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Groups, conics and recurrence relations

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1 *Introduction*

In this paper we explore some of the geometry that lies behind the real linear, second order, constant coefficient, recurrence relation

$$x_{n+2} = ax_{n+1} + bx_n, \qquad n = 0, 1, \dots, \tag{1}$$

where a and b are real numbers. Readers will be familiar with the standard method of solving this relation, and, to avoid trivial cases, we shall assume that $ab \neq 0$. The auxiliary equation $t^2 = at + b$ of (1) has two (possibly complex) solutions

$$t_1, t_2 = \tfrac{1}{2}\left(a \pm \sqrt{a^2 + 4b}\right),$$

and the most general solution of (1) is given by

(i) $x_n = At_1^n + Bt_2^n$ when $a^2 + 4b > 0$, and t_1 and t_2 are real and distinct;

(ii) $x_n = \left(\tfrac{1}{2}a\right)^n(An + B)$ when $a^2 + 4b = 0$, and $t_1 = t_2 = \tfrac{1}{2}a$;

(iii) $x_n = Ct^n + \overline{Ct^n}$ when $a^2 + 4b < 0$, and $t_1 = t = \overline{t_2}$.

Apart from some numerical examples, many discussions of recurrence relations end here. However, this is nowhere near the end of the story, and in this article we argue that if we focus on the numerical solutions of (1) to the exclusion of other ideas, then we miss a chance to show how recurrence relations are connected to a wide range of interesting and important mathematics. The beauty and power of mathematics lies in the connections between its diverse branches, and recurrence relations give us an excellent opportunity to illustrate this, and to suggest further investigations as we do throughout this paper.

2 *The underlying geometry*

From a geometric perspective, the numerical solution of (1) is sterile, for it is one-dimensional and devoid of any interesting geometry. So, to involve geometry, we make the usual change to a first-order linear recurrence relation for vectors: explicitly, we consider the sequence of

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points (x_n, x_{n+1}) in $\mathbb{R}^2$ that are given by

$$\begin{pmatrix} x_{n+1} \\ x_{n+2} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ b & a \end{pmatrix} \begin{pmatrix} x_n \\ x_{n+1} \end{pmatrix}.$$

This change of perspective moves us from the numerical calculations based on (1) to the idea of a dynamical system on $\mathbb{R}^2$ and, in particular, to a study of the iterates of the linear map

$$\begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} 0 & 1 \\ b & a \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \tag{2}$$

which is achieved by applying standard ideas from matrix algebra. A common view of the recurrence relation (1) is that it is a rule for successively finding x_2, then x_3, and so on, from a given pair (x_0, x_1). However, it is equally a rule for finding the 'backward' solution x_{-1}, then x_{-2}, and so on, so the true solution of (1) is the *doubly infinite sequence* $\ldots, x_{-1}, x_0, x_1, \ldots$. This bi-directional view should be reflected in our analysis, and it is because the matrix in (2) is non-singular. This, in turn, suggests that group theory might be useful in this discussion.

A problem can sometimes be better understood by a change of variables that reveals a connection with an apparently unrelated mathematical structure. Such a change is often the first step in making progress, and in this case we shall reveal a connection between recurrence relations, groups and conics. Throughout, we shall assume that $b < 0$, and we leave a discussion of the other cases for the interested reader to explore. We now let $y_n = x_n / \left(\sqrt{|b|}\right)^n$ and $k = a/\sqrt{|b|}$, where $\sqrt{|b|} > 0$, and we find that (1) is equivalent to

$$y_{n+2} + y_n = k y_{n+1}, \tag{3}$$

Now k is real and, replacing y_n by $(-1)^n y_n$ if necessary, we see that we may also suppose that $k > 0$. Since

$$a^2 + 4b = \left(4 - k^2\right) b, \qquad b < 0,$$

we see that if (as before) t_1 and t_2 are the roots of the auxiliary equation of (1), then

(i) t_1 and t_2 are real and distinct if, and only if, $k > 2$;

(ii) t_1 and t_2 are real and coincident if, and only if, $k = 2$;

(iii) t_1 and t_2 are distinct complex conjugates of each other if, and only if, $0 < k < 2$.

