ON A CLASS OF TRUTH-VALUE EVALUATIONS OF THE PRIMITIVE LOGIC

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Introduction

Main purpose of the present paper is to exhibit a class of truth-value evaluations of the primitive logic ***LO*** and its sentential part ***LOS***[1].

LO is the logic having two logical constants, IMPLICATION $\rightarrow$ and UNIVERSAL QUANTIFICATION ( ), and the following inference rules:

F: $\mathfrak{A}$ *is deducible from* $\mathfrak{A}$.

I: $\mathfrak{B}$ *is deducible from* $\mathfrak{A}$ *and* $\mathfrak{A} \rightarrow \mathfrak{B}$.

I*: $\mathfrak{A} \rightarrow \mathfrak{B}$ *is deducible from the fact that* $\mathfrak{B}$ *is deducible from* $\mathfrak{A}$.

U: $\mathfrak{A}(t)$ *is deducible from* $(x)\mathfrak{A}(x)$.

U*: $(x)\mathfrak{A}(x)$ *is deducible from the fact that* $\mathfrak{A}(t)$ *is deducible for any variable* t *whatever.*

LOS is the logic having the sole logical constant IMPLICATION $\rightarrow$ and the inference rules ***F***, ***I***, and ***I**** only.

The domain of truth-values can be regarded as the value-domain of an evaluation ***E*** which associates to every proposition its truth-value. Let ***B*** be the class of propositions. Then, any proposition variable p can be regarded as a variable running over ***B***, and its evaluation p^* can be regarded as a variable running over the domain ***D*** of truth-values. To logical constants, we associate combinations or operations which are so defined that ***E*** is a homomorphism of ***B*** into ***D***. Any mapping ***E*** (***B*** into ***D***) of this kind is called a SEMI-EVALUATION of the logic. The homomorphic image of a proposition $\mathfrak{p}$ by ***E*** is called the ***E***-image of $\mathfrak{p}$ and denoted by ***E***($\mathfrak{p}$) or simply by $\mathfrak{p}^*$. In the following, I will denote the ***E***-images of logical constants by the same notations as the original logical constants unless there is a fear of ambiguity.

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1) See ONO [1].

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Any semi-evaluation $\boldsymbol{E}$ of a logic is called an EVALUATION of the logic if and only if $\boldsymbol{E}(\mathfrak{p}) = 0$ holds identically whenever $\mathfrak{p}$ is a proposition provable in the logic.

In (**1**), I will prove a few theorems concerning evaluations of ***LOS*** and ***LO***.

THEOREM 1 gives a sufficient condition for that a semi-evaluation of ***LOS*** is an evaluation of it. The condition is

$\boldsymbol{E}1$: $p^* \to 0 = 0$,

$\boldsymbol{E}2$: $p^* \to p^* = 0$,

$\boldsymbol{E}3$: $0 \to p^* = p^*$,

$\boldsymbol{E}4$: $p^* \to (p^* \to q^*) = p^* \to q^*$,

$\boldsymbol{E}5$: $p^* \to (q^* \to r^*) = q^* \to (p^* \to r^*)$,

$\boldsymbol{E}6$: $p^* \to q^* = 0$ *implies* $(r^* \to p^*) \to (r^* \to q^*) = 0$.

THEOREM 2 is a theorem corresponding to THEOREM 1 for the logic ***LO***.

THEOREM 3 shows a way to construct an evaluation out of a class of evaluations satisfying the conditions in THEOREM 1 or in THEOREM 2.

EXAMPLES 1——4 of (**2**) give two extreme classes of evaluations of ***LOS*** and ***LO***. THEOREM 3 indicates a way of constructing a class of evaluations lying between them.

A sufficient condition for that any proposition $\mathfrak{p}$ satisfying $\boldsymbol{E}(\mathfrak{p}) = 0$ identically is identically true in the ordinary two-valued logic is given in THEOREM 4 (for ***LOS***) and in THEOREM 5 (for ***LO***). As well known, any proposition of the ordinary two-valued sentential logic is provable in the classical sentential logic, so Theorem 4 gives a sufficient condition for that any proposition $\mathfrak{p}$ of a sentential logic satisfying $\boldsymbol{E}(\mathfrak{p}) = 0$ is provable in the classical logic. The condition is only that the domain of the evaluation $\boldsymbol{E}$ has at least two members.

