A LEMMA ABOUT THE EPSTEIN ZETA-FUNCTION

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1. Let $h(m, n) = \alpha m^2 + 2\delta mn + \beta n^2$ be a positive definite quadratic form with determinant $\alpha\beta - \delta^2 = 1$. It may be put in the shape

$$h(m, n) = y^{-1}\{(m+nx)^2 + n^2y^2\}$$

with $y > 0$. We write (for $s > 1$)

$$G(x, y)(s) = Z_h(s) = \sum_{m=-\infty}^{\infty} \sum_{\substack{n=-\infty \\ (m,n)\neq(0,0)}}^{\infty} h(m, n)^{-s}.$$

The function $Z_h(s)$ may be analytically continued over the whole s-plane. Its only singularity is a simple pole with residue π at $s = 1$.

Let s be a fixed real number ($\neq 1$). In [**1**] it is shown that the partial derivative $G_x(x, y)$ with respect to x of the function $G(x, y)$ can be written

$$G_x(x, y) = -\frac{16\pi^{s+1}y^{\frac{1}{2}}}{\Gamma(s)}\Lambda,$$

where

$$\Lambda = \int_0^\infty \psi(\delta_t) \cosh (s-\tfrac{1}{2})t \, dt$$

with

$$\delta_t = e^{-2\pi y \cosh t}$$

and

$$\psi(\delta) = \sum_{r=1}^{\infty} r^{s+\frac{1}{2}} \sigma_{1-2s}(r)\, \delta^r \sin 2\pi rx.$$

Here

$$\sigma_\alpha(r) = \sum_{d \mid r} d^\alpha,$$

where the sum is to be taken over all positive divisors of the natural number r.

Now we have ([**1**], p. 75, Lemma 2)

LEMMA A. *$G_x(x, y) < 0$ for $0 < s \leqq 3$, $y \geqq \frac{3}{5}$ and $0 < x < \frac{1}{2}$.*

In order to prove Lemma A it is sufficient to show that the following is true.

LEMMA B. *For $0 < s \leqq 3$, $0 < \delta < 40^{-1}$, $0 < x < \frac{1}{2}$, we have $\psi(\delta) > 0$.*

The proof of Lemma B given in [1] is not correct.† (The lower bound of ω_{d_0} on p. 77 does not follow from the condition $\frac{1}{4} \leqq d_0 x < \frac{1}{2}$.) The purpose of this paper is to give a proof of Lemma B.

2. Put

$$\omega_d = \sum_{f=1}^{\infty} f^{s+\frac{1}{2}} \delta^{df} \sin 2\pi d f x.$$

Then

$$\psi(\delta) = \sum_{d=1}^{\infty} d^{\frac{1}{2}-s} \omega_d.$$

For $|z| < 1$, we have

$$\sum_{n=1}^{\infty} n^3 z^n = (z+4z^2+z^3)(1-z)^{-4}, \tag{1}$$

$$\sum_{n=1}^{\infty} n^4 z^n = (z+11z^2+11z^3+z^4)(1-z)^{-5}, \tag{2}$$

$$\sum_{n=1}^{\infty} n^5 z^n = (z+26z^2+66z^3+26z^4+z^5)(1-z)^{-6}, \tag{3}$$

$$\sum_{n=1}^{\infty} \tfrac{1}{2} n(n+1) z^n = z(1-z)^{-3}. \tag{4}$$

For $d \geqq 2$, we have, by (2),

$$\begin{aligned} |\omega_d| &\leqq \sum_{f=1}^{\infty} f^{\frac{7}{2}} \delta^{df} \leqq \sum_{f=1}^{\infty} f^4 \delta^{df} \\ &< \delta^d (1+11\,.\,40^{-2}+11\,.\,40^{-4}+40^{-6})(1-40^{-2})^{-5} < 1{\cdot}02\delta^d. \end{aligned} \tag{5}$$

