example 4.2 when $\lambda = 0.2$ and $\lambda = 0.3$.*

(iii) To obtain the expression of the mean time to failure function, using (11) we get

$$\begin{aligned}\frac{1}{m^*_{\tau_X(\boldsymbol{p})}(t)} &= \frac{\sum_{j=0}^{n}\binom{n}{j}P_j F_\lambda^j(t)\bar{F}_\lambda^{n-j}(t)}{\sum_{j=0}^{n-1}\binom{n}{j}\bar{P}_j\int_0^t F_\lambda^j(u)\bar{F}_\lambda^{n-j}(u)\,du}\\ &= \lambda\frac{\sum_{j=0}^{n}\binom{n}{j}P_j F_\lambda^j(t)\bar{F}_\lambda^{n-j}(t)}{\sum_{j=0}^{n-1}\sum_{i=j+1}^{n}(n-j)^{-1}\bar{P}_j\binom{n}{i}F_\lambda^i(t)\bar{F}_\lambda^{n-i}(t)}\\ &= \lambda\frac{\sum_{j=1}^{n}\binom{n}{j}P_j[\psi(F_\lambda(t))]^j}{\sum_{j=1}^{n}\binom{n}{j}(\sum_{i=0}^{j-1}(n-i)^{-1}\bar{P}_i)[\psi(F_\lambda(t))]^j}.\end{aligned}$$

Repeated application of Lemma 2.1 of Amini-Seresht *et al.* [2] shows that $1/m^*_{\tau_X(\boldsymbol{p})}(t)$ is non-decreasing in $t \geq 0$, i.e., $\tau_X(\boldsymbol{p})$ is $DMTTF$. Hence, the theorem proved. □

The following example shows that the condition in Theorem 4.1(i) is attainable for a particular coherent system.

**Example 4.2.** Consider a coherent system with five i.i.d. components following the exponential distribution with mean $1/\lambda$ having signature $\boldsymbol{p} = (\frac{2}{5}, \frac{3}{5}, 0, 0, 0)$. Set $R_1(j) = \bar{P}_j/\sum_{i=j}^{n-1}(n-i)^{-1}\bar{P}_i$ for $j = 0, 1$. It is plainly seen that $R_1(0) = \frac{100}{35} < 4 = R_1(1)$. Hence, by part (i) of Theorem 4.1, we conclude that $\tau_X(\boldsymbol{p})$ has the $DMRL$ property. The mean residual lifetime of $\tau_X(\boldsymbol{p})$ is obtained as

$$m_{\tau_X(\boldsymbol{p})}(t) = \frac{0.35 + 0.75(e^{\lambda t} - 1)}{\lambda(1 + 3(e^{\lambda t} - 1))}.$$

In Figure 6, we have plotted $m_{\tau_X(\boldsymbol{p})}(t)$ for $\lambda = 0.2$ and $\lambda = 0.3$ and in both cases, the desired decreasing property is preserved.

**Remark 4.3.** Navarro *et al.* [14] proved that the lifetime of a coherent system with components sharing a common exponential distribution as their lifetime is $IFR$ if and only if, $\alpha(u) = uH'(u)/H(u)$ is decreasing in $u \in (0, 1)$. In the context of Example 4.2, we see that $\alpha(u) = (12 - 10u)/(3 - 2u)$ is decreasing in $(0, 1)$ and that $\tau_X(\boldsymbol{p})$ is consequently $IFR$ and $\tau_X(\boldsymbol{p})$ is thus $DMRL$. This means that the result of Example 4.2 can be strengthened. However, in Example 4.4, we consider a mixed system consisting of i.i.d. exponential component lifetimes, where the system lifetime has $DMRL$ but no $IFR$ property.

