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Hyperbolic Lyapunov–Perron regular points and smooth invariant measures

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Abstract. For a $C^{1+\alpha}$ diffeomorphism f of a compact smooth manifold, we give a necessary and sufficient condition that guarantees that if the set of hyperbolic Lyapunov–Perron regular points has positive volume, then f preserves a smooth measure. We use recent results on symbolic coding of χ-non-uniformly hyperbolic sets and results concerning the existence of SRB measures for them.

Key words: Lyapunov–Perron regularity, smooth measures, SRB, hyperbolicity, entropy
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1. *Introduction*

In his 1998 ICM address, Viana proposed the following conjecture.

Conjecture. If a smooth map f has only non-zero Lyapunov exponents at Lebesgue almost every (a.e.) point, then f preserves an SRB measure.

SRB measures were first constructed in the 1970s by Sinai, Ruelle, and Bowen in the context of Axiom A attractors [**6**, **14**, **16**], that is, topological attractors which are uniformly hyperbolic and contain a dense set of periodic points. SRB measures are hyperbolic measures, meaning that all of their Lyapunov exponents are non-zero, whose conditional measures along almost every unstable manifold are absolutely continuous with respect to leaf volume on those unstables. Later, a somewhat similar notion of u-measures was introduced by Pesin and Sinai in [**11**], in the context of partially hyperbolic attractors. These measures are characterized by the fact that the conditional measures they generate on unstable manifolds are absolutely continuous with respect to the leaf volume, but these measures may lack some good ergodic properties due to presence of zero Lyapunov exponents in the central direction of the partially hyperbolic system.

Let us mention some recent results on the existence of SRB measures in the partially hyperbolic case. Burns *et al* [**7**] proved that under the assumptions that the Lyapunov exponents in the central direction are negative on a set of positive measure and global unstable leaves are dense, the u-measure turns out to be the unique SRB measure. Bonatti and Viana [5] proved the existence of an SRB measure assuming a splitting $TM = E^s \oplus E^{uu}$, with uniform expansion in the E^{uu}-direction and non-uniform contraction in the E^s-direction. Alves, Bonatti, and Viana [1] proved the existence of SRB measures under the assumption that the splitting is of the form $E^{ss} \oplus E^u$. This last result was generalized by Climenhaga, Dolgopyat, and Pesin [8], who only assumed the splitting to be measurable and the expansion and contractions are both non-uniform. They established the existence of SRB measures under the assumption of *effective hyperbolicity*. Ben Ovadia [4] constructed SRB measures under a certain *leaf condition*, which we will introduce and discuss later in more detail. His construction uses the theory of Markov partitions and the thermodynamical formalism developed by Sarig in [15] for countable topological Markov shifts.

One observation about all the papers mentioned above is that they deal only with the forward dynamics of the system, and so they look at the forward Lyapunov exponents, and ask, given that we have a positive volume of points with non-zero *forward* Lyapunov exponents, what can we say about the existence of the SRB measure. In this paper, we consider the *forward* and *backward* behavior of Lyapunov exponents, and ask what can we say about the natural measures for the system? So the central assumption in this paper is the positivity of the Riemannian volume of the set of hyperbolic Lyapunov–Perron regular points, which we define next.

Definition 1.1. A point $x \in M$ is Lyapunov–Perron regular for a diffeomorphism $f : M \to M$ if there exist numbers $\lambda_1(x) \leq \lambda_2(x) \leq \cdots \leq \lambda_{k(x)}(x)$ and an invariant decomposition

$$T_x M = \bigoplus_{i=1}^{k(x)} E_i(x)$$

such that for all $v \in E_i(x)$, we have

$$\lim_{n\to\pm\infty} \frac{1}{n} \log \|D_x f^n v\| = \lambda_i(x),$$
$$\lim_{n\to\pm\infty} \frac{1}{n} \log \| \det(D_x f^n)\| = \sum_{i=1}^{k(x)} \lambda_i(x),$$

and that $\lambda_i(x)$, $E_i(x)$, and $k(x)$ depend measurably on x. We denote the set of all Lyapunov–Perron regular points by $\mathcal{R}$.

Definition 1.2. (Hyperbolic Lyapunov–Perron regular points) We say that x is a *hyperbolic Lyapunov–Perron regular* point if it was a Lyapunov–Perron regular point, and if there exists an integer $1 \leq s < k(x)$ for which we have $\lambda_s(x) < 0 < \lambda_{s+1}(x)$. We denote the set of all *hyperbolic Lyapunov–Perron* regular points by $\mathcal{R}_{\text{hyp}}$.

The celebrated Oseledets' multiplicative ergodic theorem states that any invariant probability measure μ for a diffeomorphism $f : M \to M$ gives full measure to the set

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of Lyapunov–Perron regular points. That is, $\mu(\mathcal{R}) = 1$. In particular, an SRB measure μ^+ for f gives full measure to the subset $\mathcal{R}_{\text{hyp}} \subset \mathcal{R}$ of hyperbolic Lyapunov–Perron regular points. Notice that this does not necessarily imply that the the set $\mathcal{R}_{\text{hyp}}$ has positive volume. Indeed, for a point x in the basin of attraction of the SRB measure μ^+,

$$B(\mu^+) = \left\{ x \in M \mid \frac{1}{n} \sum_{k=0}^{n-1} \varphi(f^k x) \longrightarrow \int_M \varphi \, d\mu^+ \text{ for all } \varphi \in C^0(M, \mathbb{R}) \right\},$$

one has that the limit

$$\lim_{n \to \infty} \frac{1}{n} \log \|D_x f^n v\|$$

exists, but nothing can be said about this limit as $n \to -\infty$.

The conjecture of Viana mentioned above is difficult to tackle as is. On the one hand, simply assuming that points with non-zero Lyapunov exponents exist does not seem to be enough to show that SRB measures exist. On the other hand, it is hard to construct a counter example. We instead add the assumption that the set of points with non-zero Lyapunov exponents which are also Perron–Regular and satisfy a recurrence condition have positive volume. The main question with which this paper deals is:

What can we say about the dynamics of f if the set of Lyapunov–Perron regular points which are hyperbolic, has positive volume?

In this paper, we give an answer to the above question, under the assumption that the set $\mathcal{R}^{\text{PR}} \subset \mathcal{R}_{\text{hyp}}$ of *positively recurrent Lyapunov–Perron regular points* (see Definition 4.6) has a positive volume. This set consists of points that return infinitely often to a Pesin block along the orbit. The reason for considering such a set is that, under the condition of positivity of volume, one can show that the maps f and f^{-1} preserve SRB measures, see for example Theorem 5.1. The answer to the above question turns out to be that asking for the set of *positively recurrent hyperbolic Lyapunov–Perron* regular points to have positive volume is too much; the system not only preserves SRB measures, but these measures are actually absolutely continuous with respect to volume.

1.1. *Structure of the paper.* The rest of the paper is divided as follows. In §2, we will discuss a toy model and prove our main theorem in this special case. The proof here illustrates the main components of the proof of the general theorem. In §3, we introduce ergodic homoclinic components and prove the main result of this paper: we give conditions under which an SRB measure is absolutely continuous with respect to volume. In §4, we introduce the main object of this paper, the set $\mathcal{R}^{\text{PR}}_{\text{hyp}}$. We also introduce the set of non-uniformly hyperbolic points and discuss Markov partitions of this set. We end the section by proving another central proposition to our argument (Theorem 4.18). In §5, we complete the proof of our main theorem.

2. *Informal discussion of the results*

2.1. *A toy model.* Let $f : M \to M$ be a diffeomorphism of a compact smooth Riemannian manifold preserving a hyperbolic smooth measure μ. By Oseledets' multiplicative

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ergodic theorem, the set $\mathcal{R}_{\text{hyp}}$ of all hyperbolic Lyapunov–Perron regular points has positive Riemannian volume. In this paper, we consider the following problem:

Does the opposite hold, namely if $\mathcal{R}_{\text{hyp}}$ has positive volume, can we conclude that f preserves a hyperbolic smooth measure?

To illustrate our approach, we consider the following *toy problem* (the case of Anosov diffeomorphisms):

Let $f : M \to M$ be a transitive Anosov diffeomorphism, and assume that the set of Lyapunov–Perron regular points $\mathcal{R}$ has positive volume, can we conclude that f preserves a smooth measure?

The following theorem give an affirmative solution to this problem.

THEOREM 2.1. *Let $f : M \longrightarrow M$ be a transitive $C^{1+\alpha}$ Anosov diffeomorphism of compact Riemannian manifold M, and assume that* $\mathrm{Vol}(\mathcal{R}_{\text{hyp}}) > 0$*, then the system preserves a unique smooth measure μ which is equivalent to volume.*

According to Bowen [6], a transitive Anosov diffeomorphism has exactly one SRB measure, which is ergodic, and if a smooth measure is preserved, then it is the unique SRB measure. Therefore, the natural approach is to work with the unique SRB measure for *f*, μ^+, and try to prove that it is actually a smooth measure. Now if μ^+ is a smooth measure, then it is straight forward to show that the sum of Lyapunov exponents is zero. The good news is that the opposite also holds.

LEMMA 2.2. *Let μ^+ be an SRB measure for a diffeomorphism $f : M \to M$ of a compact smooth Riemannian manifold M. Assume that the sum of all Lyapunov exponents is zero at μ^+-a.e. point. Then μ^+ is a smooth measure.*

This lemma holds for general $C^{1+\alpha}$ diffeomorphisms preserving an SRB measure, we prove this lemma in §3.

