An Asymptotic Formula for a Class of Distribution Functions

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If $x_1, x_2, \ldots, x_k, \ldots$ are independent random variables each of which is subjected to a distribution law $\sigma = \sigma(x)$ independent of k and having a finite positive dispersion, then $x_1 + x_2 + \ldots + x_n$ is known to obey the Gauss law as $n \to +\infty$, no matter how $\sigma(x)$ be chosen[1]. There arises, however, the question whether it is nevertheless possible to determine the elementary law $\sigma(x)$ from the asymptotic behaviour of the distribution law of $x_1 + x_2 + \ldots + x_n$ for *very large but finite* values of n. It will be shown that the answer is affirmative under very general conditions.

Let the distribution function $\sigma(x)$ be a solution of the moment problem

$$(1) \qquad \int_{-\infty}^{+\infty} x^m \, d\sigma(x) = M_m \quad (m = 0, 1, 2, \ldots; \sigma(-\infty) = 0),$$

so that M_0 is the total probability, hence equal to 1. It is not required that (1) be a determined moment problem, *i.e.* that σ be uniquely determined by the conditions (1) if one normalises it by the requirement that $2\sigma(x) = \sigma(x+0) + \sigma(x-0)$. On excluding the case $M_2 = 0$ of the trivial distribution function $\sigma(x) = \frac{1}{2}(1 + \text{sign}\, x)$ and replacing, if necessary, $\sigma(x)$ by $\sigma(ax)$, where $a = M_2^{\frac{1}{2}} > 0$, we may suppose that $M_2 = 1$. Also, although the symmetry condition thereby imposed upon the law σ is not essential for our method, we assume for convenience that both ranges $(0, x)$ and $(-x, 0)$ of the random variable subjected to σ are equally probable, *i.e.* that

$$(2) \quad \sigma(x) + \sigma(-x) = 1; \text{ hence } \sigma(0) = \tfrac{1}{2}, \; M_{2n+1} = 0 \quad (n = 0, 1, 2, \ldots).$$

Accordingly, the characteristic function of σ,

$$(3') \qquad L(t; \sigma) = \int_{-\infty}^{+\infty} e^{itx} \, d\sigma(x) \quad (-\infty < t < +\infty),$$

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[1] *Cf.* P. Lévy, *Calcul des Probabilités*, Paris, 1925, pp. 233-235.

and its Fourier inversion[1],

$$(4') \quad \sigma(x) = \sigma(0) - \frac{1}{2\pi i}\int_{-\infty}^{+\infty} L(t;\sigma)(e^{-ixt}-1)\,t^{-1}\,dt \quad (-\infty < x < +\infty),$$

become respectively

$$(3) \qquad L(t;\sigma) = 2\int_0^{+\infty} \cos(tx)\,d\sigma(x) \qquad (-\infty < t < +\infty)$$

and

$$(4) \qquad \sigma(x) = \tfrac{1}{2} + \frac{1}{\pi}\int_0^{+\infty} L(t;\sigma)\,t^{-1}\sin(tx)\,dt.$$

We finally suppose that for some sufficiently small $\delta > 0$ and for some function $\phi(t)$

$$(5) \qquad \int^{+\infty}\{t\,\phi(t)^{1/\delta}\}^{-1}\,dt < +\infty\,, \text{ and } L(t;\sigma) = O(1/\phi(t)) \text{ as } t \to +\infty\,,$$

consideration of $t \to -\infty$ being unnecessary since $L(t;\sigma)$ is an even function, and that

$$(5a) \qquad L(t;\sigma) \to 0 \text{ as } t \to \infty\,.$$

A few remarks concerning the nature of the restriction imposed by conditions (5) and (5a) upon the behaviour of $\sigma(x)$ are not out of place. According to Lévy[2] the average of $|L(t;\sigma)|^2$ in the whole range $-\infty < t < +\infty$ always exists and is equal to the sum of the squares of all jumps of $\sigma(x)$. Hence $\sigma(x)$ is everywhere continuous if and only if the average of $|L(t;\sigma)|^2$ is zero, a condition clearly satisfied whenever (5a) is satisfied, so that σ has no discontinuity points. However (5), (5a) are sufficiently general not to require the absolute continuity of σ, *i.e.* the existence of a density of probability[3]. In fact (5) and (5a) are implied by

$$(5b) \qquad L(t;\sigma) = O(|\log t|^{-\alpha}),$$

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[1] P. Lévy, *op. cit.*, p. 167.