**Example 4.4.** Consider a mixed system with $n = 3$ components with i.i.d. lifetimes following the exponential distribution having signature $\boldsymbol{p} = (\frac{3}{10}, \frac{3}{10}, \frac{4}{10})$. Keeping the notations of Example 4.2 in

https://doi.org/10.1017/S0269964822000316 Published online by Cambridge University Press

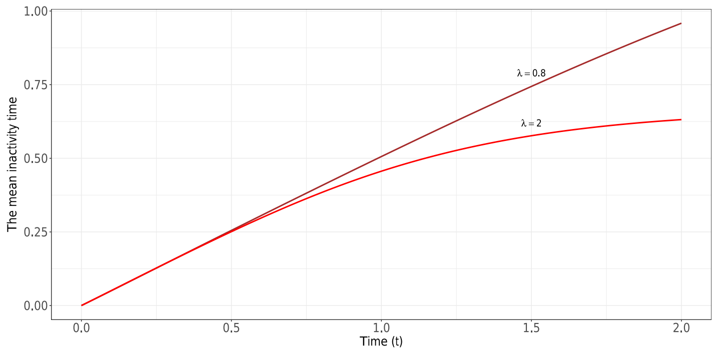


***Figure 7.*** *The mean inactivity time in Example 4.5 when $\lambda = 0.8$ and $\lambda = 2$.*

mind, we can observe that $R_1(0) = \frac{60}{65}, R_1(1) = \frac{42}{45}$ and $R_1(2) = 1$. Consequently, as it is evident $R_1(j)$ is increasing in $j$ for $j = 0, 1, 2$. Therefore, by Theorem 4.1(i), $\tau_{\boldsymbol{X}}(\boldsymbol{p})$ is DMRL. However, in light of Remark 4.3, since

$$\alpha(u) = \frac{3u^3 - 6u^2 + 12u}{u^3 - 3u^2 + 12u} \quad \text{is not decreasing over } (0, 1)$$

thus, $\tau_{\boldsymbol{X}}(\boldsymbol{p})$ is not $IFR$. Hence, the result of Theorem 4.1(i) is of particular interest.

**Example 4.5.** Consider a coherent system having four i.i.d. components with lifetimes $X_i, i = 1, 2, 3, 4$, with exponential distribution with mean $1/\lambda$, having its lifetime as

$$\tau_{\boldsymbol{X}}(\boldsymbol{p}) = \min\{\max\{X_1, X_2, X_3\}, X_4\}.$$

The associated signature is $\boldsymbol{p} = (\frac{1}{4}, \frac{1}{4}, \frac{1}{2}, 0)$. Let us set $R_2(j) = \sum_{i=0}^{j-1}(n-i)^{-1}P_i/P_j$, for $j = 1, 2, 3, 4$. We observe that $R_2(1) = 0$, $R_2(2) = \frac{2}{12}$, $R_2(3) = \frac{4}{12}$, and $R_2(4) = \frac{16}{12}$. Hence, $R_2(j)$ is non-decreasing in $j = 1, 2, 3, 4$. Thus, by part (ii) of Theorem 4.1, $\tau_{\boldsymbol{X}}(\boldsymbol{p})$ belongs to the IMIT class. The mean inactivity time of $\tau_{\boldsymbol{X}}(\boldsymbol{p})$ is obtained as

$$\tilde{m}_{\tau_{\boldsymbol{X}}(\boldsymbol{p})}(t) = \frac{1.5(e^{\lambda t} - 1) + 4(e^{\lambda t} - 1)^2 + 4(e^{\lambda t} - 1)^3}{3\lambda(1 + 3(e^{\lambda t} - 1) + 4(e^{\lambda t} - 1)^2 + (e^{\lambda t} - 1)^3)}.$$

Figure 7 presents $\tilde{m}_{\tau_{\boldsymbol{X}}(\boldsymbol{p})}(t)$ for $\lambda = 0.8$ and $\lambda = 2$. The behavior of the curves in both cases is increasing as expected.

