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PAPER

An axiomatics and a combinatorial model of creation/annihilation operators

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Abstract

A categorical axiomatic theory of creation/annihilation operators on symmetric Fock space is introduced, and the combinatorial model that motivated it is presented. Commutation relations and coherent states are considered in both frameworks.

Keywords: Biproduct; tensor product; convolution; bialgebra; Fock space; creation/annihilation operators; commutation relations; coherent states; differential categories; Leibniz rule; profunctor; combinatorial species; analytic functor

1. Introduction

This work is an investigation into the mathematical structure of creation/annihilation operators on (symmetric or bosonic) Fock space, see for example Geroch (1985). My aim is twofold: to introduce an axiomatic setting for commutation relations and coherent states, and to provide and exercise one such model of a combinatorial nature. In the spirit of Paul Dirac's credo

> "One should allow oneself to be led in the direction which the mathematics suggests . . . one must follow up a mathematical idea and see what its consequences are, even though one gets led to a domain which is completely foreign to what one started with . . . Mathematics can lead us in a direction we would not take if we only followed up physical ideas by themselves."

my hope is that the mathematical theories presented here, and the ideas that underly them, can be of use for computer science and physics.

Axiomatics. Sections 2–4 consider the axiomatics. Our starting point is the consideration of categories of spaces and linear maps. To accommodate Fock space, these should allow for the formation of superposed and of noninteracting systems. In Section 2, I respectively formalise these as compatible biproduct and symmetric monoidal structures. The linear-algebraic structure is then derived by convolution with respect to the biproduct structure. For completeness, other equivalent formalisations are also given. Of central importance to our development is the algebraic axiomatisation of biproduct structure as monoidal commutative-bialgebra structure (see Proposition 2 and Lemma 4). The resulting setting is rich enough for formalising Fock space together with creation/annihilation operators on it. Specifically, in Section 3, the Fock-space construction is axiomatised as a functor on the category of spaces and linear maps that transforms the biproduct (*i.e.* superposition) structure to the symmetric monoidal (*i.e.* noninteracting) structure. A fundamental aspect of this definition is that it lifts the biproduct commutative-bialgebra structure to a commutative-bialgebra structure on Fock space. This allows for a general definition of creation/annihilation operators (Definition 15) and embodies the essential mathematical structure of the commutation relations (Theorem 16). Section 3.2 considers coherent states on Fock space. To this end, however, one needs to specialise the discussion to Fock-space constructions with suitable comonad structure. This additional structure plays two roles: it provides a canonical notion of annihilation operator and permits the association of coherent states in Fock space to vectors (Definition 21 (2) and Theorem 22). Finally, Section 4 places creation/annihilation operators in the context of the Leibniz structure of differentiation.

Combinatorial model. Section 5 puts forward a bicategorical combinatorial model. Its combinatorial nature resides in the structure being a generalisation of that of the *combinatorial species of structures* of Joyal (1981, 1986); see Fiore et al. (2008) for details. The main consequence of this for us here is that identities, such as the commutation relations, acquire combinatorial meaning in the form of natural bijective correspondences.

The combinatorial model is based on the bicategory of profunctors (or bimodules, or distributors) as the setting for spaces and linear maps. These structures, I briefly review in Section 5.1 noting analogies with vector spaces. Combinatorial (symmetric or bosonic) Fock space is then introduced in Section 5.2. The definition mimics that of the conventional construction as a biproduct of symmetric tensor powers. After making explicit the mathematical structure of combinatorial Fock space, the commutation relation involving creation and annihilation is considered. We see here that the essence of its combinatorial content arises from the simple fact that

$$\mathfrak{S}_{n+1} \cong \mathfrak{S}_n \cup ([n] \times \mathfrak{S}_n) \qquad \text{for } [n] = \{1, \ldots, n\}$$

classifying the permutations on the set $[n+1]$ according as to whether or not they fix the element $n+1$, see (15) and (17). It is an important aspect of the theory, however, that all such calculations are done formally in the *calculus of coends* (see *e.g*. Mac Lane (1971); Loregian (2021)) within the *generalized logic* of Lawvere (1973). I further illustrate how the calculus can be seen diagrammatically.

Finally, Section 5.3 considers coherent states in the combinatorial model. Taking advantage of the duality structure available in it, a notion of exponential (in the form of a comonadic/monadic convolution) is introduced. The exponential of the creation operator of a vector at the vacuum state is shown, both algebraically and combinatorially, to yield the coherent state of the vector.

Related work. This work lies at the intersection of computer science, logic, mathematics, and physics. As such, it bears relationship with a variety of developments.

In relation to mathematical logic, the notion of comonad needed in the discussion of coherent states is as it arises in models of the *linear logic* of Girard (1987). The connection between the exponential modality of linear logic and the Fock-space construction of physics was recognised long ago by Panangaden; see for example Blute et al. (1993); Blute and Panangaden (2010). In view of subsequent developments, the connection further puts this work in the context of models of the *differential linear logic* of Ehrhard and Regnier (2003, 2006) and the *differential categories* of Blute et al. (2006). Indeed, the models to be found in Ehrhard (2002, 2005); Blute et al. (2006); Hyvernat (2009); Blute et al. (2012) all fall within the axiomatisation here. A stronger axiomatisation (of which the combinatorial model (Fiore, 2004, 2005; Fiore et al., 2024) is the motivating example) leading to fully-fledged differential structure in multiplicative biadditive intuitionistic linear logic was given by Fiore (2007b). The axiomatisation of creation/annihilation operators here may be seen as the core of the axiomatisation there for differential structure merely satisfying Leibniz rule.

An axiomatics for Fock space has independently been considered by Vicary (2008). His setting, which aims at a tight correspondence with that of Fock space on Hilbert space, is stronger than the minimalist one put forward here. As acknowledged in his work, the argument used for establishing the commutation relation between creation and annihilation is based on a private communication of mine.

The combinatorial model is closely related to the *stuff-type model* of Baez and Dolan (2001), see also Morton (2006), being both founded on species of structures. Roughly, their main difference resides in that the combinatorial model organises structure as presheaves, whilst the stuff-type model does so as bundles. A benefit of the former over the latter is that it may be developed formally within generalised logic.

In connection to mathematical physics, the stuff-type model has been related to Feynman diagrams and, in connection to mathematical logic, these have been related to the proof theory of linear logic by means of the *ϕ-calculus* of Blute and Panangaden (2010), which, in turn, has formal syntactic structure similar to that of the calculus of the combinatorial model. These intriguing relationships are worth investigating.

2. Spaces and Linear Maps

Spaces and linear maps are axiomatised by means of a category $\mathcal{S}$ equipped with compatible biproduct $(\mathrm{O}, \oplus)$ and symmetric monoidal $(\mathrm{I}, \otimes)$ structures. I review these notions and explain the linear-algebraic structure that they embody.

Biproduct structure. A category with finite coproducts and finite products is said to be *bicartesian*. One typically writes $0, +$ for the empty and binary coproducts and $1, \times$ for the empty and binary products.

An object that is both initial and terminal (*i.e.* an empty coproduct and product) is said to be a *zero object*. For a zero object O, I will write $\mathrm{O}_{A,B}$ for the map $A \to B$ given by the composite $A \to \mathrm{O} \to B$.

Definition 1. *A bicartesian category is said to have biproducts whenever:*

1. *it has a zero object* O, *and*
2. *for all objects A and B, the canonical map*

$$[\langle \mathrm{id}_A, \mathrm{O}_{A,B} \rangle, \langle \mathrm{O}_{B,A}, \mathrm{id}_B \rangle] : A + B \to A \times B$$

   is an isomorphism.

In this context, one typically writes $\oplus$ for the binary biproduct.

The proposition below gives an algebraic presentation of biproduct structure, which is crucial to our development. Recall that a *symmetric monoidal structure* $(\mathrm{I}, \otimes, \lambda, \rho, \alpha, \sigma)$ on a category $\mathcal{C}$ is given by an object $\mathrm{I} \in \mathcal{C}$, a functor $\otimes : \mathcal{C}^2 \to \mathcal{C}$, and natural isomorphisms $\lambda_C : \mathrm{I} \otimes C \cong C$, $\rho_C : C \otimes \mathrm{I} \cong C$, $\alpha_{A,B,C} : (A \otimes B) \otimes C \cong A \otimes (B \otimes C)$, and $\sigma_{A,B} : A \otimes B \cong B \otimes A$ subject to coherence conditions, see for example Mac Lane (1971).

Proposition 2. *To give a choice of biproducts in a category is equivalent to giving a symmetric monoidal structure* $(\mathrm{O}, \oplus)$ *on it together with natural transformations*

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$$\begin{tikzcd}[column sep=small]
\mathrm{O} \arrow[dr, "\mathrm{u}_A"] & & \mathrm{O} \\
& A \arrow[ur, "\mathrm{n}_A"] \arrow[dr, "\Delta_A"'] & \\
A \oplus A \arrow[ur, "\nabla_A"'] & & A \oplus A
\end{tikzcd} \tag{1}$$

such that

(1) $(A, \mathrm{u}_A, \nabla_A)$ *is a commutative monoid.*

$$\begin{tikzcd}
\mathrm{O} \oplus A \arrow[r, "\mathrm{u}_A \oplus \mathrm{id}_A"] \arrow[dr, "\cong"'] & A \oplus A \arrow[d, "\nabla_A"] & A \oplus \mathrm{O} \arrow[l, "\mathrm{id}_A \oplus \mathrm{u}_A"'] \arrow[dl, "\cong"] \\
& A &
\end{tikzcd}
\qquad
\begin{tikzcd}
A \oplus A \oplus A \arrow[r, "\mathrm{id}_A \oplus \nabla_A"] \arrow[d, "\nabla_A \oplus \mathrm{id}_A"'] & A \oplus A \arrow[d, "\nabla_A"] \\
A \oplus A \arrow[r, "\nabla_A"'] & A
\end{tikzcd} \tag{2}$$

$$\begin{tikzcd}
A \oplus A \arrow[rr, "\sigma_{A,A}"] \arrow[dr, "\nabla_A"'] & & A \oplus A \arrow[dl, "\nabla_A"] \\
& A &
\end{tikzcd} \tag{3}$$

(2) $(A, \mathrm{n}_A, \Delta_A)$ *is a commutative comonoid.*

$$\begin{tikzcd}
& A \arrow[dl, "\cong"'] \arrow[d, "\Delta_A"] \arrow[dr, "\cong"] & \\
\mathrm{O} \oplus A & A \oplus A \arrow[l, "\mathrm{n}_A \oplus \mathrm{id}_A"] \arrow[r, "\mathrm{id}_A \oplus \mathrm{n}_A"'] & A \oplus \mathrm{O}
\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[r, "\Delta_A"] \arrow[d, "\Delta_A"'] & A \oplus A \arrow[d, "\mathrm{id}_A \oplus \Delta_A"] \\
A \oplus A \arrow[r, "\Delta_A \oplus \mathrm{id}_A"'] & A \oplus A \oplus A
\end{tikzcd} \tag{4}$$

$$\begin{tikzcd}
& A \arrow[dl, "\Delta_A"'] \arrow[dr, "\Delta_A"] & \\
A \oplus A \arrow[rr, "\sigma_{A,A}"'] & & A \oplus A
\end{tikzcd} \tag{5}$$

(3)
$$\begin{aligned}
\mathrm{u}_{A \oplus B} &= \big(\mathrm{O} \cong \mathrm{O} \oplus \mathrm{O} \xrightarrow{\mathrm{u}_A \oplus \mathrm{u}_B} A \oplus B \big) \\
\mathrm{n}_{A \oplus B} &= \big(A \oplus B \xrightarrow{\mathrm{n}_A \oplus \mathrm{n}_B} \mathrm{O} \oplus \mathrm{O} \cong \mathrm{O} \big) \\
\nabla_{A \oplus B} &= \big((A \oplus B) \oplus (A \oplus B) \cong (A \oplus A) \oplus (B \oplus B) \xrightarrow{\nabla_A \oplus \nabla_B} A \oplus B \big) \\
\Delta_{A \oplus B} &= \big((A \oplus B) \xrightarrow{\Delta_A \oplus \Delta_B} (A \oplus A) \oplus (B \oplus B) \cong (A \oplus B) \oplus (A \oplus B) \big)
\end{aligned}$$

The biproduct structure induced by (1) has coproduct diagrams

$$\begin{tikzcd}
A \arrow[rr, bend left, "\amalg_1"] & \cong A \oplus \mathrm{O} \arrow[r, "\mathrm{id}_A \oplus \mathrm{u}_B"] & A \oplus B & \mathrm{O} \oplus B \cong \arrow[l, "\mathrm{u}_A \oplus \mathrm{id}_B"'] & B \arrow[ll, bend right, "\amalg_2"']
\end{tikzcd}$$

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and product diagrams

$$\begin{tikzcd} A & \cong A \oplus \mathrm{O} \arrow[l, bend right, "\pi_1"'] & A \oplus B \arrow[l, "\mathrm{id}_A \oplus \mathrm{n}_B"'] \arrow[r, "\mathrm{n}_A \oplus \mathrm{id}_B"] & \mathrm{O} \oplus B \cong \arrow[r, bend left, "\pi_2"] & B \end{tikzcd}$$

Proposition 3. *In a category with biproduct structure* $(\mathrm{O}, \oplus)$, *we have that*

$$(A \xrightarrow{\amalg_i} A \oplus A \xrightarrow{\pi_j} A) = \begin{cases} \mathrm{id}_A & \text{, if } i = j \\ \mathrm{O}_{A,A} & \text{, if } i \neq j \end{cases}$$

Lemma 4. *In a category with biproduct structure* $(\mathrm{O}, \oplus; \mathrm{u}, \nabla; \mathrm{n}, \Delta)$, *the commutative monoid and comonoid structures* $(\mathrm{u}, \nabla; \mathrm{n}, \Delta)$ *form a commutative bialgebra. That is,* u *and* ∇ *are comonoid homomorphisms and, equivalently,* n *and* Δ *are monoid homomorphisms.*

$$\begin{tikzcd} & A \arrow[dr, "\mathrm{n}_A"] & \\ \mathrm{O} \arrow[ur, "\mathrm{u}_A"] \arrow[rr, "\mathrm{id}_\mathrm{O}"'] & & \mathrm{O} \end{tikzcd} \qquad \begin{tikzcd} A \oplus A \arrow[r, "\nabla_A"] \arrow[d, "\Delta_A \oplus \Delta_A"'] & A \arrow[r, "\Delta_A"] & A \oplus A \\ A \oplus A \oplus A \oplus A \arrow[rr, "\mathrm{id}_A \oplus \sigma_{A,A} \oplus \mathrm{id}_A"'] & & A \oplus A \oplus A \oplus A \arrow[u, "\nabla_A \oplus \nabla_A"'] \end{tikzcd} \tag{6}$$

$$\begin{tikzcd} & A \arrow[dr, "\Delta_A"] & \\ \mathrm{O} \arrow[ur, "\mathrm{u}_A"] \arrow[dr, "\cong"'] & & A \oplus A \\ & \mathrm{O} \oplus \mathrm{O} \arrow[ur, "\mathrm{u}_A \oplus \mathrm{u}_A"'] & \end{tikzcd} \qquad \begin{tikzcd} & A \arrow[dr, "\mathrm{n}_A"] & \\ A \oplus A \arrow[ur, "\nabla_A"] \arrow[dr, "\mathrm{n}_A \oplus \mathrm{n}_A"'] & & \mathrm{O} \\ & \mathrm{O} \oplus \mathrm{O} \arrow[ur, "\cong"'] & \end{tikzcd} \tag{7}$$

Linear-algebraic structure. We examine the linear-algebraic structure of categories with biproduct structure. This I present in the language of enriched category theory (Kelly, 1982).