In (**2**), I will give some examples of evaluations of ***LOS*** and ***LO*** together with some remarks about them.

In the present paper, I adopt the practical way ***PD*** for describing formal deductions introduced in my paper [2]. Concerning proof-notes, I will use the nomenclature introduced in my paper [3].

(1) Theorems on evaluations of the primitive logic.

THEOREM 1. *Let* $\boldsymbol{E}$ *be any semi-evaluation of the logic* $\boldsymbol{LOS}$ *satisfying identically the following conditions:*

$\boldsymbol{E}1$: $p^* \to 0 = 0$,

$\boldsymbol{E}2$: $p^* \to p^* = 0$,

$\boldsymbol{E}3$: $0 \to p^* = p^*$,

$\boldsymbol{E}4$: $p^* \to (p^* \to q^*) = p^* \to q^*$,

$\boldsymbol{E}5$: $p^* \to (q^* \to r^*) = q^* \to (p^* \to r^*)$,

$\boldsymbol{E}6$: $p^* \to q^* = 0$ *implies* $(r^* \to p^*) \to (r^* \to q^*) = 0$.

Then, $\boldsymbol{E}$ *is an evaluation of* $\boldsymbol{LOS}$.

Proof. Let Π be any proof-note in $\boldsymbol{LOS}$ arranged in the fundamental order of steps[2], and $\mathfrak{s}$ be any step in Π. Let $\mathfrak{p}$ be any provable proposition of the step $\mathfrak{s}$ and $\{\mathfrak{p}_1, \ldots, \mathfrak{p}_n\}$ be the whole set of propositions of assumption steps of $\mathfrak{s}$. Further, let $\mathfrak{q}_1, \ldots, \mathfrak{q}_n$ be any sequence of propositional variables. Then, I assert that the $\boldsymbol{E}$-image of $\mathfrak{q}_1 \to (\mathfrak{q}_2 \to (\ldots \to (\mathfrak{q}_n \to \mathfrak{p}) \ldots))$ (this proposition is called the proposition associated to the step $\mathfrak{s}$ and is denoted simply by $\mathfrak{q}_1, \ldots, \mathfrak{q}_n \to \mathfrak{p}$ from now on) is identically equal to 0 as far as $\mathfrak{q}_1^* \to \mathfrak{p}_1^* = 0, \ldots, \mathfrak{q}_n^* \to \mathfrak{p}_n^* = 0$.

I will prove this assertion by complete induction as it holds true for *the first step* of Π because *the first step must be an assumption step of itself* and *this assertion holds true for any step having the same proposition as its assumption step* by virtue of $\boldsymbol{E}1, \boldsymbol{E}2,$ and $\boldsymbol{E}5$. So, I will prove that *this assertion holds true for any step* $\mathfrak{s}$ *of* Π *as far as it holds true for every step of* Π *standing before* $\mathfrak{s}$. *If* $\mathfrak{s}$ *is an assumption step of itself,* the assertion holds true as has been remarked above.

If $\mathfrak{s}$ *is a deduced step,* it must be deduced by the inference rule $\boldsymbol{F}$ (*Case* $\boldsymbol{F}$), deduced by the inference rule $\boldsymbol{I}$ (*Case* $\boldsymbol{I}$), or deduced by the inference rule $\boldsymbol{I}^*$ (*Case* $\boldsymbol{I}^*$).

Case $\boldsymbol{F}$, *where* $\mathfrak{s}$ *is deduced from a step* $\mathfrak{u}$ *by the inference rule* $\boldsymbol{F}$*:* The set

[2] Any step of the form $\mathfrak{r}A$ is called an *assumption step of* $\mathfrak{s}$ if and only if $\mathfrak{s}$ can be expressed in the form $\mathfrak{r}\mathfrak{u}\mathfrak{w}$. If we normalize the lengths of index-words occurring in a proof-note by adjoining suitable numbers of $\diamond$'s at their ends and arrange the steps of the proof-note according to the lexicographic order of their normalized index-words regarding $\diamond$ as the last letter, we have the *fundamental order* of steps. As for details, see ONO [3].