[2] P. Lévy, *op. cit.*, p. 171.

[3] There exists a derivative $\sigma'(x)$ up to a set of measure zero even if $\sigma(x)$ is not absolutely continuous, but

$$\text{(i)} \qquad \sigma(x) = \int_{-\infty}^{x} \sigma'(y)\,dy$$

holds if and only if $\sigma(x)$ is absolutely continuous. It is meaningless to regard $\sigma(x)$ as a density of probability if (i) is not valid.

where $\alpha > 0$ may be arbitrarily small, and there exist[1] symmetric distribution functions which satisfy (5b) but are not absolutely continuous. Conversely, the absolute continuity of σ does not imply (5) since the Riemann-Lebesgue lemma cannot be formulated by using a universal majorant which tends to zero. A sufficient condition for (5b), hence for (5) and (5a), is that there exist a density of probability satisfying a uniform Lipschitz condition of arbitrarily low index, or only the corresponding logarithmical estimate, and tending not too slowly to zero as $x \to \infty$. Another sufficient condition for (5) and (5a) is that σ satisfy the Gauss postulate for error distributions, *i.e.*, that there exist for every x a probability density which does not increase when x increases. In fact, in this case it is clear from (3), in virtue of the second mean-value theorem, that $L(t;\sigma) = O(t^{-1})$, so that (5b) is amply satisfied.

Let the random variables $x_1, x_2, \ldots, x_k, \ldots$ be such that $\sigma(x)$ represents the probability of the inequality $x_k < x$ for every k. Then if $\sigma_n(x)$ denotes the probability of the inequality $x_1+x_2+x_3+\ldots+x_n < x$, we have

$$L(t;\sigma_n) = L(t;\sigma)^n \tag{6}$$

in virtue of the supposed independence of the random variables[2]. The fundamental limit theorem of the calculus of probability[3] implies that the distribution function $\sigma(a_n x)$, where

$$a_n = M_2(\sigma_n)^{\frac{1}{2}} = n^{\frac{1}{2}} M_2^{\frac{1}{2}} = n^{\frac{1}{2}},$$

tends, as $n \to +\infty$, to the reduced Gaussian distribution function. Our purpose is to show that $\sigma_n(x)$ is capable of an infinite asymptotic development in the Poincaré sense, proceeding according to powers of $n^{-\frac{1}{2}}$. The rôle of assumption (5) is that of assuring the existence of such a development, formal treatments of which date back to Laplace[4]. The coefficient of $(n^{-\frac{1}{2}})^m$ in the asymptotic series in question is a polynomial in x having as coefficients polynomials in the moments (1) of the elementary law σ, and the coefficients of the

[1] *Cf.* D. Menchoff, "Sur l'unicité du développement trigonométrique," *Comptes Rendus*, 163 (1916), pp. 433-436.

[2] *Cf.*, *e.g.*, P. Lévy, *op. cit.*, pp. 184-185.

[3] *Ibid*, pp. 233-235.

[4] *Cf.*, *e.g.*, E. T. Whittaker and G. Robinson, *The Calculus of Observations*, London, 1924, p. 172; *cf.* also F. Zernike, *Handbuch der Physik*, 3 (1928) 450-51, where further references are also given.

latter polynomials are universal constants. The elementary laws occurring in the majority of applications satisfy (5) and are such that the Carleman condition

$$\sum_{m=0}^{+\infty} M_{2m}^{-1/(2m)} = +\infty$$

of determinateness is fulfilled. Hence we obtain a method, at least in theory, for determining the elementary law $\sigma(x)$ from the behaviour of the approximation of the iterated law to the Gauss distribution.

