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GENERALIZED MORREY SPACES OVER NONHOMOGENEOUS METRIC MEASURE SPACES

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Abstract

Let $(\mathcal{X}, d, \mu)$ be a nonhomogeneous metric measure space satisfying the so-called upper doubling and the geometric doubling conditions. In this paper, the authors give the natural definition of the generalized Morrey spaces on $(\mathcal{X}, d, \mu)$, and then investigate some properties of the maximal operator, the fractional integral operator and its commutator, and the Marcinkiewicz integral operator.

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1. Introduction

In the past ten or fifteen years, many classical results from real and harmonic analysis on $\mathbb{R}^n$ have been expanded to the spaces under nondoubling measures which only satisfy the polynomial growth condition (for example, see [2, 8, 10, 15–18, 20, 21]). However, it is obvious that the nondoubling measure may not include the well-known doubling condition that plays an important part in the assumption on homogeneous type spaces in the sense of Coifman and Weiss (see [3, 4]). Thus, in order to deal with the problem, in 2010, Hytönen in [11] introduced a new class of metric measure spaces satisfying the so-called geometric doubling and the upper doubling conditions, respectively (see Definitions 1.1 and 1.3 below), which are called nonhomogeneous metric measure spaces. Since then, many authors have proved that many known results still hold true if the underlying spaces take the place of the nonhomogeneous metric measure spaces (see [1, 6, 12–14]).

For convenience, in this paper, we always assume that $(\mathcal{X}, d, \mu)$ is a nonhomogeneous metric measure space in the sense of Hytönen [11]. In this setting, we first give a natural definition of the generalized Morrey spaces on $(\mathcal{X}, d, \mu)$, then

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we discuss the boundedness of some classical operators, which include the maximal operator, the fractional integral operator and the Marcinkiewicz integral operator. Naturally, these results not only generalize and improve the known results for the case of nondoubling measures (see [16]), but also contain the consequences in [1].

Before stating the main results of this paper, we first recall some necessary notions and notation. The following notion of geometric doubling is well known in analysis on metric spaces and was originally introduced by Coifman and Weiss in [3].

Definition 1.1. A metric space $(\mathcal{X}, d)$ is said to be geometrically doubling, if there exists some $N_0 \in \mathbb{N}$ such that, for any ball $B(x, r) \subset \mathcal{X}$, there exists a finite ball covering $\{B(x_i, \frac{r}{2})\}_i$ of $B(x, r)$ such that the cardinality of this covering is at most N_0.

Remark 1.2. Let $(\mathcal{X}, d)$ be a metric space. Hytönen in [11] showed that the following statements are mutually equivalent.

(1) $(\mathcal{X}, d)$ is geometrically doubling.
(2) For any $\epsilon \in (0, 1)$ and ball $B(x, r) \subset \mathcal{X}$, there exists a finite ball covering $\{B(x_i, \epsilon r)\}_i$ of $B(x, r)$ such that the cardinality of this covering is at most $N\epsilon^{-n}$. Here, and in what follows, N_0 is as Definition 1.1 and $n := \log_2 N_0$.
(3) For every $\epsilon \in (0, 1)$, any ball $B(x, r) \subset \mathcal{X}$ can contain at most $N\epsilon^{-n}$ centers $\{x_i\}_i$ of disjoint balls with radius ϵr.
(4) There exists $M \in \mathbb{N}$ such that any ball $B(x, r) \subset \mathcal{X}$ can contain at most $N\epsilon^{-n}$ centers $\{x_i\}_i$ of disjoint balls $\{B(x_i, \frac{r}{4})\}_{i=1}^M$.

Now we recall the following notion of upper doubling metric measure spaces from [11].