In the light of our earlier comments, we rewrite (3) in the form

$$\begin{pmatrix} y_{n+1} \\ y_{n+2} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & k \end{pmatrix} \begin{pmatrix} y_n \\ y_{n+1} \end{pmatrix},$$

and proceed to study the iterates of the map $\Phi : \mathbb{R}^2 \to \mathbb{R}^2$ given by

$$\Phi : \begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} 0 & 1 \\ -1 & k \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y \\ ky - x \end{pmatrix}. \tag{4}$$

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Now the matrix F of Φ can be written as a product of matrices, namely

$$F = \begin{pmatrix} 0 & 1 \\ -1 & k \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ k & -1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

and so we let $\Phi = \psi_2 \psi_1$, where ψ_1 and ψ_2 are given by

$$\psi_1 : \begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y \\ x \end{pmatrix}, \qquad \psi_2 \begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} 1 & 0 \\ k & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ kx - y \end{pmatrix},$$

respectively. Clearly ψ_1 is the reflection across the line $y = x$, but what is the geometry of the map ψ_2? To understand this, let us consider the conic $\mathscr{C}_K$ given by

$$\mathscr{C}_K = \left\{ (x, y) \in \mathbb{R}^2 : x^2 - kxy + y^2 = K \right\},$$

where K is some positive number. We leave the reader to check that $\mathscr{C}_K$ is an ellipse if $0 < k < 2$, a pair of parallel lines if $k = 2$, and a hyperbola if $k > 2$. We also leave the reader to check that if P, say $P = (x, y)$, is a point on $\mathscr{C}_K$, then the points $\psi_1(P)$, $\psi_2(P)$ and $\Phi(P)$ are also on $\mathscr{C}_K$, as are $\psi_1^{-1}(P)$, $\psi_2^{-1}(P)$ and $\Phi^{-1}(P)$. It follows that each of these maps takes $\mathscr{C}_K$ onto itself. Now ψ_2 leaves the first coordinate unchanged so it follows (from geometry, but the reader can also check this algebraically) that we can describe the action of ψ_2 as follows: Given a point P on $\mathscr{C}_K$, let L be the vertical line through P; then $\psi_2(P)$ is the point (other than P unless L meets $\mathscr{C}_K$ at only one point) where L meets the conic $\mathscr{C}_K$. To summarise: If P is a point on $\mathscr{C}_K$, then $\Phi(P)$ is obtained by first reflecting P in the line $y = x$ to obtain a point P', and then moving P' vertically until it meets the other point of $\mathscr{C}_K$ that lies above or below P'. We shall now put these ideas on hold, for what we have just seen is already known in another context which we shall now explain.

3 *Affine geometry*

Since we will use 2×2 non-singular matrices that are not orthogonal matrices, the following discussion takes place in the context of affine, rather than Euclidean, geometry. There is no need to embark on mental gymnastics to motivate affine geometry for, according to Felix Klein, a geometry is a group G of bijections of a set X onto itself, together with those properties of subsets of X that are invariant under the group G. Euclidean geometry consists of the set $\mathbb{R}^2$ with the group generated by all translations and all 2×2 orthogonal matrices, while affine geometry consists of $\mathbb{R}^2$ with the group generated by all translations and all 2×2 non-singular matrices.

Affine geometry is the geometry of vector spaces. A line L through the origin in $\mathbb{R}^2$ is a one-dimensional subspace of $\mathbb{R}^2$, and a line not through the origin, but parallel to L, is a coset, say $a + L$. Under affine maps, cosets map to cosets, so parallel lines map to parallel lines. Also, if a segment σ is mapped to a segment σ', then the midpoint of σ is mapped to the mid-point of σ'. Some quantities are invariant under Euclidean motions but not under affine motions (for example, lengths and angles); on the other hand, we have a far larger supply of affine motions, and therefore more flexibility in affine geometry. A conic $\mathscr{C}$ in $\mathbb{R}^2$ is the set of solutions of some equation of the form $ax^2 + by + cy^2 + dx + ey + f = 0$, and for our purposes it is crucial to note that *an affine map takes one conic to another*. In fact, any two ellipses are affine equivalent to each other, and the same is true for any two hyperbolas, and for any two parabolas. For more details see, for example, [1].

