BULL. AUSTRAL. MATH. SOC. 34A10, 34G05
VOL. 15 (1976), 421-430.

Existence of solutions to quasilinear differential equations in a Banach space

James R. Ward

Initial value problems of the form $x' + A(t, x)x = f(t, x)$, $x(0) = a$, $t \geq 0$, are considered in a real, separable, reflexive Banach space. Results concerning the existence of solutions on $[0, \infty)$ are given by considering linear systems of the form $x' + A\big(t, u(t)\big)x = f\big(t, u(t)\big)$. Here $u(t)$ belongs to a suitable function space.

1. Introduction

Let X be a real, separable, reflexive Banach space, with dual X^*. The duality mapping $F : X \to 2^{X^*}$ maps every $x \in X$ to the set $F(x) \in 2^{X^*}$ such that $(x, f) = \|f\|^2 = \|x\|^2$ for $f \in F(x)$, where (x, f) denotes the value of $f \in X^*$ at $x \in X$, and $\|\cdot\|$ denotes the norm in X or X^*. Let A be an operator defined on a set $D = D(A) \subseteq X$ mapping D into X. A is said to be accretive if for every $u, v \in D$ there is $f \in F(u-v)$ such that

$$(Au-Av, f) \geq 0 .$$

The purpose of this paper is to establish the existence of "strong" solutions to the "quasilinear" initial value problems

$$(1.1) \qquad x' + A(t, x)x = 0 , \quad t \in [0, \infty) , \quad x(0) = a ,$$

$$(1.2) \qquad x' + A(t, x)x = f(t, x) , \quad t \in [0, \infty) , \quad x(0) = a ,$$

Received 21 July 1976.

where for each $(t, u) \in [0, \infty) \times X$, $A(t, u)$ is a bounded linear accretive operator on X (in this case $A(t, u)$ is said to be a positive linear operator).

The question of existence of solutions for the related problems

$$x' + A(t)x = 0 , \quad t \in [0, T] , \quad x(0) = a , \tag{1.3}$$

$$x' + A(t)x = f(t, x) , \quad t \in [0, T] , \quad x(0) = a , \tag{1.4}$$

where $A(t)$ is bounded for each $t \in [0, T]$, is discussed in the book [2] by Daleckiĭ and Kreĭn; the case in which $A(t)$ is a possibly unbounded accretive operator has been considered by Browder in [1] and Kato in [7], among others. Fitzgibbon in [3] considered (1.3) in the case $A(t)$ is a (non-linear), accretive, weakly continuous operator. The linearization approach used here for (1.1), (1.2) is closely related to that of Kartsatos and Ward [6] for finite dimensional problems.

It would be of great interest to obtain results for (1.1) and (1.2) in case $A(t, u)$ is for each (t, u) an unbounded operator on $D \subseteq X$. The author hopes to be able to present results for this problem in a future paper. Results concerning uniqueness of solutions to (1.1) or (1.2) would also be of interest; we have none to present here.

We wish to make clear the notion of solution used here. By a (strong) solution to (1.1) $\big((1.2)\big)$ we shall mean a function $u : [0, \infty) \to X$ which is strongly continuous, locally lipschitzian, strongly differentiable almost everywhere on $[0, \infty)$, and such that $u(0) = a$ and $u(t)$ satisfies (1.1) $\big((1.2)\big)$ almost everywhere in $[0, \infty)$.

2. Main results

Our first result concerns the initial value problem (1.1); we let $\to$ $(\rightharpoonup)$ denote strong (weak) convergence in X .

THEOREM 2.1. *Assume for each* $(t, u) \in [0, \infty) \times X$, $A(t, u)$ *is a bounded linear accretive operator on* X *and the map* $(t, u, v) \to A(t, u)v$ *is weakly continuous, that is, if* $t \to t_0$, $u \rightharpoonup u_0$, $v \rightharpoonup v_0$, *then* $A(t, u)v \rightharpoonup A\big(t_0, u_0\big)v_0$; *then for each* $a \in X$ *there exists a solution* $x(t)$ *to the initial value problem* (1.1). *Moreover,* $\|x(t)\| \le \|a\|$ *for all* $t \ge 0$.

https://doi.org/10.1017/S0004972700022851 Published online by Cambridge University Press

