K. Shiraiwa and M. Kurata
Nagoya Math. J.
Vol. 82 (1981), 83–97

A GENERALIZATION OF A THEOREM OF MAROTTO

KENICHI SHIRAIWA AND MASAHIRO KURATA

In 1975, Li and Yorke [3] found the following fact. Let $f\colon I \to I$ be a continuous map of the compact interval I of the real line $\boldsymbol{R}$ into itself. If f has a periodic point of minimal period three, then f exhibits chaotic behavior. The above result is generalized by F.R. Marotto [4] in 1978 for the multi-dimensional case as follows. Let $f\colon \boldsymbol{R}^n \to \boldsymbol{R}^n$ be a differentiable map of the n-dimensional Euclidean space $\boldsymbol{R}^n$ $(n \geqq 1)$ into itself. If f has a snap-back repeller, then f exhibits chaotic behavior.

In this paper, we give a generalization of the above theorem of Marotto. Our theorem can also be regarded as a generalization of the Smale's results [6] on the transversal homoclinic point of a diffeomorphism.

§1. The Main Theorem

Let M be a smooth manifold of dimension n. We denote by $T_x(M)$ the tangent space of M at a point x of M. Let $f\colon M \to M$ be a C^1-map. The tangent map of f at $x \in M$ is denoted by $T_xf\colon T_x(M) \to T_{f(x)}(M)$.

Let $z_0 \in M$ be a fixed point of f. Then f is called a hyperbolic fixed point if all the modulus of the eigenvalues of $T_{z_0}f\colon T_{z_0}(M) \to T_{z_0}(M)$ are different from 0 and 1. Define E^s(resp. E^u) to be the direct sum of the generalized eigenspaces of $T_{z_0}f$ which correspond to the eigenvalues of modulus less than 1 (resp. greater than 1). Then E^s and E^u are $T_{z_0}f$-invariant vector subspaces of $T_{z_0}(M)$, and $T_{z_0}(M) = E^s \oplus E^u$.

Let $s = \dim E^s$ and $u = \dim E^u$. Fix a norm $\|\cdot\|$ on E^s and E^u. For $r_0 > 0$, we define $E^s(r_0) = \{x \in E^s;\ \|x\| \leqq r_0\}$ and $E^u(r_0) = \{x \in E^u;\ \|x\| \leqq r_0\}$. By stable manifold theorem, it is known that there are embeddings $\varphi^s\colon E^s(r_0) \to M$ and $\varphi^u\colon E^u(r_0) \to M$ for sufficiently small $r_0 > 0$ satisfying the following conditions.

(1) $\varphi^s(0) = z_0$ and $\varphi^u(0) = z_0$

(2) $T_0\varphi^s\colon T_0(E^s(r_0)) = E^s \to T_{z_0}(M)$ and $T_0\varphi^u\colon T_0(E^u(r_0)) = E^u \to T_{z_0}(M)$ are injections.

Received July 3, 1979.

(3) If we put $W^s_{\rm loc}(z_0) = \varphi^s(E^s(r_0))$ and $W^u_{\rm loc}(z_0) = \varphi^u(E^u(r_0))$, then there exists a neighborhood U of z_0 such that $W^s_{\rm loc}(z_0) = \{x \in U;\ \lim_{n\to\infty} f^n(x) = z_0\}$ and $W^u_{\rm loc}(z_0) = \{x \in U;\ \lim_{n\to\infty} f^{-n}(x) = z_0\}$.

We call such $W^s_{\rm loc}(z_0)$ (resp. $W^u_{\rm loc}(z_0)$) a local stable (resp. unstable) manifold of f at z_0.

Now we fix a metric d on M.

MAIN THEOREM. *Let f: $M \to M$ be a C^1-map. Let $z_0 \in M$ be a hyperbolic fixed point of f. Now we assume the following three conditions.*

(1) $u = \dim E^u > 0$.

(2) *There exist a point $z_1 \in W^u_{\rm loc}(z_0)$ ($z_1 \neq z_0$) and a positive integer m such that $f^m(z_1) \in W^s_{\rm loc}(z_0)$.*

(3) *There exists a u-dimensional disk B^u embedded in $W^u_{\rm loc}(z_0)$ such that B^u is a neighborhood of z_1 in $W^u_{\rm loc}(z_0)$, $f^m | B^u$: $B^u \to M$ is an embedding, and $f^m(B^u)$ intersects $W^s_{\rm loc}(z_0)$ transversally at $f^m(z_1)$.*

Then the following conclusion holds.