4 *A group structure on a general conic*

The next step is to understand that any non-degenerate conic $\mathscr{C}$ in $\mathbb{R}^2$ can be endowed with a binary operation $\oplus$ such that $(\mathscr{C}, \oplus)$ is a group. A proof of this is given in [2] in the *Gazette*, so here we shall simply give the geometric definition of $\oplus$ (which is also in [2]). We choose any point N on $\mathscr{C}$: this will serve as the identity (or neutral) element in the group $(\mathscr{C}, \oplus)$. Now take any two points say P and Q on $\mathscr{C}$ and form the chord PQ of $\mathscr{C}$. Next, construct the line through N and parallel to PQ; this meets $\mathscr{C}$ at another point R, say, and we define the binary operation $\oplus$ on $\mathscr{C}$ by $P \oplus Q = R = Q \oplus P$. Note that Q is the inverse P^{-1} of P if, and only if, the chord PQ is parallel to the tangent to $\mathscr{C}$ at N. It then follows that $(\mathscr{C}, \oplus)$ is an abelian group with identity element N. It is obvious that $P \oplus N = P = N \oplus P$, and once the equation of $\mathscr{C}$, and the coordinates of P and Q, are known we can obtain an explicit algebraic formula for the coordinates of $P \oplus Q$, and hence for the binary operation $\oplus$ (although we prefer geometric arguments).

Let us illustrate this with the case when $\mathscr{C}$ is the unit circle. We take N to be the point $(1, 0)$, and let $P = \exp(i\theta)$ and $Q = \exp(i\varphi)$. Then (as we leave the reader to prove from Figure 1), $P \oplus Q = \exp\left(i[\theta + \varphi]\right)$. In other words, this group operation on $\{z : |z| = 1\}$ is just the usual multiplication of complex numbers.

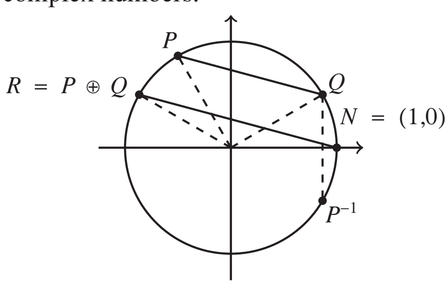

FIGURE 1: The operation $\oplus$ when the conic $\mathscr{C}$ is a circle

Now it is shown in [2] that if an affine map takes a conic $\mathscr{C}_1$ onto a conic $\mathscr{C}_2$ then it transfers (in the obvious sense) the binary operation on $\mathscr{C}_1$ to the corresponding binary operation on $\mathscr{C}_2$. In particular, as every ellipse $\mathscr{E}$ is affinely equivalent to the unit circle in $\mathbb{R}^2$, it follows that the group $(\mathscr{E}, \oplus)$ is isomorphic (indeed, conjugate in affine geometry) to the corresponding group for the unit circle which, as we have just seen, is the group of

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rotations of the circle. Thus, in an obvious sense, we may regard the group $(\mathscr{E}, \oplus)$ as *the group of affine rotations of the ellipse* $\mathscr{E}$ *onto itself*. Here it is essential to use affine, and not Euclidean, geometry.

5 *The geometric action of* Φ *when* $\mathscr{C}_K$ *is an ellipse*

We now return to the discussion in Section 2, and we shall suppose that $0 < k < 2$, so that the conic $\mathscr{C}_K$ is an ellipse which we shall now denote by $\mathscr{E}$. Our aim is to relate the geometric description of Φ given in (4) on $\mathscr{E}$ in terms of the group operation Φ on $\mathscr{E}$: see Figure 2 (where $k = 6/5$). First, we shall take N (the identity in the group) to be the point in the first quadrant where the major axis $y = x$ meets $\mathscr{E}$.