We will prove Theorem 2.1 after first presenting some preliminary results. Our method will be to first show that for each $u(t)$ in a certain class of X-valued functions there is a unique solution to the linear problem

$$x_u'(t) + A\big(t, u(t)\big)x_u(t) = 0 \ , \quad t \in [0, \infty) \ , \quad x_u(0) = a \ .$$

We then prove the existence of a solution to (1.1) by showing the existence of a fixed point for the mapping $u \to x_u$. This approach was suggested by Kartsatos in [5]. We first show

LEMMA 2.1. *Let $B(t)$ be a bounded linear operator on X for each $t \in [0, \infty)$, and assume the mapping $(t, u) \to B(t)u$ to be weakly continuous. Then for each $a \in X$ there is a unique strongly continuous solution to the initial value problem*

$$x'(t) + B(t)x(t) = 0 \ , \quad t \in [0, \infty) \ , \quad x(0) = a \ . \tag{2.1}$$

Proof. Let $C(R^+, X)$ be the linear space of strongly continuous functions mapping $[0, \infty) = R^+$ into X . For each $u \in C(R^+, X)$ the function $t \to B(t)u(t)$ is weakly continuous. Moreover, $B(t)u(t)$ is strongly measurable since for each $x^* \in X^*$, $\big(B(t)u(t), x^*\big)$ is continuous, and therefore Lebesgue measurable. The strong measurability of $B(t)u(t)$ now follows from the separability of X (Hille, Phillips, [4], p. 73).

Now for each $x \in X$, $B(t)x$ is weakly continuous and therefore locally bounded; thus for each $T > 0$ and $x \in X$, $\sup_{t\in[0,T]} \|B(t)x\| < \infty$; by the principle of uniform boundedness $\sup_{t\in[0,T]} \|B(t)\| < \infty$ and $\|B(t)\|$ is locally bounded. Let $M_t = \sup_{s\in[0,T]} \|B(s)\|$, and define the operator Q on $C(R^+, X)$ by

$$Q(u)(t) = a - \int_0^t B(s)u(s)ds \ .$$

Q is well defined since $B(s)u(s)$ is strongly measurable and $\|B(s)u(s)\| \le M_t\|u(s)\|$, so that $B(s)u(s)$ is Bochner integrable. It is obvious that $Q(u)(t)$ is locally Lipschitz continuous on $[0, \infty)$, and strongly differentiable almost everywhere with $\frac{d}{dt} Q(u)(t) = -B(t)u(t)$ almost everywhere. Furthermore, Q is a contraction considered as an operator on $C(I, X)$ if I is any sufficiently small interval contained in $[0, \infty)$. It follows by Banach's fixed point theorem that Q has a unique fixed point on $C(I, X)$, from which it follows from standard arguments that (2.1) has a unique solution on $[0, \infty)$ which is locally Lipschitz continuous.

REMARK. We wish to point out that the preceding lemma does not follow from the standard results concerning equation (2.1) such as appear in Deleckiĭ and Kreĭn [2], p. 96. In that work it is assumed that the mapping $t \to B(t)$ is uniformly measurable in the sense of Hille, Phillips [4], p. 74, and that $\|B(t)\|$ is locally Lebesgue integrable.

We will let $C_w(R^+, X)$ denote the linear space of weakly continuous functions mapping $[0, \infty)$ into X with the topology of weak uniform convergence on compact sets $\left(x_n(t) \to x(t)\right.$ weakly uniformly on I if for each $\varepsilon > 0$ and $x^* \in X^*$ there is an integer N depending only on x^* and ε such that for $n \geq N$, $\left|x^*\left(x_n(t)-x(t)\right)\right| < \varepsilon$ for $t \in I$ $\left.\right)$. C_w is a complete locally convex linear space.