(a) *There is a positive integer N such that there is a periodic point of f of minimal period p for any integer $p \geqq N$.*

(b) *There is an uncountable set S (called a scrambled set) in M satisfying the following conditions.*

(i) *S does not contain any periodic points.*

(ii) $f(S) \subset S$

(iii) $\limsup_{k\to\infty} d(f^k(x), f^k(y)) > 0$ *for any x, $y \in S$ ($x \neq y$).*

(iv) $\limsup_{k\to\infty} d(f^k(x), f^k(y)) > 0$ *for any $x \in S$ and a periodic point y.*

(v) *There is an uncountable subset S_0 contained in S such that*

$$\liminf_{k\to\infty} d(f^k(x), f^k(y)) = 0 \qquad \text{for any } x,\ y \in S_0 .$$

Remark 1. The above theorem holds if f: $M \to M$ is of class C^1 on a neighborhood of z_0 and on a neighborhood of the orbit of z_1.

Remark 2. In the above theorem $T_{z_1}f^m$ may be degenerate.

Remark 3. In case $u = \dim M$ and $f^m(z_1) = z_0$, the above theorem reduces to the theorem of Marotto.

Remark 4. If f is a diffeomorphim with $f^m(z_1) \neq z_0$, then the above assumption implies that $f^m(z_1)$ is a transversal homoclinic point.

Remark 5. Transversality condition in our assumption (3) is necessary. We have an example which shows that without the transversality condition the conclusion of the main theorem does not hold.

Throughout this paper, we work under the assumption of the main theorem and use the same notation.

§ 2. Main Lemma

Using the inverse function theorem, we know that there is an embedding Φ: $E^s(r_1) \times E^u(r_1) \to M$ such that $\Phi | E^s(r_1) \times 0 = \varphi^s$ and $\Phi | 0 \times E^u(r_1) = \varphi^u$ for sufficiently small $r_1 > 0$. Therefore, we identify $E^s(r_1) \times E^u(r_1)$ with a neighborhood of z_0 for sufficiently small r_1. By this identification, $E^s(r_1)$ (resp. $E^u(r_1)$) is identified with $W^s_{\text{loc}}(z_0)$ (resp. $W^u_{\text{loc}}(z_0)$).

Since z_0 is a hyperbolic fixed point of f by our assumption, $T_{z_0}f$ is non-degenerate. Also, we assume that $T_x f$ is non-degenerate for any point $x \in E^s(r_1) \times E^u(r_1)$ (We replace $r_1 > 0$ a smaller value if necessary).

For $x \in E^s(r_1) \times E^u(r_1)$, we write

$$T_x f = \begin{pmatrix} T^s_x & A_x \\ B_x & T^u_x \end{pmatrix}, \qquad T_x f^{-1} = \begin{pmatrix} T'^s_x & A'_x \\ B'_x & T'^u_x \end{pmatrix}$$

with respect to the product structure $E^s(r_1) \times E^u(r_1)$.

Put

$$a = \max_x \{\| T^s_x \|, \| T'^u_x \|\},$$

$$\frac{1}{b} = \max_x \{\| T^{u\,-1}_x \|, \| T'^{s\,-1}_x \|\} ,$$

and

$$k = \max_x \{\|\{A_x \|, \| B_x \|, \| A'_x \|, \| B'_x \|\}\} ,$$

where x runs over $E^s(r_1) \times E^u(r_1)$. Then, by our hyperbolicity assumption on z_0, the following inequalities hold on a sufficiently small neighborhood $E^s(r_1) \times E^u(r_1)$.

$$(*) \qquad \begin{cases} a < 1, \quad \dfrac{1}{b} < 1, \\ a + k < 1, \quad b - k > 1, \quad \text{and} \quad k < \dfrac{(b-1)^2}{4} \end{cases}$$

Since $z_1 \in W^u_{\text{loc}}(z_0)$ by our assumption, we may assume that $z_1 \in E^u(r_1)$. Choose a real number r such that $0 < r < r_1$ and $z_1 \notin E^u(r)$.

For $\sigma = s$, u, π^σ denotes the natural projection from $E^s(r_1) \times E^u(r_1)$

onto $E^\sigma(r_1)$. Since $T_x(E^s(r_1)\times E^u(r_1)) = T_x(E^s(r_1))\times T_x(E^u(r_1)) = E^s\times E^u$, $v\in T_x(E^s(r_1)\times E^u(r_1))$ is uniquely expressed as $v = v^s + v^u$, $v^\sigma \in E^\sigma$.