Definition 1.3. A metric measure space $(\mathcal{X}, d, \mu)$ is said to be upper doubling if μ is a Borel measure on $\mathcal{X}$ and there exist a dominating function $\lambda : \mathcal{X} \times (0, \infty) \to (0, \infty)$ and a positive constant C_λ such that, for each $x \in \mathcal{X}$, $r \to \lambda(x, r)$ is nondecreasing and, for all $x \in \mathcal{X}$ and $r \in (0, \infty)$,

$$\mu(B(x, r)) \le \lambda(x, r) \le C_\lambda \lambda\Big(x, \frac{r}{2}\Big). \tag{1.1}$$

Hytönen *et al.* pointed out in [12] that there exists a dominating function $\tilde{\lambda}$ related to λ satisfying the property that there exists a positive constant $C_{\tilde{\lambda}}$ such that $\tilde{\lambda} \le \lambda$, $C_{\tilde{\lambda}} \le C_\lambda$ and

$$\tilde{\lambda}(x, r) \le C_{\tilde{\lambda}} \tilde{\lambda}(y, r), \tag{1.2}$$

where $x,\ y \in \mathcal{X}$ and $d(x, y) \le r$. From now on, in this paper, we always assume that the dominating function λ, as in (1.1), satisfies (1.2).

The following coefficient $K_{B,S}$, which is analogous to Tolsa's numbers $K_{Q,R}$ in [9], was introduced [11] by Hytönen.

Definition 1.4. For all balls $B \subset S$, define

$$K_{B,S} := 1 + \int_{(2S)\setminus B} \frac{1}{\lambda(c_B, \mathrm{d}(x, c_B))} \mathrm{d}\mu(x),$$

where c_B stands for the center of the ball B.

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Next, we recall the definition of the (α,β)-doubling property given in [11].

DEFINITION 1.5. Let α, $\beta > 1$. A ball $B \subset \mathcal{X}$ is said to be (α,β)-doubling if $\mu(\alpha B) \le \beta\mu(B)$.

Hytönen in [11] pointed out that if a metric measure space $(\mathcal{X}, d, \mu)$ is upper doubling and $\beta > C_\lambda^{\log_2 \alpha} =: \alpha^\nu$, then, for any ball $B \subset \mathcal{X}$, there exists some $j \in \mathbb{Z}_+$ such that $\alpha^j\beta$ is (α, β)-doubling. In addition, assume that $(\mathcal{X}, d)$ is geometrically doubling, $\beta > \alpha^n$ with $n := \log_2 N_0$ and μ is a Borel measure on $\mathcal{X}$, which is finite on bounded sets. At the same time, Hytönen in [11] showed that for μ-almost every $x \in \mathcal{X}$, there are arbitrarily small (α,β)-doubling balls centered at x. Furthermore, the radii of these balls may be chosen to be of the form $\alpha^{-j}r$ for $j \in \mathbb{N}$ and any preassigned number $r \in (0,\infty)$. Throughout this paper, for any $\alpha > 1$ and ball B, the smallest (α, β_α)-doubling ball of the form $\alpha^j B$ with $j \in \mathbb{N}$ is denoted by $\widetilde{B}^\alpha$, where

$$\beta_\alpha := \max\{\alpha^{3n}, \alpha^{3\nu}\} + 30^n + 30^\nu.$$

If there are no special instructions about α and β, in the latter of the paper, by a *doubling ball* we always mean a $(6,\beta_6)$-doubling ball and $\widetilde{B}^6$ is simply denoted by $\widetilde{B}$.

The generalized Morrey space on $(\mathcal{X}, d, \mu)$ is defined as follows.

DEFINITION 1.6. Let $k > 1$ and $1 \le p < \infty$. Suppose that $\phi : (0,\infty) \to (0,\infty)$ is an increasing function. Then we define

$$L^{p,\phi}(\mu) := \{f \in L^p_{\rm loc}(\mu) : \|f\|_{L^{p,\phi}(\mu)} < \infty\},$$

where

$$\|f\|_{L^{p,\phi}(\mu)} = \sup_B \bigg(\frac{1}{\phi(\mu(kB))}\int_B |f(x)|^p \,d\mu(x)\bigg)^{1/p}. \tag{1.3}$$

REMARK 1.7. By means of a similar method to that used in the proof of [16, Proposition 1.2] and [1, Theorem 7], it is not difficult to show that the norm $\|f\|_{L^{p,\phi}(\mu)}$ is independent of the choice of k for $k > 1$.

Finally, we recall the notion of the ϵ-weak reverse doubling condition given in [6].