Let **Mon** (**CMon**) be the symmetric monoidal category of (commutative) monoids with respect to the universal bilinear tensor product. Recall that **Mon**-categories (**CMon**-categories) are categories all of whose homs $[A, B]$ come equipped with a (commutative) monoid structure

$$0_{A,B} \in [A, B] \quad , \quad +_{A,B} : [A, B]^2 \to [A, B]$$

such that composition is strict and bilinear; that is,

$$0_{B,C}\, f = 0_{A,C} \qquad \textit{and} \qquad f\, 0_{C,A} = 0_{C,B}$$

for all $f : A \to B$, and

$$g\,(f +_{A,B} f') = g\,f +_{A,C} g\,f' \qquad \textit{and} \qquad (g +_{B,C} g')\,f = g\,f +_{A,C} g'\,f$$

for all $f, f' : A \to B$ and $g, g' : B \to C$.

Proposition 5. *The following are equivalent.*

1. *Categories with biproduct structure.*
2. **Mon**-*categories with (necessarily enriched) finite products.*
3. **CMon**-*categories with (necessarily enriched) finite products.*

The enrichment of categories with biproduct structure $(\mathrm{O}, \oplus; \mathrm{u}, \nabla; \mathrm{n}, \Delta)$ is given by *convolution* (see *e.g.* Sweedler (1969)) as follows:

$$0_{A,B} = (A \xrightarrow{\mathrm{n}_A} \mathrm{O} \xrightarrow{\mathrm{u}_B} B) = \mathrm{O}_{A,B}$$
$$f +_{A,B} g = (A \xrightarrow{\Delta_A} A \oplus A \xrightarrow{f \oplus g} B \oplus B \xrightarrow{\Delta_B} B)$$

Proposition 6. *In a category with biproduct structure,* $\nabla_A = \pi_1 + \pi_2 : A \oplus A \to A$ *and* $\Delta_A = \amalg_1 + \amalg_2 : A \to A \oplus A$.

We now consider biproduct structure on symmetric monoidal categories. To this end, note that in a monoidal category with tensor $\otimes$ and binary products $\times$ there is a natural distributive law as follows:

$$\ell_{A,B,C} = \langle \pi_1 \otimes \mathrm{id}_C, \pi_2 \otimes \mathrm{id}_C \rangle : (A \times B) \otimes C \to (A \otimes C) \times (B \otimes C)$$

Definition 7. *A biproduct structure* $(\mathrm{O}, \oplus; \mathrm{u}, \nabla; \mathrm{n}, \Delta)$ *and a symmetric monoidal structure* $(\mathrm{I}, \otimes)$ *on a category are compatible whenever the following hold:*

$$\begin{tikzcd} & \mathrm{O} \otimes C \arrow[dr, "\mathrm{u}_A \otimes \mathrm{id}"] \arrow[dd, "\mathrm{n}_{\mathrm{O} \otimes C}"] & \\ A \otimes C \arrow[ur, "\mathrm{n}_A \otimes \mathrm{id}_C"] \arrow[dr, "\mathrm{n}_{A \otimes C}"'] & & A \otimes C \\ & \mathrm{O} \arrow[ur, "\mathrm{u}_{A \otimes C}"'] & \end{tikzcd} \qquad \begin{tikzcd} & (A \oplus A) \otimes C \arrow[dr, "\nabla_A \otimes \mathrm{id}_C"] \arrow[dd, "\ell_{A,A,C}"] & \\ A \otimes C \arrow[ur, "\Delta_A \otimes \mathrm{id}"] \arrow[dr, "\Delta_{A \otimes C}"'] & & A \otimes C \\ & (A \otimes C) \oplus (A \otimes C) \arrow[ur, "\nabla_{A \otimes C}"'] & \end{tikzcd}$$

Proposition 5 extends to the symmetric monoidal setting. Recall that a **Mon**-*enriched (symmetric) monoidal category* is a (symmetric) monoidal category with a **Mon**-enrichment for which the tensor is strict and bilinear; that is, such that

$$0_{X,Y} \otimes f = 0_{X \otimes A, Y \otimes B} \qquad \text{and} \qquad f \otimes 0_{X,Y} = 0_{A \otimes X, B \otimes Y}$$

for all $f : A \to B$, and

$$g \otimes (f + f') = g \otimes f + g \otimes f' \qquad \text{and} \qquad (g + g') \otimes f = g \otimes f + g' \otimes f$$

for all $f, f' : A \to B$ and $g, g' : X \to Y$.

Proposition 8. *The following are equivalent.*

1. *Categories with compatible biproduct and symmetric monoidal structures.*
2. **Mon**-*enriched symmetric monoidal categories with (necessarily enriched) finite products.*
3. **CMon**-*enriched symmetric monoidal categories with (necessarily enriched) finite products.*

Definition 9. *A category with compatible biproduct and symmetric monoidal structures is referred to as a category of spaces and linear maps.*

3. Fock Space

For a category of spaces and linear maps, the Fock-space construction is axiomatised as a strong symmetric monoidal functor F mapping $(\mathrm{O}, \oplus)$ to $(\mathrm{I}, \otimes)$ and, after considering such structure, I

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explain how it supports an axiomatisation of creation/annihilation operators subject to commutation relations. For F underlying a linear exponential comonad, coherent states are considered and studied in Section 3.2.

Strong-monoidal functorial structure. A *strong monoidal functor* $(F, \phi, \varphi) : (\mathcal{C}, \mathrm{I}, \otimes) \to (\mathcal{C}', \mathrm{I}', \otimes')$ between monoidal categories consists of a functor $F : \mathcal{C} \to \mathcal{C}'$, an isomorphism $\phi : \mathrm{I}' \cong F(\mathrm{I})$, and a natural isomorphism $\varphi_{A,B} : FA \otimes' FB \cong F(A \otimes B)$ subject to the coherence conditions below.

$$\begin{tikzcd}
FC \otimes' \mathrm{I}' \arrow[r, "\rho'_{FC}"] \arrow[d, "\mathrm{id}_{FC} \otimes' \phi"'] & FC \\
FC \otimes' F\mathrm{I} \arrow[r, "\varphi_{C,\mathrm{I}}"'] & F(C \otimes \mathrm{I}) \arrow[u, "F\rho_C"']
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathrm{I}' \otimes' FC \arrow[r, "\lambda'_{FC}"] \arrow[d, "\phi \otimes' \mathrm{id}_{FC}"'] & FC \\
F\mathrm{I} \otimes' FC \arrow[r, "\varphi_{\mathrm{I},C}"'] & F(\mathrm{I} \otimes C) \arrow[u, "F\lambda_C"']
\end{tikzcd}$$

$$\begin{tikzcd}
(FA \otimes' FB) \otimes' FC \arrow[r, "\alpha'_{FA,FB,FC}"] \arrow[d, "\varphi_{A,B} \otimes' \mathrm{id}_{FC}"'] & FA \otimes' (FB \otimes' FC) \arrow[r, "\mathrm{id}_{FA} \otimes' \varphi_{A,B}"] & FA \otimes' F(B \otimes C) \arrow[d, "\varphi_{A,B\otimes C}"] \\
F(A \otimes B) \otimes' FC \arrow[r, "\varphi_{A\otimes B,C}"'] & F\big((A \otimes B) \otimes C\big) \arrow[r, "F\alpha_{A,B,C}"'] & F\big(A \otimes (B \otimes C)\big)
\end{tikzcd}$$

$$\begin{tikzcd}
FA \otimes' FB \arrow[r, "\varphi_{A,B}"] \arrow[d, "\sigma'_{FA,FB}"'] & F(A \otimes B) \arrow[d, "F(\sigma_{A,B})"] \\
FB \otimes' FA \arrow[r, "\varphi_{B,A}"'] & F(B \otimes A)
\end{tikzcd}$$

Definition 10. *A strong monoidal functor* $(\mathcal{S}, \mathrm{O}, \oplus) \to (\mathcal{S}, \mathrm{I}, \otimes)$ *for a category of spaces and linear maps* $\mathcal{S}$ *is referred to as a (symmetric or bosonic) Fock-space construction.*

The Fock-space construction supports operations for *initialising* and *merging* (i, m) and for *finalising* and *splitting* (f, s).

Definition 11. *For a Fock-space construction on a category of spaces and linear maps, set:*

$$\begin{aligned}
\mathrm{i}_A &= (\mathrm{I} \cong \mathrm{FO} \xrightarrow{\mathrm{Fu}_A} \mathrm{FA})\,, & \mathrm{m}_A &= (\mathrm{FA} \otimes \mathrm{FA} \cong \mathrm{F}(A \oplus A) \xrightarrow{\mathrm{F}\nabla_A} \mathrm{FA}) \\
\mathrm{f}_A &= (\mathrm{FA} \xrightarrow{\mathrm{Fn}_A} \mathrm{FO} \cong \mathrm{I})\,, & \mathrm{s}_A &= (\mathrm{FA} \xrightarrow{\mathrm{F}\Delta_A} \mathrm{F}(A \oplus A) \cong \mathrm{FA} \otimes \mathrm{FA})
\end{aligned}$$

The commutative bialgebra structure induced by the biproduct structure yields commutative bialgebraic structure on Fock space.

Lemma 12. *For a Fock-space construction* F *on a category of spaces and linear maps, the natural transformations*

$$\begin{tikzcd}
\mathrm{I} \arrow[dr, "\mathrm{i}_A"] & & \mathrm{I} \\
& \mathrm{FA} \arrow[ur, "\mathrm{f}_A"] \arrow[dr, "\mathrm{s}_A"'] & \\
\mathrm{FA} \otimes \mathrm{FA} \arrow[ur, "\mathrm{m}_A"'] & & \mathrm{FA} \otimes \mathrm{FA}
\end{tikzcd} \tag{8}$$

form a commutative bialgebra.

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Indeed, by means of the coherence conditions of strong monoidal functors, the application of F to the diagrams (2–7) yields the commutativity of the diagrams below.

$$\begin{tikzcd}
\mathrm{I} \otimes \mathrm{F}A \arrow[r, "\mathrm{i}_A \otimes \mathrm{id}_{\mathrm{F}A}"] \arrow[dr, "\cong"'] & \mathrm{F}A \otimes \mathrm{F}A \arrow[d, "\mathrm{m}_A"] & \mathrm{F}A \otimes \mathrm{I} \arrow[l, "\mathrm{id}_{\mathrm{F}A} \otimes \mathrm{i}_A"'] \arrow[dl, "\cong"] \\
& \mathrm{F}A &
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathrm{F}A \otimes \mathrm{F}A \otimes \mathrm{F}A \arrow[r, "\mathrm{id}_{\mathrm{F}A} \otimes \mathrm{m}_A"] \arrow[d, "\mathrm{m}_A \otimes \mathrm{id}_{\mathrm{F}A}"'] & \mathrm{F}A \otimes \mathrm{F}A \arrow[d, "\mathrm{m}_A"] \\
\mathrm{F}A \otimes \mathrm{F}A \arrow[r, "\mathrm{m}_A"'] & \mathrm{F}A
\end{tikzcd}$$

$$\begin{tikzcd}
\mathrm{F}A \otimes \mathrm{F}A \arrow[rr, "\sigma_{\mathrm{F}A,\mathrm{F}A}"] \arrow[dr, "\mathrm{m}_A"'] & & \mathrm{F}A \otimes \mathrm{F}A \arrow[dl, "\mathrm{m}_A"] \\
& \mathrm{F}A &
\end{tikzcd}
\qquad
\begin{tikzcd}
& \mathrm{F}A \arrow[dl, "\mathrm{s}_A"'] \arrow[dr, "\mathrm{s}_A"] & \\
\mathrm{F}A \otimes \mathrm{F}A \arrow[rr, "\sigma_{\mathrm{F}A,\mathrm{F}A}"'] & & \mathrm{F}A \otimes \mathrm{F}A
\end{tikzcd}$$