The function $L(t;\sigma)$ has for every t derivatives of arbitrarily high order[1] which may be obtained by formal differentiation of (3), so that

$$L^{(m)}(t;\sigma) = i^m \int_{-\infty}^{+\infty} x^m e^{itx} d\sigma(x). \tag{7}$$

In fact, each of the integrals (7) is uniformly convergent with respect to t, its integrand having as a majorant[2] that of M_m.

It is clear from (3) that $|L(t;\sigma)| \leqq 1$ for every t and $= 1$ for $t = 0$. Suppose that $|L(t;\sigma)| = 1$ for a fixed t. Then

$$\int_0^{+\infty} \{1 \pm \cos(tx)\} d\sigma(x) = 0,$$

where $1 \pm \cos(tx) \geqq 0$ for every x and either $t=0$ or else $1 \pm \cos(tx) > 0$ for some x. Hence either $t = 0$ or else $\sigma(x)$ is a step-function having all its jumps at points x which form an arithmetical progression. The second case is excluded, σ being continuous in virtue of (5). Consequently[3]

$$|L(t;\sigma)| < 1 \quad \text{for every } t \neq 0. \tag{8}$$

Moreover, since the second derivative of (3) is negative at $t = 0$ in virtue of (7), and the first derivative $L'(t;\sigma)$ vanishes at $t = 0$ because of (7) and (2), we have $L'(t;\sigma) < 0$ for sufficiently small values of $t > 0$. It follows therefore from $L(0;\sigma) = 1$ that $L(t;\sigma)$ is positive and decreasing in the interval $0 < t \leqq c$ if c is sufficiently small. Let c be so chosen and put

$$K_n = \int_c^{+\infty} |t^{-1} L(t;\sigma)^n| dt. \tag{9}$$

[1] It is not true, however, that $L(t;\sigma)$ is necessarily regular-analytic along the t-axis.

[2] In virtue of the Schwarz inequality it is sufficient to consider even values of m.

[3] It may be mentioned that (8) is actually false in the second case. In fact, $L(t;\sigma)$ is then a periodic function so that $L(t;\sigma) = 1$ holds for some $t \neq 0$ since it holds for $t = 0$.

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Now $L(t;\sigma)$ has in the interval $c \leqq t < +\infty$ a positive maximum $\theta < 1$ according to (8) and (5a). On the other hand, it is clear from (5) and (9) that $K_j < +\infty$ if j is sufficiently large, so that $K_n < \theta^{n-j} K_j < +\infty$ for every $n > j$. Consequently

$$(10) \qquad \left| \int_c^{+\infty} L(t;\sigma)^n t^{-1} \sin(tx)\, dt \right| < C\theta^n, \text{ where } 0 < \theta < 1,$$

for every x and for every $n > j$, where θ and $C = K_j/\theta^j$ depend only upon c.

Since $L(0;\sigma) = 1$ and $L(t;\sigma)$ is positive and decreasing in the range $0 < t \leqq c$, the function

$$(11) \qquad s = s(t) = \{-\log L(t;\sigma)\}^{\frac{1}{2}}$$

is positive and increasing in this range so that there exists an inverse function $t = t(s)$, where $0 \leqq s \leqq d$ and $d = \{-\log L(c;\sigma)\}^{\frac{1}{2}}$. Now the derivative $L'(t;\sigma)$ is negative at every point of the range $0 < t \leqq c$ and vanishes at $t = 0$ only in the first order in virtue of $L''(0;\sigma) = -1$; hence the function $s = s(t)$ vanishes at $t = 0$ exactly in the first order, and consequently $r(t) = t/s(t)$ is positive at $t = 0$. Upon placing, for a fixed value of x,

$$(12) \qquad \pi f(x;s) = \sin(xt(s))\, \dot{t}(s)/t(s) \qquad (0 \leqq s \leqq d),$$

where the dot denotes differentiation with respect to s, it follows from the Bürmann-Lagrange rule[1] that all derivatives of $f(x;s)$ with respect to s exist not only in the range $0 < s \leqq d$, but at $s = 0$ as well, and, moreover, that the derivatives are given by the explicit formula

$$(13) \qquad \left\{\frac{\partial^n f(x;s)}{\partial s^n}\right\}_{s=0} = \frac{1}{\pi}\left\{\frac{\partial^n}{\partial t^n}\frac{\sin(tx)\, r(t)^{n+1}}{t}\right\}_{t=0}.$$

