

STRONG CONVERGENCE OF SELECTIONS IMPLIED BY WEAK

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In some situations weak convergence in \mathcal{L}_1 , implies strong convergence. Let $P, L : T \rightarrow Co(\mathbb{R}^d)$ be measurable multifunctions ($Co(\mathbb{R}^d)$ being the set of closed, convex subsets of \mathbb{R}^d), the values $L(t)$ affine sets and $W(t) = P(t) \cap L(t)$ extremal faces of $P(t)$. Let p_k be integrable selections of P , $\tilde{p}_k(t)$ the projection of $p_k(t)$ on $L(t)$ and $\bar{p}_k(t)$ on $W(t)$. We prove that if $p_k - \tilde{p}_k$ converges weakly to zero then $p_k - \bar{p}_k$ converges to zero in measure. We give also some extensions of this theorem. As applications to differential inclusions we investigate convergence of derivatives of convergent sequences of solutions and we describe solutions which are in some sense isolated. Finally we discuss what can be said about control functions u when the corresponding trajectories of $\dot{x} = f(t, x, u)$ are convergent to some trajectory.

1. PRELIMINARY REMARKS ON MEASURABLE MULTIFUNCTIONS

Multifunctions are maps whose values are sets. There exist exhaustive references to the fundamental properties of multifunctions - see [1, 3]. We shall discuss here, as briefly as possible, those that are essential to the results given in the sequel.

Throughout the paper, T will be a space endowed with a σ -field \mathcal{L} and a σ -finite, complete measure m .

For a metric space Y by $Cl(Y)$ we mean the collection of its closed subsets; $Co(\mathbb{R}^d)$ is the subfamily of $Cl(\mathbb{R}^d)$ composed of convex sets.

For $A, B \subseteq Y$ we put:

$$e(A, B) = \sup\{\text{dist}(a, B) \mid a \in A\}$$
$$H(A, B) = \max\{e(A, B), e(B, A)\}.$$

H is called Hausdorff metric when applied to A, B closed.

A multifunction $P : T \rightarrow Cl(Y)$ is said to be *measurable* if for every open $U \subset Y$ the set $\{t \in T; P(t) \cap U \neq \emptyset\}$ is measurable. A fundamental property of measurable multifunctions, for Y separable and complete, is the existence of a measurable selection, that means a measurable function $f(t) \in P(t)$ - known as the Theorem of Kuratowski and Ryll-Nardzewski - and its stronger version, due to Castaing, stating that there is

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a sequence of measurable selections f_n such that the sets $\{f_n(t); n \in N\}$ are dense in $P(t)$ a.e. in T . All this can be found in [3].

If Y is separable and complete then there is a very useful characterisation of measurability of a multifunction through the measurability of its graph $G(P) = \{(t, y); y \in P(t)\}$ in the product σ -field $\mathcal{L} \times \mathcal{B}$, where \mathcal{B} is the Borel σ -field in Y . Applying it one can prove easily many useful properties, for example the measurability of denumerable intersections of measurable multifunctions. Moreover, if $G(P)$ is measurable and if the values $P(t)$ are non-empty then P has a measurable selection even if the $P(t)$ are not closed. All this is given in detail in [3, chapter 3.4].

If the domain X of a multifunction $P: X \rightarrow Cl(Y)$ is a metric space then we can speak about other kinds of regularity of P . It is *upper semicontinuous* (usc) if the set $\{x \in X; P(x) \cap C \neq \emptyset\}$ is closed for every closed $c \subseteq Y$. For compact-valued multifunctions upper semicontinuity is equivalent to the implication $x_n \rightarrow x \rightarrow e(P(x_n), P(x)) \rightarrow 0$, for every x . Continuity is understood with respect to the Hausdorff metric in the range.

Especially important are convex-valued multifunctions. Before discussing their properties we recall some facts about convex sets – a reference for these is [8]. The scalar product of $x, y \in R^d$ will be denoted by $\langle x, y \rangle$. If $C \in Co(R^d)$, $y, z \in R^d$ then

$$\begin{aligned} \text{supp}(C, y) &= \sup\{\langle x, y \rangle; x \in C\}, \\ S(C, y) &= \{x \in R^d; \langle x, y \rangle = \text{supp}(C, y)\}, \\ N(C, z) &= \{u \in R^d; \forall x \in C \langle x - z, u \rangle \leq 0\}. \end{aligned}$$

$N(C, z)$ is the normal cone to C at the point z . It is closed convex and for $z \in C, y \in N(C, z)$ we have $z \in C \cap S(C, y)$.

A *face* of $C \in Co(R^d)$ is any subset D such that if $a, b \in C, a \neq b$ and $\lambda a + (1 - \lambda)b \in D$ for some $\lambda \in (0, 1)$ then $a, b \in D$. Faces are closed and convex. If D is a k -dimensional affine set L in R^d for which $DC \cap L$.

The notion of lexicographical order in R^d is strictly connected with the above. Recall that the lexicographical order induced by the set of linearly independent vectors $\{e_1, \dots, e_k\} = E$ is defined by: $x \leq_E y$ if and only if the first nonzero product $\langle e_j, y - x \rangle$ is positive. This relation is transitive and for every $x, y \in R^d$ either $x \leq_E y$ or $y \leq_E x$. But $x \leq_E y$ and $y \leq_E x$ imply $x = y$ only if $k = d$.