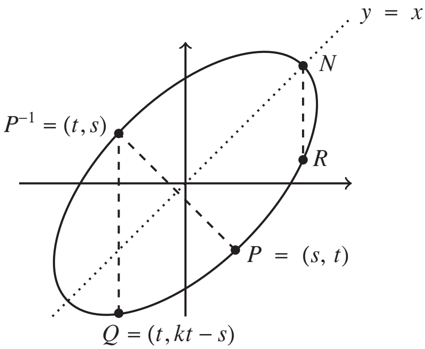

FIGURE 2: $P^{-1} \oplus Q = R$, or $Q = RP$

We now recall the geometric description of the map Φ given in Section 2. We started with any point P on $\mathscr{E}$, and then reflected P in the line $y = x$ to reach a point P'. Since the line joining P to its reflection is parallel to the tangent at N, we find that the reflected point P is the inverse P^{-1} in the group $(\mathscr{E}, \oplus)$. Next we move P^{-1} vertically until it meets $\mathscr{E}$ again at a point which we label Q. It follows immediately (see Section 4) that $P^{-1} \oplus Q = R$, where R is the point where the vertical line through N (which is the identity in the group) meets $\mathscr{E}$. This shows (recall that the group is abelian) that $Q = R \oplus P$, so that the map $\Phi : P \to Q$ is actually the map $P \to R \oplus P$. We have now proved the following result.

Theorem 1: In the notation above, the map Φ is the map $P \to R \oplus P$. In particular, the orbit of P under the forward and backward iterates of Φ is the coset $\langle R \rangle P$ in the group $(\mathscr{E}, \oplus)$, where $\langle R \rangle$ is the cyclic subgroup of $(\mathscr{E}, \oplus)$ that is generated by R.

In simpler terms, this result shows that if we select a starting point (y_0, y_1) for the recurrence relation (3), then this point lies on an ellipse $\mathscr{E}$ that is invariant under Φ, and that the set of points $\{(y_n, y_{n+1}) : n \in \mathbb{Z}\}$ is the set of points $\{R^n \oplus P : n \in \mathbb{Z}\}$ in $\mathscr{E}$.

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6 *Periodicity*

We shall now consider the periodic solutions of the recurrence relation (3), and the periodicity (or the lack of it) of Φ. In general, a function f of a set X onto itself may have points of different periods; for example, the permutation $(1\ 2\ 3)(4\ 5)$ of $\{1, 2, 3, 4, 5\}$ has two points of period two, and three points of period three. In contrast to this, if we ignore the origin, Φ cannot have cycles of different lengths: if *one* point of $\mathscr{E}$ first returns to is original position after m applications of Φ then the same is true of *every* point of $\mathscr{E}$. Indeed, as Φ is given by the matrix F (see (4)), say, it is simply a matter of finding which (if any) integers m are such that F^m is the identity matrix. A standard result in linear algebra says that a square matrix with distinct eigenvalues is diagonalisable, and as F has distinct eigenvalues this essentially solves our problem. However, in the present context it is perhaps more helpful to regard this result as providing the affine map of the ellipse onto a circle. As $0 < k < 2$, Φ has no fixed points on $\mathscr{E}$, so there are no periodic points of period one. We now give a necessary and sufficient condition on θ, where $k = 2\cos\theta$, for Φ to be periodic.

Theorem 2: Let m be a positive integer with $m \geqslant 2$. Then $F^m = I$ (the identity matrix) if, and only if, $\theta = 2\pi q/m$ for some integer q that is coprime with m.

Proof: Since $k = 2\cos\theta = \zeta + 1/\zeta$, where $\zeta = e^{i\theta}$, we see that

$$F = \begin{pmatrix} 0 & 1 \\ -1 & k \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & \zeta + 1/\zeta \end{pmatrix}.$$

Now F has eigenvalues ζ and $1/\zeta$, and these are distinct, as otherwise $k = \zeta$ implies $|k| = 1$, and we have excluded this case. Next, with $U = \begin{pmatrix} 1 & -\zeta \\ -\zeta & 1 \end{pmatrix}$, the reader can check that

$$UFU^{-1} = \begin{pmatrix} \zeta & 0 \\ 0 & 1/\zeta \end{pmatrix}, \qquad UF^mU^{-1} = \begin{pmatrix} \zeta^m & 0 \\ 0 & 1/\zeta^m \end{pmatrix}.$$

It is now clear that $F^m = I$ if, and only if, $UF^mU^{-1} = I$, and this is so if. and only if, $\zeta^m = 1$ or, equivalently, $\theta = 2\pi q/m$ for some integer q.

Theorem 2 shows that if θ/π is irrational, then the successive images of each point of $\mathscr{E}$ move around $\mathscr{E}$, never returning to any point that has been visited before. On the other hand, if θ/π is rational, say $2q/m$ with q and m coprime, then every solution is periodic with period m and so, under repeated applications of the matrix F, each point on the ellipse $\mathscr{E}$ cycles through a finite set of exactly m points on $\mathscr{E}$.