Setting

$$\chi_n(x) = \tfrac{1}{2} + \frac{1}{\pi}\int_0^c L(t;\sigma)^n t^{-1} \sin(tx)\, dt,$$

we have from (11) and (12)

$$(14) \qquad \chi_n(x) = \tfrac{1}{2} + \int_0^d \exp(-ns^2) f(x;s)\, ds.$$

This function $\chi_n(x)$ admits for every fixed x an asymptotic development[2]

$$(15) \qquad \tfrac{1}{2} + \sum_{k=1}^{+\infty} P_k(x)\, n^{-\frac{1}{2}k},$$

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[1] *Cf.* P. L. Tchebychef, *Oeuvres*, vol. 1, St. Pétersburg, 1899, pp. 251-270, where analyticity of the functions is not required.

[2] *Cf.* A. Wintner, "On the asymptotic formulae of Riemann and of Laplace," *Proceedings of the National Academy of Sciences*, 20 (1934), pp. 57-62.

where

$$(16) \qquad P_k = P_k(x) = \frac{\Gamma(\frac{1}{2}k)}{2\,\Gamma(k)} \left\{\frac{\partial^{k-1} f(x;\, s)}{\partial s^{k-1}}\right\}_{s=0}$$

or

$$(17) \qquad P_k = P_k(x) = \frac{\Gamma(\frac{1}{2}k)}{2\pi\,\Gamma(k)} \left\{\frac{\partial^{k-1}}{\partial t^{k-1}} \frac{\sin(tx)\, r(t)^k}{t}\right\}_{t=0}.$$

Thus

$$(18) \quad P_{2k+1} = \frac{\Gamma(k+\frac{1}{2})}{2\pi\,\Gamma(2k+1)} \sum_{\nu=0}^{k} \binom{2k}{2\nu} (-1)^{k-\nu} \frac{x^{2(k-\nu)+1}}{2(k-\nu)+1} \left\{\frac{d^{2\nu} r(t)^{2k+1}}{dt^{2\nu}}\right\}_{t=0}$$

by the Leibniz rule for differentiation, while $P_{2k} = 0$ for every x because of (2). In particular

$$P_1(x) = (2\pi)^{-\frac{1}{2}} x,$$

$$P_3(x) = (2\pi)^{-\frac{1}{2}} \left\{-\frac{x_3}{6} + (M_4 - 3)\,\frac{x}{8}\right\},$$

$$P_5(x) = (2\pi)^{-\frac{1}{2}} \left\{\frac{x^5}{40} - 5(M_4 - 3)\frac{x^3}{48} + (35M_4^2 - 8M_6 - 90M_4 + 75)\frac{x}{384}\right\}.$$

It is clear from (10) that the above asymptotic expansion of (14) is also an asymptotic development of

$$(19) \qquad \tfrac{1}{2} + \frac{1}{\pi} \int_0^{+\infty} L(t;\, \sigma)^n\, t^{-1} \sin(tx)\, dt.$$

Moreover, upon applying (4) to $\sigma_n(x)$ instead of $\sigma(x)$, we see from (6) that (19) is exactly $\sigma_n(x)$. Therefore (15) is an asymptotic development of $\sigma_n(x)$.

It may be mentioned that (15) can in certain cases be a convergent series. For example, if $\sigma(x)$ obey the Gauss law the asymptotic development (15) for $\sigma_n(x)$ is found from (18) to be the convergent power-series representation of $\sigma_n(x)$.