A point $x \in C \subseteq R^d$ is said to be *maximal* in C if $y \leq_E x$ for all $y \in C$. If $C \in Co(R^d)$ then the set of all maximal points in C is a face of dimension less than or equal to $d - k$.

Conversely, if D is a $(d - k)$ -dimensional face of a convex, closed set C then there

exists a set $\{e_1, \dots, e_k\} = E$ of orthonormal vectors such that D is the set of maximal points in C with respect to the order \leq_E .

We shall often put a dot in the place of a variable for example $f(t, \cdot)$ is the function $g(x) = f(t, x)$, where t is fixed.

LEMMA 1.1. *If $P: T \rightarrow Co(\mathbb{R}^d)$ and $p: T \rightarrow \mathbb{R}^d$ are measurable then $N(P(\cdot), p(\cdot))$ is a measurable multifunction.*

PROOF: Let q_n be measurable selections of P with $\{q_n(t) \mid n \in \mathbb{N}\}$ dense in $P(t)$ a.e. in T . We put

$$M_n(t) = \{y \in \mathbb{R}^d; \langle y, q_n(t) - p(t) \rangle \leq 0\}.$$

The multifunctions M_n are measurable and $N(P(t), p(t)) = \bigcap \{M_n(t) \mid n \in \mathbb{N}\}$ which proves the assertion. ■

Now let a measurable multifunction $P: T \rightarrow Co(\mathbb{R}^d)$ be fixed. We shall discuss some properties of multifunctions whose values are faces of $P(t)$.

Let $e_1, \dots, e_{d-k}: T \rightarrow \mathbb{R}^d$, where $0 \leq k < d$, be measurable functions with the values $\{e_1(t), \dots, e_{d-k}(t)\} = E(t)$ linearly independent a.e. in T . We denote by $W(t)$ the set of all maximal points in $P(t)$ with respect to the order $\leq_{E(t)}$.

LEMMA 1.2. *The $W(t)$ are faces of $P(t)$ of dimension less than or equal to k and the multifunction W is measurable.*

PROOF: We put $W_0(t) = P(t)$ and define by recurrence

$$W_j(t) = W_{j-1}(t) \cap S(W_{j-1}(t), e_j(t)).$$

Then $W(t) = W_{d-k}(t)$ and thus W is measurable. ■

Conversely, the following lemma holds.

LEMMA 1.3. *If $W: T \rightarrow Co(\mathbb{R}^d)$ is measurable and its values are faces of $P(t)$ of dimension k , where $0 \leq k \leq d$, then there exist measurable functions $e_1, \dots, e_{d-k}: T \rightarrow \mathbb{R}^d$ with values $\{e_1(t), \dots, e_{d-k}(t)\} = E(t)$ orthonormal a.e. in T and such that $W(t)$ are the sets of maximal points in $P(t)$ with respect to $\leq_{E(t)}$.*

PROOF: In the case $k = d$ we have $E(t) = \emptyset$ and the theorem is trivial. $W(t) = P(t)$ and all points in $P(t)$ are maximal.

We fix $k < d$. Let q be some measurable selection of W . We put $W_0(t) = P(t)$ and $L_0(t) = \mathbb{R}^d$. We show now how to find W_j , L_j and e_j if W_{j-1} , L_{j-1} are known - the W_j will be faces of $P(t)$ lying in affine sets $L_j(t)$, and the $e_j(t)$ will be vectors normal to $W_{j-1}(t)$ at $q(t)$. As long as $j < d - k$ $q(t)$ belongs to the boundary of $W_{j-1}(t)$ with respect to the euclidean topology in $L_{j-1}(t)$, so the intersections

$$N(W_{j-1}(t), q(t)) \cap S_{d-1} \cap (L_{j-1}(t) - q(t))$$

are non-empty and define a measurable, closed-value multifunction (S_{d-1} is the unit sphere in R^d). We can pick up its measurable selection e_j and put $L_j(t) = L_{j-1}(t) \cap S(W_{j-1}(t), e_j(t))$. The functions e_1, \dots, e_{d-k} satisfy the desired properties. ■

Let us remark that the multifunction $L(t) = L_{d-k}(t)$ from the above proof is measurable, its values are k -dimensional affine subsets of R^d and $W(t) = P(t) \cap L(t)$. Our assumption on the dimension of $W(t)$ implies that L is the unique multifunction having these properties.

Lemma 1.3 can be slightly modified to yield:

LEMMA 1.4. *If $L: T \rightarrow Co(R^d)$ is a measurable multifunction whose values are affine sets of dimension k and $W(t) = P(t) \cap L(t)$ are extremal faces of $P(t)$ then there exist measurable functions $e_1, \dots, e_{d-k}: T \rightarrow R^d$ with values $\{e_1(t), \dots, e_{d-k}(t)\} = E(t)$ orthonormal, orthogonal to the linear subspace parallel to $L(t)$ and such that $W(t)$ are the sets of maximal points in $P(t)$ with respect to $\leq_{E(t)}$.*

(Note that the dimension of $W(t)$ may be for some t less than k).