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7 *Chebyshev polynomials*

Although we did not mention this when we introduced (3), our motivation for studying the recurrence relation (3) came from the extremely important class of Chebyshev polynomials, so let us now briefly discuss these. The Chebyshev polynomials are certain solutions of the polynomial recurrence relation

$$p_{n+2}(z) + p_n(z) = 2zp_{n+1}(z), \tag{5}$$

where z is a complex number, and which the reader should compare with (3). Specifically, the Chebyshev polynomials $T_0, T_1, T_2, \ldots$ of the first kind, and the Chebyshev polynomials $U_0, U_1, U_2, \ldots$ of the second kind, are solutions of the recurrence relation (5) generated from the initial terms

$$T_0(z) = 1, \qquad T_1(z) = z;$$
$$U_0(z) = 1, \qquad U_1(z) = 2z.$$

For readers who are not familiar with the Chebyshev polynomials, we note that

$$T_0(z) = 1, \qquad T_1(z) = z, \qquad T_2(z) = 2z^2 - 1, \qquad T_3(z) = 4z^3 - 3z, \qquad \ldots$$

and

$$U_0(z) = 1, \qquad U_1(z) = 2z, \qquad U_2(z) = 4z^2 - 1, \qquad U_3(z) = 8z^3 - 4z, \qquad \ldots .$$

Now consider a sequence $z_0, z_1, \ldots$ defined by

$$z_n = \alpha T_n(w) + \beta U_n(w), \qquad n = 0, 1, \ldots ,$$

where α, β and w are complex numbers that will be defined shortly. Then the z_n satisfy the relations

$$z_0 = \alpha + \beta,$$

$$z_1 = w(\alpha + 2\beta),$$

$$z_{n+2} + z_n = 2wz_{n+1}, \qquad n = 0, 1, \ldots .$$

Now (as we have seen above) any recurrence relation $x_{n+2} = ax_{n+1} + bx_n$ can be reduced (by a change of variable, see (3)) to the form

$$y_{n+2} + y_n = ky_{n+1},$$

so if we put $w = k/2$ and choose α and β such that equations

$$y_0 = \alpha + \beta,$$

$$y_1 = w(\alpha + 2\beta)$$

(this can always be done) then we find that $y_n = z_n$ or, equivalently,

$$y_n = \alpha T_n(\tfrac{1}{2}k) + \beta U_n(\tfrac{1}{2}k). \tag{6}$$

In conclusion, *the Chebyshev polynomial solutions T_n and U_n of the single polynomial recurrence relation* (5) *simultaneously solve all recurrence relations $x_{n+2} = ax_{n+1} + bx_n$* for all choices of a, b, x_0 and x_1.

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In fact, there is a better way to look at Chebyshev polynomials, for the polynomials T_n and U_n are intimately connected to the trigonometric functions by the relations

$$T_n(\cos z) = \cos nz, \qquad U_n(\cos z) = \frac{\sin(n+1)z}{\sin z},$$

and these and other formulae for the Chebyshev polynomials can easily be found in the literature. For example, the trigonometric identity

$$\cos(n+2)z + \cos nz = 2\cos z\cos(n+1)z$$

shows that the T_n satisfy (5). Now, broadly speaking, identities satisfied by the trigonometric functions imply identities between the Chebyshev polynomials, and these then induce *combinatorial identities for the solutions of the most general linear recurrence relation*, and especially those that satisfy the initial conditions $z_0 = 0$ and $z_1 = 1$. We are all familiar with a multitude of identities between the Fibonacci numbers F_n, but it should be more widely known that these are just very special cases of far more general results. Consider, for example, the well-known Cassini identity

$$F_{n+1}F_{n-1} - F_n^2 = (-1)^n.$$

Now for *all complex* θ we have

$$\sin\left([n+2]\theta\right)\sin\left(n\theta\right) = \sin^2\left([n+1]\theta\right)\sin^2\theta. \tag{7}$$

This shows that

$$U_{n+1}\left(\cos\theta\right)U_{n-1}\left(\cos\theta\right) = U_n\left(\cos\theta\right)^2 - 1,$$

and, if we let $\cos\theta = -i/2$ and use the known fact [4] that $F_{n+1} = i^n U_n(-i/2)$, we obtain the Cassini identity for the Fibonacci numbers. However, by taking different values of $\cos\theta$ we obtain Cassini's identity for every other recurrence relation, so Cassini's identity is really (7), and the restriction to Fibonacci numbers is entirely unnecessary. For more information and examples of this type, see [3, 4, 5].