Using this, a natural strategy would be to prove that the sum of Lyapunov exponents of μ^+ is zero. Let us denote the Lyapunov exponents of μ^+ by $\chi_1(\mu^+) \leq \cdots \leq \chi_n(\mu^+)$, then by Ruelle's inequality and the entropy formula for μ^+, we have that

$$\sum_{i=1}^{n} \chi_i(\mu^+) \leq 0.$$

Since f^{-1} is also an Anosov diffeomorphism, it has an SRB measure μ^-, which is characterized by absolutely continuous conditional measures along local stable manifolds. As is the case of the measure μ^+, if *f* preserves a smooth measure, then it is in fact μ^-, and hence $\mu^+ = \mu^-$. Note that in general, μ^- is not equal to μ^+; in fact, this happens only in a very specific situation.

LEMMA 2.3. *Let $f : M \to M$ be a $C^{1+\alpha}$ diffeomorphism, and let μ be an SRB measure for both f and f^{-1}, then μ is a smooth measure.*

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The lemma is an immediate consequence of Lemma 2.1 above and the entropy formula, and holds for general $C^{1+\alpha}$, not only Anosov diffeomorphisms, again we will prove it in §3 of this paper.

This naturally leads us to trying to show that $\mu^+ = \mu^-$. To do this, assume that there exists a point $x \in M$ whose forward Lyapunov exponents are controlled by μ^+, that is,

$$\chi_i^+(x) = \chi_i(\mu^+) \quad \text{for all } i = 1, \ldots, n,$$

and whose backward Lyapunov exponents are controlled by μ^-, that is,

$$\chi_i^-(x) = \chi_i(\mu^-) \quad \text{for all } i = 1, \ldots, n.$$

This by itself does not give anything, but if in addition we know that $x \in \mathcal{R}$, we can immediately conclude that

$$\chi_{n-i+1}(\mu^-) = -\chi_i(\mu^+)$$

for all $i = 1, \ldots, n$. So in particular, we see that

$$\sum_i \chi_i(\mu^+) = -\sum_i \chi_i(\mu^-). \tag{2.1}$$

Using now Ruelle's inequality applied to both f and f^{-1}, we obtain that

$$\sum_i \chi_i(\mu^+) \leq 0 \quad \text{and} \quad \sum_i \chi_i(\mu^-) \leq 0.$$

Combing this with equation (2.1), yields

$$\sum_i \chi_i(\mu^+) = 0 = \sum_i \chi_i(\mu^-).$$

Using Lemma 1, we see that both μ^+ and μ^- are smooth measures, and hence, by uniqueness, we conclude that $\mu^+ = \mu^-$ is the unique smooth measure preserved by f.

It is therefore enough to prove the existence of such a point x, the proof of which is not difficult in this case and we sketch it here. First, since $\mathrm{Vol}(\mathcal{R}) > 0$, we can prove, using the absolute continuity properties of the stable and unstable manifolds, that there exists a point (actually a positive volume of points) such that:

(1) x is a Lyapunov–Perron regular point;
(2) the local unstable manifold $V^u_{\mathrm{loc}}(x)$ at x contains a positive leaf volume of Lyapunov–Perron regular points;
(3) the local stable manifold $V^s_{\mathrm{loc}}(x)$ at x contains a positive leaf volume of Lyapunov–Perron regular points.

Now, since μ^+ and μ^- are SRB measures for f and f^{-1} respectively, we know that for μ^+/μ^--almost all points x^+/x^-, almost all points on the local unstable manifold $V^u_{\mathrm{loc}}(x^+)$ with respect to leaf volume are Lyapunov–Perron regular. Furthermore, almost every such point has Lyapunov exponents controlled by μ^+. That is, setting

$$\mathcal{R}(\mu) = \{y \in \mathcal{R} \mid \chi_i(x) = \chi_i(\mu) \text{ for } i = 1, \ldots, n\},$$

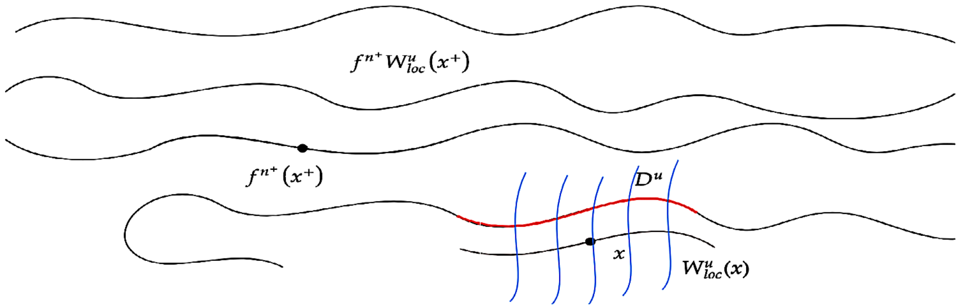

FIGURE 1. The images of the unstable manifold $W^u_{\text{loc}}(x^+)$ become more and more dense in the space under the map f. Therefore, for some $n^+ > 0$, one can find $\mathbf{D}^u \subset f^{n^+} W^u_{\text{loc}}(x^+)$ which is very close to the unstable manifold $W^u_{\text{loc}}(x)$, so that $\mathcal{D}^u$ intersects all the local stable manifolds of points in $W^u_{\text{loc}}(x)$.

we have that

$$m_{V^u_{\text{loc}}(x^+)}(V^u_{\text{loc}}(x^+) \cap \mathcal{R}(\mu^+)) = 1 \quad \text{and} \quad m_{V^s_{\text{loc}}(x^-)}(V^s_{\text{loc}}(x^-) \cap \mathcal{R}(\mu^-)) = 1.$$

Now, we note that when f is Anosov, then

$$\bigcup_{n\geq 0} f^n(V^u_{\text{loc}}(x^+))$$

is dense in M and hence, there is an $n \geq 0$ such that $f^n(V^u_{\text{loc}}(x^+))$ contains an embedded disk D^u which is really close to the local unstable $V^u_{\text{loc}}(x)$ in the C^1 topology as in Figure 1 below.

Therefore, the holonomy map

$$\Gamma^s : V^u_{\text{loc}}(x) \to D^u$$

given by

$$\Gamma^s(z) := W^s_{\text{loc}}(z) \cap D^u$$

is well defined. One can show that (using Lemma 3.11, which we prove in §3) D^u satisfies

$$m_{D^u}(D^u \cap \mathcal{R}(\mu^+)) = 1.$$

Now, the holonomy map Γ^s has the absolute continuity property. Therefore, we can see that $(\Gamma^s)^{-1}(D^u \cap \mathcal{R}(\mu^+))$ has full leaf volume in $V^u_{\text{loc}}(x)$.

Although the forward Lyapunov exponents of two points on the same stable manifolds coincide, it is not, in general, true that points on the stable manifold of a Lyapunov–Perron regular point are also Lyapunov–Perron regular. This is because *a priori* we can not compare their backward behavior. So, in general, the pre-image of $D^u \cap \mathcal{R}(\mu^+)$ under the holonomy map does not necessarily lie in $\mathcal{R}(\mu^+)$. However, in our situation, we have chosen x so that its unstable local manifold contains a positive leaf volume of Lyapunov–Perron regular points and hence,

$$m_{V^u_{\text{loc}}(x)}(V^u_{\text{loc}}(x) \cap \mathcal{R} \cap (\Gamma^s)^{-1}(D^u \cap \mathcal{R}(\mu^+))) > 0.$$

Now the set $V^u_{\mathrm{loc}}(x) \cap \mathcal{R} \cap (\Gamma^s)^{-1}(D^u \cap \mathcal{R}(\mu^+))$ consists of Lyapunov–Perron regular points, which lie on the stable manifold of some point in $\mathcal{R}(\mu^+)$, using Lemma 3.13 (which we prove in §3); therefore, we see that we have in fact $m_{V^u_{\mathrm{loc}}(x)}(V^u_{\mathrm{loc}}(x) \cap \mathcal{R}(\mu^+)) > 0$. Since $x \in \mathcal{R}$, we see that this implies, using Lemma 3.13, that the Lyapunov exponents of x with respect to f coincide with the Lyapunov exponents of μ^+.

Similarly, we see that

$$m_{V^s_{\mathrm{loc}}(x)}(v^s_{\mathrm{loc}}(x) \cap \mathcal{R}(\mu^-)) > 0$$

and that this again implies that that the Lyapunov exponents of x with respect to f^{-1} coincide with the Lyapunov exponents of μ^-, and that is exactly what we needed to show.

So, at least in this toy model, we see that the positivity of volume of the set of hyperbolic Lyapunov–Perron regular points can only be explained by the fact that the system preserves a smooth measure. This paper addresses the problem in the more general case of non-uniformly hyperbolic systems.

2.2. *The main result.* The approach we have outlined for the toy model cannot be immediately generalized to the case of non-uniformly hyperbolic systems for a few reasons. First, we need to start with SRB measures for f and f^{-1}, whose existence is not trivial for general systems. Second, even if we know that the system admits two SRB measures μ^+ and μ^- for f and f^{-1} respectively, the images under f of a μ^+-typical local unstable manifold, $V^u_{\mathrm{loc}}(x^+)$, do not necessarily fill in the space M, and in particular, we may not be able to bring $f^n(V^u_{\mathrm{loc}}(x^+))$ close to $V^u_{\mathrm{loc}}(x)$ for any $n \geq 0$. Third, even if we overcome the first two obstacles, we still need to relate the unstable $f^n(V^u_{\mathrm{loc}}(x^+))$ and $V^u_{\mathrm{loc}}(x)$ using local stable manifolds, since the holonomy maps, used in the proof of the model case, may not be well defined. This last problem arises from the non-uniformity of the sizes of unstable and stable manifolds for general non-uniformly hyperbolic systems.