LEMMA 2.2. *Suppose for each* $(t, u) \in [0, \infty) \times X$ *that* $A(t, u)$ *is a bounded linear operator on* X *and that the mapping* $(t, u, v) \to A(t, u)v$ *from* $[0, \infty) \times X \times X$ *into* X *is weakly continuous. Then for each* $u \in C_w(R^+, X)$ *and* $a \in X$ *there is a unique strongly continuous solution* x_u *to the initial value problem*

$$x_u'(t) + A\big(t, u(t)\big)x_u(t) = 0 , \quad t \in [0, \infty) , \quad x_u(0) = a . \tag{2.2}$$

Proof. For $u \in C_w(R^+, X)$ let $B_u(t) = A\big(t, u(t)\big)$. Then it is easy to see that B_u satisfies the hypotheses of Lemma 2.1, since if $(t, y) \to \left(t_0, y_0\right)$ then $u(t) \to u\left(t_0\right)$ and

$$A\big(t, u(t)\big)y = B_u(t)y \to B_u\left(t_0\right)y_0 = A\left(t_0, u\left(t_0\right)\right)y .$$

This lemma thus follows from Lemma 2.1.

Proof of Theorem 2.1. Let $a \in X$, $a \neq 0$, and let

https://doi.org/10.1017/S0004972700022851 Published online by Cambridge University Press

$$S = \{x \in C_w \mid \|x(t)\| \le \|a\| \quad \text{for} \quad t \ge 0\} .$$

S is a closed and convex subset of C_w . By Lemma 2.1 for each $u(t) \in S$ there is a unique function $x_u(t)$ which is a strong solution to the initial value problem

$$x' + A(t, u(t))x = 0 , \quad t \ge 0 , \quad x(0) = a .$$

Since $x_u(t)$ is strongly continuous it is weakly continuous also, and $x_u(t)$ may be considered as a function in $C_w(R^+, X)$. Let Φ be the map $\Phi(u) = x_u$. We will show that Φ has a fixed point in S .

Let $u \in S$; then we have that each $x_u(t)$ is Lipschitz continuous; thus $\|x_u(t)\|$ is Lipschitz continuous and therefore differentiable almost everywhere with (Kato [7], p. 510) for some $f \in F(x_u(t))$,

$$\tfrac{1}{2} \frac{d}{dt} \|x_u(t)\|^2 = (x_u'(t), f) = -(A(t, u(t))x_u(t), f) \le 0 ,$$

almost everywhere. Thus

$$\tfrac{1}{2} \frac{d}{dt} \|x_u(t)\|^2 = \|x_u(t)\| \cdot \frac{d}{dt} \|x_u(t)\| \le 0$$

and we have

$$\frac{d}{dt} \|x_u(t)\|^2 \le 0 ,$$

and upon integrating this inequality we have

$$\|x_u(t)\|^2 \le \|x_u(0)\|^2 = \|a\|^2 . \tag{2.3}$$

Thus $\|x_u(t)\| \le \|a\|$ and $x_u \in S$ for all $u \in S$, or $\Phi(S) \subseteq S$. Moreover, $\Phi(S)$ is an equicontinuous family of functions since, for any $u \in S$ and $t, t_0 \ge 0$, we have

$$\|x_u(t)-x_u(t_0)\| = \left\|\int_{t_0}^{t} A(s, u(s))x_u(s)ds\right\| \le \left|\int_{t_0}^{t} M\|x_u(s)\|ds\right| \le |t-t_0|M\|a\|$$

where

$$M = \sup_{\Omega} \|A(s,\, u)\| \,,$$

$$\Omega = \{(s,\, u) \mid 0 \le s \le \max(t,\, t_0),\ \|u\| \le \|a\|\} \,.$$

M is finite because of the fact that weakly continuous functions in a reflexive space are locally bounded, together with an application of the principle of uniform boundedness as in Lemma 2.1. This also shows that the $x_u(t)$ satisfy a uniform local Lipschitz condition.

We can now show that Φ is a continuous mapping from $S \subseteq C_w(R^+,\, X)$ into itself. Let $\{u_n\} \subseteq S$ with $u_n(t) \to u(t)$ weakly uniformly on $[0,\, T)$ for each $T > 0$, and let $x_n = \Phi(u_n)$. Then $x_n \in S$ and the family $\{x_n(t)\}_{n=1}^{\infty}$ is equicontinuous on R^+. It follows (see Szép [9], p. 200) that $\{x_n(t)\}_{n=1}^{\infty}$ contains a weakly fundamental subsequence $\{x_{n_k}(t)\}_{k=1}^{\infty}$ which converges weakly uniformly on compact sets to a function $x_0(t) \in S$. Let $\bar{x} = \Phi(u)$. We claim $\bar{x}(t) = x_0(t)$. Since $x_0(t)$ is weakly continuous it is strongly measurable, and $\|x_0(t)\| \le \|a\|$, so that $\|x_0(t)\|$ is locally Lebesgue integrable, and for each $x^* \in X^*$ we have by dominated convergence