$\{\mathfrak{p}_1, \ldots, \mathfrak{p}_n\}$ of the propositions of assumption steps of $\underline{\mathfrak{s}}$ includes the set of the propositions of assumption steps of $\underline{\mathfrak{u}}$. Hence, by virtue of **E1, E2,** and **E5,** the expression $\mathfrak{q}_1^*, \ldots, \mathfrak{q}_n^* \rightarrow \mathfrak{p}^*$ is identically equal to 0 as far as $\mathfrak{q}_i^* \rightarrow \mathfrak{p}_i^* = 0$ for $i = 1, \ldots, n$.

Case ***I**, where $\underline{\mathfrak{s}}$ is deduced from the steps $\underline{\mathfrak{u}}$ and $\underline{\mathfrak{v}}$*: Let the propositions of $\underline{\mathfrak{u}}$ and $\underline{\mathfrak{v}}$ be $\mathfrak{h}$ and $\mathfrak{h} \rightarrow \mathfrak{k}$, respectively. Then, the proposition of $\underline{\mathfrak{s}}$ is $\mathfrak{k}$. Now, the set $\{\mathfrak{p}_1, \ldots, \mathfrak{p}_n\}$ includes the set of the propositions of assumption steps of $\underline{\mathfrak{u}}$ as well as that of $\underline{\mathfrak{v}}$. Accordingly, for any specification of truth-values of variables satisfying $\mathfrak{q}_i^* \rightarrow \mathfrak{p}_i^* = 0$ for every $i = 1, \ldots, n$, the formulas $\mathfrak{q}_1^*, \ldots, \mathfrak{q}_n^* \rightarrow \mathfrak{h}^* = 0$ and $\mathfrak{q}_1^*, \ldots, \mathfrak{q}_n^* \rightarrow (\mathfrak{h}^* \rightarrow \mathfrak{k}^*) = 0$ hold by virtue of our induction assumption, **E4**, and **E5**. The latter implies $(\mathfrak{q}_1^*, \ldots, \mathfrak{q}_n^* \rightarrow \mathfrak{h}^*) \rightarrow (\mathfrak{q}_1^*, \ldots, \mathfrak{q}_n^* \rightarrow \mathfrak{k}^*) = 0$ by virtue of **E4, E5,** and **E6.** Hence, $0 \rightarrow (\mathfrak{q}_1^*, \ldots, \mathfrak{q}_n^* \rightarrow \mathfrak{k}^*)$ must be equal to 0 for this specification. On the other hand, this must be equal to $\mathfrak{q}_1^*, \ldots, \mathfrak{q}_n^* \rightarrow \mathfrak{k}^*$ according to **E3**. Consequently, $\mathfrak{q}_1^*, \ldots, \mathfrak{q}_n^* \rightarrow \mathfrak{k}^* = 0$ holds for this specification.

Case ***I*****, where $\underline{\mathfrak{s}}$ is deduced from the fact that the step $\underline{\mathfrak{s}}\epsilon$ is deducible from the step $\underline{\mathfrak{s}}A$*: Let the propositions of $\underline{\mathfrak{s}}A$ and $\underline{\mathfrak{s}}\epsilon$ be $\mathfrak{h}$ and $\mathfrak{k}$, respectively. Then, the proposition of $\underline{\mathfrak{s}}$ is $\mathfrak{h} \rightarrow \mathfrak{k}$. The propositions associated to $\underline{\mathfrak{s}}\epsilon$ and $\underline{\mathfrak{s}}$ are of the forms

$$\mathfrak{q}_1, \ldots, \mathfrak{q}_n, \mathfrak{q}_{n+1} \rightarrow \mathfrak{k} \quad \text{and} \quad \mathfrak{q}_1, \ldots, \mathfrak{q}_n \rightarrow (\mathfrak{h} \rightarrow \mathfrak{k}),$$