DEFINITION 1.8. Let $\epsilon \in (0,\infty)$. A dominating function λ is said to satisfy the ϵ-weak reverse doubling condition if, for all $s \in (0, 2\mathrm{diam}(\mathcal{X}))$ and $a \in (1, 2\mathrm{diam}(\mathcal{X})/s)$, there exists a number $C(a) \in [1,\infty)$, depending only a and $\mathcal{X}$, such that,

$$\lambda(x, as) \ge C(a)\lambda(x, s) \quad x \in \mathcal{X} \tag{1.4}$$

and, moreover,

$$\sum_{k=1}^{\infty} \frac{1}{[C(a^k)]^\epsilon} < \infty. \tag{1.5}$$

The organization of this paper is as follows. Section 2 is devoted to the study of the maximal operator. In Section 3, we will prove the boundedness of the fractional integral operator and its commutator on $(\mathcal{X}, d, \mu)$. We will establish the boundedness of the Marcinkiewicz integral operator in Section 4. Throughout this paper, C represents a positive constant that is independent of the main parameters, but may be different from line to line.

2. Maximal operator $M_{r,\rho}$

In this section, we will investigate the boundedness of the maximal operator given by

$$M_{r,\rho}f(x) := \sup_{B\ni x}\Big(\frac{1}{\mu(\rho B)}\int_B |f(y)|^r \mathrm{d}\mu(y)\Big)^{1/r}, \tag{2.1}$$

where $\rho > 0$ and $r \in [1, \infty)$.

We shall need the following lemma from [1].

Lemma 2.1. *Let $\rho > 0$, $r \in [1, \infty)$ and $p > 1$. Then the maximal operator $M_{r,\rho}$, as in (2.1), is bounded on $L^p(\mu)$.*

Now we will formulate the main result of this section.

Theorem 2.2. *Let $\rho > 1$, $1 < r < p < \infty$ and let $\phi : (0, \infty) \to (0, \infty)$ be an increasing function. Suppose that $M_{r,\rho}$ is as in (2.1), the mapping $t \mapsto \phi(t)/t$ is almost decreasing and there is a constant $C > 0$ such that*

$$\frac{\phi(t)}{t} \le C\frac{\phi(s)}{s} \tag{2.2}$$

for $s \ge t$. Then there exists a positive constant $C > 0$, such that, for any $f \in L^{p,\phi}(\mu)$,

$$\|M_{r,\rho}f\|_{L^{p,\phi}(\mu)} \le C\|f\|_{L^{p,\phi}(\mu)}.$$

Proof. For convenience, we assume that $\rho = 2$, as in (2.1), and $k = 12$, as in (1.3). Let B be a fixed doubling ball. Then we only need to estimate

$$\Big(\frac{1}{\phi(\mu(12B))}\int_B [M_{r,2}(f)(x)]^p \mathrm{d}\mu(x)\Big)^{1/p} \le C\|f\|_{L^{p,\phi}(\mu)}. \tag{2.3}$$

To estimate (2.3), decompose f as

$$f(x) = f_1(x) + f_2(x),$$

where $f_1(x) := f(x)\chi_{6B}(x)$ and $f_2(x) := f(x)\chi_{\mathcal{X}\setminus 6B}(x)$. Write

$$\begin{aligned}
&\Big(\frac{1}{\phi(\mu(12B))}\int_B [M_{r,2}(f)(x)]^p \mathrm{d}\mu(x)\Big)^{1/p}\\
&\le \Big(\frac{1}{\phi(\mu(12B))}\int_B [M_{r,2}(f_1)(x)]^p \mathrm{d}\mu(x)\Big)^{1/p}\\
&\quad + \Big(\frac{1}{\phi(\mu(12B))}\int_B [M_{r,2}(f_2)(x)]^p \mathrm{d}\mu(x)\Big)^{1/p}\\
&=: \mathrm{D}_1 + \mathrm{D}_2.
\end{aligned}$$

By applying Lemma 2.1, it is not difficult to obtain that

$$\mathrm{D}_1 \le C\Big(\frac{1}{\phi(\mu(12B))}\int_{6B} |f(x)|^p \mathrm{d}\mu(x)\Big)^{1/p} \le C\|f\|_{L^{p,\phi}(\mu)}.$$