Now let p be a measurable selection of a measurable multifunction $P: T \rightarrow Co(R^d)$. We shall discuss the properties of the multifunction $W(t)$ whose values are faces of $P(t)$ containing $p(t)$ and of minimal dimension. The following property is useful.

LEMMA 1.5. *Let $Q, L: T \rightarrow Co(R^d)$ be measurable multifunctions, the values $L(t)$ affine sets and $Q(t) \subseteq L(t)$ a.e. in T . Then the multifunction $\partial_{L(t)}Q(t)$ is measurable. (Its values are the boundaries of $Q(t)$ with respect to the euclidean topology in $L(t)$.)*

PROOF: Let q_n be measurable selections of L such that the sets $\{q_n(t) \mid n \in \mathbb{N}\}$ are dense in $L(t)$. The functions $v_n(t) = \text{dist}(q_n(t), Q(t))$ are measurable. Putting $B_n(t) = B(q_n(t), v_n(t))$ if $v_n(t) > 0$ and $B_n(t) = \emptyset$ otherwise, we can express the boundary in the following way

$$\partial_{L(t)}Q(t) = \overline{\bigcup_{n \in \mathbb{N}} (B_n(t) \cap Q(t))}$$

and this implies the required measurability. ■

On the basis of the above property and using finite induction the following lemma can be proved.

LEMMA 1.6. *Let $P: T \rightarrow Co(R^d)$ be measurable, p its measurable selection and $W(t)$ the face of $P(t)$ containing $p(t)$ of minimal dimension. Then W is measurable and the $U_k = \{t \in T; \dim(W(t)) = k\}$ are measurable for $k = 0, 1, \dots, d$.*

2. THE MAIN RESULT

We shall prove now the basic result of the paper and give some of its direct consequences.

Throughout this section $P, L: T \rightarrow Co(\mathbb{R}^d)$ will be measurable multifunctions; the measure on T is, as before, σ -finite and complete. We assume that the $L(t)$ are affine sets, and that the $W(t) = P(t) \cap L(t)$ are extremal faces of $P(t)$.

For any function $p: T \rightarrow \mathbb{R}^d$, by $\tilde{p}(t)$ we shall mean the projection of $p(t)$ on $L(t)$ and by $\bar{p}(t)$ the projection on $W(t)$. If p is measurable then both \tilde{p} and \bar{p} are also measurable.

Speaking of weak convergence, compactness, etcetera we shall always mean the weak topology in $\mathcal{L}_1(T, \mathbb{R}^d)$.

We shall consider a sequence p_n of integrable selections of P . The following theorem will be the basis for all other results in this paper.

THEOREM 2.1. *Assume that W has an integrable selection and that the sequence of selections p_n of P is weakly precompact. Then the weak convergence to zero of $p_n - \tilde{p}_n$ implies convergence in measure to zero of $p_n - \bar{p}_n$.*

PROOF: The proof can be reduced to the case when $\dim(L(t))$ does not depend on t , since T can be divided into measurable sets $\{t \in T; \dim(L(t)) = j\}$, $j = 0, 1, \dots, d$. So, let us suppose that $\dim(L(t))$ is constant and equal to k .

Let the functions $e_1, \dots, e_{d-k}: T \rightarrow \mathbb{R}^d$ be as in Lemma 1.4.

We can write $p_n - \bar{p}_n = (p_n - \tilde{p}_n) + (\tilde{p}_n - \bar{p}_n)$. We shall prove first that $p_n - \tilde{p}_n \rightarrow 0$ in measure and then the same for $\tilde{p}_n - \bar{p}_n$.

We put $f_{n,j}(t) = \langle e_j(t), p_n(t) - \tilde{p}_n(t) \rangle$. Then

$$p_n(t) - \tilde{p}_n(t) = \sum_{j=1}^{d-k} f_{n,j}(t)e_j(t).$$

We shall prove first by induction on j that the sequences $f_{n,j}$ tend to zero in measure, as $n \rightarrow \infty$.

The functions $f_{n,1}$ are non-positive so for every $K \subset T$ of finite measure the weak convergence to zero of their restrictions to K implies convergence in measure to zero on K . In view of Lemma 2.3, cited at the end of this section, for any fixed $\alpha, \epsilon > 0$ there exists a set $K \subseteq T$ of finite measure such that

$$\int_{T \setminus K} |f_{n,1}| dm < \frac{\alpha \epsilon}{2}$$

for all $n \in \mathbb{N}$, so $m(\{t \in T \setminus K; |f_{n,1}(t)| > \alpha\}) < \epsilon/2$. From this and the convergence in measure on K we get $m(\{t \in T; |f_{n,1}(t)| > \alpha\}) < \epsilon$ for n sufficiently great.

Now, let us suppose that the sequences $f_{n,1}, \dots, f_{n,j-1}$ converge in measure to zero as $n \rightarrow \infty$, where $1 \leq j < d - k$. We prove first that $f_{n,j} \rightarrow 0$ in measure for any fixed $K \subseteq T$ of finite measure

Let us fix $\alpha, \epsilon > 0$. In view of the boundedness of p_n in the norm of \mathcal{L}_1 , there is a closed ball B in \mathbb{R}^d , centred at 0, such that putting $B_n = \{t \in K; p_n(t) \in B\}$ we have $m(K \setminus B_n) < \epsilon/3$ for all n .