$$\begin{aligned} x^*[x_0(t)-\bar{x}(t)] &= \lim_{k\to\infty} x^*\left[\int_0^t A(s,\, u(s))\bar{x}(s)-A(s,\, u_{n_k}(s))x_{n_k}(s)ds\right] \\ &= \lim_{k\to\infty} \int_0^t x^*[A(s,\, u(s))\bar{x}(s)-A(s,\, u_{n_k}(s))x_{n_k}(s)]ds \\ &= \int_0^t x^*[A(s,\, u(s))(\bar{x}(s)-x_0(s))ds] \\ &= x^* \int_0^t A(s,\, u(s))(\bar{x}(s)-x_0(s))ds \end{aligned}$$

Thus we have, for $0 \le t \le T$,

$$x_0(t) - \bar{x}(t) = \int_0^t A(s,\, u(s))(\bar{x}(s)-x_0(s))ds$$

and

$$\|x_0(t)-\bar{x}(t)\| \le M_T \int_0^t \|\bar{x}(s)-x_0(s)\|ds$$

where $M_T = \sup_{0\le s\le T} \|A(s, u(s))\|$. By Gronwall's inequality $\|x_0(t)-\bar{x}(t)\| = 0$ on $[0, T]$, for arbitrary $T > 0$.

Thus $x_0 = \bar{x}$ and

$$x_0(t) = w - \lim_{k\to\infty} \Phi(u_{n_k})(t) = \Phi(u_0)(t) .$$

Since we could have started with any subsequence of $\{u_n(t)\}$ instead of $u_n(t)$ itself, it follows that $\Phi(u_n)(t) \rightharpoonup \Phi(u_0)(t)$ weakly uniformly on compact subsets of R^+ , and Φ is continuous on S .

If $\{x_n(t)\}$ is any sequence in $\Phi(S)$ then $\{x_n(t)\}$ is uniformly bounded and equicontinuous, and thus contains a subsequence weakly uniformly convergent on compact sets of R^+ . Thus $\Phi(S)$ is relatively compact in C_w . It now follows by the Schauder Theorem (Smart, [8]) that Φ has a fixed point x in S , $\Phi(x) = x$. Since all functions in $\Phi(S)$ are strongly Lipschitz continuous and strongly differentiable almost everywhere on $[0, \infty)$, this applies also to the fixed point x , and $x(t)$ is therefore a strong solution to $x' + A(t, x)x = 0$, $t \ge 0$, $x(0) = a$. This completes the proof of Theorem 2.1.

We shall now consider the initial value problem

$$x' + A(t, x)x = f(t, x) , \quad t \ge 0 , \quad x(0) = a$$

in the Banach space X . First we need

LEMMA 2.3. *Let* $B(t)$ *satisfy the hypotheses of Lemma* 2.1, *and let* $f(t)$ *be a weakly continuous function on* R^+ *into* X . *Then for each* $a \in X$ *there is a unique strong solution to the initial value problem*

$$x' + B(t)x = f(t) , \quad t \ge 0 , \quad x(0) = a .$$

Proof. Since $f(t)$ is weakly continuous and X is separable, $f(t)$ is strongly measurable and locally Bochner integrable (Hille, Phillips, [4], p. 73). The proof then follows the same argument as the proof of Lemma 2.1 by considering the operator

https://doi.org/10.1017/S0004972700022851 Published online by Cambridge University Press

$$Q(u)(t) = a - \int_0^t B(s)u(s)ds + \int_0^t f(s)ds \ ;$$

we therefore omit the details.

If the function $f(t, x)$ in equation (1.2) is a small perturbation, we will still have solutions for any $a \in X$, as the following shows.