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Now we turn to D_2. Suppose that Q is a doubling ball. For every $x \in B$, a geometric observation shows that

$$\begin{aligned} M_{r,2}(f_2)(x) &= \sup_{x\in Q}\Big(\frac{1}{\mu(2Q)}\int_Q |f_2(y)|^r \, d\mu(y)\Big)^{1/r} \\ &\le \sup_{\substack{x\in Q\\ B\subset Q}}\Big(\frac{1}{\mu(\frac{4}{3}Q)}\int_Q |f(y)|^r d\mu(y)\Big)^{1/r}. \end{aligned} \tag{2.4}$$

By the Hölder inequality, (2.2) and (2.4),

$$\begin{aligned} D_2 &\le \Big(\frac{\mu(B)}{\phi(\mu(12B))}\Big)^{1/p} \sup_{\substack{x\in Q\\ B\subset Q}}\Big(\frac{1}{\mu(\frac{4}{3}Q)}\int_Q |f(y)|^r d\mu(y)\Big)^{1/r} \\ &\le \Big(\frac{\mu(B)}{\phi(\mu(12B))}\Big)^{1/p} \sup_{\substack{x\in Q\\ B\subset Q}} \mu\Big(\frac{4}{3}Q\Big)^{-1/r}\Big(\int_Q |f(y)|^p d\mu(y)\Big)^{1/p} \mu(Q)^{1/r-1/p} \\ &\le C\Big(\frac{\mu(B)}{\phi(\mu(B))}\Big)^{1/p} \sup_{\substack{x\in Q\\ B\subset Q}}\Big(\frac{\phi(\mu(\frac{4}{3}Q))}{\mu(\frac{4}{3}Q)}\Big)^{1/p} \|f\|_{L^{p,\phi}(\mu)} \\ &\le C\|f\|_{L^{p,\phi}(\mu)}, \end{aligned}$$

which, together with D_1, implies (2.3) and hence completes the proof of Theorem 2.2. □

3. Fractional integral operator and its commutator

In 2008, Sawano obtained the boundedness of the fractional integral operator on a generalized Morrey space with nondoubling measure in [16]. Further, in 2013, the authors obtained the boundedness of the fractional integral operator on Morrey spaces in metric measure spaces (see [9]). In recent years, Sawano *et al.* have obtained some behaviours of the generalized fractional integral operators and the generalized fractional maximal operators (see [5, 9], respectively). Based on these, we will prove the boundedness of the fractional integral operator and its commutator on the generalized Morrey space $L^{p,\phi}(\mu)$.

Now we recall the notion of 'Regular Bounded Mean Oscillations' RBMO(μ) given in [11].

Definition 3.1. Let $\nu > 1$. A function $f \in L^1_{\mathrm{loc}}(\mu)$ is claimed to be in the space RBMO(μ) if there exist a positive constant C and, for any ball $B \subset \mathcal{X}$, a number f_B such that

$$\frac{1}{\mu(\nu B)}\int_B |f(x) - f_B| \, d\mu(x) \le C \tag{3.1}$$

and, for any two balls B and R such that $B \subset R$,

$$|f_B - f_R| \le C K_{B,R}. \tag{3.2}$$

The infimum of the constants C satisfying (3.1) and (3.2) is defined to be the RBMO(μ) norm of f and denoted by $\|f\|_{\mathrm{RBMO}(\mu)}$.

Next, we recall the definition of the fractional integral operator from [1].

For $0 < \alpha < 1$, $f \in L_b^\infty(\mu)$ and $x \in \mathcal{X}$, the fractional integral operator I_α is defined by

$$I_\alpha f(x) = \int_{\mathcal{X}} \frac{f(y)}{[\lambda(x, d(x,y))]^{1-\alpha}} \mathrm{d}\mu(y), \tag{3.3}$$

where $L_b^\infty(\mu)$ is the space of all $L^\infty(\mu)$ functions with bounded support. Let $b \in \mathrm{RBMO}(\mu)$. Then the commutator of the fractional integral operator $[I_\alpha, b]$ is defined by,

$$[I_\alpha, b](f)(x) := b(x)I_\alpha(f)(x) - I_\alpha(bf)(x). \tag{3.4}$$

In [19], Sihwaningrum and Sawano proved that the fractional integral operator is bounded on Morrey spaces in metric measure spaces. However, by applying [19, Theorem 1.2], it is not difficult to see that the fractional integral operator on the generalized Morrey space over $(\mathcal{X}, d, \mu)$ is still valid. Thus, we have the following claim.