$$\begin{tikzcd}
& \mathrm{F}A \arrow[dl, "\cong"'] \arrow[d, "\mathrm{s}_A"] \arrow[dr, "\cong"] & \\
\mathrm{I} \otimes \mathrm{F}A & \mathrm{F}A \otimes \mathrm{F}A \arrow[l, "\mathrm{f}_{\mathrm{F}A} \otimes \mathrm{id}_{\mathrm{F}A}"] \arrow[r, "\mathrm{id}_{\mathrm{F}A} \otimes \mathrm{f}_A"'] & \mathrm{F}A \otimes \mathrm{I}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathrm{F}A \arrow[r, "\mathrm{s}_A"] \arrow[d, "\mathrm{s}_A"'] & \mathrm{F}A \otimes \mathrm{F}A \arrow[d, "\mathrm{id}_A \otimes \mathrm{s}_A"] \\
\mathrm{F}A \otimes \mathrm{F}A \arrow[r, "\mathrm{s}_A \otimes \mathrm{id}_A"'] & \mathrm{F}A \otimes \mathrm{F}A \otimes \mathrm{F}A
\end{tikzcd}$$

$$\begin{tikzcd}
& \mathrm{F}A \arrow[dr, "\mathrm{f}_A"] & \\
\mathrm{I} \arrow[ur, "\mathrm{i}_A"] \arrow[rr, "\mathrm{id}_\mathrm{I}"'] & & \mathrm{I}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathrm{F}A \otimes \mathrm{F}A \arrow[r, "\mathrm{m}_A"] \arrow[d, "\mathrm{s}_A \otimes \mathrm{s}_A"'] & \mathrm{F}A \arrow[r, "\mathrm{s}_A"] & \mathrm{F}A \otimes \mathrm{F}A \\
\mathrm{F}A \otimes \mathrm{F}A \otimes \mathrm{F}A \otimes \mathrm{F}A \arrow[rr, "\mathrm{id}_{\mathrm{F}A} \otimes \sigma_{\mathrm{F}A,\mathrm{F}A} \otimes \mathrm{id}_{\mathrm{F}A}"'] & & \mathrm{F}A \otimes \mathrm{F}A \otimes \mathrm{F}A \otimes \mathrm{F}A \arrow[u, "\mathrm{m}_A \otimes \mathrm{m}_A"']
\end{tikzcd}$$

$$\begin{tikzcd}
& \mathrm{F}A \arrow[dr, "\mathrm{s}_A"] & \\
\mathrm{I} \arrow[ur, "\mathrm{i}_A"] \arrow[dr, "\cong"'] & & \mathrm{F}A \otimes \mathrm{F}A \\
& \mathrm{I} \otimes \mathrm{I} \arrow[ur, "\mathrm{i}_A \otimes \mathrm{i}_A"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
& \mathrm{F}A \arrow[dr, "\mathrm{f}_A"] & \\
\mathrm{F}A \otimes \mathrm{F}A \arrow[ur, "\mathrm{m}_A"] \arrow[dr, "\mathrm{f}_A \otimes \mathrm{f}_A"'] & & \mathrm{I} \\
& \mathrm{I} \otimes \mathrm{I} \arrow[ur, "\cong"'] &
\end{tikzcd}$$

Proposition 13. *For a Fock-space construction* $(\mathrm{F}, \phi, \varphi)$, *the isomorphism* $\varphi_{A,B}$ *has inverse* $(\mathrm{F}\pi_1 \otimes \mathrm{F}\pi_2)\, \mathrm{s}_{A \oplus B}$.

Proof. Follows from the commutativity of

$$\begin{tikzcd}
& \mathrm{F}(A \oplus B) \otimes \mathrm{F}(A \oplus B) \arrow[r, "\mathrm{F}\pi_1 \otimes \mathrm{F}\pi_2"] \arrow[d, "\cong" ', "\varphi_{A \oplus B, A \oplus B}"] & \mathrm{F}A \otimes \mathrm{F}B \arrow[d, "\cong"', "\varphi_{A,B}"] \\
\mathrm{F}(A \oplus B) \arrow[ur, "\mathrm{s}_{A \oplus B}"] \arrow[r, "\mathrm{F}\Delta_{A \oplus B}"'] \arrow[rr, bend right, "\mathrm{id}_{\mathrm{F}(A \oplus B)}"'] & \mathrm{F}\big((A \oplus B) \oplus (A \oplus B)\big) \arrow[r, "\mathrm{F}(\pi_1 \oplus \pi_2)"'] & \mathrm{F}(A \oplus B)
\end{tikzcd}$$

□

Proposition 14. *For a Fock-space construction* F, *we have that* $\mathrm{F}(0_{A,B}) = \mathrm{i}_B\, \mathrm{f}_A$ *and that* $\mathrm{F}(f + g) = \mathrm{m}_B\, (\mathrm{F}f \otimes \mathrm{F}g)\, \mathrm{s}_A : \mathrm{F}A \to \mathrm{F}B$ *for all* $f, g : A \to B$.

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3.1 Creation/annihilation operators

Definition 15. *Let* F *be a Fock-space construction. For natural transformations* $\eta_A : A \to \mathrm{F}A$ *and* $\varepsilon_A : \mathrm{F}A \to A$, *define the associated creation (or raising) natural transformation* $\overline{\eta}$ *and annihilation (or lowering) natural transformation* $\underline{\varepsilon}$ *as*

$$\overline{\eta}_A = \left(A \otimes \mathrm{F}A \xrightarrow{\eta_A \otimes \mathrm{id}_{\mathrm{F}A}} \mathrm{F}A \otimes \mathrm{F}A \xrightarrow{\mathrm{m}_A} \mathrm{F}A \right)$$

$$\underline{\varepsilon}_A = \left(\mathrm{F}A \xrightarrow{\mathrm{s}_A} \mathrm{F}A \otimes \mathrm{F}A \xrightarrow{\varepsilon_A \otimes \mathrm{id}_{\mathrm{F}A}} A \otimes \mathrm{F}A \right)$$

The above form for creation and annihilation operators is non-standard. More commonly, see for example Geroch (1985), the literature deals with creation operators $\overline{\eta}^v_A : \mathrm{F}A \to \mathrm{F}A$ for vectors $v : \mathrm{I} \to A$ and annihilation operators $\underline{\varepsilon}^{v'}_A : \mathrm{F}A \to \mathrm{F}A$ for covectors $v' : A \to \mathrm{I}$. In the present setting, these are derived as follows:

$$\overline{\eta}^v_A = \left(\mathrm{F}A \cong \mathrm{I} \otimes \mathrm{F}A \xrightarrow{v \otimes \mathrm{id}_{\mathrm{F}A}} A \otimes \mathrm{F}A \xrightarrow{\overline{\eta}_A} \mathrm{F}A \right)$$

$$\underline{\varepsilon}^{v'}_A = \left(\mathrm{F}A \xrightarrow{\underline{\varepsilon}_A} A \otimes \mathrm{F}A \xrightarrow{v' \otimes \mathrm{id}_{\mathrm{F}A}} \mathrm{I} \otimes \mathrm{F}A \cong \mathrm{F}A \right)$$

Theorem 16 (Commutation Theorem). *Let* F *be a Fock-space construction on a category of spaces and linear maps. For natural transformations* $\eta_A : A \to \mathrm{F}A$ *and* $\varepsilon_A : \mathrm{F}A \to A$, *their associated creation and annihilation natural transformations* $\overline{\eta}_A : A \otimes \mathrm{F}A \to \mathrm{F}A$ *and* $\underline{\varepsilon}_A : \mathrm{F}A \to A \otimes \mathrm{F}A$ *satisfy the commutation relations:*

1. $\underline{\varepsilon}_A \, \overline{\eta}_A = (\varepsilon_A \, \eta_A \otimes \mathrm{id}_{\mathrm{F}A}) + (\mathrm{id}_A \otimes \overline{\eta}_A)(\sigma_{A,A} \otimes \mathrm{id}_{\mathrm{F}A})(\mathrm{id}_A \otimes \underline{\varepsilon}_A) \; : A \otimes \mathrm{F}A \to A \otimes \mathrm{F}A$
2. $\overline{\eta}_A \, (\mathrm{id}_A \otimes \overline{\eta}_A) = \overline{\eta}_A \, (\mathrm{id}_A \otimes \overline{\eta}_A) \, (\sigma_{A,A} \otimes \mathrm{id}_{\mathrm{F}A}) \; : A \otimes A \otimes \mathrm{F}A \to \mathrm{F}A$
3. $(\mathrm{id}_A \otimes \underline{\varepsilon}_A) \, \underline{\varepsilon}_A = (\sigma_{A,A} \otimes \mathrm{id}_{\mathrm{F}A})(\mathrm{id}_A \otimes \underline{\varepsilon}_A) \, \underline{\varepsilon}_A \; : \mathrm{F}A \to A \otimes A \otimes \mathrm{F}A$

It follows as a corollary that

$$\begin{aligned} \underline{\varepsilon}^{v'}_A \, \overline{\eta}^v_A &= \left(\mathrm{F}A \cong \mathrm{I} \otimes \mathrm{F}A \xrightarrow{(v' \varepsilon_A \eta_A v) \otimes \mathrm{id}_{\mathrm{F}A}} \mathrm{I} \otimes \mathrm{F}A \cong \mathrm{F}A \right) + \left(\mathrm{F}A \xrightarrow{\overline{\eta}^v_A \, \underline{\varepsilon}^{v'}_A} \mathrm{F}A \right) \\ \overline{\eta}^u_A \, \overline{\eta}^v_A &= \overline{\eta}^v_A \, \overline{\eta}^u_A \\ \underline{\varepsilon}^{u'}_A \, \underline{\varepsilon}^{v'}_A &= \underline{\varepsilon}^{v'}_A \, \underline{\varepsilon}^{u'}_A \end{aligned} \tag{9}$$

for all $u, v : \mathrm{I} \to A$ and $u', v' : A \to \mathrm{I}$.

The proof of the Commutation Theorem depends on the following lemma.

Lemma 17. *For a Fock-space construction* F, *the following hold for all natural transformations* $\eta_A : A \to \mathrm{F}A$ *and* $\varepsilon_A : \mathrm{F}A \to A$.

1. $\eta_{A \oplus A} \, \Delta_A = (\mathrm{F}\amalg_1 + \mathrm{F}\amalg_2) \, \eta_A : A \to \mathrm{F}(A \oplus A)$ *and* $\nabla_A \, \varepsilon_{A \oplus A} = \varepsilon_A \, (\mathrm{F}\pi_1 + \mathrm{F}\pi_2) :$ $\mathrm{F}(A \oplus A) \to A$.
2. *Product rules:*

$$\mathrm{s}_A \, \eta_A = (A \cong A \otimes \mathrm{I} \xrightarrow{\eta_A \otimes \mathrm{i}_A} \mathrm{F}A \otimes \mathrm{F}A) + (A \cong \mathrm{I} \otimes A \xrightarrow{\mathrm{i}_A \otimes \eta_A} \mathrm{F}A \otimes \mathrm{F}A)$$

$$\varepsilon_A \, \mathrm{m}_A = (\mathrm{F}A \otimes \mathrm{F}A \xrightarrow{\varepsilon_A \otimes \mathrm{f}_A} A \otimes \mathrm{I} \cong A) + (\mathrm{F}A \otimes \mathrm{F}A \xrightarrow{\mathrm{f}_A \otimes \varepsilon_A} \mathrm{I} \otimes A \cong A)$$

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3. *Constant rules:*

$$\mathrm{f}_A\ \eta_A = 0_{A,\mathrm{I}} : A \to \mathrm{I} \ , \quad \varepsilon_A\ \mathrm{i}_A = 0_{\mathrm{I},A} : \mathrm{I} \to A$$

Proof. For the first and third items, I only detail the proof of one of the identities; the other identity being established dually.

One calculates as follows:

(1) $\eta_{A\oplus A}\ \Delta_A = \eta_{A\oplus A}\ (\amalg_1 + \amalg_2) = \eta_{A\oplus A}\ \amalg_1 + \eta_{A\oplus A}\ \amalg_2 = \mathrm{F}(\amalg_1)\ \eta_A + \mathrm{F}(\amalg_2)\ \eta_A$
$= (\mathrm{F}\amalg_1 + \mathrm{F}\amalg_2)\ \eta_A$.