THEOREM 2.2. *Let X, X^*, and $A(t, u)v$ be as in Theorem 2.1, and let $f : R^+ \times X \to X$ be weakly continuous with $f(t, 0) = 0$. If there is a function $g(t) \in L_1(R^+)$ with $\|g\|_{L_1} = c < 1$ such that for all $u \in X$ and $t \in R^+$ the inequality*

$$\|f(t, u)\| \le g(t)\|u\|$$

holds, then for each $a \in X$ there is a solution $x(t)$ to the initial value problem

$$x' + A(t, x)x = f(t, x) \ , \quad t \ge 0 \ , \quad x(0) = a \ ,$$

with $\|x(t)\| \le (1-c)^{-1}\|a\|$.

Proof. Let $a \in X$, let $p = (1-c)^{-1}\|a\|$, and let

$$S = \{u \in C_w(R^+, X) \mid \|u(t)\| \le p\} \ .$$

It follows from Lemma 2.3 that for each $u \in S$ there is a unique strong solution $x_u(t)$ to the problem

$$x_u' + A\big(t, u(t)\big)x_u = f\big(t, u(t)\big) \ , \quad t \ge 0 \ , \quad x_u(0) = a \ .$$

Moreover, since $A(t, u)v$ is accretive in v, we have as in Theorem 2.1 for some $f \in F\big(x_u(t)\big)$,

$$\begin{aligned}\|x_u(t)\| \frac{d}{dt} \|x_u(t)\| &= \big(x_u'(t), f\big) = \big(-A\big(t, u(t)\big)x_u(t), f\big) + \big(f\big(t, u(t)\big), f\big) \\ &\le \big(f\big(t, u(t)\big), f\big) \le \|f\big(t, u(t)\big)\|\|f\| = \|f\big(t, u(t)\big)\|\|x_u(t)\| \ ,\end{aligned}$$

so that for all t for which $x_u(t) \ne 0$ we have

$$\frac{d}{dt} \|x_u(t)\| \le \|f\big(t, u(t)\big)\|$$

almost everywhere, and

https://doi.org/10.1017/S0004972700022851 Published online by Cambridge University Press

$$\|x_u(t)\| \le \|x_u(0)\| + \int_0^t g(s)\|u(s)\|ds \le \|a\| + cp = p .$$

Thus $x_u \in S$ for all $u \in S$. As in Theorem 2.1 we can show that functions in the family $\{x_u(t) \mid u \in S\}$ satisfy a uniform Lipschitz condition in t and that the family is relatively compact in the space $C_w(R^+, X)$. In a manner similar to that used in Theorem 2.1, we can also show that the mapping $u \to x_u$ from S into S is continuous in the topology inherited from $C_w(R^+, X)$. Thus the mapping has a fixed point, which is a solution to the initial value problem (1.2).

References

[1] Felix E. Browder, "Non-linear equations of evolution and nonlinear accretive operators in Banach spaces", *Bull. Amer. Math. Soc.* 73 (1967), 867-874.

[2] Ju.L. Daleckiĭ and M.G. Kreĭn, *Stability of solutions of differential equations in Banach space* (translated by S. Smith. Translations of Mathematical Monographs, 43. American Mathematical Society, Providence, Rhode Island, 1974).

[3] W.E. Fitzgibbon, "Nonlinear differential equations in reflexive Banach spaces", *Bull. Austral. Math. Soc.* 10 (1974), 31-37.

[4] Einar Hille, Ralph S. Phillips, *Functional analysis and semi-groups*, revised edition (Amer. Math. Soc. Colloquium Publications, 31. American Mathematical Society, Providence, Rhode Island, 1957).

[5] Athanassios G. Kartsatos, "Banach space-valued solutions of differential equations containing a parameter", *Arch. Rational Mech. Anal.* 57 (1974), 142-149.

[6] A.G. Kartsatos and J.R. Ward, "Boundedness and existence of periodic solutions of quasi-linear systems", *J. Inst. Math. Appl.* 15 (1975), 187-194.

[7] Tosio Kato, "Nonlinear semigroups and evolution equations", *J. Math. Soc. Japan* 19 (1967), 508-520.

[8] D.R. Smart, *Fixed point theorems* (Cambridge Tracts in Mathematics, **66**. Cambridge University Press, Cambridge, 1974).

[9] A. Szép, "Existence theorem for weak solutions of ordinary differential equations in reflexive Banach spaces", *Stud. Sci. Math. Hungar.* **6** (1971), 197-203.

Department of Mathematics,
Pan American University,
Edinburg,
Texas,
USA.