respectively. The **E**-image of the former is identically equal to 0 as far as $\mathfrak{q}_1^* \rightarrow \mathfrak{p}_1^* = 0, \ldots, \mathfrak{q}_n^* \rightarrow \mathfrak{p}_n^* = 0$, and $\mathfrak{q}_{n+1}^* \rightarrow \mathfrak{h}^* = 0$. By taking $\mathfrak{q}_{n+1}^* = \mathfrak{h}^*$ and taking **E2** into consideration, we can see that

$$\mathfrak{q}_1^*, \ldots, \mathfrak{q}_n^*, \mathfrak{h}^* \rightarrow \mathfrak{k}^* = 0 \quad \textit{i.e.} \quad \mathfrak{q}_1^*, \ldots, \mathfrak{q}_n^* \rightarrow (\mathfrak{h}^* \rightarrow \mathfrak{k}^*) = 0$$

holds as far as $\mathfrak{q}_1^* \rightarrow \mathfrak{p}_1^* = 0, \ldots, \mathfrak{q}_n^* \rightarrow \mathfrak{p}_n^* = 0$ hold.

THEOREM 2. *Let* ***E*** *be any semi-evaluation of the logic* ***LO*** *satisfying identically the following conditions:*

E1: $p^* \rightarrow 0 = 0$,

E2: $p^* \rightarrow p^* = 0$,

E3: $0 \rightarrow p^* = p^*$,

E4: $p^* \rightarrow (p^* \rightarrow q^*) = p^* \rightarrow q^*$,

E5: $p^* \rightarrow (q^* \rightarrow r^*) = q^* \rightarrow (p^* \rightarrow r^*)$,

E6: $p^* \rightarrow q^* = 0$ *implies* $(r^* \rightarrow p^*) \rightarrow (r^* \rightarrow q^*) = 0$,

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E 7: $(x)p^*(x) \to p^*(t) = 0$,

E 8: *If* $u^* \to p^*(t) = 0$ *for every* t, *then* $u^* \to (x)p^*(x) = 0$.

Then, **E** *is an evaluation of* **LO**.

Proof. Let Π be any proof-note of **LO** arranged in the fundamental order of steps, and $\underline{s}$ be any step in Π. Let $\mathfrak{p}$ be the proposition of the step $\underline{s}$, and $\{\mathfrak{p}_1, \ldots, \mathfrak{p}_n\}$ be the whole set of propositions of the assumption steps of $\underline{s}$ skipping over denomination steps[3].

Let $\mathfrak{q}_1, \ldots, \mathfrak{q}_n$ be any sequence of proposition variables. Then, I can assert as in the proof of the preceding theorem that the **E**-image of the proposition $\mathfrak{q}_1, \ldots, \mathfrak{q}_n \to \mathfrak{p}$ associated to the step $\underline{s}$ is identically equal to 0 as far as $\mathfrak{q}_1^* \to \mathfrak{p}_1^* = 0, \ldots, \mathfrak{q}_n^* \to \mathfrak{p}_n^* = 0$. This assertion can be proved also by *complete induction*, since *this assertion can be proved easily for any assumption step* $\underline{s}$. Accordingly, let us assume that *the assertion holds true for any step of* Π *standing before* $\underline{s}$. If $\underline{s}$ is a *deduced step*, it must be deduced by the inference rule **F**, **I**, or **I*** (*Case* **F**, *Case* **I**, or *Case* **I***, respectively), or it is deduced by the inference rules **U** or **U*** (*Case* **U** or *Case* **U***, respectively). In *Cases* **F**, **I**, or **I***, we can prove our assertion for the step $\underline{s}$ quite similarly as we have proved the corresponding facts in the proof of the preceding theorem. So, I will prove our assertion here only in the *Cases* **U** and **U***.