(2)
$$\begin{aligned}
\mathrm{s}_A\ \eta_A &= (\mathrm{F}\pi_1 \otimes \mathrm{F}\pi_2)\ \mathrm{s}_{A\oplus A}\ \mathrm{F}(\Delta_A)\ \eta_A \\
&\qquad \text{, by definition of s and Proposition 13} \\
&= (\mathrm{F}\pi_1 \otimes \mathrm{F}\pi_2)\ \mathrm{s}_{A\oplus A}\ (\mathrm{F}\amalg_1 + \mathrm{F}\amalg_2)\ \eta_A \\
&\qquad \text{, by naturality of } \eta \text{ and item (1) of this lemma} \\
&= (\mathrm{F}\pi_1 \otimes \mathrm{F}\pi_2)\ ((\mathrm{F}\amalg_1 \otimes \mathrm{F}\amalg_1) + (\mathrm{F}\amalg_2 \otimes \mathrm{F}\amalg_2))\ \mathrm{s}_A\ \eta_A \\
&\qquad \text{, by naturality of s} \\
&= ((\mathrm{id}_{\mathrm{F}A} \otimes \mathrm{i}_A\mathrm{f}_A) + (\mathrm{i}_A\mathrm{f}_A \otimes \mathrm{id}_{\mathrm{F}A}))\ \mathrm{s}_A\ \eta_A \\
&\qquad \text{, by Proposition 3 and the definitions of i and f} \\
&= (A \cong A \otimes \mathrm{I} \xrightarrow{\eta_A \otimes \mathrm{i}_A} \mathrm{F}A \otimes \mathrm{F}A) + (A \cong \mathrm{I} \otimes A \xrightarrow{\mathrm{i}_A \otimes \eta_A} \mathrm{F}A \otimes \mathrm{F}A) \\
&\qquad \text{, by the comonoid structure of (f, s)}
\end{aligned}$$

$$\begin{aligned}
\varepsilon_A\ \mathrm{m}_A &= (\pi_1 + \pi_2)\ \varepsilon_{A\oplus A}\ \varphi_{A,A} \\
&\qquad \text{, by definition of s and naturality of } \varepsilon \\
&= (\varepsilon_A\ \mathrm{F}(\pi_1)\ \varphi_{A,A}) + (\varepsilon_A\ \mathrm{F}(\pi_2)\ \varphi_{A,A}) \\
&\qquad \text{, by bilinearity of composition and naturality} \\
&= (\mathrm{F}A \otimes \mathrm{F}A \xrightarrow{\varepsilon_A \otimes \mathrm{f}_A} A \otimes \mathrm{I} \cong A) + (\mathrm{F}A \otimes \mathrm{F}A \xrightarrow{\mathrm{f}_A \otimes \varepsilon_A} \mathrm{I} \otimes A \cong A) \\
&\qquad \text{, by definition of f and coherence of F}
\end{aligned}$$

(3) $\mathrm{f}_A\ \eta_A = (A \xrightarrow{\eta_A} \mathrm{F}A \xrightarrow{\mathrm{Fn}_A} \mathrm{FO} \cong \mathrm{I}) = (A \xrightarrow{\mathrm{n}_A} \mathrm{O} \xrightarrow{\eta_{\mathrm{O}}} \mathrm{FO} \cong \mathrm{I})$. ☐

Proof of the Commutation Theorem. (1) By means of Lemma 17 (2), the commutativity of the diagram

$$\begin{tikzcd}
A \otimes \mathrm{F}A \arrow[d, "\eta_A \otimes \mathrm{id}_{\mathrm{F}A}"'] \arrow[dr, "\bar{\eta}_A"] \arrow[dd, bend right=60, "(\mathrm{s}_A\ \eta_A)\otimes \mathrm{s}_A"'] & & A \otimes \mathrm{F}A \\
\mathrm{F}A \otimes \mathrm{F}A \arrow[r, "\mathrm{m}_A"] \arrow[d, "\mathrm{s}_A \otimes \mathrm{s}_A"'] & \mathrm{F}A \arrow[ur, "\underline{\varepsilon}_A"] \arrow[r, "\mathrm{s}_A"] & \mathrm{F}A \otimes \mathrm{F}A \arrow[u, "\varepsilon_A \otimes \mathrm{id}_{\mathrm{F}A}"'] \\
\mathrm{F}A \otimes \mathrm{F}A \otimes \mathrm{F}A \otimes \mathrm{F}A \arrow[rr, "\mathrm{id}_{\mathrm{F}A} \otimes \sigma_{\mathrm{F}A,\mathrm{F}A} \otimes \mathrm{id}_{\mathrm{F}A}"'] & & \mathrm{F}A \otimes \mathrm{F}A \otimes \mathrm{F}A \otimes \mathrm{F}A \arrow[u, "\mathrm{m}_A \otimes \mathrm{m}_A"'] \arrow[uu, bend right=60, "(\varepsilon_A\ \mathrm{m}_A)\otimes \mathrm{m}_A"']
\end{tikzcd}$$

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shows that $\underline{\varepsilon}_A\ \overline{\eta}_A$ equals

$$
\begin{aligned}
&\Big(\ A\otimes \mathrm{FA}\cong A\otimes \mathrm{I}\otimes \mathrm{FA} \xrightarrow{\eta_A\otimes \mathrm{i}_A\otimes \mathrm{s}_A} \mathrm{FA}\otimes \mathrm{FA}\otimes \mathrm{FA}\otimes \mathrm{FA} \\
&\qquad \xrightarrow{\mathrm{id}_{\mathrm{FA}}\otimes \sigma_{\mathrm{FA},\mathrm{FA}}\otimes \mathrm{id}_{\mathrm{FA}}} \mathrm{FA}\otimes \mathrm{FA}\otimes \mathrm{FA}\otimes \mathrm{FA} \xrightarrow{\varepsilon_A\otimes \mathrm{f}_A\otimes \mathrm{m}_A} A\otimes \mathrm{I}\otimes \mathrm{FA}\cong A\otimes \mathrm{FA}\ \Big) \\
&+ \\
&\Big(\ A\otimes \mathrm{FA}\cong \mathrm{I}\otimes A\otimes \mathrm{FA} \xrightarrow{\mathrm{i}_A\otimes \eta_A\otimes \mathrm{s}_A} \mathrm{FA}\otimes \mathrm{FA}\otimes \mathrm{FA}\otimes \mathrm{FA} \\
&\qquad \xrightarrow{\mathrm{id}_{\mathrm{FA}}\otimes \sigma_{\mathrm{FA},\mathrm{FA}}\otimes \mathrm{id}_{\mathrm{FA}}} \mathrm{FA}\otimes \mathrm{FA}\otimes \mathrm{FA}\otimes \mathrm{FA} \xrightarrow{\varepsilon_A\otimes \mathrm{f}_A\otimes \mathrm{m}_A} A\otimes \mathrm{I}\otimes \mathrm{FA}\cong A\otimes \mathrm{FA}\ \Big) \\
&+ \\
&\Big(\ A\otimes \mathrm{FA}\cong A\otimes \mathrm{I}\otimes \mathrm{FA} \xrightarrow{\eta_A\otimes \mathrm{i}_A\otimes \mathrm{s}_A} \mathrm{FA}\otimes \mathrm{FA}\otimes \mathrm{FA}\otimes \mathrm{FA} \\
&\qquad \xrightarrow{\mathrm{id}_{\mathrm{FA}}\otimes \sigma_{\mathrm{FA},\mathrm{FA}}\otimes \mathrm{id}_{\mathrm{FA}}} \mathrm{FA}\otimes \mathrm{FA}\otimes \mathrm{FA}\otimes \mathrm{FA} \xrightarrow{\mathrm{f}_A\otimes \varepsilon_A\otimes \mathrm{m}_A} \mathrm{I}\otimes A\otimes \mathrm{FA}\cong A\otimes \mathrm{FA}\ \Big) \\
&+ \\
&\Big(\ A\otimes \mathrm{FA}\cong \mathrm{I}\otimes A\otimes \mathrm{FA} \xrightarrow{\mathrm{i}_A\otimes \eta_A\otimes \mathrm{s}_A} \mathrm{FA}\otimes \mathrm{FA}\otimes \mathrm{FA}\otimes \mathrm{FA} \\
&\qquad \xrightarrow{\mathrm{id}_{\mathrm{FA}}\otimes \sigma_{\mathrm{FA},\mathrm{FA}}\otimes \mathrm{id}_{\mathrm{FA}}} \mathrm{FA}\otimes \mathrm{FA}\otimes \mathrm{FA}\otimes \mathrm{FA} \xrightarrow{\mathrm{f}_A\otimes \varepsilon_A\otimes \mathrm{m}_A} \mathrm{I}\otimes A\otimes \mathrm{FA}\cong A\otimes \mathrm{FA}\ \Big)
\end{aligned}
$$

which, in turn, by the bialgebra laws and Lemma 17 (3), equals

$$((\varepsilon_A\ \eta_A)\otimes \mathrm{id}_{\mathrm{FA}}) + 0_{A\otimes \mathrm{FA},A\otimes \mathrm{FA}} + 0_{A\otimes \mathrm{FA},A\otimes \mathrm{FA}} + ((\mathrm{id}_A\otimes \overline{\eta}_A)(\sigma_{A,A}\otimes \mathrm{id}_{\mathrm{FA}})(\mathrm{id}_A\otimes \underline{\varepsilon}_A))$$

(2) & (3) The arguments crucially rely on the commutativity of the Fock-space bialgebra structure. Since the two arguments are dual of each other, I only consider one of them.

$$
\begin{tikzcd}
A\otimes A\otimes \mathrm{FA} \arrow[r, "\mathrm{id}_A\otimes\eta_A\otimes\mathrm{id}_{\mathrm{FA}}"] \arrow[d, "\sigma_{A,A}\otimes\mathrm{id}_{\mathrm{FA}}"'] & A\otimes \mathrm{FA}\otimes \mathrm{FA} \arrow[rr, "\mathrm{id}_{\mathrm{FA}}\otimes\mathrm{m}_A"] \arrow[d, "\sigma_{A,\mathrm{FA}}\otimes\mathrm{id}_{\mathrm{FA}}"'] \arrow[dr, "\eta_A\otimes\mathrm{id}_{\mathrm{FA}}\otimes\mathrm{id}_{\mathrm{FA}}"] & & A\otimes \mathrm{FA} \arrow[dr, "\eta_A\otimes\mathrm{id}_{\mathrm{FA}}"] & \\
A\otimes A\otimes \mathrm{FA} \arrow[r, "\eta_A\otimes\mathrm{id}_A\otimes\mathrm{id}_{\mathrm{FA}}"'] \arrow[d, "\mathrm{id}_A\otimes\eta_A\mathrm{id}_{\mathrm{FA}}"'] & \mathrm{FA}\otimes A\otimes \mathrm{FA} \arrow[dr, "\mathrm{id}_{\mathrm{FA}}\otimes\eta_A\otimes\mathrm{id}_{\mathrm{FA}}"'] & \mathrm{FA}\otimes \mathrm{FA}\otimes \mathrm{FA} \arrow[rr, "\mathrm{id}_{\mathrm{FA}}\otimes\mathrm{m}_A"] \arrow[d, "\sigma_{\mathrm{FA},\mathrm{FA}}\otimes\mathrm{id}_{\mathrm{FA}}"'] \arrow[dr, "\mathrm{m}_A\otimes\mathrm{id}_{\mathrm{FA}}"] & & \mathrm{FA}\otimes \mathrm{FA} \arrow[d, "\mathrm{m}_A"] \\
A\otimes \mathrm{FA}\otimes \mathrm{FA} \arrow[rr, "\eta_A\otimes\mathrm{id}_{\mathrm{FA}}\otimes\mathrm{id}_{\mathrm{FA}}"'] \arrow[dr, "\mathrm{id}_A\otimes\mathrm{m}_A"] & & \mathrm{FA}\otimes \mathrm{FA}\otimes \mathrm{FA} \arrow[r, "\mathrm{m}_A\otimes\mathrm{id}_{\mathrm{FA}}"'] \arrow[dr, "\mathrm{id}_{\mathrm{FA}}\otimes\mathrm{m}_A"] & \mathrm{FA}\otimes \mathrm{FA} \arrow[r, "\mathrm{m}_A"] & \mathrm{FA} \\
& A\otimes \mathrm{FA} \arrow[rr, "\eta_A\otimes\mathrm{id}_{\mathrm{FA}}"'] & & \mathrm{FA}\otimes \mathrm{FA} \arrow[ur, "\mathrm{m}_A"'] &
\end{tikzcd}
$$

□

Analogously, one can establish the following laws of interaction between the creation/annihilation operators and the bialgebra structure.

Proposition 18. *For a Fock-space construction* F, *the following hold for all natural transformations* $\eta_A : A \to \mathrm{FA}$ *and* $\varepsilon_A : \mathrm{FA} \to A$.

1. *Leibniz rules:*

$$
\begin{aligned}
\mathrm{s}_A\ \overline{\eta}_A &= ((\overline{\eta}_A\otimes \mathrm{id}_{\mathrm{FA}}) + (\mathrm{id}_{\mathrm{FA}}\otimes \overline{\eta}_A)\,(\sigma_{A,\mathrm{FA}}\otimes \mathrm{id}_{\mathrm{FA}}))\,(\mathrm{id}_A\otimes \mathrm{s}_A) : A\otimes \mathrm{FA}\to \mathrm{FA}\otimes \mathrm{FA} \\
\underline{\varepsilon}_A\ \mathrm{m}_A &= (\mathrm{id}_A\otimes \mathrm{m}_A)\,((\underline{\varepsilon}_A\otimes \mathrm{id}_{\mathrm{FA}}) + (\sigma_{\mathrm{FA},A}\otimes \mathrm{id}_{\mathrm{FA}})\,(\mathrm{id}_{\mathrm{FA}}\otimes \underline{\varepsilon}_A)) : \mathrm{FA}\otimes \mathrm{FA}\to A\otimes \mathrm{FA}
\end{aligned}
$$

2. *Constant rules:*

$$\mathrm{f}_A\,\overline{\eta}_A = 0_{A\otimes \mathrm{F}A,\mathrm{I}} \quad , \qquad \mathrm{i}_A\,\underline{\varepsilon}_A = 0_{\mathrm{I},A\otimes \mathrm{F}A}$$

3.2 Coherent states

Our discussion of coherent states is within the framework of categorical models of linear logic, see for example Melliès (2009).