Claim. Let $0 < \alpha < 1$, $1 < p < q < \infty$, $1/q = 1/p - \alpha$ and ϕ satisfy (2.2). Suppose that λ satisfies the ϵ-weak reverse doubling condition. Then I_α, as in (3.3), is bounded from $L^{p,\phi}(\mu)$ into $L^{q,\phi^{q/p}}(\mu)$.

Now we state the main theorem in this section.

Theorem 3.2. *Make the same assumption as in the above Claim. Then the commutator $[I_\alpha, b]$, as in (3.4), is bounded from $L^{p,\phi}(\mu)$ into $L^{q,\phi^{q/p}}(\mu)$.*

To prove Theorem 3.2, we need the following lemma, which was given in [6].

Lemma 3.3. *Let $b \in \mathrm{RBMO}(\mu)$, $\alpha \in (0,1)$, $1 < p < \frac{1}{\alpha}$ and $\frac{1}{q} = \frac{1}{p} - \alpha$. If λ satisfies the ϵ-weak reverse doubling condition for some $\epsilon \in (0, \min\{\alpha,\ 1-\alpha, \frac{1}{q}\})$, then $[I_\alpha, b]$ is bounded from $L^p(\mu)$ into $L^q(\mu)$.*

Proof of Theorem 3.2. We decompose $f = f_1 + f_2$, as in the proof of Theorem 2.2, where $f_1 = f\chi_{9B}$ and $f_2 = f - f_1$. Then

$$\|[I_\alpha, b]f\|_{L^{q,\phi^{q/p}}(\mu)} \le \|[I_\alpha, b]f_1\|_{L^{q,\phi^{q/p}}(\mu)} + \|[I_\alpha, b]f_2\|_{L^{q,\phi^{q/p}}(\mu)} =: \mathrm{H}_1 + \mathrm{H}_2.$$

Applying Lemma 3.3, it is not difficult to get that

$$\begin{aligned} H_1 &\le C\|b\|_{\mathrm{RBMO}}(\mu) \frac{1}{[\phi(\mu(12B))]^{1/p}} \Big(\int_{6B} |f(x)|^p \mathrm{d}\mu(x) \Big)^{1/p} \\ &\le C\|b\|_{\mathrm{RBMO}(\mu)} \|f\|_{L^{p,\phi}(\mu)}. \end{aligned}$$

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For $x \in B$, by the Hölder inequality, $\frac{1}{q} = \frac{1}{p} - \alpha$, (1.4) and [7, Corollary 3.3],