We will tackle these issues in order, the first of which is the existence of SRB measures for both f and f^{-1}. The main difference between the toy model and the general non-uniformly hyperbolic case is that now, it is not obvious if SRB measures exist, even when $\mathrm{Vol}(\mathcal{R}_{\mathrm{hyp}}) > 0$. Recently Ben Ovadia [**4**] proved the existence of SRB measures for f under a *leaf condition*. We will expand on this later, but roughly speaking, the idea is that if there exists an unstable manifold with positive leaf volume of points on it which are hyperbolic and *positively recurrent* (meaning that the unstable and stable manifolds return to a uniform size frequently along the forward orbit), then the system admits an SRB measure μ^+. We will use this to prove the existence of SRB measures μ^+ and μ^- for f and f^{-1}. Assuming that the set $\mathcal{R}^{\mathrm{PR}} \subset \mathcal{R}_{\mathrm{hyp}}$ of positively recurrent points has positive volume, we will describe $\mathcal{R}^{\mathrm{PR}}$ more precisely, but for now, we can say that it consists of hyperbolic Lyapunov–Perron regular points whose stable and unstable manifolds return to uniform sizes frequently along the forward and backward orbits.

To deal with the second problem, we divide the set $\mathcal{R}^{\mathrm{PR}}$ into possibly countably many subsets, such that each of these subsets have forward and backward images that lie very close to two hyperbolic points, and then make sure that the SRB measures we constructed lie in the *ergodic homocolinic classes* (introduced in [12]) of these two hyperbolic points. This will allow us to conclude that each of these subsets, which we obtained by dividing $\mathcal{R}^{\mathrm{PR}}$, lie in the basin of attraction of the two SRB measures, and with a little bit more work, we will also be able to overcome the third obstacle.

Thus we get our the main theorem of this paper.

THEOREM 2.4. (Main Theorem) *Let $f : M \longrightarrow M$ be a $C^{1+\alpha}$ diffeomorphism of a compact Riemannian manifold M, and assume that the set of hyperbolic positively recurrent Lyapunov–Perron regular points $\mathcal{R}^{\mathrm{PR}}_{\mathrm{hyp}}$ has positive volume, then f preserves a smooth measure.*

Our main theorem is an immediate consequence of the following more technical theorem, which we will prove in §5 of this paper.

THEOREM 2.5. *Let $f : M \to M$ be a C^{1+} diffeomorphism, and assume that* $\mathrm{Vol}(\mathcal{R}^{\mathrm{PR}}_{\mathrm{hyp}}) > 0$. *Then there exists at most countably many ergodic absolutely continuous measures $\mu_1, \mu_2, \ldots$ such that* Vol-*a.e. point in $\mathcal{R}^{\mathrm{PR}}_{\mathrm{hyp}}$ is Tsuji regular for μ_n, for some* $n \geq 1$.

Tsuji regularity was first defined in [17]. We include the definition here for completeness.

Definition 2.6. [17] Let $f : M \longrightarrow M$ be a $C^{1+\alpha}$ diffeomorphism on a smooth Riemannian manifold, and let μ be an ergodic invariant measure. We say that a point $x \in M$ is *forward Tsuji regular* for μ if the following hold.

(1) $(1/n) \sum_{k=0}^{n-1} \delta_{f^k x} \longrightarrow \mu$ in the sense of the weak topology.

(2) The Lyapunov exponents of x with respect to f coincide with the Lyapunov exponents of μ.

We say that $x \in M$ is *forward Tsuji regular* for μ if it is *forward Tsuji regular* for μ with respect to f^{-1}. We say that $x \in M$ is *Tsuji regular* for μ if it is both forward and backward Tsuji regular.

The condition $\mathrm{Vol}(\mathcal{R}^{\mathrm{PR}}) > 0$, as we will see, guarantees the existence of the SRB measures. The next step is to relate the SRB measures to each other, but here we encounter another problem. It is possible that the system admits countably many ergodic SRB measures for f and countably many ergodic SRB measures for f^{-1} and the question then is:

Which ones are related to which ones?

We approach this issue by examining the construction of the SRB measures more closely, and then construct two specific SRB measures μ^+ and μ^- so that they are related almost by construction. To do that, we will have to deal with the symbolic representation of the dynamics on the set $\mathcal{R}^{\mathrm{PR}}$, and we will have to consider the more general set RWT_χ which was introduced in [4].

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The next issue we will deal with is:

Why should there be an $n \geq 0$ *such that* $f^n(V^u_{\mathrm{loc}}(x^+))$ *and the local unstable manifold of another chosen point* $x, V^u_{\mathrm{loc}}(x)$ *, are close to each other so that the stable holonomy map as in the toy model could be defined?*

We will resolve this by considering homoclinic ergodic classes as introduced in [12]. We will choose $x \in \mathcal{R}^{\mathrm{PR}}$ so that it is in both of the homoclinic ergodic classes supporting μ^+ and μ^-, and then using arguments inspired by those in [4], we will replicate the proof of the toy model case.

The last issue we will tackle is to show that the resulting absolutely continuous measures μ^+ and μ^- are actually equal to each other. Along the way, we will prove an analog of Lemmas 1 and 2 for general non-uniformly hyperbolic systems.

3. *The entropy argument*

The main goal of this section is to prove the following proposition.

PROPOSITION 3.1. *Let* P^+ *and* P^- *be two hyperbolic periodic points. Let* R^+ *and* R^- *be two rectangles at* P^+ *and* P^- *respectively. Let* $x_0 \in \mathcal{R}_{\mathrm{hyp}}$ *be a hyperbolic Lyapunov–Perron regular point. Let us assume the following items.*

(1) *Each of the homoclinic ergodic classes* $H(P^+)$ *and* $H(P^-)$ *supports a unique SRB measure* μ^+ *for* f *and* μ^- *for* f^{-1} *respectively.*
(2) *There is an unstable* W^+ *in the rectangle* R^+ *for which we have:*
  (a) $m_{W^+}(W^+ \cap R^+ \cap \mathcal{R}_{\mathrm{hyp}}) > 0$;
  (b) $f^{-n^+}(W^+) \subset V^u_{\mathrm{loc}}(x_0)$, *for some* $n^+ > 0$.
(3) *There is a stable* W^- *in the rectangle* R^- *for which we have:*
  (a) $m_{W^-}(W^- \cap R^- \cap \mathcal{R}_{\mathrm{hyp}}) \geq 0$;
  (b) $f^{n^-}(W^-) \subset V^s_{\mathrm{loc}}(x_0)$, *for some* $n^- \geq 0$.

Then both μ^+ *and* μ^- *are absolutely continuous with respect to volume.*

This proposition is one of the key component of the proof of the main theorem. We will use it to prove that the SRB measures we construct later are absolutely continuous with respect to volume.

3.1. *u-measures and s-measures*

Definition 3.2. (Global unstable manifolds) Let $x \in \mathcal{R}_{\mathrm{hyp}}$, then the *global unstable manifold* at x, which we denote by $W^u(x)$, is defined to be

$$W^u(x) := \bigcup_{n \geq 0} f^n(V^u_{\mathrm{loc}}(f^{-n}(x))).$$

This set is an immersed submanifold of M (see [2]).

Definition 3.3. (Partition subordinate to unstable manifolds) Let $f : M \longrightarrow M$ be a $C^{1+\alpha}$, and let μ be an invariant measure with at least one non-zero Lyapunov exponent, then we say that a measurable partition ξ^u is subordinate to unstable manifolds if it satisfies the following properties:

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(1) $\xi^u(x)$ is an open subset of $W^u(x)$, for μ-a.e. $x \in M$;
(2) $\bigvee_{n\geq 0} f^{-n}\xi^u = \epsilon$, the partition by points;
(3) $f^{-1}\xi^u \geq \xi^u$;
(4) $\bigwedge_{n\geq 0} f^n\xi^u = W^u$, where W^u is the partition by global unstable manifolds.

Definition 3.4. (u/s-measure) An invariant measure μ is a u-measure if, given any measurable partition ξ subordinate to unstable manifolds, the conditional measures μ_x^u along $\xi(x)$ are absolutely continuous with respect to leaf volume for μ-a.e. x. A measure μ is an s-measure if it is a u-measure for f^{-1}. A u-measure which is also hyperbolic is called an SRB measure for *f*. Similarly, an s-measure which is also hyperbolic is called an SRB measure for f^{-1}.

There are many ways to characterize SRB measures, but one of the most remarkable characterizations is the following theorem of Ledrappier and Young [9].

PROPOSITION 3.5. (The entropy formula) *Let $f : M \to M$ be a $C^{1+\alpha}$ diffeomorphism. Then a measure μ is an SRB measure for f if and only if the Pesin entropy formula holds*

$$h_\mu(f) = \int_M \sum_i \lambda_i^+(x)\, d\mu(x),$$

where $\lambda_i^+(x) = \max\{\lambda_i(x), 0\}$. *Similarly, μ is an SRB measure for f^{-1} if and only if*

$$h_\mu(f) = \int_M \sum_i -\lambda_i^-(x)\, d\mu(x),$$

where $\lambda_i^-(x) = \min\{\lambda_i(x), 0\}$.