Definition 19. *A linear Fock-space construction is one equipped with linear exponential comonad structure* (ϵ,δ) *in the form of natural transformations* $\epsilon_A : \mathrm{F}A \to A$ *and* $\delta_A : \mathrm{F}A \to \mathrm{FF}A$ *such that*

$$\begin{tikzcd}
 & \mathrm{F}A \arrow[dl, "\mathrm{id}_{\mathrm{F}A}"'] \arrow[d, "\delta_A"] \arrow[dr, "\mathrm{id}_{\mathrm{F}A}"] & \\
\mathrm{F}A & \mathrm{FF}A \arrow[l, "\varepsilon_{\mathrm{F}A}"] \arrow[r, "\mathrm{F}\varepsilon_A"'] & \mathrm{F}A
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathrm{F}A \arrow[r, "\delta_A"] \arrow[d, "\delta_A"'] & \mathrm{FF}A \arrow[d, "\delta_{\mathrm{F}A}"] \\
\mathrm{FF}A \arrow[r, "\mathrm{F}\delta_A"] & \mathrm{FF}A
\end{tikzcd}$$

and subject to the coherence conditions

$$\begin{tikzcd}
 & \mathrm{FFO} \arrow[dr, "F\mathrm{n}_{\mathrm{FO}}"] & \\
\mathrm{FO} \arrow[ur, "\delta_{\mathrm{O}}"] \arrow[rr, "\mathrm{id}_{\mathrm{FO}}"'] & & \mathrm{FO}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathrm{F}A\otimes \mathrm{F}B \arrow[rr, "\delta_A\otimes\delta_B"] \arrow[d, "\varphi_{A,B}"'] & & \mathrm{FF}A\otimes \mathrm{FF}B \arrow[d, "\varphi_{\mathrm{F}A,\mathrm{F}B}"] \\
\mathrm{F}(A\oplus B) \arrow[r, "\delta_{A\oplus B}"'] & \mathrm{FF}(A\oplus B) \arrow[r, "\mathrm{F}\langle \mathrm{F}\pi_1,\mathrm{F}\pi_2\rangle"'] & \mathrm{F}(\mathrm{F}A\oplus \mathrm{F}B)
\end{tikzcd}$$

Definition 20. *Let* F *be a linear Fock-space construction. A coherent state* γ *is a map* $\mathrm{I}\to \mathrm{F}A$ *such that*

1. $\underline{\epsilon}_A\,\gamma = (\mathrm{I}\cong \mathrm{I}\otimes \mathrm{I} \xrightarrow{v\otimes\gamma} A\otimes \mathrm{F}A)$ *for some* $v : \mathrm{I}\to A$,
2. $\mathrm{f}_A\,\gamma = \mathrm{id}_{\mathrm{I}}$, *and*
3. $\mathrm{s}_A\,\gamma = (\mathrm{I}\cong \mathrm{I}\otimes \mathrm{I} \xrightarrow{\gamma\otimes\gamma} \mathrm{F}A\otimes \mathrm{F}A)$.

Definition 21. *Let* F *be a linear Fock-space construction.*

1. *The Kleisli extension* $u^{\#} : \mathrm{F}X \to \mathrm{F}A$ *of* $u : \mathrm{F}X\to A$ *is defined as* $\mathrm{F}(u)\circ\delta_X$.
2. *The extension* $\widetilde{v} : \mathrm{I}\to \mathrm{F}A$ *of* $v : \mathrm{I}\to A$ *is the composite* $\mathrm{I}\cong \mathrm{FO} \xrightarrow{\delta_{\mathrm{O}}} \mathrm{FFO}\cong \mathrm{FI} \xrightarrow{\mathrm{F}v} \mathrm{F}A$.

For instance, $\widetilde{0_{\mathrm{I},A}} = \mathrm{i}_A : \mathrm{I}\to \mathrm{F}A$.

Theorem 22. *For every* $v : \mathrm{I}\to A$, *the extension* $\widetilde{v} : \mathrm{I}\to \mathrm{F}A$ *is a coherent state.*

The theorem arises from the following facts.

Proposition 23. *Let* F *be a linear Fock-space construction.*

1. *For* $f : A\to B$, $\mathrm{f}_B\circ \mathrm{F}(f) = \mathrm{f}_A : \mathrm{F}A\to \mathrm{I}$.
2. $\mathrm{f}_{\mathrm{F}A}\,\delta_A = \mathrm{f}_A : \mathrm{F}A\to \mathrm{I}$.
3. $\mathrm{s}_{\mathrm{F}A}\,\delta_A = (\delta_A\otimes\delta_A)\,\mathrm{s}_A : \mathrm{F}A\to \mathrm{FF}A\otimes \mathrm{FF}A$.

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4. *For* $u : \mathrm{FX} \to A$, $\underline{\epsilon}_A \circ u^{\#} = (u \otimes u^{\#})\, s_X$.
5. $s_O = (\mathrm{FO} \cong \mathrm{I} \cong \mathrm{I} \otimes \mathrm{I} \cong \mathrm{FO} \otimes \mathrm{FO})$.

We conclude the section by recording a property that will be useful at the end of the paper.

Proposition 24. *Let* $\eta_A : A \to \mathrm{F}A$ *be a natural transformation for a linear Fock-space construction* F. *For* $v : \mathrm{I} \to A$,

$$(\overline{\eta}^{v}_{A})^{\#}\, \mathrm{i}_A = \widetilde{\eta_A\, v} \tag{10}$$

4. Leibniz Structure

This section shows that the axiomatisation of creation (resp. annihilation) operators of Section 3.1 may be seen from the viewpoint of the theory of differential (resp. codifferential) categories (Blute et al., 2006) as arising from a derivation that merely satisfies Leibniz rule.

4.1 Leibniz transformation

Key ingredients of differential categories are additive and comonadic modality structures. The additive structure is provided by **CMon**-enriched symmetric monoidal categories; while, in the present context, the required modality structure is just functorial.

Definition 25. *For a symmetric monoidal category* $\mathcal{C}$, *a commutative comonoid (resp. bialgebra) in the category of endofunctors on* $\mathcal{C}$ *equipped with the pointwise symmetric monoidal structure is referred to as a functorial commutative coalgebra (resp. bialgebra).*

For a **CMon**-enriched symmetric monoidal category $(\mathcal{C}, 0, +, \mathrm{I}, \otimes)$ and a functorial commutative coalgebra $(F : \mathcal{C} \to \mathcal{C}, f : F \to \mathrm{I}, s : F \to F \otimes F)$, we consider natural transformations satisfying the *Leibniz Rule* of deriving transformations in differential categories (Blute et al., 2006).

Definition 26. *A Leibniz transformation is a natural transformation* $\mathrm{d}_A : A \otimes FA \to FA$ *satisfying:*

$$\begin{tikzcd} A \otimes FA \arrow[rr, "\mathrm{d}_A"] \arrow[d, "\mathrm{id}_A \otimes s_A"'] & & FA \arrow[d, "s_A"] \\ A \otimes FA \otimes FA \arrow[rr, "(\mathrm{d}_A \otimes \mathrm{id}_A) + (\mathrm{id}_A \otimes \mathrm{d}_A)\,(\sigma_{A,FA} \otimes \mathrm{id}_{FA})"'] & & FA \otimes FA \end{tikzcd}$$

Remark 27.

1. *A Leibniz transformation satisfies the Constant Rule (Blute et al., 2020, Lemma 3):*

$$\begin{tikzcd} A \otimes FA \arrow[rr, "\mathrm{d}_A"] \arrow[dr, "0"'] & & FA \arrow[dl, "f_A"] \\ & \mathrm{I} & \end{tikzcd}$$

2. *The commutation relation of a Leibniz transformation with itself is the Interchange Rule of deriving transformations in differential categories (Blute et al., 2006).*

As in Theorem 16 (3), we have the following.

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Proposition 28. *For every natural transformation $\varepsilon_A : FA \to A$, the induced annihilation natural transformation $\underline{\varepsilon}_A = (\varepsilon_A \otimes \mathrm{id}_{FA})\, s_A : FA \to A \otimes FA$ satisfies the commutation relation with itself:*

$$(\mathrm{id}_A \otimes \underline{\varepsilon}_A)\, \underline{\varepsilon}_A \;=\; (\sigma_{A,A} \otimes \mathrm{id}_{\mathrm{F}A})(\mathrm{id}_A \otimes \underline{\varepsilon}_A)\, \underline{\varepsilon}_A \;\; : \mathrm{F}A \to A \otimes A \otimes \mathrm{F}A$$

The commutation relation between Leibniz and annihilation transformations is available under a linearity condition.

Definition 29. *For an endomorphism α on A, a natural transformation $\varepsilon_A : FA \to A$ is said to be α-linear whenever $\varepsilon_A\, \mathrm{d}_A = (A \otimes FA \xrightarrow{\alpha \otimes f_A} A \otimes \mathrm{I} \cong A)$.*

Lemma 30. *For every Leibniz transformation $\mathrm{d}_A : A \otimes FA \to FA$ and α-linear natural transformation $\varepsilon_A : FA \to A$, the Leibniz transformation d_A and the annihilation natural transformation $\underline{\varepsilon}_A$ satisfy the commutation relation:*

$$\underline{\varepsilon}_A\, \mathrm{d}_A \;=\; (\alpha \otimes \mathrm{id}_{FA}) + (\mathrm{id}_A \otimes \mathrm{d}_A)\, (\sigma_{A,A} \otimes \mathrm{id}_{FA})\, (\mathrm{id}_A \otimes \underline{\varepsilon}_A) : A \otimes FA \to A \otimes FA$$

4.2 Leibniz coderelictión

I now place the development of the previous section in the context of creation/annihilation operators by considering Leibniz transformations that arise as creation natural transformations. In differential linear logic (Ehrhard and Regnier, 2006; Ehrhard, 2018) and in differential categories (Blute et al., 2006, 2020), this corresponds to how deriving transformations arise from coderelictions.

For $(F : \mathcal{C} \to \mathcal{C}, i : \mathrm{I} \to F, m : F \otimes F \to F, f : F \to \mathrm{I}, s : F \to F \otimes F)$ a functorial commutative bialgebra, we consider natural transformations satisfying the *Product Rule* of coderelictions (Blute et al., 2006).

Definition 31. *A Leibniz codereliction is a natural transformation $\eta_A : A \to FA$ satisfying:*

$$\begin{tikzcd} A \arrow[rr, "\eta_A"] \arrow[dr, "{\substack{(A \cong A \otimes \mathrm{I} \xrightarrow{\eta_A \otimes i_A} FA \otimes FA) \\ +\, (A \cong \mathrm{I} \otimes A \xrightarrow{i_A \otimes \eta_A} FA \otimes FA)}}"'] & & FA \arrow[dl, "s_A"] \\ & FA \otimes FA & \end{tikzcd}$$

Remark 32. *A Leibniz codereliction satisfies the Constant Rule (Blute et al., 2020, Lemma 6):*

$$\begin{tikzcd} A \arrow[rr, "\eta_A"] \arrow[dr, "0"'] & & FA \arrow[dl, "f_A"] \\ & \mathrm{I} & \end{tikzcd}$$

Lemma 33. *For every Leibniz codereliction $\eta_A : A \to FA$, the induced creation natural transformation $\overline{\eta}_A = m_A\, (\eta_A \otimes \mathrm{id}_{FA}) : A \otimes FA \to FA$ is a Leibniz transformation.*

Leibniz derelictions are defined dually.

Definition 34. *A Leibniz dereliction is a natural transformation $\varepsilon_A : FA \to A$ satisfying:*

$$\begin{tikzcd} & FA \otimes FA \arrow[dl, "m_A"'] \arrow[dr, "{\substack{(FA \otimes FA \xrightarrow{\varepsilon_A \otimes f_A} A \otimes \mathrm{I} \cong A) \\ +\, (FA \otimes FA \xrightarrow{f_A \otimes \varepsilon_A} \mathrm{I} \otimes A \cong A)}}"] & \\ FA \arrow[rr, "\varepsilon_A"'] & & A \end{tikzcd}$$

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Remark 35. *A Leibniz dereliction satisfies the Constant Rule:*

$$\begin{tikzcd} & \mathrm{I} \arrow[dl, "i_A"'] \arrow[dr, "0"] & \\ FA \arrow[rr, "\varepsilon_A"'] & & A \end{tikzcd}$$

Lemma 36. *For every Leibniz codereliction $\eta_A : A \to FA$, every Leibniz dereliction $\varepsilon_A : FA \to A$ is $(\varepsilon_A \eta_A)$-linear for the creation Leibniz transformation $\overline{\eta}_A : A \otimes FA \to FA$.*

The above together with Lemma 30 and its dual provide the following result.

Corollary 37. *For every Leibniz codereliction $\eta_A : A \to FA$ and every Leibniz dereliction $\varepsilon_A : FA \to A$, the creation and annihilation natural transformations $\overline{\eta}_A : A \otimes FA \to FA$ and $\underline{\varepsilon}_A : FA \to A \otimes FA$ satisfy the commutation relations:*

1. $\underline{\varepsilon}_A\, \overline{\eta}_A \;=\; (\varepsilon_A\, \eta_A \otimes \mathrm{id}_{\mathrm{F}A}) + (\mathrm{id}_A \otimes \overline{\eta}_A)(\sigma_{A,A} \otimes \mathrm{id}_{\mathrm{F}A})(\mathrm{id}_A \otimes \underline{\varepsilon}_A) \;: A \otimes \mathrm{F}A \to A \otimes \mathrm{F}A$
2. $\overline{\eta}_A\, (\mathrm{id}_A \otimes \overline{\eta}_A) \;=\; \overline{\eta}_A\, (\mathrm{id}_A \otimes \overline{\eta}_A)\, (\sigma_{A,A} \otimes \mathrm{id}_{\mathrm{F}A}) \;: A \otimes A \otimes \mathrm{F}A \to \mathrm{F}A$
3. $(\mathrm{id}_A \otimes \underline{\varepsilon}_A)\, \underline{\varepsilon}_A \;=\; (\sigma_{A,A} \otimes \mathrm{id}_{\mathrm{F}A})(\mathrm{id}_A \otimes \underline{\varepsilon}_A)\, \underline{\varepsilon}_A \;: \mathrm{F}A \to A \otimes A \otimes \mathrm{F}A$

In this framework, Lemma 17 (2) can be recast as follows.

Lemma 38. *For a Fock-space construction* F *on a category of spaces and linear maps, every natural transformation $\eta_A : A \to \mathrm{F}A$ is a Leibniz codereliction and every natural transformation $\varepsilon_A : \mathrm{F}A \to A$ is a Leibniz dereliction.*

The Commutation Theorem (Theorem 16) may be then seen to follow from the previous lemma and corollary.

5. Combinatorial Model

I introduce and study a model for Fock space with creation/annihilation operators that arises in the setting of *generalised species of structures* (Fiore, 2005, 2006b; Fiore et al., 2008). These are a categorical generalisation of both the structural combinatorial theory of species of structures (Joyal, 1981; Bergeron et al., 1998) and the relational model of linear logic.