$$\begin{aligned}
|[I_\alpha, b](f_2)(x)| &\le C \sum_{k=1}^{\infty} \int_{6^{k+1}B \setminus 6^k B} \frac{|f(y)|}{[\lambda(x, d(x,y))]^{1-\alpha}} |b(x) - b(y)| \,\mathrm{d}\mu(y) \\
&\le C \sum_{k=1}^{\infty} |b(x) - m_{\widetilde{6^{k+1}B}}(b)| \int_{6^{k+1}B} \frac{|f(y)|}{[\lambda(x, d(x,y))]^{1-\alpha}} \,\mathrm{d}\mu(y) \\
&\quad + C \sum_{k=1}^{\infty} \int_{6^{k+1}B} \frac{|f(y)|}{[\lambda(x, d(x,y))]^{1-\alpha}} |b(y) - m_{\widetilde{6^{k+1}B}}(b)| \,\mathrm{d}\mu(y) \\
&\le C \sum_{k=1}^{\infty} |b(x) - m_{\widetilde{6^{k+1}B}}(b)| \frac{\mu(6^{k+1}B)^{1-\frac{1}{p}}}{[\lambda(c_B, 6^k r_B)]^{1-\alpha}} \bigg(\int_{6^{k+1}B} |f(y)|^p \,\mathrm{d}\mu(y) \bigg)^{1/p} \\
&\quad + C \sum_{k=1}^{\infty} \frac{1}{[\lambda(c_B, 6^k r_B)]^{1-\alpha}} \bigg(\int_{6^{k+1}B} |f(y)|^p \,\mathrm{d}\mu(y) \bigg)^{1/p} \\
&\quad \times \bigg(\int_{6^{k+1}B} |b(y) - m_{\widetilde{6^{k+1}B}}(b)|^{p'} \,\mathrm{d}\mu(y) \bigg)^{1/p'} \\
&\le C \|f\|_{L^{p,\phi}(\mu)} [\lambda(c_B, r_B)]^{\alpha-1} \sum_{k=1}^{\infty} [\|b\|_{\mathrm{RBMO}(\mu)} + |b(x) - m_{\widetilde{6^{k+1}B}}(b)|] \\
&\quad \times \frac{\mu(6^{k+1}B)}{[C(6^k)]^{1-\alpha}} \bigg[\frac{\phi(\mu(2 \times 6^{k+1}B))}{\mu(6^{k+1}B)} \bigg]^{1/p}.
\end{aligned}$$

Further, by applying (1.1), (1.5) and (2.2),

$$\begin{aligned}
\mathrm{H}_2 &\le C \|b\|_{\mathrm{RBMO}(\mu)} \|f\|_{L^{p,\phi}(\mu)} [\lambda(c_B, r_B)]^{\alpha-1} \frac{[\mu(B)]^{1/q}}{[\phi(\mu(12B))]^{1/p}} \\
&\quad \times \sum_{k=1}^{\infty} \frac{\mu(6^{k+1}B)}{[C(6^k)]^{1-\alpha}} \bigg[\frac{\phi(\mu(2 \times 6^{k+1}B))}{\mu(6^{k+1}B)} \bigg]^{1/p} \\
&\quad + C \|f\|_{L^{p,\phi}(\mu)} \frac{[\lambda(c_B, r_B)]^{\alpha-1}}{[\phi(\mu(12B))]^{1/p}} \sum_{k=1}^{\infty} \frac{\mu(6^{k+1}B)}{[C(6^k)]^{1-\alpha}} \bigg[\frac{\phi(\mu(2 \times 6^{k+1}B))}{\mu(6^{k+1}B)} \bigg]^{1/p} \\
&\quad \times \bigg(\int_B |b(x) - m_{\widetilde{6^{k+1}B}}(f)|^q \,\mathrm{d}\mu(x) \bigg)^{1/q} \\
&\le C \|b\|_{\mathrm{RBMO}(\mu)} \|f\|_{L^{p,\phi}(\mu)} [\lambda(c_B, r_B)]^{\alpha-1} \frac{[\mu(B)]^{1/q}}{[\mu(12B)]^{1/p}} \lambda(c_B, r_B) \\
&\quad \times \sum_{k=1}^{\infty} \frac{1+k}{[C(6^k)]^{1-\alpha}} \bigg[\frac{[\mu(12B)]}{[\phi(\mu(12B))]} \bigg]^{1/p} \bigg[\frac{\phi(\mu(2 \times 6^{k+1}B))}{\mu(6^{k+1}B)} \bigg]^{1/p}
\end{aligned}$$

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$$\begin{aligned}
&\le C\|b\|_{\mathrm{RBMO}(\mu)}\|f\|_{L^{p,\phi}(\mu)}[\lambda(c_B, r_B)]^{\alpha-1}[\lambda(c_B, r_B)]^{1/q-1/p}\lambda(c_B, r_B)\\
&\quad\times\sum_{k=1}^{\infty}\frac{1+k}{[C(6^k)]^{1-\alpha}}\bigg[\frac{[\mu(12B)]}{[\phi(\mu(12B))]}\bigg]^{1/p}\bigg[\frac{\phi(\mu(2\times 6^{k+1}B))}{\mu(6^{k+1}B)}\bigg]^{1/p}\\
&\le C\|b\|_{\mathrm{RBMO}(\mu)}\|f\|_{L^{p,\phi}(\mu)},
\end{aligned}$$

where we used the fact that

$$|m_{\widetilde{B}}(b) - m_{\widetilde{6^{k+1}B}}(b)| \le k\|b\|_{\mathrm{RBMO}(\mu)}.$$