LEMMA 3.6. *Let $f : M \to M$ be a $C^{1+\alpha}$ diffeomorphism and let μ be an SRB measure for f. Let ξ^u be a measurable partition subordinate to unstable manifolds. Then for μ-a.e $x \in M$, the conditional measure $\mu_{\xi^u(x)}$ and normalized leaf Riemannian volume $m_{\xi^u(x)}$ are equivalent.*

Proof. In [9, Corollary 6.14], the authors show that for μ-a.e. $x \in M$, one has that the conditional measure $\mu_{\xi^u(x)}$ is absolutely continuous with respect to the Riemmanian volume along the global unstable manifold $V^u(X)$, $m_{\xi^u(x)}$, with a density ρ which is strictly positive on $\xi^u(x)$. By straightforward measure theory, one can show that this implies that the two measures $\mu_{\xi^u(x)}$ and $m_{\xi^u(x)}$ are equivalent. □

The main goal of this paper is to establish that a diffeomorphism preserves a measure absolutely continuous with respect to volume under some conditions. The strategy we are following is to to establish the existence of a measure μ which is an SRB measure for both f and f^{-1} at the same time. The goal for the remainder of this section is to establish some conditions under which an SRB measure for f is also an SRB measure for f^{-1}, which implies that it is actually absolutely continuous with respect to volume.

LEMMA 3.7. *A hyperbolic measure μ is absolutely continuous with respect to volume if μ is both an SRB measure for f and for f^{-1}.*

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To the best of our knowledge, this statement appears in the paper [11] and in the book [2], but an explicit proof has not been given, so for completeness, we provide a proof of the statement.

Proof of the lemma. Let $E \subset M$ be such that $\mu(E) > 0$, and let $x \in E$ be a density point for x such that $m^u_x(E) > 0$ and $m^s_x(E) > 0$, where m^u_x and m^s_x are leaf volumes along the unstable and stable manifold of x respectively; such a point exists since the conditional measures μ^u_x and μ^s_x of μ along the unstable and stable manifolds are equivalent to leaf volumes for μ-a.e. $x \in M$.

Now locally, volume can be written as the product of the two leaf volumes m^u_x and m^s_x, that is, Vol $\sim m^u_x \times m^s_x$. Hence, we see that $\mathrm{Vol}(E) > 0$, which implies that $\mu <<$ Vol. □

The following observation is a simple corollary of the above lemma, the entropy formula, and Ruelle's inequality.

LEMMA 3.8. *An ergodic SRB measure μ for f satisfies*

$$\sum_i \lambda_i(f, \mu) \leq 0$$

and equality holds if and only if it is absolutely continuous with respect to volume.

Proof. The inequality is an immediate consequence of the Pesin entropy formula and the Ruelle inequality, following the argument used in the proof of Lemma 2.3. Since μ is an SRB measure, we have

$$h_\mu(f) = \sum_i \lambda_i^+(x) d.$$

Now, since the average of the sum of the Lyapunov exponents is zero, we see that

$$\sum_i \lambda_i^+(x) = \sum_i -\lambda_i^-(x) d,$$

and since $h_\mu(f) = h_\mu(f^{-1})$,

$$h_\mu(f^{-1}) = h_\mu(f) = \sum_i \lambda_i^+(x) = \sum_i -\lambda_i^-(x).$$

Hence, μ is also an SRB measure for f^{-1}, and therefore, absolutely continuous with respect to volume. □

3.2. *Global manifolds and rectangles.* We will need the following definition of hyperbolic rectangles.

Definition 3.9. Let f be a $C^{1+\alpha}$ diffeomorphism of a smooth compact Riemannian manifold M. Let $0 < \lambda < 1 < \nu$, $\epsilon > 0$, j be an integer between 1 and $dim(M) - 1$, and $\ell > 0$. Consider the Pesin block $\Lambda^\ell := \Lambda^\ell_{\lambda,\nu,\epsilon,j}$. We say that $R \subset M$ is a δ-rectangle at a

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point $w \in \Lambda^\ell$ if $w \in R \subset B(w, \delta) \cap \Lambda^{\psi(\ell)}$, and for any $x, y \in R$, we have

$$V^s(x) \cap V^u(y) \in R,$$

where $V^s(z)$ and $V^u(z)$ are the local stable land unstable manifolds for $z \in \Lambda^\ell$, and $\psi(\ell)$ is defined so that for any $x \in \Lambda^\ell$ and $y \in \Lambda^\ell \cap B(x, \delta_\ell)$, one has $V^s(x) \cap V^u(u) \in \Lambda^{\psi(\ell)}$, where δ_ℓ is the scale at which we see local products in Λ^ℓ.

We say that an unstable manifold W^+ is in R if it is the local unstable manifold of a point in R. We say that a stable manifold W^- is in R if it is the local stable manifold of a point in R.

Definition 3.10. Let μ be an ergodic invariant measure. We define the set $\mathcal{R}(\mu)$ by

$$\mathcal{R}(\mu) = \{x \in \mathcal{R} \mid \lambda_i(x) = \lambda_i(\mu)\}.$$

This is an invariant set, and one has by Oseledets' multiplicative ergodic theorem that $\mu(\mathcal{R}(\mu)) = 1$.

We will need the following lemma to show that global unstable manifolds intersect the set $\mathcal{R}(\mu)$ in a set of full leaf volume when μ is an SRB measure.

LEMMA 3.11. *Let μ be an ergodic SRB measure for f. Let E be an invariant set of non-zero μ measure. Then for μ-a.e. point $x \in M$, the following is satisfied: any compact subset D of the global unstable manifold $W^u(x)$ intersects E in a set of full measure inside D, with respect to the Riemannian volume $m_{W^u(x)}$ on $W^u(x)$.*

Proof. Step 1. First, since μ is ergodic and E is invariant of positive μ measure, we know that $\mu(E) = 1$. Now, let ξ^u be a measurable partition subordinate to unstable manifolds. Then, since μ is an SRB measure, we know from Lemma 3.6 that for μ-a.e. point $x \in M$, one has that $\mu_{\xi^u(x)}$ is equivalent to $m_{\xi^u(x)}$,the leaf volume restricted to $\xi^u(X)$. Now, since $\mu(E) = 1$, we know that for μ-a.e. $x \in M$, we have that $\mu_{\xi^u(x)}(\xi^u(x) \cap E) = 1$, and since $\mu_{\xi^u(x)}$ and $mu_{\xi^u(x)}$ are equivalent for μ-a.e. point x, we see that this implies that we have $m_{\xi^u(x)}(\xi^u(x) \cap E) = 1$ for μ-a.e $x \in M$.

Step 2. From [2, Proposition 9.4.1], we know that there is a measurable function $\beta : M \longrightarrow (0, r_0)$, such that for μ-a.e. $x \in M$, $\xi^u(x)$ contains the neighborhood of x defined by $\{y \in W^u(x) \mid d_{W^u(x)}(x, y) < \beta(x)\}$. Let us fix $r_1 > 0$ to be small enough so that the measurable set

$$\{w \in M \mid \beta(w) \geq r_1\}$$

has positive μ-measure. Then by ergodicity of μ, we see that for μ-a.e. $x \in M$, one can find infinitely many times $n \geq 0$ for which we have the following.

(1) $m_{\xi^u(f^{-n}x)}(\xi^u(f^{-n}x) \cap E) = 1$.

(2) $\beta(f^{-n}x) \geq r_1$.

However, we know that for all large enough times $n > 0$, we have

$$f^{-n}(V^u_{\text{loc}}(x)) \subset \left\{y \in W^u(f^{-n}x) \mid d_{W^u(f^{-n}x)}(f^{-n}x, y) < \tfrac{1}{2}r_1\right\}.$$

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Therefore, for a time n large enough, which also satisfies (1) and (2), we see that we actually have

$$f^{-n}\left(V^u_{\text{loc}}(x)\right) \subset \xi^u(f^{-n}x),$$

and from condition (1) above, we see that we have $m_{f^{-n}(V^u_{\text{loc}}(x))}(f^{-n}(V^u_{\text{loc}}(x)) \cap E) = 1$, and since the pushforward of the measure $m_{f^{-n}(V^u_{\text{loc}}(x))}$ under the map f^n is equivalent to the leaf volume $m_{V^u_{\text{loc}}(x)}$, and since E is invariant, we see that this implies that $m_{V^u_{\text{loc}}(x)}(V^u_{\text{loc}}(x) \cap E) = 1$.

Step 3. Now, since D is a compact subset of $W^u(x)$, we can find a time n, for which we have $f^{-n}(D) \subset V^u_{\text{loc}}(f^{-n}x)$. From steps 1 and 2 above, we see that this implies that $E \cap f^{-n}(D)$ has full measure inside $f^{-n}(D)$ with respect to the leaf measure $m_{f^{-n}(V^u_{\text{loc}}(x))}$, for μ-a.e. $x \in M$, which in turn implies the lemma. □

The main lemma in this subsection is the following.

LEMMA 3.12. *Let μ be an SRB measure for f, let R be a hyperbolic rectangle, and let D^u be a local transversal to all stable manifolds in the rectangle R. Assume furthermore that m_{D^u}-a.e. point in D^u is in $\mathcal{R}(\mu)$. Then for any unstable W^+ in R, we have that m_{W^+}-a.e. point in $W^+ \cap R \cap \mathcal{R}_{\text{hyp}}$ is in $\mathcal{R}(\mu)$.*

This lemma allows us to control the Lyapunov exponents of Lyapunov–Perron regular points lying in a hyperbolic rectangle through the Lyapunov exponents of an SRB measure, and is central to the proof of the main proposition of this section. To prove this lemma, we will the following two lemmas.

LEMMA 3.13. [17, Proposition 16(3)] *Let $x_0 \in \mathcal{R}_{\text{hyp}}$ be a hyperbolic Lyapunov–Perron regular point. Assume that x is a point such that we have one of the following satisfied:*

(1) $x \in V^u_{\text{loc}}(x_0) \cap \mathcal{R}_{\text{hyp}}$;

(2) $x \in V^s_{\text{loc}}(x_0) \cap \mathcal{R}_{\text{hyp}}$.

Then the points x_0 and x have the same Lyapunov exponents with respect to f.