Case **U**, *where the step* $\underline{s}$ *is deduced from a step* $\underline{r}$ *by the inference rule* **U**: The propositions of the steps $\underline{r}$ and $\underline{s}$ are of the forms $(x)\mathfrak{p}(x)$ and $\mathfrak{p}(t)$. Because any assumption step of $\underline{r}$ is also an assumption step of $\underline{s}$, $\mathfrak{q}_1^*, \ldots, \mathfrak{q}_n^* \to (x)\mathfrak{p}^*(x)$ must be identically equal to 0 as far as $\mathfrak{q}_1^* \to \mathfrak{p}_1^* = 0$, $\ldots, \mathfrak{q}_n^* \to \mathfrak{p}_n^* = 0$ according to **E** 1, **E** 4, and **E** 5. On the other hand,

$$(\mathfrak{q}_1^*, \ldots, \mathfrak{q}_n^* \to (x)\mathfrak{p}^*(x)) \to (\mathfrak{q}_1^*, \ldots, \mathfrak{q}_n^* \to \mathfrak{p}^*(t)) = 0$$

by virtue of **E** 6 and **E** 7, so $0 \to (\mathfrak{q}_1^*, \ldots, \mathfrak{q}_n^* \to \mathfrak{p}^*(t)) = 0$, therefore $\mathfrak{q}_1^*, \ldots, \mathfrak{q}_n^* \to \mathfrak{p}^*(t) = 0$, according to **E** 3, holds true as far as $\mathfrak{q}_1^* \to \mathfrak{p}_1^* = 0$, $\ldots, \mathfrak{q}_n^* \to \mathfrak{p}_n^* = 0$.

Case **U***, *where the step* $\underline{s}$ *is deducible from the fact that* $\underline{s}\epsilon$ *is deducible from the denomination* $\underline{s}A$ *of the form* $\forall t!$, *i.e. from the fact that* $\underline{s}\epsilon$ *is deducible for any*

[3] In proof-notes of **LO**, some steps are denominations of the form $\forall t!$. These steps are skipped over in the present proof.

variable t whatever. The propositions of the steps $\mathfrak{s}$ and $\mathfrak{s}\epsilon$ are propositions of the forms $(x)\mathfrak{p}(x)$ and $\mathfrak{p}(t)$, respectively. Because any assumption step of $\mathfrak{s}\epsilon$ except for the denomination $\mathfrak{s}A$ is also an assumption step of $\mathfrak{s}$, we can well assume according to ***E*1**, ***E*4**, and ***E*5** that $\mathfrak{q}_1^*, \ldots, \mathfrak{q}_n^* \to \mathfrak{p}^*(t) = 0$ holds for any variable t whatever as far as $\mathfrak{q}_1^* \to \mathfrak{p}_1^* = 0, \ldots, \mathfrak{q}_n^* \to \mathfrak{p}_n^* = 0$. Consequently, $\mathfrak{q}_1^*, \ldots, \mathfrak{q}_n^* \to (x)\mathfrak{p}^*(x)$ must be identically equal to 0 as far as $\mathfrak{q}_1^* \to \mathfrak{p}_1^* = 0, \ldots, \mathfrak{q}_n^* \to \mathfrak{p}_n^* = 0$, according to ***E*8**.

THEOREM 3. *Let $\{\boldsymbol{E}_i;\, i \in \boldsymbol{J}\}$ be any family of evaluations of* ***LOS*** *(or* ***LO****) satisfying* ***E*1 — *E*6** *(or* ***E*1 — *E*8****) identically. Let $\boldsymbol{D}_i$ be the domain of truth values of the evaluation $\boldsymbol{E}_i$, and 0 of $\boldsymbol{D}_i$ be denoted by 0_i. I will denote the $\boldsymbol{E}_i$–image of $\mathfrak{p}$ by $\boldsymbol{E}_i(\mathfrak{p})$ or $\mathfrak{p}^i$ and the $\boldsymbol{E}_i$–image of $\to$ and ( ) by $\xrightarrow{i}$ and $(\ )^i$, respectively. Let us now define a new semi-evaluation* ***E*** *as follows:*

1) *The domain* ***D*** *of truth-values of* ***E*** *is formed by functions p^* defined over* ***J*** *satisfying $p^*(i) \in \boldsymbol{D}_i$,*