We define an auxiliary function

$$w(t, \delta) = \max \left\{ \langle e_j(t), y - \text{proj}_{W(t)}y \rangle; y \in P(t) \cap B, \right. \\ \left. \langle e_i(t), y - \text{proj}_{W(t)}y \rangle \geq -\delta, \text{ for } i = 1, \dots, j - 1 \right\},$$

where $t \in K, \delta > 0$. For every fixed δ the function $w(\cdot, \delta)$ is measurable and for every fixed $t \in K, \lim_{\delta \rightarrow 0^+} w(t, \delta) = 0$. Using Egorov's Theorem we can fix $\delta_0 > 0$ and a measurable set $Z \subseteq K$ such that $m(K \setminus Z) < \epsilon/3$ and $w(t, \delta) \leq \alpha$ for $t \in Z$ and $0 < \delta \leq \delta_0$.

Now let

$$D_n = \bigcup_{i=1}^{j-1} \{t \in K \mid f_{n,i}(t) \geq -\delta_0\}.$$

The induction hypothesis implies that $m(K \setminus D_n) < \epsilon/3$ for n greater than some n_0 .

Let us put $E_n = B_n \cap Z \cap D_n$. For $n \geq n_0$ we have $m(K \setminus E_n) < \epsilon$ and, if $t \in E_n$, then the inequality $f_{n,j}(t) \leq \alpha$ holds. Thus the positive parts of $f_{n,j}$ converge on K in measure to zero and this implies, together with the weak convergence of $f_{n,j}$, that the $f_{n,j}$ converge to zero in measure.

To prove the convergence in measure on all T we proceed exactly as in the case of $f_{n,1}$. We find, for fixed $\alpha, \epsilon > 0$, a set $K \subseteq T$ of finite measure such that

$$\int_{T \setminus K} |f_{n,j}| \, dm < \frac{\alpha \epsilon}{2}$$

for all n and then use the convergence in measure on K .

The proof of the convergence $\bar{p}_n - \bar{p}_n \rightarrow 0$ in measure is almost exactly the same as the second induction step above - instead of w one has to use the auxiliary function

$$\bar{w}(t, \delta) = \max \left\{ \left| \text{proj}_{L(t)}y - \text{proj}_{W(t)}y \right|; y \in B \cap P(t), \right. \\ \left. \left| y - \text{proj}_{L(t)}y \right| \geq \delta \right\}$$

where B is a suitable ball. (That is the only part of the proof where the existence of an integrable selection of W is needed.) ■

Theorem 2.1 could be formulated without using L and the projection \bar{p} . The assumption would be then the weak convergence of $p_n - \bar{p}_n$ to zero. To prove it in this case one can divide T onto disjoint, measurable subsets on which the dimension of $W(t)$ is constant. On each one there exists a measurable, affine valued multifunction L with $\dim(L(t)) = \dim(W(t))$, $W(t) = P(t) \cap L(t)$ and Theorem 2.1 can be applied.

The assumption in Theorem 2.1 that p_n are selections of P can be weakened to some kind of convergence of p_n to P .

DEFINITION: A sequence of measurable functions $p_n: K \rightarrow \mathbb{R}^d$, where $K \subseteq T$, converges on K in measure to a measurable multifunction $P: K \rightarrow Cl(\mathbb{R}^d)$ if for every $\alpha > 0$

$$\lim_{n \rightarrow \infty} m(\{t \in K \mid \text{dist}(p_n(t), P(t)) \geq \alpha\}) = 0.$$

We can give now the following extension of Theorem 2.1.

COROLLARY 2.2. *Let all the assumptions of Theorem 2.1 be satisfied except the condition that the p_n are selections of P . We suppose instead the convergence of p_n to P in measure on every set $K \subset T$ of finite measure. Then the assertion of Theorem 2.1 holds.*

PROOF: Let $q_n(t)$ be the projection of $p_n(t)$ on $P(t)$. The existence of integrable selection of P and the weak precompactness of $\{p_n; n \in \mathbb{N}\}$ imply that the q_n are integrable and the set $\{q_n; n \in \mathbb{N}\}$ is weakly precompact. From the equality $q_n - \bar{q}_n = (q_n - p_n) + (p_n - \bar{p}_n) + (\bar{p}_n - \bar{q}_n)$ we get, using the assumption of convergence of p_n to P , the weak convergence of $q_n - \bar{q}_n$ to zero. Theorem 2.1 and the equality $p_n - \bar{p}_n = (p_n - q_n) + (q_n - \bar{q}_n) + (\bar{q}_n - \bar{p}_n)$ imply that $p_n - \bar{p}_n \rightarrow 0$ in measure. ■

We cite now a lemma on weak compactness in \mathcal{L}_1 that was used in this section.

LEMMA 2.3. (Theorem 8.9 in [4]) *A set (f) of integrable functions is weakly precompact in \mathcal{L}_1 if and only if the following conditions are satisfied:*

- (a) *it is norm-bounded on \mathcal{L}_1 ;*
- (b) *it is uniformly integrable;*
- (c) *for every $\epsilon > 0$ there is a set $K \subset T$ of finite measure such that*

$$\int_{T \setminus K} |f| \, dm \leq \epsilon$$

for all f in this set.