Remark 3.14. For more details on Lemma 3.13, see [17, Proposition 16(3)], and in particular the proof where the author uses this lemma to prove item (3) in proposition 16.

We also need the following well-known inclination lemma.

LEMMA 3.15. (Inclination lemma) *Let $f : M \longrightarrow M$ be a C^1 diffeomorphism on a Riemannian manifold M. Let p be a fixed hyperbolic point. Let $B \subset W^u(p)$ be an embedded C^1 disk of the same dimension as $W^u(p)$. Let D be a C^1 embedded disk transversal to the $W^s(p)$ of the same dimension as $W^u(p)$. Then for any $\epsilon > 0$, there is an $n_0 \geq 0$, such that for an $n \geq n_0$, $f^n(D)$ contains a C^1 embedded disk which is ϵ close to B in the C^1-topology.*

Now we are ready to prove Lemma 3.12.

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Proof of Lemma 3.12. Let us fix an unstable W^+ in R. Then we can define the stable holonomy map

$$\Gamma^s : W^+ \cap R \longrightarrow D^u \cap R$$

defined by $\Gamma^s(x) := V^s_{\mathrm{loc}}(x) \cap D^u$. We can see that this is a well-defined map since D^u is a transversal to stable manifolds in R. This map is absolutely continuous with respect to the Riemmanian volume on W^+ and D^u, that is, $(\Gamma^s)^*(m_{D^u|_{D^u \cap R}})$ is equivalent to $m_{W^+}|_{W^+ \cap R}$. Now, let us consider the $(\Gamma^s)^{-1}(D^u \cap R \cap \mathcal{R}(\mu)) \subset W^+ \cap R$. We see that since $D^u \cap \mathcal{R}(\mu)$ has full measure with respect to the measure m_{D^u}, it has full measure with respect to the restriction $m_{D^u}|_{D^u \cap R}$, and by absolute continuity of the stable holonomy map Γ^s, this implies that $(\Gamma^s)^{-1}(D^u \cap R \cap \mathcal{R}(\mu)))$ has full measure with respect to the restriction $m_{W^+}|_{W^+ \cap R}$. This implies that $(\Gamma^s)^{-1}(D^u \cap R \cap \mathcal{R}(\mu)) \cap \mathcal{R}_{\mathrm{hyp}}$ has full measure inside $W^+ \cap R \cap \mathcal{R}_{\mathrm{hyp}}$ with respect to $m_{W^+}|_{W^+ \cap R}$. Now, the set $(\Gamma^s)^{-1}(D^u \cap R \cap \mathcal{R}(\mu)) \cap \mathcal{R}_{\mathrm{hyp}}$ consists of all those points in $W^+ \cap R$ which are in $\mathcal{R}_{\mathrm{hyp}}$, and at the same time are on the stable manifold of a point in $\mathcal{R}(\mu)$. Using Lemma 3.13, we see that this implies that these points are in $\mathcal{R}(\mu)$, which implies the lemma. □

3.3. *Ergodic homoclinic classes.* In [12], the authors introduce ergodic homoclinic classes.

Definition 3.16. [12] Let $P \in M$ be a hyperbolic periodic point and let

$$H^u(P) := \{x \in \mathcal{R}_{\mathrm{hyp}} \mid W^s(x) \pitchfork W^u(\mathcal{O}(P)) \neq \emptyset\},$$
$$H^s(P) := \{x \in \mathcal{R}_{\mathrm{hyp}} \mid W^u(x) \pitchfork W^s(\mathcal{O}(P)) \neq \emptyset\}.$$

We define the ergodic homoclinic class of P by $H(P) = H^u(P) \cap H^s(P)$.

The reason why these homoclinic classes were introduced is to better understand the ergodic components of SRB and smooth measures, precisely, the authors of [15] and [16] prove the following result.

PROPOSITION 3.17. *Let μ be an SRB measure for a $C^{1+\alpha}$ diffeomorphism $f : M \to M$. Then the following statements hold:*

(1) *if $\mu(H(P)) > 0$, then the restriction $\mu|_{H(P)}$ is an ergodic SRB measure;*
(2) *given any ergodic SRB measures μ_1 and μ_2 such that $\mu_1(H(P)) > 0$ and $\mu_2(H(P)) > 0$, one has $\mu_1 = \mu_2$.*

There are two main reasons we wish to work with ergodic homoclinic classes in our case. First, they allow us to extend the transitivity argument in the proof of the toy model. The following proposition captures this precisely.

Remark 3.18. For a given point x inside a Pesin block Λ_ℓ, we denote by $W^u_\ell(x)$ and $W^s_\ell(x)$ the maximal dimensional local unstable and stable manifolds at x, with size bounded below by $e^{-\ell}$.

LEMMA 3.19. *Let P be a hyperbolic periodic point. Let R be rectangle containing P. Let μ be an ergoidic SRB measure supported on the ergodic homoclinic class $H(P)$. Then there*

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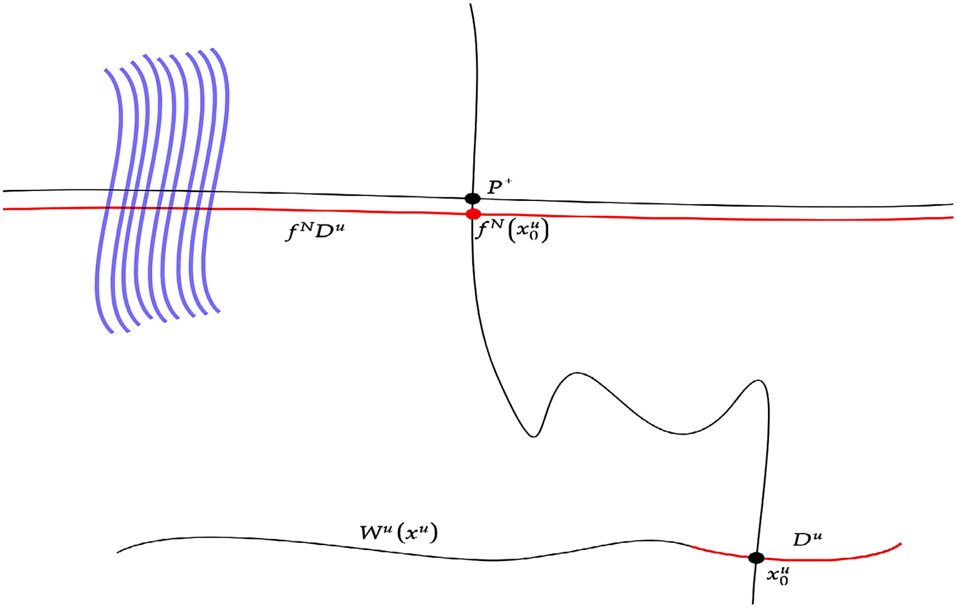

FIGURE 2. The inclination lemma implies that, for a large N, $f^N(\mathcal{D}^u)$ is very close to the unstable manifold of the periodic point P. Since the unstable manifold of the point P intersects all the stable manifolds of points in the rectangle R, we see that $f^N(\mathcal{D}^u)$ does too.

is a local transversal D^u for all stable manifolds in R, such that m_{D^u}-a.e. point in D^u is in $\mathcal{R}(\mu)$.

Proof. Since the set $\mathcal{R}(\mu)$ is invariant for f, and has full measure with respect to R, we see that μ-a.e. point $x \in M$ satisfies the conclusion of Lemma 3.11. Let us fix such a point and denote it by x^u. Since μ is supported on $\mu(H(P)) = 1$, we can assume without loss of generality that $x^u \in H(P)$. Let x_0^u be a point in $W^u(x^u) \pitchfork W^s(P)$, where this point exists by the assumption that $x^u \in H(P)$. Let $D \subset W^u(x^u)$ be an embedded disk containing the intersection point x_0^u, and let $p \geq 0$ be the period of the hyperbolic periodic point P. Then for any $\epsilon > 0$, the inclination lemma allows us to find $N > 0$ such that $f^{Np}(D)$ contains an embedded disk D^u, such that $d_{C^1}(D^u, V_{\mathrm{loc}}^u(P)) < \epsilon$ (see Figure 2). By taking $\epsilon > 0$ small enough, we can ensure that D^u is transversal for all stable manifolds in R, since $V_{\mathrm{loc}}^u(P)$ is a transversal to all stable manifolds in R. □

Now, we are ready to prove the min proposition of this section, Proposition 3.1.