2) *0 of* ***D*** *is defined by $0(i) = 0_i$,*

3) $p^*(i) = p^i$,

4) $(p^* \to q^*)(i) = p^i \xrightarrow{i} q^i$,

5) $((x)p^*(x))(i) = (x)^i p^i(x)$ *(for* ***LO*** *only).*

Then, ***E*** *is an evaluation of* ***LOS*** *(or* ***LO****).*

Proof. It can be proved without difficulty that ***E*1 — *E*6** (or ***E*1 — *E*8**) hold for the semi-evaluation ***E***. Hence, ***E*** is an evaluation of ***LOS*** (or ***LO***) according to THEOREM 1 (or THEOREM 2).

THEOREM 4. *Let* ***E*** *be any evaluation of* ***LOS*** *satisfying* ***E*1 — *E*6**, *and let us assume that the domain* ***D*** *of truth-values of* ***E*** *has at least two members. Then, any proposition whose* ***E***–*image is identically equal to 0 is provable in the classical sentential logic.*

Proof. In the domain ***D***, there are 0 and a member other than 0 because ***D*** is assumed to have at least two members. The member of ***D*** other than 0 is denoted by 1. Then, by virtue of ***E*1**, ***E*2**, and ***E*3**, we can prove

$$0 \to 0 = 1 \to 0 = 1 \to 1 = 0 \quad \text{and} \quad 0 \to 1 = 1.$$

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Namely, the combination $\rightarrow$ behaves just as the IMPLICATION of the ordinary two valued logic with respect to the pair $\{0, 1\}$ of truth-values. It is also remarkable that $\{0, 1\}$ is closed with respect to the combination $\rightarrow$.

Let $\mathfrak{p}$ be any proposition whose $\boldsymbol{E}$-image $\mathfrak{p}^*$ is identically equal to 0. Then, $\mathfrak{p}$ is also identically equal to 0 for any specification of variables which specify them to 0 or 1. Accordingly, $\mathfrak{p}^*$ is identically equal to 0 with respect to the ordinary evaluation of two-valued sentential logic. Because the ordinary two-valued sentential logic is nothing but the classical sentential logic, so $\mathfrak{p}$ must be provable in the classical sentential logic.

THEOREM 5. *Let $\boldsymbol{E}$ be any evaluation of $\boldsymbol{LO}$ satisfying $\boldsymbol{E}1$ — $\boldsymbol{E}8$, and let us assume that the domain $\boldsymbol{D}$ of truth-values of $\boldsymbol{E}$ has a member* 1 *such that*

1) $1 \neq 0$,

2) $1 \rightarrow u^* = 0$ *implies* $u^* \rightarrow 1 = 0$ *or* $u^* = 0$.

Then, any proposition whose $\boldsymbol{E}$-image is identically equal to 0 *is also identically true in the usual two-valued logic.*

Proof. Let $\mathfrak{p}$ be any proposition whose $\boldsymbol{E}$-image is identically equal to 0 with respect to an evaluation $\boldsymbol{E}$ satisfying the above mentioned condition. Let us take up the sub-domain $\boldsymbol{D}_0 = \{u^*; 1 \rightarrow u^* = 0\}$. Then, clearly $0, 1 \in \boldsymbol{D}_0$. Moreover, $\boldsymbol{D}_0$ is closed with respect to the logical operations $\rightarrow$ and $(\ )$. Hence, we have an evaluation $\boldsymbol{E}_0$ of $\boldsymbol{LO}$ satisfying $\boldsymbol{E}1$ — $\boldsymbol{E}8$ by restricting its value domain to $\boldsymbol{D}_0$. In the domain $\boldsymbol{D}_0$, we can replace every member of $\boldsymbol{D}_0$ other than 0 by 1 keeping $\boldsymbol{E}1$ — $\boldsymbol{E}8$ and $\boldsymbol{p}^* = 0$ identically true. Hence, $\boldsymbol{p}^* = 0$ holds identically for the usual evaluation of the two valued logic.

(2) Examples and remarks.