Our combinatorial model conforms to the axiomatics of the previous section by being an example of its generalisation from categories to *bicategories* (Bénabou, 1967), by which I roughly mean the categorical setting where all structural identities hold up to canonical coherent isomorphism. I will not dwell on this here but refer the reader to Fiore et al. (2024).

5.1 The bicategory of profunctors

Our setting for spaces and linear maps will be the *bicategory of profunctors* $\mathcal{P}$**rof**, for which see for example Lawvere (1973); Bénabou (2000). A *profunctor* (or *bimodule*, or *distributor*) $\mathbb{A} \mathrel{\mapstochar\rightarrow} \mathbb{B}$ between small categories $\mathbb{A}$ and $\mathbb{B}$ is a functor $\mathbb{A}^\circ \times \mathbb{B} \to \mathcal{S}$**et**. It might be useful to think of these as category-indexed set-valued matrices.

The bicategory $\mathcal{P}$**rof** has objects given by small categories, maps given by profunctors, and 2-cells given by natural transformations. The profunctor composition $TS : \mathbb{A} \mathrel{\mapstochar\rightarrow} \mathbb{C}$ of $S : \mathbb{A} \mathrel{\mapstochar\rightarrow} \mathbb{B}$

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and $T : \mathbb{B} \dashrightarrow \mathbb{C}$ is given by the matrix-multiplication formula

$$TS\,(a,c) = \int^{b\in\mathbb{B}} S(a,b) \times T(b,c) \tag{11}$$

where $\times$ and $\int$, respectively, denote the cartesian product and coend operations. The associated identity profunctors $I_{\mathbb{C}}$ are the hom-set functors $\mathbb{C}^{\circ} \times \mathbb{C} \to \boldsymbol{\mathcal{S}et} : (c', c) \mapsto \mathbb{C}(c', c)$.

The notion of coend and its properties (see *e.g.* Mac Lane (1971); Loregian (2021)) is central to the calculus of this section. A coend is a colimit arising as a coproduct under a quotient that establishes compatibility under left and right actions. Technically, the coend $\int^{z\in\mathbb{C}} H(z,z) \in \boldsymbol{\mathcal{S}et}$ of a functor $H : \mathbb{C}^{\circ} \times \mathbb{C} \to \boldsymbol{\mathcal{S}et}$ can be presented as the following coequaliser:

$$\begin{array}{c}
(f : x \to y, h) \longmapsto \big(x, H(f, \mathrm{id}_x)(h)\big) \\[1ex]
\coprod_{f:x\to y \text{ in } \mathbb{C}} H(y,x) \rightrightarrows \coprod_{z\in\mathbb{C}} H(z,z) \twoheadrightarrow \int^{z\in\mathbb{C}} H(z,z) \\[1ex]
(f : x \to y, h) \longmapsto \big(y, H(\mathrm{id}_y, f)(h)\big)
\end{array}$$

As for (11), then, $TS\,(a,c)$ consists of equivalence classes of triples in $\coprod_{b\in\mathbb{B}} S(a,b) \times T(b,c)$ under the equivalence relation generated by identifying $(b, s, T(f, \mathrm{id}_{b'})(t'))$ and $(b', S(\mathrm{id}_a, f)(s), t')$ for all $f : b \to b'$ in $\mathbb{B}$, $s \in S(a,b)$, $t' \in T(b', c)$. Note also that, for all $P : \mathbb{C}^{\circ} \to \boldsymbol{\mathcal{S}et}$, there is a canonical natural isomorphism

$$P(c) \cong \int^{z\in\mathbb{C}} P(z) \times \mathbb{C}(c,z) \tag{12}$$

known as the *density formula* (Mac Lane, 1971) or *Yoneda lemma* (Kelly, 1982) that essentially embodies the unit laws of profunctor composition with the identities.

The bicategory $\boldsymbol{\mathcal{P}rof}$ not only has compatible biproduct and symmetric monoidal structures but is in fact a *compact closed bicategory*, see Day and Street (1997). The biproduct structure is given by the empty and binary coproduct of categories (*i.e.* $\mathrm{O} = 0$ and $\oplus = +$), and the tensor product structure is given by the empty and binary product of categories (*i.e.* $\mathrm{I} = 1$ and $\otimes = \times$).

Remark 39. *The analogy of profunctors between categories as matrices between bases can be also phrased as an analogy between cocontinuous functors between presheaf categories and linear transformations between free vector spaces.*

As it is well known, the free small-colimit completion of a small category $\mathbb{C}$ is the functor category $\boldsymbol{\mathcal{S}et}^{\mathbb{C}^{\circ}}$ of (contravariant) presheaves on $\mathbb{C}$ and natural transformations between them. The universal map is the Yoneda embedding $\mathbb{C} \hookrightarrow \boldsymbol{\mathcal{S}et}^{\mathbb{C}^{\circ}} : z \mapsto |\, z\,\rangle$ where

$$|\, z\,\rangle : \mathbb{C}^{\circ} \to \boldsymbol{\mathcal{S}et} : c \mapsto \mathbb{C}(c,z)$$

The use of Dirac's ket notation in this context is justified by regarding presheaves as vectors and noticing that the isomorphism (12) above amounts to the following one

$$P \cong \int^{z\in\mathbb{C}} P_z \cdot |\, z\,\rangle$$

in $\boldsymbol{\mathcal{S}et}^{\mathbb{C}^{\circ}}$ expressing every presheaf as a colimit of the basis vectors (referred to as representable presheaves in categorical terminology). Associated to this representation, the notion of linearity for transformations corresponds to that of cocontinuity (i.e. colimit preservation) for functors. Indeed, the bicategory of profunctors is biequivalent to the 2-category with objects consisting of small categories, morphisms from $\mathbb{A}$ to $\mathbb{B}$ given by cocontinuous functors $\boldsymbol{\mathcal{S}et}^{\mathbb{A}} \to \boldsymbol{\mathcal{S}et}^{\mathbb{B}}$, and 2-cells given by natural transformations. The biequivalence associates a profunctor $T : \mathbb{A} \dashrightarrow \mathbb{B}$ with the cocontinuous functor $\mathrm{Fun}(T) : \boldsymbol{\mathcal{S}et}^{\mathbb{A}} \to \boldsymbol{\mathcal{S}et}^{\mathbb{B}} : P \mapsto \int^{b\in\mathbb{B}} \big[\int^{a\in\mathbb{A}} P_a \times T(a,b)\big] \cdot |\, b\,\rangle$*, whilst the* *profunctor* $\mathrm{Pro}(F) : \mathbb{A} \mathrel{-\!\!\!\mapsto} \mathbb{B}$ *underlying a cocontinous functor* $F : \mathcal{S}\mathbf{et}^{\mathbb{A}} \to \mathcal{S}\mathbf{et}^{\mathbb{B}}$ *has entry* $F \mid a\,\rangle_b$ *at* $(a, b) \in \mathbb{A}^\circ \times \mathbb{B}$*. In particular, note the following:*

$$
\begin{aligned}
&\mathrm{Fun}(\mathrm{Pro}\ F)(P)\\
&\quad = \int^{b\in\mathbb{B}} \Big[\int^{a\in\mathbb{A}} P_a \times F \mid a\,\rangle_b \Big] \cdot \mid b\,\rangle \cong \int^{a\in\mathbb{A}} P_a \cdot \Big(\int^{b\in\mathbb{B}} F \mid a\,\rangle_b \cdot \mid b\,\rangle \Big)\\
&\quad \cong \int^{a\in\mathbb{A}} P_a \cdot F \mid a\,\rangle \cong F\Big(\int^{a\in\mathbb{A}} P_a \cdot \mid a\,\rangle \Big), \text{ by cocontinuity}\\
&\quad \cong F(P)\\
&\mathrm{Pro}(\mathrm{Fun}\ T)(a, b)\\
&\quad = \Big(\int^{y\in\mathbb{B}} \Big[\int^{x\in\mathbb{A}} \mid a\,\rangle_x \times T(x, y) \Big] \cdot \mid y\,\rangle \Big)_b \cong \Big(\int^{y\in\mathbb{B}} T(a, y) \cdot \mid y\,\rangle \Big)_b \cong T(a, b)
\end{aligned}
$$

5.2 Combinatorial Fock space

I introduce the combinatorial Fock-space construction.

Definition 40. *The combinatorial Fock space of a small category* $\mathbb{C}$ *is the small category*

$$\mathsf{F}\,\mathbb{C} = \coprod_{n\in\mathbb{N}} \mathbb{C}^n /\!/_{\mathfrak{S}_n}$$

where $\mathbb{C}^n /\!/_{\mathfrak{S}_n}$ *has objects given by n-tuples of objects of* $\mathbb{C}$ *and hom-sets*

$$\mathbb{C}^n /\!/_{\mathfrak{S}_n}(\vec{c}, \vec{z}) = \coprod_{\sigma\in\mathfrak{S}_n} \prod_{1\le i\le n} \mathbb{C}(c_i, z_{\sigma i})$$

It is a very important part of the general theory, for which see Fiore (2005) and Fiore et al. (2008), that the combinatorial Fock-space construction is the *free symmetric (strict) monoidal completion*; the unit and tensor product being respectively given by the empty tuple and tuple concatenation, and denoted as $(\,)$ and $\cdot$.

Proposition 41. *Hom-sets in combinatorial Fock space satisfy the following combinatorial laws.*

1. $\mathsf{F}\mathbb{A}(\vec{u} \cdot \vec{v}, \vec{x} \cdot \vec{y})$
   $\cong \int^{\vec{a},\vec{b},\vec{c},\vec{d}\in\mathsf{F}\mathbb{A}} \mathsf{F}\mathbb{A}(\vec{u}, \vec{a} \cdot \vec{b}) \times \mathsf{F}\mathbb{A}(\vec{v}, \vec{c} \cdot \vec{d}) \times \mathsf{F}\mathbb{A}(\vec{a} \cdot \vec{c}, \vec{x}) \times \mathsf{F}\mathbb{A}(\vec{b} \cdot \vec{d}, \vec{y})$
2. $\mathsf{F}\mathbb{A}((\,), (\,)) \cong 1 \;,\; \mathsf{F}\mathbb{A}((a), (x)) \cong \mathbb{A}(a, x)$

   $\mathsf{F}\mathbb{A}((\,), (a)) \cong 0 \;,\; \mathsf{F}\mathbb{A}((a), (\,)) \cong 0$
3. $\mathsf{F}\mathbb{A}((\,), \vec{x} \cdot \vec{y}) \cong \mathsf{F}\mathbb{A}((\,), \vec{x}) \times \mathsf{F}\mathbb{A}((\,), \vec{y})$,

   $\mathsf{F}\mathbb{A}(\vec{x} \cdot \vec{y}, (\,)) \cong \mathsf{F}\mathbb{A}(\vec{x}, (\,)) \times \mathsf{F}\mathbb{A}(\vec{y}, (\,))$
4. $\mathsf{F}\mathbb{A}((a), \vec{x} \cdot \vec{y}) \cong \big(\mathsf{F}\mathbb{A}((a), \vec{x}) \times \mathsf{F}\mathbb{A}((\,), \vec{y}) \big) + \big(\mathsf{F}\mathbb{A}((\,), \vec{x}) \times \mathsf{F}\mathbb{A}((a), \vec{y}) \big)$

   $\mathsf{F}\mathbb{A}(\vec{x} \cdot \vec{y}, (a)) \cong \big(\mathsf{F}\mathbb{A}(\vec{x}, (a)) \times \mathsf{F}\mathbb{A}(\vec{y}, (\,)) \big) + \big(\mathsf{F}\mathbb{A}(\vec{x}, (\,)) \times \mathsf{F}\mathbb{A}(\vec{y}, (a)) \big)$
5. $\mathsf{F}(\mathbb{A} + \mathbb{B})\,(\mathsf{F}\amalg_1(\vec{a}) \cdot \mathsf{F}\amalg_2(\vec{b}), \mathsf{F}\amalg_1(\vec{x}) \cdot \mathsf{F}\amalg_2(\vec{y})) \;\cong\; \mathsf{F}\mathbb{A}(\vec{a}, \vec{x}) \times \mathsf{F}\mathbb{B}(\vec{b}, \vec{y})$

I proceed to describe the structure of the combinatorial Fock space.