Proof of Proposition 3.1. Using Lemma 3.19, we see that we can find a local transversal D^u to all stable manifolds in R^+, and such that m_{D^u}-a.e. point in D^u is in $\mathcal{R}(\mu^+)$. Using Lemma 3.12, we see that this implies that m_{W^+}-a.e. point in $W^+ \cap R^+ \cap \mathcal{R}_{\mathrm{hyp}}$ is in $\mathcal{R}(\mu^+)$, and by condition (2) in the proposition, we see that we have $m_{W^+}(W^+ \cap \mathcal{R}(\mu^+)) > 0$. By replacing f by f^{-1}, we see that the same exact argument implies that we also have $m_{W^-}(W^- \cap \mathcal{R}(\mu^-)) > 0$. Now, using the fact that $\mathcal{R}(\mu^+)$ and $\mathcal{R}(\mu^-)$ are invariant sets for f, and conditions (2) and (3) in the proposition, we see

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that $m_{V^u_{\mathrm{loc}}(x_0)}(V^u_{\mathrm{loc}}(x_0) \cap \mathcal{R}(\mu^+)) > 0$ and $m_{V^s_{\mathrm{loc}}(x_0)}(V^s_{\mathrm{loc}}(x_0) \cap \mathcal{R}(\mu^-)) > 0$. Using Lemma 3.13, we see that the Lyapunov exponents of x_0 with respect to f coincide with the Lyapunov exponents of f with respect to the SRB measure μ^+, and that the Lyapunov exponents of x_0 with respect to f^{-1} coincide with the Lyapunov exponents of f^{-1} with respect to the measure μ^-. By the regularity of the point x_0, we know that the Lyapunov exponents of x_0 with respect to f^{-1} are the negative of the Lyapunov exponents of x_0 with respect to f. In particular, this means that we have

$$\sum_i \lambda_i(f, \mu^+) = -\sum_i \lambda_i(f^{-1}, \mu^-);$$

however, using Lemma 3.8, we see that we have

$$\sum_i \lambda_i(f, \mu^+) \le 0$$

and that we also have

$$\sum_i \lambda_i(f^{-1}, \mu^-) \le 0.$$

Therefore, we see that we end up with

$$\sum_i \lambda_i(f, \mu^+) = 0 = \sum_i \lambda_i(f^{-1}, \mu^-).$$

Using Lemma 3.8 again, we see that this implies that both μ^+ and μ^- are absolutely continuous with respect to volume. □

4. *Non-uniform hyperbolicity*

The second reason for considering ergodic homoclinic classes is that they have a convenient symbolic description which we will describe in this section.

4.1. *Lyapunov–Perron regularity.*

Our main object in this paper is the set of hyperbolic Lyapunov–Perron regular points $\mathcal{R}_{\mathrm{hyp}}$, which we define next.

Definition 4.1. A point $x \in M$ is Lyapunov–Perron regular for a diffeomorphism $f : M \to M$ if there exist numbers $\lambda_1(x) < \lambda_2(x) < \cdots < \lambda_{k(x)}(x)$ and an invariant decomposition

$$T_x M = \bigoplus_{i=1}^{k(x)} E_i(x)$$

such that for all $v \in E_i(x)$, we have

$$\lim_{n\to\pm\infty} \frac{1}{n} \log \|D_x f^n v\| = \lambda_i(x),$$

$$\lim_{n\to\pm\infty} \frac{1}{n} \log \|\det(D_x f^n)\| = \sum_{i=1}^{k(x)} \lambda_i(x),$$

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and so that $\lambda_i(x)$, $E_i(x)$, and $k(x)$ depend measurably on x. We denote the set of all such Lyapunov–Perron regular points by $\mathcal{R}$.

Definition 4.2. A point $x \in \mathcal{R}$ is hyperbolic if $\lambda_i(x) \neq 0$ for all $i = 1, \ldots, k(x)$ and $\lambda_1(x) < 0 < \lambda_{k(x)}(x)$. We denote the set of all hyperbolic Lyapunov–Perron regular points by $\mathcal{R}_{\text{hyp}}$. Given $\chi > 0$, we define $\mathcal{R}_\chi \subset \mathcal{R}_{\text{hyp}}$ by

$$\mathcal{R}_\chi = \{x \in \mathcal{R}_{\text{hyp}} \mid |\lambda_i(x)| > \chi \quad \text{for all } i = 1, \ldots, k(x)\}.$$

We call it the set of χ-hyperbolic Lyapunov–Perron regular points.

Remark 4.3. We stress that, while by Oseledets theorem, for a given hyperbolic invariant measure μ, almost all points with respect to μ are in $\mathcal{R}_{\text{hyp}}$, we can easily find that $\mathcal{R}_{\text{hyp}}$ has zero measure with respect to volume, even in the case of Anosov systems. In fact, as we pointed out in the informal discussion above, in the case of an Anosov map, $\text{Vol}(\mathcal{R}_{\text{hyp}}) > 0$ implies that the system preserves a smooth measure, so in general, one can expect that the volume of $\mathcal{R}_{\text{hyp}}$ is zero.

Remark 4.4. A simple observation is that χ-hyperbolic regular points exhaust $\mathcal{R}_{\text{hyp}}$, that is,

$$\mathcal{R}_{\text{hyp}} = \bigcup\nolimits_{\chi > 0} \mathcal{R}_\chi.$$

We also note that each $\mathcal{R}_\chi$ is invariant under the action of f. In this paper, we will fix a small $\chi > 0$ and work with $\mathcal{R}_\chi$, then using the fact that they exhaust $\mathcal{R}_{\text{hyp}}$, we will be able to obtain our main theorem.

Definition 4.5. We define the set $\mathcal{R}_{\text{hyp}}^{\text{PR}}$ of *positively recurrent hyperbolic Lyapunov–Perron regular* points to be the set of points $x \in \mathcal{R}_{\text{hyp}}$, for which there is a Pesin level set Λ^{r_x} for which we have the following two recurrence conditions satisfied:

(1) $\limsup_{n\to\infty}(1/n) \sum_{k=0}^{n-1} 1_{\Lambda^{r_x}} \circ f^k(x) > 0$;
(2) $\limsup_{n\to\infty}(1/n) \sum_{k=0}^{n-1} 1_{\Lambda^{r_x}} \circ f^{-k}(x) > 0$.

Remark 4.6. The definition of positively recurrent hyperbolic Lyapunov–Perron regular points is directly by the definition of RWT_χ^{PR} in [4]. In fact, we have that $\mathcal{R}_\chi^{\text{PR}} \subset RWT_\chi^{\text{PR}}$. The main difference here is that our definition concerns Lyapunov–Perron regular points, and is also symmetric in time, while the definition of RWT_χ^{PR} is concerned mainly with the forward dynamics of the system.

In the course of the proof of the the main theorem, we will have to use stable and unstable holonomy maps, and one problem is that $\mathcal{R}_\chi^{\text{PR}}$ is not invariant under these holonomy maps, so we will find ourselves forced to consider the larger sets of points RWT_χ^{PR}, which we introduce next.

4.2. *Non-uniform hyperbolicity.* In [4], the author introduces the set RWT_χ. This set is larger than the set $\mathcal{R}_\chi$ of χ-Lyapunov–Perron regular points. The author observed that the whole theory of invariant manifolds does not require Lyapunov–Perron regularity, instead we only need some notion of hyperbolicity. For many purposes, this set is much more

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flexible to work with than $\mathcal{R}_\chi$, for the simple reason that we can work with holonomy maps on theses sets while we cannot do that with $\mathcal{R}_\chi$.

Definition 4.7. [4, Definition 2.1] A point $x \in M$ is χ-summable if it satisfies the following condition: There is a unique splitting $T_x M = H^s(x) \oplus H^u(x)$ such that

$$\sup_{v \in H^s(x), \|v\|=1} \sum_{n \geq 0} e^{2n\chi} \|D_x f^n v\|^2 < \infty,$$

$$\sup_{v \in H^u(x), \|v\|=1} \sum_{n \geq 0} e^{2n\chi} \|D_x f^{-n} v\|^2 < \infty.$$

This set is denoted by $\chi - summ$.

A point $x \in M$ is χ-hyperbolic if it satisfies the following condition: There is a unique splitting $T_x M = H^s(x) \oplus H^u(x)$ such that

$$\limsup_{n \to \infty} \frac{1}{n} \log(\|D_x f^n v\|) < -\chi \quad \text{for all } v \in H^s(x),$$

$$\limsup_{n \to \infty} \frac{1}{n} \log(\|D_x f^{-n} v\|) < -\chi \quad \text{for all } v \in H^u(x).$$

A measure carried by $\chi - hyp$ is called a χ-hyperbolic measure.

THEOREM 4.8. [4, Theorem 2.2] *For all $x \in \chi - summ$, there exists an invertible linear map $C_\chi(x) : \mathbb{R}^d \to T_x M$ such that $D_\chi(x) = C_\chi(f(x))^{-1} \circ D_x f \circ C_\chi(x)$ has the block decomposition*

$$D_\chi(x) = \begin{pmatrix} D_\chi{}^s(x) & 0 \\ 0 & D_\chi^u(x) \end{pmatrix}$$

with respect to the decomposition $T_x M = H^s(x) \oplus H^u(x)$, where $\|D_\chi^s(x)v\| \leq e^{-\chi}\|v\|$ and $\|D_\chi^u(x)w\| \geq e^{\chi}\|v\|$.

Definition 4.9. [4, Definition 2.3] Let $\epsilon > 0$, and let $x \in \chi - summ$, then

$$Q_{\epsilon_\chi}(x) := \max\{Q \in \{e^{-\ell\epsilon_\chi/3}\}_{\ell \geq 0} \mid Q \leq 3^{-6/\beta}\epsilon^{90/\beta}\|C_\chi(x)^{-1}\|^{-48/\beta}\}.$$

Definition 4.10. [4, Definition 2.4] Let $\epsilon > 0$. A point $x \in \chi - summ$ is called ϵ-weakly temperable if there is a function $q : \{f^n(x)\}_{n \in \mathbb{Z}} \longrightarrow (0, \epsilon_\chi] \cap \{e^{(-\ell\epsilon_\chi)/3}\}_{\ell \geq 0}$ such that
(1) $e^{-\epsilon_\chi} \leq q \circ f/q \leq e^{\epsilon_\chi}$;
(2) $q \circ f^n(x) \leq Q_{\epsilon_\chi} \circ f^n(x)$, for all $n \in \mathbb{Z}$.
We call an ϵ-weakly temperable point x recurrently ϵ-weakly temperable if we also have $\limsup_{n \to \infty} q(f^{\pm}(x)) > 0$.