In the assertions of Theorem 2.1 and Corollary 2.2 we need the convergence in measure. However, weak compactness in \mathcal{L}_1 and convergence in measure of a sequence imply its strong convergence in \mathcal{L}_1 .

Remarks. Theorem 1 in [11] corresponds to the special case of our Theorem 2.1 when the affine sets $L(t)$ are 0-dimensional, that means $L(t) = W(t) = \{p_0(t)\}$ and $p_0(t)$ are extremal points of $P(t)$. The theorem says then that the weak convergence of p_k to p_0 implies their strong convergence. The proof differs from the one given above and is based, roughly speaking, on the fact that if c is an extremal point of a closed, convex set C then for any $r > 0$ it can be strongly separated from the convex hull of $C \setminus B(c, r)$.

In [2] Balder considered the case of functions with values in reflexive and separable Banach spaces. He proves a theorem analogous to that above of Visintin - the convergence of p_k that he gets is weaker than strong but stronger than weak. The proof is based on the use of Young measures.

In [6, 7] Olech gave results which are close to those discussed above. He proved that if c is an extremal point of the closure of $I = \int_{[0,1]} P(t)dt$ (the integral in the sense of Aumann, $P: [0,1] \rightarrow Cl(\mathbb{R}^d)$ measurable), p, q are integrable selections of P and $\left|c - \int_{[0,1]} p(t)dt\right|, \left|c - \int_{[0,1]} q(t)dt\right|$ are small then $\int_{[0,1]} |p(t) - q(t)| dt$ is small. This implies that if p_k, p_0 are integrable selections of $P, \int_{[0,1]} p_k(t)dt \rightarrow \int_{[0,1]} p_0(t)dt$ and $c = \int_{[0,1]} p_0(t)dt$ then $p_k \rightarrow p_0$ strongly in \mathcal{L}_1 . We note that instead of weak convergence it suffices to assume here this convergence of the integrals over all the space. The cost of this is the assumption of extremality of c in I . Thus c is maximal in I with respect to some total lexicographical order and $p_0(t)$ must be maximal in $P(t)$ with respect to the same, constant order. In our case the order may change with t . Olech gave two proofs of his theorem. One is based on the use of lexicographical order [6] and the other [7] on the property of extremal points which we mentioned in relation with the proof of Visintin.

3. APPLICATIONS TO DIFFERENTIAL INCLUSION

We shall give two applications. Firstly we shall treat convergent sequences of solutions of a differential inclusion and see what can be said about the convergence of their derivatives. Then we shall discuss some special kinds of solutions which are, in a sense, isolated.

We shall consider a multifunction $F: I \times \mathbb{R}^d \rightarrow Co(\mathbb{R}^d)$, where $I = [0,1]$, and solutions of the differential inclusion

$$(3.1) \quad \dot{x} \in F(t, x)$$

in the sense of Caratéodory (absolutely continuous functions $x: I \rightarrow \mathbb{R}^d$ satisfying $\dot{x}(t) \in F(t, x(t))$ a.e. in I).

Also, F will satisfy the following conditions:

- (a1) for every $\epsilon > 0$, there is a closed subset $I_\epsilon \subseteq I$ such that $m(I \setminus I_\epsilon) < \epsilon$

and the restriction of F to $I_\epsilon \times \mathbb{R}^d$ is upper semicontinuous;

- (b1) there exist integrable functions $g_1, g_2: I \rightarrow \mathbb{R}^+$ such that for every $t \in I, x \in \mathbb{R}^d, y \in F(t, x)$ the inequality $|y| \leq g_1(t)|x| + g_2(t)$ holds.

Condition (a1) is, in some sense, minimal for differential inclusions with convex-valued right-hand side. This is explained in [5, 9].

Assumption (b1) implies that if $\{x_\gamma; \gamma \in \Gamma\}$, where Γ is some set, is a family of solutions of (3.1) and the set of initial conditions $\{x_\gamma(0)\}$ is bounded then there exists an integrable function $g: I \rightarrow \mathbb{R}^+$ such that for every $\gamma \in \Gamma$ $|\dot{x}_\gamma(t)| \leq g(t)$ a.e. in I .

Now let x_0, x_n , for $n \in \mathbb{N}$, be solutions of (3.1) and $e_1, \dots, e_{d-k}: I \rightarrow \mathbb{R}^d$ measurable functions with values $\{e_1(t), \dots, e_{d-k}(t)\} = E(t)$ orthonormal a.e. in I . By (a1) the multifunction $P(t) = F(t, x_0(t))$ is measurable.

Let $K \subseteq I$ be measurable.