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§ *5.2.1.* For a profunctor $T : \mathbb{A} \mathrel{-\!\!\!\mapsto} \mathbb{B}$, the profunctor $\mathsf{F}\,T : \mathsf{F}\,\mathbb{A} \mathrel{-\!\!\!\mapsto} \mathsf{F}\mathbb{B}$ is given by

$$\mathsf{F}T\,(\vec{x},\vec{y}) \;=\; \int^{\vec{z}\in \mathsf{F}(\mathbb{A}^\circ\times\mathbb{B})} \big(\textstyle\prod_{z_i\in\vec{z}} T\,z_i\big) \times \mathsf{F}\mathbb{A}(\vec{x},\mathsf{F}\pi_1\vec{z}) \times \mathsf{F}\mathbb{B}(\mathsf{F}\pi_2\vec{z},\vec{y})$$

so that

$$\mathsf{F}\,T\,((a_1,\ldots,a_m),(b_1,\ldots,b_n)) \cong \begin{cases} \coprod_{\sigma\in\mathfrak{S}_m}\prod_{1\le i\le m} T(a_i,b_{\sigma i}) & \text{, if } m=n \\ 0 & \text{, otherwise} \end{cases}$$

§ *5.2.2.* There are canonical natural coherent equivalences as follows:

$$\phi : 1 \simeq \mathsf{F}\,0 \qquad , \qquad \phi(\,*,(\,)\,) = 1$$

$$\varphi_{\mathbb{A},\mathbb{B}} : \mathsf{F}\mathbb{A}\times\mathsf{F}\mathbb{B} \simeq \mathsf{F}(\mathbb{A}+\mathbb{B})\ , \qquad \varphi_{\mathbb{A},\mathbb{B}}(\,(\vec{x},\vec{y}),\vec{z}\,) = \mathsf{F}(\mathbb{A}+\mathbb{B})(\mathsf{F}\amalg_1(\vec{x})\cdot\mathsf{F}\amalg_2(\vec{y}),\vec{z})$$

§ *5.2.3.* The pseudo commutative bialgebra structure (8) consists of:

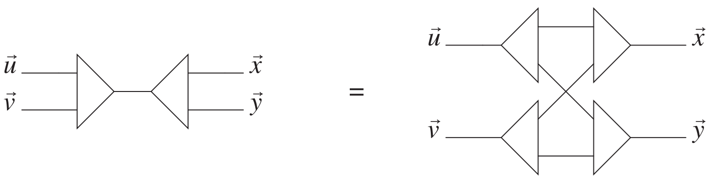
$$\mathrm{i}_{\mathbb{A}} : 1 \mathrel{-\!\!\!\mapsto} \mathsf{F}\mathbb{A} \qquad , \qquad \mathrm{i}_{\mathbb{A}}(*,\vec{a}) = \mathsf{F}\mathbb{A}((\,),\vec{a})$$

$$\mathrm{m}_{\mathbb{A}} : \mathsf{F}\mathbb{A}\times\mathsf{F}\mathbb{A} \mathrel{-\!\!\!\mapsto} \mathsf{F}\mathbb{A}\ , \qquad \mathrm{m}_{\mathbb{A}}((\vec{x},\vec{y}),\vec{z}) = \mathsf{F}\mathbb{A}(\vec{x}\cdot\vec{y},\vec{z})$$

$$\mathrm{f}_{\mathbb{A}} : \mathsf{F}\mathbb{A} \mathrel{-\!\!\!\mapsto} 1 \qquad , \qquad \mathrm{f}_{\mathbb{A}}(\vec{a},*) = \mathsf{F}\mathbb{A}(\vec{a},(\,))$$

$$\mathrm{s}_{\mathbb{A}} : \mathsf{F}\mathbb{A} \mathrel{-\!\!\!\mapsto} \mathsf{F}\mathbb{A}\times\mathsf{F}\mathbb{A} \quad , \qquad \mathrm{s}_{\mathbb{A}}(\vec{z},(\vec{x},\vec{y})) = \mathsf{F}\mathbb{A}(\vec{z},\vec{x}\cdot\vec{y})$$

The bialgebra law for $\mathrm{m}_{\mathbb{A}}\,\mathrm{s}_{\mathbb{A}}$ arises from the combinatorial law of Proposition 41 (1), which is a formal expression for the diagrammatic law:



§ *5.2.4.* The linear exponential pseudo comonad structure is given by

$$\epsilon_{\mathbb{A}} : \mathsf{F}\mathbb{A} \mathrel{-\!\!\!\mapsto} \mathbb{A} \quad , \qquad \epsilon_{\mathbb{A}}(\vec{x},a) = \mathsf{F}\mathbb{A}(\vec{x},(a))$$

$$\delta_{\mathbb{A}} : \mathsf{F}\mathbb{A} \mathrel{-\!\!\!\mapsto} \mathsf{F}\mathsf{F}\mathbb{A}\ , \qquad \delta_{\mathbb{A}}(\vec{a},\alpha) = \mathsf{F}\mathbb{A}(\vec{a},\alpha^\bullet)$$

where $(\vec{a}_1,\ldots,\vec{a}_n)^\bullet = \vec{a}_1\cdot\ldots\cdot\vec{a}_n \in \mathsf{F}\mathbb{A}$ for $\vec{a}_i \in \mathsf{F}\mathbb{A}$.

The laws of Proposition 41 (4) exhibit the combinatorial context of the identities of Proposition 17 (2).

§ *5.2.5.* The bicategory ***Prof*** admits a *duality*, by which a small category $\mathbb{A}$ is mapped to its opposite category $\mathbb{A}^\circ$ and a profunctor $T : \mathbb{A} \mathrel{-\!\!\!\mapsto} \mathbb{B}$ to the profunctor $T^\circ : \mathbb{B}^\circ \mathrel{-\!\!\!\mapsto} \mathbb{A}^\circ$ with $T^\circ(\vec{y},\vec{x}) = T(\vec{x},\vec{y})$. Thereby, the pseudo *comonadic* structure of the combinatorial Fock-space construction can be turned into pseudo *monadic* structure (η,μ) by setting $\eta_{\mathbb{A}} = (\epsilon_{\mathbb{A}^\circ})^\circ$ and $\mu_{\mathbb{A}} = (\delta_{\mathbb{A}^\circ})^\circ$. Specifically, we have:

$$\eta_{\mathbb{A}} : \mathbb{A} \mathrel{-\!\!\!\mapsto} \mathsf{F}\mathbb{A} \quad , \qquad \eta_{\mathbb{A}}(a,\vec{x}) = \mathsf{F}\mathbb{A}((a),\vec{x})$$

$$\mu_{\mathbb{A}} : \mathsf{F}\mathsf{F}\mathbb{A} \mathrel{-\!\!\!\mapsto} \mathsf{F}\mathbb{A}\ , \qquad \mu_{\mathbb{A}}(\alpha,\vec{a}) = \mathsf{F}\mathbb{A}(\alpha^\bullet,\vec{a})$$

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§ *5.2.6.* The structure results in canonical creation and annihilation operators:

$$\overline{\eta}_{\mathbb{A}} : \mathbb{A} \times \mathsf{F}\mathbb{A} \dashrightarrow \mathsf{F}\mathbb{A}\ , \quad \overline{\eta}_{\mathbb{A}}((a,\vec{x}),\vec{y})) = \mathsf{F}\mathbb{A}(\vec{x}\cdot(a),\vec{y})$$

$$\underline{\epsilon}_{\mathbb{A}} : \mathsf{F}\mathbb{A} \dashrightarrow \mathbb{A} \times \mathsf{F}\mathbb{A}\ , \quad \underline{\epsilon}_{\mathbb{A}}(\vec{x},(a,\vec{y})) = \mathsf{F}\mathbb{A}(\vec{x},(a)\cdot\vec{y})$$

so that, for $V : 1 \dashrightarrow \mathbb{A}$ and $V' : \mathbb{A} \dashrightarrow 1$, we have

$$\begin{aligned} \overline{\eta}^{V}_{\mathbb{A}}(\vec{x},\vec{y}) &\cong \int^{a\in\mathbb{A}} V_a \times \mathsf{F}\mathbb{A}(\vec{x}\cdot(a),\vec{y}) \qquad \text{for } V_a = V(*,a) \\ \underline{\epsilon}^{V'}_{\mathbb{A}}(\vec{x},\vec{y}) &\cong \int^{a\in\mathbb{A}} V'_a \times \mathsf{F}\mathbb{A}(\vec{x},(a)\cdot\vec{y}) \qquad \text{for } V'_a = V(a,*) \end{aligned} \tag{13}$$

yielding the functorial forms

$$(\mathrm{Fun}\,\overline{\eta}^{V}_{\mathbb{A}})(X) \cong \int^{a\in\mathbb{A},\vec{z}\in\mathsf{F}\mathbb{A}} [\, V_a \times X_{\vec{z}}\,] \cdot |\, \vec{z}\cdot(a)\,\rangle$$

$$(\mathrm{Fun}\,\underline{\epsilon}^{V'}_{\mathbb{A}})(X) \cong \int^{a\in\mathbb{A},\vec{z}\in\mathsf{F}\mathbb{A}} [\, V'_a \times X_{(a)\cdot\vec{z}}\,] \cdot |\, \vec{z}\,\rangle$$

Identity (9) then becomes

$$\mathrm{Fun}(\underline{\epsilon}^{V'}_{\mathbb{A}}\,\overline{\eta}^{V}_{\mathbb{A}})\,(X) \cong \langle V, V'\rangle \cdot X + \int^{a,b\in\mathbb{A},\vec{z}\in\mathsf{F}\mathbb{A}} [\, V_a \times V'_b \times X_{(b)\cdot\vec{z}}\,] \cdot |\, \vec{z}\cdot a\,\rangle$$

where $\langle V, V'\rangle = \int^{a\in\mathbb{A}} V_a \times V'_a$.

§ *5.2.7.* In the current setting, the axiomatic proof of the commutation relation for $\underline{\epsilon}_{\mathbb{A}}\,\overline{\eta}_{\mathbb{A}}$ acquires formal combinatorial content made explicit by the following chain of isomorphisms:

$$\begin{aligned}
&\underline{\epsilon}_{\mathbb{A}}\,\overline{\eta}_{\mathbb{A}}((a,\vec{x}),(b,\vec{y})) \\
&\quad\cong \mathsf{F}\mathbb{A}(\vec{x}\cdot(a),(b)\cdot\vec{y}) \qquad\qquad (14)\\
&\quad\cong \int^{\vec{z}_1,\vec{z}_2,\vec{z}_3,\vec{z}_4\in\mathsf{F}\mathbb{A}} \mathsf{F}\mathbb{A}(\vec{x},\vec{z}_1\cdot\vec{z}_2) \times \mathsf{F}\mathbb{A}((a),\vec{z}_3\cdot\vec{z}_4) \times \mathsf{F}\mathbb{A}(\vec{z}_1\cdot\vec{z}_3,(b)) \times \mathsf{F}\mathbb{A}(\vec{z}_2\cdot\vec{z}_4,\vec{y}) \\
&\quad\cong \int^{\vec{z}_1,\vec{z}_2,\vec{z}_3,\vec{z}_4\in\mathsf{F}\mathbb{A}} \mathsf{F}\mathbb{A}(\vec{x},\vec{z}_1\cdot\vec{z}_2) \\
&\qquad\qquad \times\ [\mathsf{F}\mathbb{A}((a),\vec{z}_3) \times \mathsf{F}\mathbb{A}((\,),\vec{z}_4) + \mathsf{F}\mathbb{A}((\,),\vec{z}_3) \times \mathsf{F}\mathbb{A}((a),\vec{z}_4)] \\
&\qquad\qquad \times\ [\mathsf{F}\mathbb{A}(\vec{z}_1,(b)) \times \mathsf{F}\mathbb{A}(\vec{z}_3,(\,)) + \mathsf{F}\mathbb{A}(\vec{z}_1,(\,)) \times \mathsf{F}\mathbb{A}(\vec{z}_3,(b))] \\
&\qquad\qquad \times\ \mathsf{F}\mathbb{A}(\vec{z}_2\cdot\vec{z}_4,\vec{y}) \\
&\quad\cong [\mathsf{F}\mathbb{A}(\vec{x},(b)\cdot\vec{y}) \times \mathsf{F}\mathbb{A}((a),(\,))] + [\mathsf{F}\mathbb{A}(\vec{x},\vec{y}) \times \mathsf{F}\mathbb{A}((a),(b))] \\
&\qquad + [\int^{\vec{z}_2\in\mathsf{F}\mathbb{A}} \mathsf{F}\mathbb{A}(\vec{x},(b)\cdot\vec{z}_2) \times \mathsf{F}\mathbb{A}(\vec{z}_2\cdot(a),\vec{y})] + [\mathsf{F}\mathbb{A}((\,),(b)) \times \mathsf{F}\mathbb{A}(\vec{x}\cdot(a),\vec{y})] \\
&\quad\cong [\mathbb{A}(a,b) \times \mathsf{F}\mathbb{A}(\vec{x},\vec{y})] + [\int^{\vec{z}\in\mathsf{F}\mathbb{A}} \mathsf{F}\mathbb{A}(\vec{x},(b)\cdot\vec{z}) \times \mathsf{F}\mathbb{A}(\vec{z}\cdot(a),\vec{y})] \qquad\qquad (15)\\
&\quad\cong I_{\mathbb{A}\times\mathsf{F}\mathbb{A}}((a,\vec{x}),(b,\vec{y})) \\
&\qquad + \int^{\vec{z}\in\mathsf{F}\mathbb{A},c,d\in\mathbb{A}} \mathsf{F}\mathbb{A}(\vec{x},(c)\cdot\vec{z}) \times (\mathbb{A}\times\mathbb{A})\,((a,c),(d,b)) \times \mathsf{F}\mathbb{A}(\vec{z}\cdot(d),\vec{y}) \\
&\quad\cong (I_{\mathbb{A}\times\mathsf{F}\mathbb{A}} + (I_{\mathbb{A}} \times \overline{\eta}_{\mathbb{A}})\,(\sigma_{\mathbb{A},\mathbb{A}} \times I_{\mathsf{F}\mathbb{A}})\,(I_{\mathbb{A}} \times \underline{\epsilon}_{\mathbb{A}}))\,((a,\vec{x}),(b,\vec{y}))
\end{aligned}$$