Definition 4.11. [4, Definition 2.4] and [4, Definition 7.7] The set RWT_χ is defined to be the set of all χ-summable and recurrently ϵ_χ-weakly temperable, where $\epsilon_\chi > 0$ is a fixed small constant. The set of *positively recurrent* points in RWT_χ is defined by

$$RWT_\chi^{\rm PR} = \left\{ x \in RWT_\chi \mid \text{there exists } r_x > 0 \text{ such that } \limsup_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} 1_{\Lambda^{r_x}} \circ f^k(x) > 0 \right\}.$$

We emphasize the dependence of RWT_χ and $RWT_\chi^{\rm PR}$ on the map f by writing $RWT_\chi(f)$ and $RWT_\chi^{\rm PR}(f)$.

Remark 4.12. An easy example that illustrates the difference between $\mathcal{R}_\chi$ and RWT_χ is the case of Anosov diffeomorphisms. Typically, it has two distinct SRB measures μ^+ and μ^- for f and f^{-1} respectively. The basins of attraction of these two SRB measures as $n \to \infty$ and as $n \to -\infty$ have full Lebesgue volume respectively, so we see that for almost all points, the forward and backward Lyapunov exponents are different, and therefore we see that $\mathcal{R}_{\rm hyp}$ has zero Lebesgue volume. However, we can see that for $\chi > 0$ small enough, one has $RWT_\chi = M$.

4.3. *The coding of the dynamics on RWT_χ.* The following theorem, proved in [15] for the surface diffeomorphism case, and in [3] for the general case, is going to be a corner stone in the proof of our main result.

PROPOSITION 4.13. [3, Theorem 1.1] *Let $f : M \to M$ be a $C^{1+\alpha}$ diffeomorphism. Then there exists a locally compact graph $(\mathcal{P}, \mathcal{E})$ that induces a topological Markov shift $\hat{\Sigma}^\#$ and a map $\hat{\pi} : \Sigma^\# \to RWT\chi$ satisfying:*

(1) *$\hat{\pi}$ is Holder continuous with respect to the word metric;*
(2) *$\hat{\pi} \circ \sigma = f \circ \hat{\pi}$;*
(3) *$\hat{\pi}$ is a finite-to-one map;*
(4) *$\hat{\pi}(\Sigma^\#)$ has full measure with respect to any χ-hyperbolic measure.*

This result is a far reaching generalization of Bowen's result for Axiom A systems [6]. While the approach follows the lines of the argument by Bowen, the technical details are much more intricate than in the Axiom A case. To be able to use this result, we will need a more precise understanding of the topological Markov shift and how it should be interpreted, so for the rest of this subsection, we will give a more detailed description of this coding and some of its properties that we are going to use.

4.3.1. *The first coding.* Let $\rho > 0$ and $r > 0$ be such that the exponential map $\exp_x : B_r(0) \to B_\rho(x)$ is a diffeomorphism. Let $Q_\epsilon(x)$ be the constant defined in [3, Definition 2.12], then we make the following definition.

Definition 4.14. (Pesin Charts) The Pesin chart at x,

$$\psi_x^\eta : [-\eta, \eta]^n \to M,$$

is defined by $\psi_x^\eta := \exp_x \circ C_\chi(x)$. A pair $\psi_x^{p^s,p^u} := (\psi_x^{p^s}, \psi_x^{p^u})$ of Pesin charts is called a double-Pesin chart.

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PROPOSITION 4.15. [3, Theorem 3.2] *There exists a countable and locally finite graph* $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ *that induces a topological Markov shift* $\Sigma = \{\underline{u} \in \mathcal{V}^{\mathbb{Z}} \mid (u_i, u_{i+1}) \in \mathcal{E},$ *for all* $i \in \mathbb{Z}\}$ *such that there is a map* $\pi : \Sigma \to M$ *satisfying:*

(1) $f \circ \pi = \pi \circ \sigma$, *where* $\sigma : \Sigma \to \Sigma$ *is the shift map;*

(2) π *is Holder continuous with respect to the word metric on* Σ*;*

(3) $\pi(\Sigma^{\#})$ *carries all invariant* χ*-hyperbolic measures of f, where*

$$\Sigma^{\#} = \big\{\underline{u} \in \Sigma \mid \text{there exists } a, b \in \mathcal{V} \text{ such that } u_n = a,\ u_{-m} = b \text{ for infinitely many } n, m \geq 0\big\}.$$

The edges of the graph are a countable subset of double charts, and an edge corresponds to the overlapping relation $\psi_x^{\eta} \to \psi_y^{\eta'}$. This map is typically infinite-to-one, but we can use a Bowen–Sinai refinement procedure to get a Markov partition, as done in [3, 15].

4.3.2. *The Markov partition.* For any $v \in \mathcal{V}$, we define $Z[v] := \pi([v] \cap \Sigma^{\#})$ and let $\mathcal{Z} := \{Z[v] \mid v \in \mathcal{V}\}$. An essential property of $\mathcal{Z}$ is that it is locally finite, that is, $\#\{Z' \in \mathcal{Z} \mid Z' \cap Z \neq \emptyset\} < \infty$ for all $Z \in \mathcal{Z}$. This allows us to perform a Bowen–Sinai refinement on $\mathcal{Z}$ to obtain a partition $\mathcal{P}$, which induces a TMS $(\hat{\Sigma}, \sigma)$ and a factor map $\hat{\pi} : \hat{\Sigma} \to M$ satisfying:

(1) $\hat{\pi}$ is holder continuous with respect to the word metric;

(2) $\hat{\pi}|_{\hat{\Sigma}^{\#}}$ is finite-to-one;

(3) for all $\underline{R} \in \hat{\Sigma}^{\#}$, we have $\hat{\pi}(\underline{R}) \in \overline{R}_0$;

(4) the image of $\hat{\Sigma}^{\#}$ carry all χ-hyperbolic measures.

Again, from the fact that $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ is locally compact, one can prove that the new graph is also locally compact, and from local finiteness of $\mathcal{Z}$, we can see that $\#\{Z \in \mathcal{Z} \mid R \in Z\} < \infty$.

The map $\hat{\pi} : \hat{\Sigma}^{\#} \to RWT_{\chi}$ is define by

$$\hat{\pi}(\underline{R}) = \bigcap_{n \in \mathbb{Z}} \overline{(f^n(R_{-n}) \cap \cdots \cap f^{-n}(R_n))}.$$

One also has the following proposition.

PROPOSITION 4.16. [4, Proposition 3.8] $\hat{\pi}(\hat{\Sigma}^{\#}) = \pi(\Sigma^{\#}) = RWT_{\chi}$.

Definition 4.17. [4, Definitions 4.3 and 4.23] For a chain $\underline{R} \in \hat{\Sigma}_L$, the unstable of $\underline{R}$ is defined to be $V^u(\underline{R}) := W(R_0) \cap V^u(\underline{u})$ for some $\underline{u}$ covering $\underline{R}$ (see [4, Definition 4.2] for more details).

Similarly, for a chain $\underline{R} \in \hat{\Sigma}_R$, the stable of $\underline{R}$ is defined to be $V^s(\underline{R}) := W(R_0) \cap V^s(\underline{u})$ for some $\underline{u}$ covering $\underline{R}$ (see [4, Definition 4.23] for more details).

Here, $\hat{\Sigma}_L := \{(R_i)_{i \leq 0} \mid (R_i)_{i \in \mathbb{Z}} \in \hat{\Sigma}\}$ and $\hat{\Sigma}_R := \{(R_i)_{i \geq 0} \mid (R_i)_{i \in \mathbb{Z}} \in \hat{\Sigma}\}$.

In [4, Corollary 4.6], the author shows that $V^u(\underline{R})$ and $V^s(\underline{R})$ are actually open submanifolds of M.

The following definition and lemma can be found in [4].

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Definition 4.18. [4, Definition 3.15] We say that $R \sim S$ if there exists a path $R \to \cdots \to S$ from R to S and a path $S \to \cdots \to R$ from S to R. On the sub alphabet of letters satisfying $R \sim R$, the relation $\sim$ is an equivalence relation, and we denote the equivalence class of R by $\langle R \rangle$. We define the maximal irreducible component corresponding to R to be $\langle R \rangle^{\mathbb{Z}} \subset \hat{\Sigma}$.

The importance of maximal irreducible components to us is that they correspond to subsets of ergodic homoclinic components as stated in the following lemma.

LEMMA 4.19. *Given $R \in \mathcal{P}$ for which $R \sim R$, there exists a hyperbolic periodic point P such that $\hat{\pi}(\langle R \rangle^{\mathbb{Z}}) \subset H(P)$.*

Remark 4.20. In fact, as shown in [4], one can prove that given any hyperbolic measure μ, we have $H(P) = \hat{\pi}(\langle R \rangle^{\mathbb{Z}}) \bmod \mu$.

The following theorem is central to the proof of the main theorem. We will use it to show that the SRB measures we are going to construct are related in exactly the same way as in the assumptions of the main theorem in §3.