THEOREM 3.1. *If the solutions x_n converge pointwise to the solution x_0 and the derivatives $\dot{x}_0(t)$ for $t \in K$ are maximal in $F(t, x_0(t))$ with respect to the lexicographical order induced by $E(t)$ then for every $j = 1, \dots, d - k$ the sequence $\langle e_j(\cdot), \dot{x}_n(\cdot) - \dot{x}_0(\cdot) \rangle$ converges in measure to zero on K .*

PROOF: For every $r \in I$ we have $x_n(t) \rightarrow x_0(t)$ as $n \rightarrow \infty$, so $e(F(t, x_n(t)), P(t)) \rightarrow 0$. This implies that the sequence \dot{x}_n converges in measure to the multifunction P . The assumption (b1) and the convergence of x_n to x_0 imply the weak convergence of $\dot{x}_n - \dot{x}_0$ to zero. To end the proof it suffices to apply Corollary 2.2. ■

An important special case of Theorem 3.1 is when $k = 0$ - this means that $\dot{x}_0(t)$ are extremal points of $F(t, x_0(t))$ a.e. in I .

COROLLARY 3.2. *If the solutions x_n converge to the solution x_0 and $\dot{x}_0(t) \in \text{Ext}(F(t, x_0(t)))$ a.e. in I then $\dot{x}_n \rightarrow \dot{x}_0$ in measure. (Ext denotes all extremal points of a set.)*

Now we shall discuss a situation that may occur when the sets $F(t, x_0(t))$, along a solution x_0 , are much bigger than the sets $F(t, x)$ for x near to $x_0(t)$.

So, let x_0 be a solution of (3.1) and consider the following condition:

- (c1) almost everywhere in I there exist $r(t) > 0$ such that

$$\dot{x}_0(t) \notin \text{conv} \bigcup \{F(t, x); 0 < |x - x_0(t)| \leq r(t)\}.$$

(conv stands for closed, convex hull.)

THEOREM 3.3. *Let the conditions (a1), (b1), (c1) hold. If x_n is a sequence of solutions of (3.1) convergent pointwise to the solution x_0 then*

$$\lim_{n \rightarrow \infty} m(\{t \in I; x_n(t) \neq x_0(t)\}) = 0.$$

To prove this theorem we shall need a lemma in which we give to the condition (a3) a form suitable for applying Theorem 3.1.

LEMMA 3.4. Under assumptions (a1), (b1), (c1) there exist measurable functions $q: I \rightarrow R^+$, $u: I \rightarrow R^d$ such that $|u(t)| = 1$ and

$$h(t) = \langle \dot{x}_0(t), u(t) \rangle - \text{supp} \left(\text{conv} \bigcup \{F(t, x); 0 < |x - x_0(t)| \leq q(t)\}, u(t) \right) > 0$$

a.e. in I .

PROOF: Let $q^*: I \rightarrow R^+$ be the measurable function which is greater than or equal to r and less than or equal to any other measurable function which is greater than r . We put $q(t) = q^*(t)/2$. The set $C = \{t \in I; r(t) > q(t)\}$ has the outer Lebesgue measure $m^*(C)$ equal to 1. Moreover, as $m^*(\{t \in I; q(t) > 0\}) = 1$, so $q(t) > 0$ a.e. in I .

Let us prove that the multifunction

$$Z(t) = \text{conv} \bigcup \{F(t, x); 0 < |x - x_0(t)| \leq q(t)\}$$

is measurable. First we show that for a fixed $\beta > 0$ the multifunction

$$Z_\beta(t) = \bigcup \{F(t, x); \beta \leq |x - x_0(t)| \leq q(t)\}$$

has $\mathcal{L} \times \mathcal{B}$ measurable graph. To do this let us fix for a moment $n \in \mathbf{N}$ and let $J_n \subset I$ be a closed set such that $m(I \setminus J_n) \leq 1/n$, the restriction of q to J_n is continuous and of F to $J_n \times R^d$ upper semicontinuous with respect to (t, x) . The set

$$A_n = \{(t, x) \in J_n \times R^d; \beta \leq |x - x_0(t)| \leq q(t)\}$$

is compact so the graph $G(F |_{A_n})$ is also compact in $I \times R^d \times R^d$. Let $\Psi: I \times R^d \times R^d \rightarrow I \times R^d$ be the projection defined by $\Psi(t, x, y) = (t, y)$. $G(Z_\beta |_{J_n}) = \Psi(G(F |_{A_n}))$ so $G(Z_\beta |_{J_n})$ is compact. This implies that the graph of the restriction of Z_β to $\bigcup \{J_n; n \in \mathbf{N}\}$ is $\mathcal{L} \times \mathcal{B}$ measurable in $I \times R^d$.

Now consider the multifunction

$$\tilde{Z}(t) = \bigcup \{F(t, x); 0 < |x - x_0(t)| \leq q(t)\}.$$

Applying the above we deduce that there is a measurable set $J \subset I$ such that $m(I \setminus J) = 0$ and the graph $G(\tilde{Z} |_J)$ is $\mathcal{L} \times \mathcal{B}$ measurable. This implies finally that the multifunction $Z(t) = \text{conv} \tilde{Z}(t)$ is measurable.