Put

$$u = \pi - 2\pi x. \tag{6}$$

Then, for $d \geqq 2$, we also have, by (3),

$$\begin{aligned} |\omega_d| &\leqq \sum_{f=1}^{\infty} f^{s+\frac{1}{2}} \delta^{df} |\sin dfu| < du \sum_{f=1}^{\infty} f^{s+\frac{3}{2}} \delta^{df} < du \sum_{f=1}^{\infty} f^5 \delta^{df} \\ &< \delta^d du (1+26\,.\,40^{-2}+66\,.\,40^{-4}+26\,.\,40^{-6}+40^{-8})(1-40^{-2})^{-6} \\ &< 1{\cdot}03\, \delta^d du. \end{aligned} \tag{7}$$

On applying partial summation, we obtain

$$4\omega_d \sin^2(\pi d x) = \sum_{f=1}^{\infty} g_f\{(f+1)\sin 2\pi d x - \sin 2\pi (f+1) d x\}, \tag{8}$$

where

$$g_f = f^{s+\frac{1}{2}} \delta^{df} - 2(f+1)^{s+\frac{1}{2}}\, \delta^{d(f+1)} + (f+2)^{s+\frac{1}{2}} \delta^{d(f+2)}. \tag{9}$$

† It has been pointed out by Professor G. Emersleben that, according to the original paper of Epstein [2], all the Zeta-functions have the value -1 at $s=0$. Therefore $G_x(x, y)$ and $G_y(x, y)$ vanish at $s=0$, so that Statement R and Lemmas 1 and 2 in [1] hold true only for $s>0$ and not for $s \geqq 0$. This fact can, of course, also be seen for example from the expression (4) in [1], since $1/\Gamma(s)=0$ for $s=0$.

3. We suppose first that

$$0 < x \leqq \tfrac{3}{8}.$$

Clearly, there exists a natural number d_0 such that

$$\tfrac{1}{8} \leqq d_0 x \leqq \tfrac{3}{8}. \tag{10}$$

Now it is easy to see that $g_f > 0$. (See [1], p. 76, (12).) By (8), this implies that $\omega_d > 0$ for $0 < dx < \frac{1}{2}$. Hence $\omega_d > 0$ for $d \leqq d_0$, so that

$$\psi(\delta) \geqq \sum_{d=d_0}^{\infty} d^{\frac{3}{2}-s}\omega_d \geqq d_0^{\frac{3}{2}-s}\omega_{d_0} - \sum_{d=d_0+1}^{\infty} d^{\frac{3}{2}-s} \,|\, \omega_d \,| \,. \tag{11}$$

We shall now determine a lower bound for ω_{d_0}. By (10),

$$\sin 2\pi d_0 x \geqq 2^{-\frac{1}{2}},$$

so that (8) and (9) imply

$$\begin{aligned} \omega_{d_0} &\geqq \omega_{d_0} \sin^2 (\pi d_0 x) \\ &= \tfrac{1}{4} \sum_{f=1}^{\infty} g_f \{(f+1) \sin 2\pi\, d_0 x - \sin 2\pi(f+1)\, d_0 x\} \\ &\geqq \tfrac{1}{4} \sum_{f=1}^{\infty} g_f \{(f+1) 2^{-\frac{1}{2}} - 1\} \\ &= \tfrac{1}{4}(2^{\frac{1}{2}} - 1)\, \delta^{d_0} + \tfrac{1}{4}(1 - 2^{-\frac{1}{2}}) 2^{s+\frac{1}{2}}\, \delta^{2d_0} \\ &> \tfrac{1}{4}(2^{\frac{1}{2}} - 1)\, \delta^{d_0} \\ &> 0{\cdot}1\, \delta^{d_0}. \end{aligned} \tag{12}$$

By (5), (11), (12), we have

$$d_0^{s-\frac{3}{2}}\, \delta^{-d_0} \psi(\delta) > 0{\cdot}1 - 1{\cdot}02 \sum_{d=d_0+1}^{\infty} \left(\frac{d}{d_0}\right)^{\frac{3}{2}-s} \delta^{d-d_0} \,.$$