PROPOSITION 4.21. *Assume that* $\mathrm{Vol}(\mathcal{R}_\chi^{\mathrm{PR}}) > 0$. *Then there exists at most countably many hyperbolic periodic points $P_1^+, P_2^+, \ldots$ and $P_1^-, P_2^-, \ldots$ such that the following hold.*

(1) *The local unstable manifold of each P_n^+ contains a positive volume of points in $RWT_\chi^{\mathrm{PR}}(f)$ for all $n \geq 1$.*
(2) *The local stable manifold of each P_n^- contains a positive volume of points in $RWT_\chi^{\mathrm{PR}}(f^{-1})$ for all $n \geq 1$.*
(3) Vol-*a.e. $x \in \mathcal{R}^{\mathrm{PR}} y_\chi$ satisfy conditions (2) and (3) of Proposition 3.1 with respect to P_n^+ and P_n^-, for some $n \geq 1$.*

Proof. Step 1. Let $x \in \mathcal{R}_\chi^{\mathrm{PR}}$. Then by definition, we know that there is $r_x > 0$ and a Pesin level set Λ^{r_x} for which the following two conditions are satisfied:

$$\limsup_{n\to\infty} \frac{\#\{0 \leq k < n \mid f^k(x) \in \Lambda^{r_x}\}}{n} > 0;$$

$$\limsup_{n\to\infty} \frac{\#\{0 \leq k < n \mid f^{-k}(x) \in \Lambda^{r_x}\}}{n} > 0.$$

Now, by [4, Claim 7.6], we know that $\mathcal{R}_\chi^{\mathrm{PR}} \cap \Lambda^{r_x}$ is contained in at most finitely many partition elements R in the Markov partition $\mathcal{P}$. Therefore, we can find two partition elements R^+ and R^- in the Markov partition $\mathcal{P}$, for which we have:

$$\limsup_{n\to\infty} \frac{\#\{0 \leq k < n \mid f^k(x) \in R^+\}}{n} > 0;$$

$$\limsup_{n\to\infty} \frac{\#\{0 \leq k < n \mid f^{-k}(x) \in R^-\}}{n} > 0.$$

Let us define $\mathcal{R}_{R^-,R^+}^{\mathrm{PR}}$ to be the subset of $\mathcal{R}_\chi^{\mathrm{PR}}$ that satisfies the two upper density conditions above. Since $\mathcal{P}$ is countable, there are at most countably many partition

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elements $R_1^+, R_2^+, \ldots$ and $R_1^-, R_2^-, \ldots$ so that Lebesgue almost every point in $\mathcal{R}_\chi^{\rm PR}$ lies in the union $\bigcup_n \mathcal{R}_{R_n^-, R_n^+}$. Since we assume that $\mathrm{Vol}(\mathcal{R}_\chi^{\rm PR}) > 0$, we can assume without loss of generalization that the volume of each one of the sets $\mathcal{R}_{R_n^-, R_n^+}$ is positive. We now fix one of the sets constructed above and call it $\mathcal{R}^\pm$.

Step 2. Next, let R_0 be an element of the Markov partition $\mathcal{P}$, such that $\mathcal{R}^\pm \cap R_0$ has positive volume. We can see that $\mathcal{R}^\pm \cap R_0$ is exhausted by images of cylinders in $\hat{\Sigma}^\#$ of the form

$$C = \{\underline{S} \in \hat{\Sigma}^\# \mid S_{-n^-} = R^-, \ldots, S_0 = R_0, \ldots S_{n^+} = R^+\}.$$

The number of such cylinders is countable; therefore, $\mathcal{R}_\chi^\pm \cap R_0$ is contained in the image of at most countably many such cylinders, each of which has positive volume. Let us fix one of these rectangles, and denote its image by $\hat{C}$. Then by absolute continuity of local stable and unstable manifolds, we can show that for Vol-a.e point $x \in \hat{C} \cap \mathcal{R}^\pm$, we have

$$m_{V^u_{\rm loc}(x)}(V^u_{\rm loc}(x) \cap \hat{C} \cap \mathcal{R}^\pm) > 0,$$

$$m_{V^s_{\rm loc}(x)}(V^s_{\rm loc}(x) \cap \hat{C} \cap \mathcal{R}^\pm) > 0.$$

Fix such a point $x_0 \in \hat{C} \cap \mathcal{R}_\chi^\pm$. Since for all $y \in V^u_{\rm loc}(x_0) \cap \hat{C} \cap \mathcal{R}^\pm$, we have $\dim(V^u_{\rm loc}(x_0)) = \dim(V^u(\underline{R}(y)))$, we see that we can choose $\underline{R} \in \hat{\Sigma}^\circ \cap [R_0]$, such that

$$m_{V^u(\underline{R})}(R_0 \cap \hat{C} \cap \mathcal{R}^\pm) > 0$$

and such that the partition R^+ appears infinitely many times in the future of $\underline{R}$. Let us define $\underline{R}' \in \hat{\Sigma}_L$ to be the concatenation $\underline{R} \cdot (S_1, \ldots, R^+)$. Then we know that $f^{-n^+}(V^u(\underline{R}')) \supset V^u(\underline{R}) \cap \hat{C} \cap \mathcal{R}^\pm$. Hence by the invariance of the set $\mathcal{R}^\pm$, we see that this means that we have

$$m_{V^u(\underline{R}')}(R^+ \cap \mathcal{R}^\pm) > 0.$$

Step 3. Now, since R^+ is a recurring symbol for all the points in $R^+ \cap \mathcal{R}^\pm$, and since $V^u(\underline{R}')$ gives a positive leaf measure to this set, we can see that we can construct a periodic chain $\underline{P}_L^+ \in \hat{\Sigma}_L \cap [R^+]$. We also have that the stable holonomy map

$$\Gamma^s : V^u(\underline{R}') \cap \mathcal{R}^\pm \longrightarrow V^u(\underline{P}_L^+)$$

is well defined. By the absolute continuity property of stable holonomies, we see that $V^u(\underline{P}_L^+)$ gives $\Gamma^s(V^u(\underline{R}') \cap \mathcal{R}^\pm)$ a positive leaf measure. Now, for any point $y \in \Gamma^s(V^u(\underline{R}') \cap \mathcal{R}^\pm)$, we know that $y \in V^s(\underline{S})$, where $\underline{S} \in \hat{\Sigma}_R^\circ \cap [R^+]$, and such that $\limsup_{n\to\infty} \#\{0 \leq k < n \mid S_k = R^+\}/n > 0$. Hence, by [4, Claim 7.6], we see that this implies that $y \in RWT_\chi^{\rm PR}(f)$. Therefore, we see that $V^u(\underline{P}_L^+)$ gives $RWT_\chi^{\rm PR}$ positive leaf volume. Now, we can continue the chain $\underline{P}_L^+$ to the right periodically to get a periodic chain $\underline{P}^+ \in \hat{\Sigma} \cap R^+$, and we define the periodic point P^+ to be the image of $\underline{P}^+$ under the coding map.

Step 4. Now, we can repeat steps 2 and 3 above for the map f^{-1} to get the periodic point P^-. We see that all the conditions in the proposition above are satisfied. □

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5. *Proof of the main theorem*

Now we are ready to prove the main theorem by replicating the argument in the proof of the toy model. The first step is the following theorem proved in [4], asserting the existence of an ergodic SRB measure supported on the image of a maximal connected component of $\Sigma^{\#}$ under the assumption of a leaf condition: that unstable leafs of points in the image give positive leaf volume to RWT_{χ}^{PR}. The following theorem is a reformulation of [4, Theorem 7.9].

THEOREM 5.1. [4, Theorem 7.9] *Let $\tilde{\Sigma} \subset \hat{\Sigma}^{\#}$ be a maximal connected component, and let P be a hyperbolic periodic point in the image $\hat{\pi}(\tilde{\Sigma})$, such that the set $V^u_{\mathrm{loc}}(P) \cap RWT_{\chi}^{\mathrm{PR}}$ has positive leaf volume. Then f preserves an ergodic SRB measure μ^+ supported on $\hat{\pi}(\tilde{\Sigma}) \subset H(P)$.*

Proof of the main theorem. Step 1. Let us fix a pair of periodic hyperbolic points P_n^+ and P_n^- constructed in Proposition 4.21. Then properties (1) and (2) of Proposition 4.21 allow us to use Theorem 5.1 to construct two SRB measures μ^+ and μ^- for f and f^{-1} respectively, supported on the ergodic homoclinic components $H(P_n^+)$ and $H(P_n^-)$ respectively. Now, let us take a point $x \in \mathcal{R}_{\chi}^{\mathrm{PR}}$ satisfying condition (3) of Proposition 4.21 for P_n^+ and P_n^-. We see now that all the conditions of Proposition 3.1 are satisfied, and therefore, each of the SRB measures μ_n^+ and μ_n^- are in fact absolutely continuous with respect to volume.

Step 2. A consequence of the proof of Proposition 4.21 and the construction therein is that there is a hyperbolic rectangle R_0, such that $R_0 \cap \mathcal{R}_{\chi}^{\mathrm{PR}}$ has positive volume, and that there are two hyperbolic rectangles R^+ and R at P_n^+ and P_n^-, and two times $n^+ > 0$ and $n^- > 0$ for which we have $f^{n^+}(R_0) \subset R^+$ and $f^{-n^-}(R_0) \subset R^-$. Now, since μ_n^+ is an SRB for both f and f^{-1} (as we saw from step 1), we can show that for μ^+-a.e. point $x^+ \in H(P^+)$, one can find $N > 0$ and $N' > 0$ such that $f^N(V^u(x^+))$ contains an embedded C^1 disk which is transversal to every stable in R^+, and such that $f^{-N'}(V^s(x^+))$ contains an embedded C^1 disk which is transversal to every unstable in R^+, and such that each of the transversals have a full measure of Tsuji regular points. This follows from an argument similar to the proof of Lemma 3.19, and from Lemma 3.12. By using the fact the holonomy maps are absolutely continuous, we immediately see that this implies that Vol-a.e. point in $f^{n^+}(R_0) \cap R^+$ is forward and backward Tsuji regular with respect to the measure μ_n^+. A similar argument can be made for μ_n^- by replacing f by f^{-1} in the argument above. This implies two things: first, that Vol-a.e. point in $\mathcal{R}_{\chi}^{\mathrm{PR}}$ is Tsuji regular for both μ_n^+ and μ_n^- for some $n \geq 1$. Second, that we in fact have $\mu_n^+ = \mu_n^-$, since all the points in R_0 are Tsuji regular for both measures μ_n^+ and μ_n^-. □

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