Now let $z(t) = \dot{x}_0(t) - \text{proj}_{Z(t)} \dot{x}_0(t)$. For $t \in C$ we have $z(t) \neq 0$. As z is measurable and $m^*(C) = 1$ so $z(t) \neq 0$ a.e. in I . We can thus put $u(t) = z(t)/|z(t)|$. The functions q and u have the required properties. ■

PROOF OF THEOREM 3.3: We modify first the multifunction F along the solution x_0 . For $x = x_0(t)$ we put

$$\hat{F}(t, x) = F(t, x) \cap \{y \in \mathbb{R}^d; \langle y, u(t) \rangle \leq \langle \dot{x}_0(t), u(t) \rangle\}$$

and $\hat{F}(t, x) = F(t, x)$ if $x \neq x_0(t)$. \hat{F} has exactly the same properties as F , $\dot{x} \in \hat{F}(t, x)$ has the same solutions as (3.1) and $\dot{x}_0(t)$ lies on the support hyperplane to $\hat{F}(t, x_0(t))$ orthogonal to $u(t)$. This allows us to apply Theorem 3.1 for $k = d - 1$, that means for one function $e_1 = u$, and get the convergence of $\langle u(\cdot), \dot{x}_n(\cdot) - \dot{x}_0(\cdot) \rangle$ in measure to zero.

Now let us fix $\epsilon > 0$. The function h is measurable and positive valued so there exists $\delta > 0$ such that the measure of the set $D_\delta = \{t \in I; h(t) \leq \delta\}$ is less than $\epsilon/3$. Let

$$U_n = \{t \in I; |\langle u(t), \dot{x}_n(t) - \dot{x}_0(t) \rangle| \geq \delta\},$$

$$V_n = \{t \in I; |x_n(t) - x_0(t)| \geq q(t)\}.$$

We can choose such k that $m(U_n) \leq \epsilon/3$ and $m(V_n) \leq \epsilon/3$ for $n \geq k$.

Let $n \geq k$. Then $m(D_\delta \cup U_n \cup V_n) < \epsilon$ and we shall see that for $t \in I \setminus (D_\delta \cup U_n \cup V_n)$ the equality $x_n(t) = x_0(t)$ holds. Let us suppose the contrary, that is $x_n(t) \neq x_0(t)$. As $t \notin V_n$ so $|x_n(t) - x_0(t)| < q(t)$. Thus $\langle u(t), \dot{x}_0(t) - \dot{x}_n(t) \rangle \geq h(t)$. $t \notin D$ implies $h(t) > \delta$, hence $\langle u(t), \dot{x}_0(t) - \dot{x}_n(t) \rangle > \delta$. On the other hand $t \notin U_n$, so $|\langle u(t), \dot{x}_0(t) - \dot{x}_n(t) \rangle| < \delta$. This contradiction proves that the inequality $x_n(t) \neq x_0(t)$ is not possible and ends the proof. ■

Here is the most simple example of the situation described in Theorem 3.3.

Example 3.5. We define $F: I \times \mathbb{R} \rightarrow Co(\mathbb{R})$ as follows: $F(t, x) = \{1\}$ if $x \neq 0$ and $[0, 1]$ if $x = 0$. In fact F does not depend on t . The special solution is $x_0(t) = 0$.

4. AN APPLICATION TO CONTROL SYSTEMS

In this section we shall apply Corollary 3.2 to control systems of the form

$$(4.1) \quad \dot{x} = f(t, x, u)$$

where $f: I \times \mathbb{R}^d \times \mathbb{R}^w \rightarrow \mathbb{R}^d$, $I = [a, b]$. We assume that f satisfies the following conditions:

$$(a2) \quad f(\cdot, x, u,) \text{ measurable for all } x, u;$$

- (b2) $f(t, \cdot, u)$ Lipschitzian for all t, u with a Lipschitz constant $M(t)$ such that the function $M(\cdot)$ is integrable;
- (c2) $f(t, x, \cdot)$ continuous for all t, x ;
- (d2) $|f(t, x, u)| \leq g_1(t)|x| + g_2(t)$, where g_1, g_2 integrable.

For each (t, x) the control parameter u , which may appear in $f(t, x, u)$, must belong to a set $U(t, x)$; the multifunction $U: I \times \mathbb{R}^d \rightarrow \text{Comp}(\mathbb{R}^w)$ will satisfy the conditions:

- (e2) for every $\varepsilon > 0$ there is a closed set $I_\varepsilon \subseteq I$ for which $m(I \setminus I_\varepsilon) < \varepsilon$ and the restriction of U to $I_\varepsilon \times \mathbb{R}^d$ is upper semicontinuous with respect to (t, x) ;
- (f2) $|u| \leq g_3(t)$ for $u \in U(t, x)$, where g_3 is integrable.

For a fixed measurable function $u: I \rightarrow \mathbb{R}^w$ the differential equation

$$(4.2) \quad \dot{x} = f(t, x, u(t))$$

has, for any initial condition $x(a)$, a unique solution $x: I \rightarrow \mathbb{R}^d$. If a.e. in I the condition $u(t) \in U(t, x(t))$ holds then we say that the pair $(x(\cdot), u(\cdot))$ is admissible.

We put

$$F(t, x) = \{f(t, x, u) \mid u \in U(t, x)\}.$$

F is a multifunction defined on $I \times \mathbb{R}^d$ and with values in $\text{Comp}(\mathbb{R}^d)$. Its properties will be given in Lemma 4.2.