**Example 4.6.** Consider a coherent system having four i.i.d. components with lifetimes following exponential distribution with mean $1/\lambda$, and signature $\boldsymbol{p} = (\frac{1}{4}, \frac{3}{4}, 0, 0)$. Let $R_3(j) = P_j/\sum_{i=0}^{j-1}(n-i)^{-1}\bar{P}_i$, for $j = 1, 2, 3, 4$. It follows that $R_3(1) = 1$ and $R_3(j) = 2$ for $j = 2, 3, 4$. Thus, $R_2(j)$ is non-decreasing in $j = 1, 2, 3, 4$. Using part (iii) of Theorem 4.1, it implies that $\tau_{\boldsymbol{X}}(\boldsymbol{p})$ has the DMTTF property. The mean time to failure of $\tau_{\boldsymbol{X}}(\boldsymbol{p})$ is obtained as

$$m^*_{\tau_{\boldsymbol{X}}(\boldsymbol{p})}(t) = \frac{1 + 3(e^{\lambda t} - 1) + 2(e^{\lambda t} - 1)^2 + 0.5(e^{\lambda t} - 1)^3}{\lambda(1 + 6(e^{\lambda t} - 1) + 4(e^{\lambda t} - 1)^2 + (e^{\lambda t} - 1)^3)}.$$

The graphs of $m^*_{\tau_{\boldsymbol{X}}(\boldsymbol{p})}(t)$ when $\lambda = 0.1$ and $\lambda = 0.2$ have been plotted in Figure 8. In both cases, the expected monotone behavior readily seen.

In the following result, we consider two coherent systems with different distribution of component lifetimes, having i.i.d. exponential distributions with different parameters, but with the same, signature

https://doi.org/10.1017/S0269964822000316 Published online by Cambridge University Press

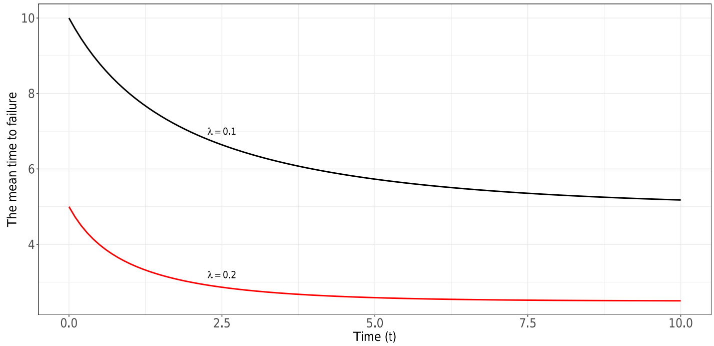


***Figure 8.*** *The mean time to failure in Example 4.6 for values $\lambda = 0.1$ and $\lambda = 0.2$.*

vector. Then, we provide sufficient conditions on the common signature vector for making stochastic comparisons according to different stochastic orderings given in Definition 2.2.

**Theorem 4.7.** *Let $\tau_{\mathbf{X}}(\boldsymbol{p})$ be the lifetime of a coherent system with signature vector $\boldsymbol{p}$ and with i.i.d. component lifetimes $X_1, \ldots, X_n$, with common exponential distribution with mean $1/\lambda$. Let $\tau_{\mathbf{Y}}(\boldsymbol{p})$ be the lifetime of an another coherent system with signature vector $\boldsymbol{p}$ and with i.i.d. component lifetimes $Y_1, \ldots, Y_n$ with common exponential distribution with mean $1/\eta$, such that $\lambda > \eta$. If*

(i) $\bar{P}_j / \sum_{i=j}^{n-1} (n-i)^{-1} \bar{P}_i$ *is non-decreasing in $j = 0, 1, \ldots, n-1$, then $\tau_{\mathbf{X}}(\boldsymbol{p}) \leq_{MRL} \tau_{\mathbf{Y}}(\boldsymbol{p})$;*
(ii) $P_j / \sum_{i=0}^{j-1} (n-i)^{-1} P_i$ *is non-decreasing in $j = 1, \ldots, n$, then $\tau_{\mathbf{X}}(\boldsymbol{p}) \leq_{MIT} \tau_{\mathbf{Y}}(\boldsymbol{p})$;*
(iii) $P_j / \sum_{i=0}^{j-1} (n-i)^{-1} \bar{P}_i$ *is non-decreasing in $j = 1, \ldots, n$, then $\tau_{\mathbf{X}}(\boldsymbol{p}) \leq_{MTTF} \tau_{\mathbf{Y}}(\boldsymbol{p})$.*