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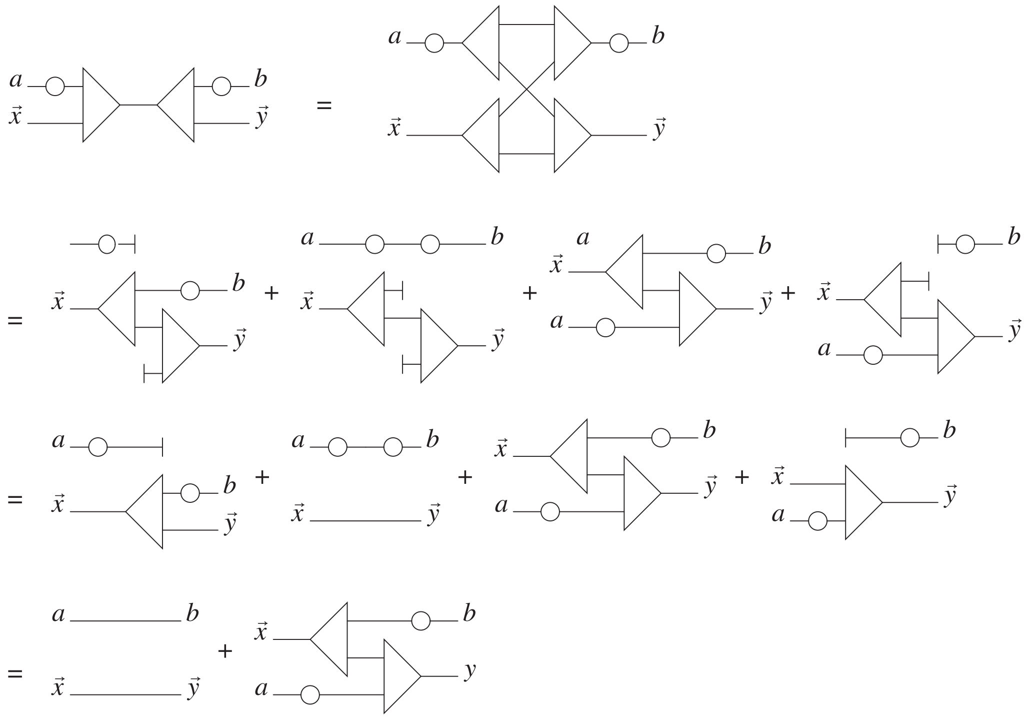
This formal derivation can be pictorially represented as follows:



5.3 Coherent states

In this section, I will indistinguishably regard profunctors $1 \rightarrow\!\!\!\!\!\!\dashv\;\; \mathbb{A}$ as (covariant) presheaves in $\mathcal{S}\mathbf{et}^{\mathbb{A}}$ and vice versa. Thus, according to Definition 21 (2), every $V \in \mathcal{S}\mathbf{et}^{\mathbb{A}}$ has a coherent state extension $\widetilde{V} \in \mathcal{S}\mathbf{et}^{\mathsf{F}\mathbb{A}}$. A calculation shows this to be given as

$$\widetilde{V} \cong \int^{\vec{a} \in \mathsf{F}\mathbb{A}} \big(\textstyle\prod_{a_i \in \vec{a}} V_{a_i} \big) \cdot | \, \vec{a} \, \rangle$$

The combinatorial version of the coherent state property of Definition 20 (1) enjoyed by $\widetilde{V}$ according to Theorem 22 yields the isomorphism

$$(\mathrm{Fun}\, \underline{\epsilon}_{\mathbb{A}})(\widetilde{V})_{(a,\vec{x})} \cong V_a \times \widetilde{V}_{\vec{x}}$$

from which we obtain the functorial form

$$(\mathrm{Fun}\, \underline{\epsilon}_{\mathbb{A}})(\widetilde{V}) \cong \int^{a \in \mathbb{A}, \vec{x} \in \mathsf{F}\mathbb{A}} \big(V_a \times \textstyle\prod_{x_i \in \vec{x}} V_{x_i} \big) \cdot | \, (a, \vec{x}) \, \rangle$$

I now proceed to introduce a notion of *exponential* (as parameterised by algebras) and show how, when applied to the creation operator (with respect to the free algebra), generalises the coherent state extension. The definition of exponential is based on that given in Vicary (2008, Section 4).

I have remarked in Section 2.2.5 that (F, η, μ) is a pseudo monad on the bicategory of profunctors. Pseudo algebras for it consist of profunctors $M : \mathsf{F}\mathbb{A} \rightarrow\!\!\!\!\!\!\dashv\;\; \mathbb{A}$ equipped with natural isomorphisms

$$\begin{tikzcd} \mathbb{A} \arrow[r, "\eta_{\mathbb{A}}"] \arrow[dr, "I_{\mathbb{A}}"'] & \mathsf{F}\mathbb{A} \arrow[d, "M"] \arrow[dl, phantom, "\cong"] \\ & \mathbb{A} \end{tikzcd} \qquad \begin{tikzcd} \mathsf{FF}\mathbb{A} \arrow[r, "\mathsf{F}M"] \arrow[d, "\mu_{\mathbb{A}}"'] \arrow[dr, phantom, "\cong"] & \mathsf{F}\mathbb{A} \arrow[d, "M"] \\ \mathsf{F}\mathbb{A} \arrow[r, "M"'] & \mathbb{A} \end{tikzcd}$$

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subject to coherence conditions, see *e.g.* Blackwell et al. (1989). These pseudo algebras provide the right notion of *unbiased commutative promonoidal category*, generalising the notion of *symmetric promonoidal category* (Day, 1970), viz. commutative pseudo monoids in the bicategory of profunctors, to biequivalent structures specified by n-ary operations $M^{(n)} : \mathbb{A}^n /\!/_{\mathfrak{S}_n} \relbar\!\!\shortmid\!\!\rightarrow \mathbb{A}$ for all $n \in \mathbb{N}$ that are commutative and associative with unit $M^{(0)}$ up to coherent isomorphism. The most common examples of pseudo F-algebras arise from small symmetric monoidal categories, say $(\mathbb{M}, 1, \odot)$, by letting $\mathbb{M}^\star : \mathsf{F}\mathbb{M} \relbar\!\!\shortmid\!\!\rightarrow \mathbb{M}$ be given by $\mathbb{M}^\star((x_1, \ldots, x_n), x) = \mathbb{M}(x_1 \odot \cdots \odot x_n, x)$, so that $\mathbb{M}^\star((\,), x) = \mathbb{M}(1, x)$. In particular, the free pseudo algebra $\mu_\mathbb{A} : \mathsf{F}\mathbb{A} \relbar\!\!\shortmid\!\!\rightarrow \mathbb{A}$ on $\mathbb{A}$ is obtained by this construction on the free symmetric monoidal category $(\mathsf{F}\mathbb{A}, (\,), \cdot)$ on $\mathbb{A}$.

Definition 42. *Let* $M : \mathsf{F}\mathbb{A} \relbar\!\!\shortmid\!\!\rightarrow \mathbb{A}$ *be a pseudo* F*-algebra. For* $T : \mathsf{F}\mathbb{X} \relbar\!\!\shortmid\!\!\rightarrow \mathbb{A}$, *define* $\exp_M(T) = M\,T^\# : \mathsf{F}\mathbb{X} \relbar\!\!\shortmid\!\!\rightarrow \mathbb{A}$.

In particular, for $V \in \mathcal{S}\mathbf{et}^\mathbb{A}$, we have that

$$\exp_M(V) = \int^{a\in\mathbb{A}} \Big[\int^{\vec{x}\in\mathsf{F}\mathbb{A}} \big(\textstyle\prod_{x_i\in\vec{x}} V_{x_i} \big) \times M(\vec{x}, a) \Big] \cdot |\, a \,\rangle$$

Proposition 43. *For a pseudo* F*-algebra* $M : \mathsf{F}\mathbb{A} \relbar\!\!\shortmid\!\!\rightarrow \mathbb{A}$,

$$\exp_M(0_1, \mathbb{A}) \cong M^{(0)}$$

and

$$\exp_M(S+T) \cong \Big(\mathsf{F}\mathbb{X} \xrightarrow{\mathrm{s}_{\mathsf{F}\mathbb{X}}} \mathsf{F}\mathbb{X} \times \mathsf{F}\mathbb{X} \xrightarrow{\exp_M(S)\times\exp_M(T)} \mathsf{F}\mathbb{A} \times \mathsf{F}\mathbb{A} \xrightarrow{\mathrm{m}_{\mathsf{F}\mathbb{A}}} \mathsf{F}\mathbb{A} \Big)$$

for all $S, T : \mathsf{F}\mathbb{X} \relbar\!\!\shortmid\!\!\rightarrow \mathbb{A}$.

Note that the notion of exponential with respect to free algebras is a form of *comonadic/monadic convolution*, as for $T : \mathsf{F}\mathbb{X} \relbar\!\!\shortmid\!\!\rightarrow \mathsf{F}\mathbb{A}$, the definition of $\exp_{\mu_\mathbb{A}}(T)$ amounts to the composite

$$\mathsf{F}\mathbb{X} \xrightarrow{\delta_\mathbb{X}} \mathsf{F}\mathsf{F}\mathbb{X} \xrightarrow{\mathsf{F}\,T} \mathsf{F}\mathsf{F}\mathbb{A} \xrightarrow{\mu_\mathbb{A}} \mathsf{F}\mathbb{A} \tag{16}$$

Theorem 44. *For* $V \in \mathcal{S}\mathbf{et}^\mathbb{A}$,

$$\exp_{\mu_\mathbb{A}}(\overline{\eta}^V_\mathbb{A})\,\mathrm{i}_\mathbb{A} \cong \widetilde{V}$$

Proof. A simple algebraic proof follows:

$$\begin{aligned} \exp_{\mu_\mathbb{A}}(\overline{\eta}^V_\mathbb{A})\,\mathrm{i}_\mathbb{A} &= \mu_\mathbb{A}\,(\overline{\eta}^V_\mathbb{A})^\#\,\mathrm{i}_\mathbb{A} \cong \mu_\mathbb{A}\,\widetilde{\eta_A\,V} \quad \text{, by (10)} \\ &\cong \mu_\mathbb{A}\,\mathsf{F}(\eta_\mathbb{A})\,\widetilde{V} \cong \widetilde{V} \quad \text{, by a monad law} \end{aligned}$$

□

I conclude the paper with a formal combinatorial proof of this result. Observe first that for the composite (16), we have:

$$\begin{aligned} &(\mu_\mathbb{A}\,\mathsf{F}(T)\,\delta_\mathbb{X})\,(\vec{x}, \vec{a}) \\ &\cong \int^{\xi\in\mathsf{F}\mathsf{F}\mathbb{X},\alpha\in\mathsf{F}\mathsf{F}\mathbb{A}} \int^{\vec{z}\in\mathsf{F}(\mathsf{F}\mathbb{X}^\circ\times\mathsf{F}\mathbb{A})} \big(\textstyle\prod_{z_i\in\vec{z}} Tz_i \big) \times \mathsf{F}\mathsf{F}\mathbb{X}(\xi, \mathsf{F}\pi_1\vec{z}) \times \mathsf{F}\mathsf{F}\mathbb{A}(\mathsf{F}\pi_2\vec{z}, \alpha) \\ &\qquad\qquad \times \mathsf{F}\mathbb{X}(\vec{x}, \xi^\bullet) \times \mathsf{F}\mathbb{A}(\alpha^\bullet, \vec{a}) \\ &\cong \int^{\vec{z}\in\mathsf{F}(\mathsf{F}\mathbb{X}^\circ\times\mathsf{F}\mathbb{A})} \big(\textstyle\prod_{z_i\in\vec{z}} Tz_i \big) \times \mathsf{F}\mathbb{X}(\vec{x}, [\mathsf{F}\pi_1\vec{z}]^\bullet) \times \mathsf{F}\mathbb{A}([\mathsf{F}\pi_2\vec{z}]^\bullet, \vec{a}) \end{aligned}$$

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and hence that

$$\begin{aligned}
&(\mu_{\mathbb{A}}\ \mathsf{F}(T)\ \delta_{\mathbb{X}}\ \mathrm{i}_{\mathbb{X}})\ (\vec{a}) \\
&\quad\cong \int^{\vec{z}\in\mathsf{F}(\mathsf{F}\mathbb{X}^{\circ}\times\mathsf{F}\mathbb{A})} \big(\textstyle\prod_{z_i\in\vec{z}} Tz_i\big) \times \mathsf{F}\mathbb{X}\big(\,(\,)\,, [\mathsf{F}\pi_1\vec{z}]^{\bullet}\big) \times \mathsf{F}\mathbb{A}\big([\mathsf{F}\pi_2\vec{z}]^{\bullet}, \vec{a}\big) \\
&\quad\cong \int^{\vec{z}\in\mathsf{FF}\mathbb{A}} \big(\textstyle\prod_{z_i\in\vec{z}} T((\,), z_i)\big) \times \mathsf{F}\mathbb{A}(\vec{z}^{\,\bullet}, \vec{a})
\end{aligned}$$

Then, according to (13),

$$\begin{aligned}
&(\mu_{\mathbb{A}}\ \mathsf{F}(\overline{\eta}^{V}_{\mathbb{A}})\ \delta_{\mathbb{A}}\ \mathrm{i}_{\mathbb{A}})\ (\vec{a}) \\
&\quad\cong \int^{\vec{z}\in\mathsf{FF}\mathbb{A}} \big(\textstyle\prod_{z_i\in\vec{z}} \int^{x\in\mathbb{A}} V_x \times \mathsf{F}\mathbb{A}((x), z_i)\big) \times \mathsf{F}\mathbb{A}(\vec{z}^{\,\bullet}, \vec{a}) \\
&\quad\cong \int^{\vec{z}\in\mathsf{FF}\mathbb{A}} \int^{x_{z_i}\in\mathbb{A}\,(z_i\in\vec{z})} \big(\textstyle\prod_{z_i\in\vec{z}} V_{x_{z_i}}\big) \times \big(\textstyle\prod_{z_i\in\vec{z}} \mathsf{F}\mathbb{A}((x_{z_i}), z_i)\big) \times \mathsf{F}\mathbb{A}(\vec{z}^{\,\bullet}, \vec{a}) \\
&\quad\cong \int^{\vec{x}\in\mathsf{F}\mathbb{A}} \big(\textstyle\prod_{x_i\in\vec{x}} V_{x_i}\big) \times \mathsf{F}\mathbb{A}(\lfloor\vec{x}\rfloor^{\bullet}, \vec{a}) \\
&\quad\cong \textstyle\prod_{x_i\in\vec{a}} V_{x_i}
\end{aligned}$$

where, for $a_i \in \mathbb{A}$, $\lfloor(a_1, \ldots, a_n)\rfloor = \big(\,(a_1), \ldots, (a_n)\big) \in \mathsf{FF}\mathbb{A}$; so that, for $\vec{a} \in \mathsf{F}\mathbb{A}$, $\lfloor\vec{a}\rfloor^{\bullet} = \vec{a}$.

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