For $t \in I$, $x \in \mathbb{R}^d$, $v \in \mathbb{R}^w$ we shall denote by $\Phi(t, x, v)$ the set of all $u \in U(t, x)$ such that $f(t, x, u) = v$, that means

$$\Phi(t, x, v) = (f(t, x, \cdot))^{-1}(v) \cap U(t, x).$$

Thus we have defined a multifunction $\Phi: I \times \mathbb{R}^d \times \mathbb{R}^d \rightarrow \text{Comp}(\mathbb{R}^w)$.

Now we fix an admissible pair (x_0, u_0) .

THEOREM 4.1. *Let us suppose that a.e. in I*

$$(4.3) \quad \dot{x}_0(t) \in \text{Ext}(\text{conv}(F(t, x_0(t))))$$

and let (x_n, u_n) be a sequence of admissible pairs such that the solutions x_n are pointwise convergent to x_0 . Then the sequence of functions

$$h_n(t) = \text{dist}(u_n(t), \Phi(t, x_0(t), \dot{x}_0(t)))$$

is convergent in measure to zero.

Before proving this theorem we shall give three necessary lemmas. Let us remark that in view of (d2) all the values of the functions x_n, x_0 are contained in some closed ball $B \subseteq \mathbb{R}^d$.

LEMMA 4.2. *The restriction $F: I \times B \rightarrow \text{Comp}(\mathbb{R}^d)$ is integrably bounded and satisfies the condition (a1).*

PROOF: The integral boundedness is implied by (d2) and the boundedness of B .

To prove (a1) let us fix an $\varepsilon > 0$ and let I_ε be such a closed subset of I that the following conditions are true:

- (a3) $m(I \setminus I_\varepsilon) < \varepsilon$;
- (b3) $f|_{I_\varepsilon \times B \times \mathbb{R}^w}$ is continuous with respect to (t, x, u) (this can be achieved due to (a2)-(d2) and the Theorem of Scorza-Dragoni [10]);
- (c3) the set $f(I_\varepsilon \times B \times \mathbb{R}^w)$ is precompact in \mathbb{R}^d ;
- (d3) the restriction of U to $I_\varepsilon \times \mathbb{R}^d$ is upper semicontinuous with respect to (t, x) .

The last condition implies that the union of $U(t, x)$, for $(t, x) \in I_\varepsilon \times B$, is compact.

By (c3) the graph $G(F|_{I_\varepsilon \times B})$ is bounded in $I \times B \times \mathbb{R}^d$. We shall show that it is closed, thus compact. The compactness of the graph implies the upper semicontinuity of a multifunction and this will complete our proof.

So let $(t_n, x_n, v_n) \rightarrow (t_0, x_0, v_0)$, where $t_n \in I_\varepsilon$, $x_n \in B$, $v_n \in F(t_n, x_n)$. There exist $u_n \in U(t_n, x_n)$ such that $v_n = f(t_n, x_n, u_n)$. The sequence u_n has a convergent subsequence $u'_n \rightarrow u_0 \in U(t_0, x_0)$. From (b3) we have $f(t_0, x_0, u_0) = v_0$ and thus $v_0 \in F(t_0, x_0)$. So the graph $G(F|_{I_\varepsilon \times B})$ is actually closed. ■

In the following lemmas X, Y, Z will denote metric spaces.

LEMMA 4.3. *Let $\psi: X \times Y \rightarrow Z$ be a continuous function. Then the multifunction $\Psi: X \times Z \rightarrow Cl(Y)$ defined by $\Psi(x, z) = (\psi(x, \cdot))^{-1}(z)$ has closed graph.*

Its proof is immediate as is the proof of the next lemma.

LEMMA 4.4. *Let $\Psi: X \rightarrow \text{Comp}(Y)$ be upper semicontinuous, and let the sequence of measurable functions $w_n: I \rightarrow X$ converge in measure to a measurable function w_0 . Then the sequence of functions $\mathcal{H}_n(t) = e(\Psi(w_n(t)), \Psi(w_0(t)))$ converges in measure to zero.*

PROOF OF THEOREM 4.1: From Lemma 4.2 we deduce that the multifunction $\text{conv}(F(t, x))$ satisfies on $\mathbb{1} \times B$ the conditions (a1), (b1). By (4.3) Corollary 3.2 implies that the sequence \dot{x}_n is convergent to \dot{x}_0 in measure.

We fix $\varepsilon > 0$ and let I_ε be a closed subset of I for which conditions (a3)-(d3) are satisfied. From Lemma 4.2 we get the upper semicontinuity of $\Phi|_{I_\varepsilon \times B \times \mathbb{R}^d}$ with respect to (t, x, v) .

The sequence of functions $z_n(t) = (t, x_n(t), \dot{x}_n(t))$ converges in measure to the function $z_0(t) = (t, x_0(t), \dot{x}_0(t))$. Lemma 4.4 implies then that the $\phi_n(t) =$

$e(\Phi(t, x_n(t), \dot{x}_n(t)), \Phi(t, x_0(t), \dot{x}_0(t)))$ converge in measure to zero. We have $u_n(t) \in \Phi(t, x_n(t), \dot{x}_n(t))$ thus h_n converge on I_ε to zero in measure. As ε was arbitrary, we get the convergence of $h_n \rightarrow 0$ in measure on all I . ■

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