*Proof.* We only provide the proof of assertion (i) since the proofs of other two assertions are quite similar. From assumption that $\lambda > \eta$, we have, $F_\lambda(t) \geq F_\eta(t)$, for all $t \geq 0$. Thus, if $\psi(x) = x/(1-x)$, then $\psi(F_\lambda(t)) \geq \psi(F_\eta(t))$, for all $t \geq 0$. In the spirit of the proof of part (i) of Theorem 4.1, since

$$\lambda \frac{\sum_{j=0}^{n-1} \binom{n}{j} \bar{P}_j [\psi(x)]^j}{\sum_{j=0}^{n-1} \binom{n}{j} (\sum_{i=j}^{n-1} (n-i)^{-1} \bar{P}_i) [\psi(x)]^j}$$

is non-decreasing in $x$, thus it follows, for all $t \geq 0$, that

$$\begin{aligned} \frac{1}{m_{\tau_{\mathbf{X}}(\boldsymbol{p})}(t)} &= \lambda \frac{\sum_{j=0}^{n-1} \binom{n}{j} \bar{P}_j [\psi(F_\lambda(t))]^j}{\sum_{j=0}^{n-1} \binom{n}{j} (\sum_{i=j}^{n-1} (n-i)^{-1} \bar{P}_i) [\psi(F_\lambda(t))]^j} \\ &\geq \eta \frac{\sum_{j=0}^{n-1} \binom{n}{j} \bar{P}_j [\psi(F_\lambda(t))]^j}{\sum_{j=0}^{n-1} \binom{n}{j} (\sum_{i=j}^{n-1} (n-i)^{-1} \bar{P}_i) [\psi(F_\lambda(t))]^j} \\ &\geq \eta \frac{\sum_{j=0}^{n-1} \binom{n}{j} \bar{P}_j [\psi(F_\eta(t))]^j}{\sum_{j=0}^{n-1} \binom{n}{j} (\sum_{i=j}^{n-1} (n-i)^{-1} \bar{P}_i) [\psi(F_\eta(t))]^j} = \frac{1}{m_{\tau_{\mathbf{Y}}(\boldsymbol{p})}(t)}. \end{aligned}$$

Hence, the theorem proved. □

## 5. Concluding remarks

The classical problem of whether certain classes of lifetime distributions are preserved in the formation of coherent systems has been discussed in this work. Four classes of lifetime distributions, namely the $DMRL$, $IMIT$, $DMTTF$, and $ILR$ classes, have been considered. Considering a coherent system with possibly dependent identically distributed component lifetimes, sufficient conditions are given on the

https://doi.org/10.1017/S0269964822000316 Published online by Cambridge University Press

distortion function of the lifetime of the system for establishing the preservation of the classes of the $IMIT$ and $DMTTF$ in the formation of the system. Examples have been presented to show that the $IMIT$ property is preserved under the formation of a series system with $n$ components with i.i.d. lifetimes and also that the $IMIT$ property is preserved under the formation of a series system with two dependent components. In addition, some examples have been provided to show that the $DMTTF$ property is preserved under the formation of a parallel system with $n$ components having i.i.d. lifetimes and that the $DMTTF$ property is also preserved under the formation of a parallel system with two dependent components. Sufficient conditions are also provided for the signature vector of a coherent system with i.i.d. lifetime of components to preserve the $ILR$ class under the formation of the coherent system. The obtained sufficient conditions for the preservation of the $DMRL$, $IMIT$, and $DMTTF$ classes may not be necessary, as seen in the partial observations. The second part of the paper has focused on the preservation of the $DMRL$, $IMIT$, and $DMTTF$ classes under the formation of a coherent system with components having i.i.d. exponentially distributed lifetimes. To this end, convenient and tractable conditions have been founded on the signature vector of the coherent system to preserve the corresponding properties.