Combining the estimates for H_1 and H_2, the proof of Theorem 3.2 is complete. □

4. Marcinkiewicz integral operator

In this section, we will discuss the boundedness of the Marcinkiewicz integral operator. First, we recall the notion of the Marcinkiewicz integral operator (see [13]).

Definition 4.1. Let $K(x, y)$ be a locally integrable function on $(\mathcal{X}\times\mathcal{X})\setminus\{(x, x) : x \in \mathcal{X}\}$. Assume that there exists a positive constant C such that, for all $x, y \in \mathcal{X}$ with $x \ne y$,

$$|K(x, y)| \le C\frac{d(x, y)}{\lambda(x, d(x, y))} \tag{4.1}$$

and, for all $y, y' \in \mathcal{X}$,

$$\int_{d(x,y)\ge 2d(y,y')}[|K(x, y) - K(x, y')| + |K(y, x) - K(y', x)|]\frac{\mathrm{d}\mu(x)}{\mathrm{d}(x, y)} \le C.$$

Associated with the above kernel K, the Marcinkiewicz integral $\mathcal{M}(f)$ is defined by

$$\mathcal{M}(f)(x) := \bigg(\int_0^{\infty}|T[\chi_{B(x,t)}f](x)|^2\frac{\mathrm{d}t}{t^3}\bigg)^{1/2} \quad x \in \mathcal{X}, \tag{4.2}$$

where $T[\chi_{B(x,t)}f](x) = \int_{d(x,y)\le t}K(x, y)f(y)\,\mathrm{d}\mu(y)$ and $B(x, t) := \{y \in \mathcal{X} : d(x, y) \le t\}$.

Theorem 4.2. *Let $1 < p < \infty$ and let $\mathcal{M}$ be as in (4.2). Suppose that ϕ is a function satisfying (2.2). Then $\mathcal{M}$ is bounded on $L^{p,\phi}(\mu)$: that is, there exists a positive constant C, such that for all $f \in L^{p,\phi}(\mu)$,*

$$\|\mathcal{M}(f)\|_{L^{p,\phi}(\mu)} \le C\|f\|_{L^{p,\phi}(\mu)}.$$

To prove the above theorem, we need the following lemma, which was given in [13].

Lemma 4.3. *Suppose that $\mathcal{M}$, as in (4.2), is bounded on $L^{p_0}(\mu)$ space for some $p_0 \in (1, \infty)$. Then $\mathcal{M}$ is bounded on $L^p(\mu)$ spaces for all $p \in (1, \infty)$.*

Proof of Theorem 4.2. For a fixed doubling ball B, $f \in L^{p,\phi}(\mu)$, let $f(x) = f_1(x) + f_2(x)$, where $f_1(x) = f\chi_{6B}(x)$. Write

$$\|\mathcal{M}(f)\|_{L^{p,\phi}(\mu)} \le \|\mathcal{M}(f_1)\|_{L^{p,\phi}(\mu)} + \|\mathcal{M}(f_2)\|_{L^{p,\phi}(\mu)} =: \mathrm{I} + \mathrm{II}.$$

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By the $L^p(\mu)$-boundedness of $\mathcal{M}$, it is not difficult to get

$$\mathrm{I} \le C\Big(\frac{1}{\phi(\mu(12B))}\int_{6B}|f(x)|^p\,\mathrm{d}\mu(x)\Big)^{1/p} \le C\|f\|_{L^{p,\phi}(\mu)}.$$

For II. Denote the center and radius of B by c_B and r_B, respectively. Decompose

$$\begin{aligned}\mathcal{M}(f_2)(x) &\le \Big(\int_0^{d(c_B,y)+r_B}|T[\chi_{B(x,t)}f_2](x)|^2\frac{\mathrm{d}t}{t^3}\Big)^{1/2}\\ &\quad+\Big(\int_{d(c_B,y)+r_B}^{\infty}|T[\chi_{B(x,t)}f_2](x)|^2\frac{\mathrm{d}t}{t^3}\Big)^{1/2}\\ &=: \mathrm{II}_1+\mathrm{II}_2.\end{aligned}$$