8 *Golden triples*

In this section we comment briefly on the rational points of a conic and some of the associated number theory. In [6] the author defines four sets of integer triples as follows: an integer triple (a, b, c) is

a Pythagorean triple	if $a^2 + b^2 = c^2$;
a golden triple	if $a^2 + ab - b^2 = c^2$;
an Eisenstein triple	if $a^2 - ab + b^2 = c^2$;
an aureate triple	if $a^2 - ab - b^2 = c^2$.

The set $\mathcal{G}$ of golden triples in which $c \neq 0$ is discussed in detail in [6], where it is shown that if we define a multiplication, denoted by $*$, of golden triples by the rule

$$\left(a_1, b_1, c_1\right) * \left(a_2, b_2, c_2\right) = \left(a_1a_2 + b_1b_2, b_1b_2 + a_1b_2 + a_2b_1, c_1c_2\right),$$

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then, with this multiplication, $\mathcal{G}$ is an abelian group with identity $(1, 0, 1)$, and where the inverse of (a, b, c) is $\left(\frac{a+b}{c^2}, -\frac{b}{c^2}, \frac{1}{c}\right)$. Now a triple (a, b, c) of integers with $c \neq 0$ is a golden triple if, and only if, the point $\left(\frac{a}{c}, \frac{b}{c}\right)$ lies on the hyperbola $\mathcal{H}$ whose equation is $x^2 + xy - y^2 = 1$. It is then noted in [6] that the binary relation $*$ on $\mathcal{G}$ induces a binary operation (which we continue to denote by $*$) on $\mathcal{H}$, and then $(\mathcal{H}, *)$ is a group with respect to the operation

$$(a_1, b_1) * (a_2, b_2) = (a_1a_2 + b_1b_2, b_1b_2 + a_1b_2 + a_2b_1).$$

In fact, this operation $*$ on $\mathcal{H}$ given in [6] is precisely the binary operation $\oplus$ on $\mathcal{H}$ that was described earlier in [2], and it is therefore directly available without being introduced through number theory. To verify this it is sufficient to construct an affine map Θ of the hyperbola $\mathcal{H}_0$ onto the hyperbola $\mathcal{H}$, where

$$\mathcal{H}_0 \{(u, v) : uv = 1\}, \qquad \mathcal{H} \{(x, y) : x^2 + xy - y^2 = 1\};$$

and show that the operation $\oplus$ on $\mathcal{H}_0$ transfers to the operation $*$ on $\mathcal{H}$. We have chosen $\mathcal{H}_0$ here because it is parametrised by the points $(t, 1/t)$, where $t \neq 0$, and if $N(1, 1)$ then the operation $\oplus$ on $\mathcal{H}_0$ takes a particularly simple form, namely

$$\left(t, \frac{1}{t}\right) \oplus \left(s, \frac{1}{s}\right) = \left(st, \frac{1}{st}\right)$$

(see [2]). Thus we need to show that

$$\Theta\left(\left(s, \frac{1}{s}\right) \oplus \left(t, \frac{1}{t}\right)\right) = \Theta\left(s, \frac{1}{s}\right) * \Theta\left(t, \frac{1}{t}\right),$$

or, equivalently,

$$\Theta\left(\left(st, \frac{1}{st}\right)\right) = \Theta\left(s, \frac{1}{s}\right) * \Theta\left(t, \frac{1}{t}\right).$$

Let us now construct the map Θ, but first we note that it is more efficient to express $*$ in matrix form, namely

$$(a_1, b_1) * (a_2, b_2) = (a_3, b_3),$$

where

$$a_3 = (a_1, b_1)(a_2, b_2)^t, \qquad b_3 = (a_1, b_1) J (a_2, b_2)^t; \qquad J = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}.$$

and X^t denotes the transpose of X.