**Acknowledgments.** The authors are very grateful to three anonymous reviewers for their comments and suggestions which lead to this improved version of the paper.

## References

[1] Ahmad, I.A., Kayid, M., & Pellerey, F. (2005). Further results involving the MIT order and the IMIT class. *Probability in the Engineering and Informational Sciences* 19(3): 377–395.
[2] Amini-Seresht, E., Khaledi, B.E., & Kochar, S. (2020). Some new results on stochastic comparisons of coherent systems using signatures. *Journal of Applied Probability* 57(1): 156–173.
[3] Barlow, R.E. & Proschan, F. (1975). *Statistical theory of reliability and life testing: Probability models*. Tallahassee: Florida State University.
[4] Belzunce, F., Ruiz, J.M., & Ruiz, C. (2002). On preservation of some shifted and proportional orders by systems. *Statistics and Probability Letters* 60: 141–154.
[5] Hall, W.J. & Wellner, J.A. (1981). Mean residual life. *Statistics and Related Topics* 169: 184.
[6] Kayid, M. & Ahmad, I.A. (2004). On the mean inactivity time ordering with reliability applications. *Probability in the Engineering and Informational Sciences* 18(3): 395–409.
[7] Kayid, M. & Izadkhah, S. (2018). Testing behavior of the mean inactivity time. *Journal of Testing and Evaluation* 46(6): 2649–2653.
[8] Kayid, M., Ahmad, I.A., Izadkhah, S., & Abouammoh, A.M. (2013). Further results involving the mean time to failure order, and the decreasing mean time to failure class. *IEEE Transactions on Reliability* 62(3): 670–678.
[9] Knopik, L. (2005). Some results on the ageing class. *Control and Cybernetics* 34(4): 1175–1180.
[10] Lindqvist, B.H. & Samaniego, F.J. (2019). Some new results on the preservation of the NBUE and NWUE aging classes under the formation of coherent systems. *Naval Research Logistics* 66(5): 430–438.
[11] Nanda, A., Jain, K., & Singh, H. (1998). Preservation of some partial orderings under the formation of coherent systems. *Statistics and Probability Letters* 39: 123–131.
[12] Navarro, J. (2018). Preservation of DMRL and IMRL aging classes under the formation of order statistics and coherent systems. *Statistics and Probability Letters* 137: 264–268.
[13] Navarro, J. (2021). *Introduction to system reliability theory*. Springer International Publishing.
[14] Navarro, J., del Aguila, Y., Sordo, M.A., & Suarez-Llorens, A. (2014). Preservation of reliability classes under the formation of coherent systems. *Applied Stochastic Models in Business and Industry* 30(4): 444–454.
[15] Navarro, J., del Aguila, Y., Sordo, M.A., & Suarez-Llorens, A. (2016). Preservation of stochastic orders under the formation of generalized distorted distributions. Applications to coherent systems. *Methodology and Computing in Applied Probability* 18: 529–545.
[16] Rychlik, T. & Szymkowiak, M. (2021). Properties of system lifetime in the classical model with iid exponential component lifetimes. In *Advances in statistics - Theory and applications*. Cham: Springer, pp. 43–66.
[17] Samaniego, F.J. (1985). On closure of the IFR class under formation of coherent systems. *IEEE Transactions on Reliability* 34(1): 69–72.
[18] Shaked, M., Shanthikumar, J.G. (eds) (2007). *Stochastic orders*. New York, NY: Springer New York.

**Cite this article:** Izadkhah S, Amini-Seresht E and Balakrishnan N (2023). Preservation properties of some reliability classes by lifetimes of coherent and mixed systems and their signatures. *Probability in the Engineering and Informational Sciences* **37**, 943–960. https://doi.org/10.1017/S0269964822000316