Here

$$\left(\frac{d}{d_0}\right)^{\frac{3}{2}-s} < \left(\frac{d}{d_0}\right)^{\frac{3}{2}} \leqq 2^{\frac{3}{2}}(d-d_0)(d-d_0+1).$$

Hence we obtain, by (4),

$$\begin{aligned} d_0^{s-\frac{3}{2}} \delta^{-d_0} \psi(\delta) &> 0{\cdot}1 - 1{\cdot}02 \,.\, 2^{\frac{3}{2}} \sum_{k=1}^{\infty} \tfrac{1}{2} k(k+1)\, \delta^k \\ &= 0{\cdot}1 - 1{\cdot}02 \,.\, 2^{\frac{3}{2}}\, \delta(1-\delta)^{-3} \\ &> 0{\cdot}1 - 1{\cdot}02 \,.\, 2^{\frac{3}{2}} \,.\, 40^{-1}(1-40^{-1})^{-3} \\ &> 0. \end{aligned}$$

4. We suppose now that

$$\tfrac{3}{8} < x < \tfrac{1}{2}.$$

We shall use the notation (6). Then

$$\sin 2\pi x = \sin u > u\left(1-\frac{u^2}{6}\right) > u\left(1-\frac{\pi^2}{96}\right) > 0{\cdot}89\, u.$$

Using this estimate we obtain

$$\begin{aligned}\omega_1 &\geqq \delta \sin u - \sum_{f=2}^{\infty} f^{s+\frac12}\delta^f \,|\sin fu\,| \\ &> 0{\cdot}89\,\delta u - \sum_{f=2}^{\infty} f^{s+\frac12}\,\delta^f fu \\ &> \delta u\left\{0{\cdot}89 - 2^{-\frac12}\sum_{f=2}^{\infty} f^5\delta^{f-1}\right\}. \qquad (13)\end{aligned}$$

Here we have, by (3),

$$\begin{aligned}2^{-\frac12}\sum_{f=2}^{\infty} f^5\delta^{f-1} &= 2^{-\frac12}\{40(40^{-1}+26\,.\,40^{-2}+66\,.\,40^{-3}+26\,.\,40^{-4}+40^{-5})(1-40^{-1})^{-6}-1\} \\ &< 2^{-\frac12}\{(1+28\,.\,40^{-1})(1-6\,.\,40^{-1})^{-1}-1\} = 2^{-\frac12} < 0{\cdot}71.\end{aligned}$$

On substituting this estimate in (13) we obtain

$$\omega_1 > 0{\cdot}18\,\delta u. \qquad (14)$$

On the other hand, we have, by (1) and (7),

$$\begin{aligned}\sum_{d=2}^{\infty} d^{\frac12-s}\,|\,\omega_d\,| &< 1{\cdot}03\,\delta u \sum_{d=2}^{\infty} d^{\frac12-s}\,\delta^{d-1} \\ &< 1{\cdot}03\,.\,2^{-\frac12}\,\delta u \sum_{d=2}^{\infty} d^3\delta^{d-1} \\ &< 1{\cdot}03\,.\,2^{-\frac12}\,\delta u\{40(40^{-1}+4\,.\,40^{-2}+40^{-3})(1-40^{-1})^{-4}-1\} \\ &< 1{\cdot}03\,.\,2^{-\frac12}\,.\,0{\cdot}22\,\delta u \\ &< 0{\cdot}17\,\delta u. \qquad (15)\end{aligned}$$

The assertion follows from (14) and (15).

REFERENCES

1. J. W. S. Cassels, On a problem of Rankin about the Epstein Zeta-function, *Proc. Glasgow Math. Assoc.* **4** (1959), 73–80.

2. P. Epstein, Zur Theorie allgemeiner Zetafunktionen, *Math. Ann.* **56** (1903), 615–644.

3. R. A. Rankin, A minimum problem for the Epstein Zeta-function, *Proc. Glasgow Math. Assoc.* **1** (1953), 149–158.

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