Now let

$$\varphi = \frac{1+\sqrt{5}}{2}, \qquad \psi = \frac{1-\sqrt{5}}{2}, \qquad \varphi\psi = -1, \qquad \varphi + \psi = 1,$$

so that φ is the golden ratio. Now consider the map $(x, y) \to (u, v)$ given by $u = x + \varphi y$ and $v = x + \psi y$. Then

$$uv = (x + \varphi y)(x + \psi y) = x^2 + xy - y^2,$$

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and we see that the inverse map $(u, v) \to (x, y)$, which we take to be our map Θ, takes the hyperbola $\mathcal{H}_0$ bijectively onto the hyperbola $\mathcal{H}$. It is best to write this in matrix form, namely

$$(x, y) = \Theta(u, v) = (u, v)M, \qquad (u, v) = \Theta^{-1}(x, y) = (x, y)M^{-1},$$

where

$$M = \frac{1}{\sqrt{5}}\begin{pmatrix} -\psi & 1 \\ \varphi & -1 \end{pmatrix}, \qquad M^{-1} = \begin{pmatrix} 1 & 1 \\ \varphi & \psi \end{pmatrix}.$$

It follows that we need to show that

$$\left(st, \frac{1}{st}\right)M = \left(s, \frac{1}{s}\right)M * \left(t, \frac{1}{t}\right)M,$$

or, equivalently, that

$$\left(st, \frac{1}{st}\right)M = (U, V)$$

$$U = \left(s, \frac{1}{s}\right)MM^t\left(t, \frac{1}{t}\right)^t;$$

$$V = \left(s, \frac{1}{s}\right)MJM^t\left(t, \frac{1}{t}\right)^t.$$

The verification of these equations is a matter of simple algebra which we leave to the reader.

Finally, the discussion in [6] is concerned with solutions of various Diophantine equations, and the fact that these are related to the rational points on a conic. It is important to note that rational points on a conic are not necessarily transferred to rational points under an affine map (see [2]), and, with this in mind, it seems worthwhile to record the next result (which, although expressed here in a slightly different form, is Lemma 2.1 in [7]).

Theorem 3: Let (A, B, C), where $C \neq 0$, be an integral solution of the Diophantine equation

$$\alpha X^2 + \beta XY + \gamma Y^2 = \delta Z^2, \tag{8}$$

where α, β, γ and δ are integers with $\beta^2 \neq 4\alpha\gamma$ and $\delta \neq 0$. Then any integral solution (X, Y, Z) of (8) is a rational multiple of some vector (A_1, B_1, C_1), where m and n are some coprime integers with $m \geqslant 0$, and where

$$A_1 = \gamma n(An - 2Bm) - (\alpha A + \beta B)m^2;$$

$$B_1 = \alpha m(Bm - 2An) - (\gamma B + \beta A)n^2;$$

$$C_1 = \pm C\left(\alpha m^2 + \beta mn + \gamma n^2\right).$$

If $\alpha = \delta = \gamma = 1$ and $\beta = 0$, we obtain $X^2 + Y^2 = Z^2$, and we find that every Pythagorean triple is a rational multiple of some vector

$$\left(n^2 - m^2,\ 2mn,\ \pm\left(m^2 + n^2\right)\right),$$

where m and n are coprime. It is then easy to see that the Pythagorean triple

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is actually an integer multiple of this vector, and so we see that the familiar result on Pythagorean triples is just one of a whole family of similar results. It is noteworthy that the proof of Theorem 3 in [7] is entirely elementary, and consist of nothing more than finding where a line with rational coordinates, and that passes through the point (A, B, C), meets the given conic. Thus this result is in the same spirit, and at the same level of difficulty, as the construction of the group operation $\oplus$.

9 *Lyness cycles*

We end with the briefest of notes about another circumstance in which the recurrence relation (3) seems to make an appearance. In the recent paper [8] in the *Gazette*, Stan Dolan discusses the idea of a Z-curve, and a corresponding 'unfolding map', associated with a recurrence relation

$$x_{n+2} + x_n = f(x_{n+1}),$$

where f is some suitable (but fairly general) function. These ideas can be found in some earlier unpublished notes by Christopher Zeeman on Lyness cycles (see [8]), and the author thanks Stan Dolan for his valuable contributions during many discussions on parts of this paper.

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