From (1.1) and (1.2), we deduce that, for any ball B, $y\notin kB$ with $k\in(1,\infty)$ and $x\in B$,

$$\lambda(c_B,d(y,c_B))\sim\lambda(y,d(y,c_B))\sim\lambda(x,d(x,y)),$$

which, together with the Hölder inequality and (4.1), gives

$$\begin{aligned}\mathrm{II}_1 &\le C\int_{\mathcal{X}}\frac{d(x,y)}{\lambda(x,d(x,y))}|f_2(y)|\Big(\int_{d(x,y)}^{d(c_B,y)+r_B}\frac{\mathrm{d}t}{t^3}\Big)^{1/2}\mathrm{d}\mu(y)\\ &\le C\int_{\mathcal{X}}\frac{d(x,y)}{\lambda(x,d(x,y))}|f_2(y)|\Big(\frac{r_B}{[d(x,y)]^3}\Big)^{1/2}\mathrm{d}\mu(y)\\ &\le C\sum_{k=1}^{\infty}\int_{6^{k+1}B\setminus 6^kB}\frac{r_B^{1/2}|f(y)|}{[d(c_B,y)]^{1/2}\lambda(c_B,d(c_B,y))}\,\mathrm{d}\mu(y)\\ &\le C\sum_{k=1}^{\infty}6^{-k/2}\frac{1}{\lambda(c_B,6^kr_B)}\int_{6^{k+1}B}|f(y)|\,\mathrm{d}\mu(y)\\ &\le C\sum_{k=1}^{\infty}6^{-k/2}\Big[\frac{\phi(\mu(2\times6^{k+1}B))}{\mu(2\times6^{k+1}B)}\Big]^{1/p}\|f\|_{L^{p,\phi}(\mu)}.\end{aligned}$$

By means of a similar method to that used in the proof of II_1,

$$\begin{aligned}\mathrm{II}_2 &\le C\int_{\mathcal{X}\setminus6B}\frac{1}{\lambda(c_B,d(c_B,y))}|f(y)|\,\mathrm{d}\mu(y)\\ &\le C\sum_{k=1}^{\infty}\frac{1}{\lambda(c_B,6^kr_B)}\int_{2^{k+1}B}|f(y)|\,\mathrm{d}\mu(y)\\ &\le C\sum_{k=1}^{\infty}\frac{\mu(6^{k+1}B)^{1-1/p}}{\lambda(c_B,6^kr_B)}\phi(\mu(2\times6^{k+1}B))^{1/p}\|f\|_{L^{p,\phi}(\mu)}\\ &\le C\sum_{k=1}^{\infty}\Big[\frac{\phi(\mu(2\times6^{k+1}B))}{\mu(2\times6^{k+1}B)}\Big]^{1/p}\|f\|_{L^{p,\phi}(\mu)}.\end{aligned}$$

Combining the estimates for II_1 and II_2,

$$\mathcal{M}(f_2)(x) \le C\|f\|_{L^{p,\phi}(\mu)} \sum_{k=1}^{\infty}(6^{-k/2}+1)\left[\frac{\phi(\mu(2\times 6^{k+1}B))}{\mu(2\times 6^{k+1}B)}\right]^{1/p}.$$

Using this and (2.2), we can conclude that

$$\begin{aligned}\mathrm{II} &\le C\|f\|_{L^{p,\phi}(\mu)} \sum_{k=1}^{\infty} 6^{-k/2}\left(\frac{\mu(12B)}{\phi(\mu(12B))}\right)^{1/p}\left[\frac{\phi(\mu(2\times 6^{k+1}B))}{\mu(2\times 6^{k+1}B)}\right]^{1/p}\\ &\le C\|f\|_{L^{p,\phi}(\mu)}.\end{aligned}$$

Thus, we have completed the proof of Theorem 4.2. □

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