Remark 1. The domain $\boldsymbol{D}$ of truth-values satisfying the conditions $\boldsymbol{E}1$ — $\boldsymbol{E}6$ or $\boldsymbol{E}1$ — $\boldsymbol{E}8$ is almost a partly ordered system with the minimum member 0 with respect to the ordering $p \geq q$ defined by $p \rightarrow q = 0$. This relation $\geq$ is really reflexive and transitive, but we can not assert that $p \geq q$ and $q \geq p$ imply $p = q$. Naturally, $p \geq 0$ is always true, and more over, we can deduce $p = 0$ from $0 \geq p$, for this relation $\geq$.

EXAMPLE 1. Let $\boldsymbol{D}$ be any semi-lattice having for any pair of its members the union of the pair and satisfying the following condition: *For any pair of members* p^* *and* q^* *of* $\boldsymbol{D}$, *there exists a member* u^* *satisfying*

1) $p^* \cup u^* \geqq q^*$,

2) *For any* w^*, $p^* \cup w^* \geqq q^*$ *implies* $w^* \geqq u^*$.

The member u^* *uniquely determined by* p^* *and* q^* *is denoted by* $p^* \rightarrow q^*$.

The member $p^* \rightarrow p^*$ of $\boldsymbol{D}$ can be proved to be independent of p^*. It is proved to be the minimum member of $\boldsymbol{D}$, so I will denote it by 0. With respect to the special member 0 and the combination $\rightarrow$, the conditions $\boldsymbol{E}1$—$\boldsymbol{E}6$ can be proved to hold. So, we can define an evaluation $\boldsymbol{E}_s$ of $\boldsymbol{LOS}$ by making use of the domain $\boldsymbol{D}$.

EXAMPLE 2. Let $\boldsymbol{V}$ be any domain of objects and let $\boldsymbol{D}$ be any semi-lattice having for any number of its members the union of them and satisfying the condition for defining the combination $\rightarrow$ in EXAMPLE 1. We deal with functions of any number of variables running over $\boldsymbol{V}$ and having $\boldsymbol{D}$ as their value domain. Composite functional expressions can be constructed starting from expressions of the form $f^*(x, \ldots, z)$ (elementary formulas) by the combination $\rightarrow$ and the operators of the form (x) which stands for $\underset{x \in \boldsymbol{V}}{U}$. Just as in EXAMPLE 1, we can prove that $\boldsymbol{D}$ has its minimum member 0. With respect to this minimum member 0, the combination $\rightarrow$, and the operators of the form (x), the conditions $\boldsymbol{E}1$—$\boldsymbol{E}8$ hold true. So, we can define an evaluation $\boldsymbol{E}_p$ of $\boldsymbol{LO}$ by making use of $\boldsymbol{D}$.

EXAMPLE 3. Let $\boldsymbol{D}$ be any partly ordered system having the minimum member 0. Then, we can define two operators $\underset{\sim}{0}$ and $\underset{\sim}{1}$ over $\boldsymbol{D}$ by $\underset{\sim}{0} \,.\, p^* = 0$ and $\underset{\sim}{1} \,.\, p^* = p^*$. We can further define two-variable function $\boldsymbol{X}$ over $\boldsymbol{D}$ by that $\boldsymbol{X}(p^*, q^*)$ is the operator $\underset{\sim}{0}$ if $p^* \geqq q^*$ and it is the operator $\underset{\sim}{1}$ otherwise. The value domain of $\boldsymbol{X}$ is the pair-set of the operators $\underset{\sim}{0}$ and $\underset{\sim}{1}$. By making use of this function $\boldsymbol{X}$, I will define a combination $\rightarrow$ of members of $\boldsymbol{D}$ by $p^* \rightarrow q^* = \boldsymbol{X}(p^*, q^*) \,.\, q^*$. By this combination $\rightarrow$, we obtain a member of $\boldsymbol{D}$ from any ordered pair of members of $\boldsymbol{D}$. With respect to the minimum member 0 and the combination $\rightarrow$, the conditions $\boldsymbol{E}1$—$\boldsymbol{E}6$ hold. So, we can define an evaluation $\boldsymbol{E}^s$ of $\boldsymbol{LOS}$ by making use of $\boldsymbol{D}$.

EXAMPLE 4. Let $\boldsymbol{V}$ be any domain of objects and let $\boldsymbol{D}$ be any semi-lattice having for any number of its members the union of them. We deal

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with functions of any number of variables running over $\boldsymbol{V}$ and having $\boldsymbol{D}$ as their value-domain. We can define the combination $\rightarrow$ over $\boldsymbol{D}$ just as we have done in EXAMPLE 3. By this combination $\rightarrow$, we obtain a member of $\boldsymbol{D}$ for any ordered pair of members of $\boldsymbol{D}$. Composite functional expressions can be constructed starting from expressions of the form $f^*(x, \ldots, z)$ (elementary formulas) by the combination $\rightarrow$ and the operators of the form (x) which stands for $\bigcup_{x \in V}$. As the union of nullset (a subset of $\boldsymbol{D}$) must be the minimum member of $\boldsymbol{D}$, the domain surely has its minimum member 0. With respect to the minimum member 0, the combination $\rightarrow$, and the operators of the form (x), the conditions $\boldsymbol{E1}$—$\boldsymbol{E8}$ hold true. So, we can define an evaluation $\boldsymbol{E}^p$ of $\boldsymbol{LO}$ by making use of $\boldsymbol{D}$.

Remark 2. EXAMPLE 1 (or EXAMPLE 2) and EXAMPLE 3 (or EXAMPLE 4) show two extremities of evaluations of $\boldsymbol{LOS}$ (or $\boldsymbol{LO}$). To show the difference, let us take $\boldsymbol{D}$ as the class of positive (including 0) valued functions $p^*, q^*, \ldots$ of a variable x running over the closed interval $[0, 1]$. We regard here $p^* \geq q^*$ as denoting $(x)(p^*(x) \geq q^*(x))$, the range of the quantification variable x is $[0,1]$. Take for example $p^*(x) = x$ and $q^*(x) = 1 - x$. Let $\underset{\cdot}{\rightarrow}$ and $\dot{\rightarrow}$ be the $\boldsymbol{E}_s$-image (or $\boldsymbol{E}_p$-image) and $\boldsymbol{E}^s$-image (or $\boldsymbol{E}^p$-image) of $\rightarrow$ of $\boldsymbol{LOS}$ (or $\boldsymbol{LO}$). Then,

$$(p^* \underset{\cdot}{\rightarrow} q^*)(x) = 1 - x \quad \text{in} \quad 0 \leq x \leq \frac{1}{2}$$

$$= 0 \quad \text{in} \quad \frac{1}{2} < x \leq 1.$$

On the other hand,

$$(p^* \dot{\rightarrow} q^*)(x) = 1 - x \quad \text{everywhere in } [0,1].$$

Intermediate evaluations can be constructed by dividing $[0, 1]$ into a number of intervals, defining $\boldsymbol{E}_s$ (or $\boldsymbol{E}_p$) for every intervals, and combining these evaluations by making use of THEOREM 3. For example, let $\boldsymbol{E}^*$ be the combined evaluations of the $\boldsymbol{E}_s$- or $\boldsymbol{E}_p$-evaluations for the intervals $\left[0, \frac{1}{3}\right]$, $\left(\frac{1}{3}, \frac{2}{3}\right)$, and $\left[\frac{2}{3}, 1\right]$. Then,

$$(p^* \underset{\cdot}{\dot{\rightarrow}} q^*)(x) = 1 - x \quad \text{in} \quad 0 \leq x < \frac{2}{3}$$

$$= 0 \quad \text{in} \quad \frac{2}{3} \leq x \leq 1,$$

where $\dashrightarrow$ is the $\boldsymbol{E}^*$-image of $\rightarrow$ of ***LOS*** (or ***LO***). Evidently, $\boldsymbol{E}^*$-image of any proposition lies between $\boldsymbol{E}_s$- and $\boldsymbol{E}^s$-images (or $\boldsymbol{E}_p$- and $\boldsymbol{E}^p$-images) of the same proposition.

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