For a σ-dimensional disk D^σ in $E^\sigma(r_1)$, we denote its boundary in $E^\sigma(r_1)$ by ∂D^σ.

MAIN LEMMA. *Assume the same conditions of the main theorem. Let B^u be an u-dimensional disk in $E^u(r_1)$, and let B^s be an arbitrary s-dimensional disk with the origin 0. If ψ: $B^s\times B^u \to E^s(r_1)\times E^u(r_1)$ is an embedding such that $\psi|0\times B^u$ is the inclusion map $B^u\subset E^u(r_1)$, then for any $\varepsilon > 0$ and $L > 0$ there exists a positive integer $N(\psi,\varepsilon,L)$ satisfying the following conditions.*

For any integer $n\geqq N(\psi,\varepsilon,L)$, there is an embedding $\phi = \phi(\psi,\varepsilon,L,n)$: $E^s(r)\times B^u \to E^s(r_1)\times E^u(r_1)$ satisfying the following eight conditions.

(2.1) $\phi(E^s(r)\times y)\subset \psi(B^s\times y)$ *for* $y\in B^u$.

(2.2) $f^{-n}(\phi(E^s(r)\times B^u))\subset E^s(r)\times E^u(r)$.

(2.3) $f^{-n}(\phi(\partial E^s(r)\times B^u))\subset \partial E^s(r)\times E^u(r)$.

(2.4) $\pi^s f^{-n}\phi(x\times B^u) = x$ *for* $x\in E^s(r)$.

(2.5) $\|v^u\| < \varepsilon\|v^s\|$ *for any non-zero v in* $T(f^{-n}\phi(E^s(r)\times y))$, $y\in B^u$.

(2.6) $\|(Tf^{-n}(v))^s\| > L\|v^s\|$ *for any non-zero v in* $T(\phi(E^s(r)\times y))$, $y\in B^u$.

(2.7) $\|(Tf^n(v))^s\| < \varepsilon\|(Tf^n(v))^u\|$ *and*

(2.8) $\|(Tf^n(v))^u\| > L\|v^u\|$ *for any non-zero v in* $T(f^{-n}\phi(x\times B^u))$, $x\in E^s(r)$.

Proof. Let r_2 and ε_1 be real numbers such that $r < r_2 < r_1$ and $0 < \varepsilon_1 < \min\{\varepsilon, r, r_2 - r, r_1 - r_2\}$. Since the set $\{\|v^u\|/\|v^s\|;\ v\neq 0\in T_y(\psi(B^s\times y)),\ y\in B^u\}$ is bounded, there exists an integer $N_1 > 0$ such that some s-dimensional disk in $f^{-n}\psi(B^s\times y)$ is ε_1 C^1-close to $E^s(r_2)$ for any $n\geqq N_1$ and $y\in B^u$. This is proved by the same argument as in the proof of the λ-lemma in J. Palis [5], if we note that the λ-lemma holds uniformly with respect to a disk family $\{\psi(B^s\times y);\ y\in B^u\}$.

The above fact implies that there exists a neighborhood V of B^u in $B^s\times B^u$ such that $f^{-n}(\psi((B^s\times y)\cap V))$ is an s-dimensional disk, which is ε_1 C^1-close to $E^s(r_1)$ and $\pi^s f^{-n}(\psi((B^s\times y)\cap V))\supset E^s(r)$ for any $y\in B^u$. If we take $\varepsilon_1 > 0$ small enough, then $f^{-n}(\psi((B^s\times y)\cap V))$ intersects with $x\times E^u(r)$ ($x\in E^s(r)$) in a single point for each $y\in B^u$. We denote this point by $\chi(x,y)$. Now, define ϕ: $E^s(r)\times B^u\to E^s(r_1)\times E^u(r_1)$ by

$$\phi(x, y) = f^n(\chi(x, y)) \qquad \text{for } (x, y) \in E^s(r) \times B^u .$$

Then, the conditions (2.1)~(2.5) are clearly satisfied by the definition of ϕ.

If N_1 is large enough, then there is a neighborhood W of B^u in $\psi(B^s \times B^u)$ such that for any non-zero v in $T_z(\psi(B^s \times y))$ with $y \in B^u$ and $z \in W \cap \psi(B^s \times y)$ the inequality

$$\|(Tf^{-N_1}(v))^u\| / \|(Tf^{-N_1}(v))^s\| < 1$$

holds. Then,

$$\begin{aligned} \|(Tf^{-N_1-1}(v))^s\| &\geqq \|(T'^s)^{-1}\|^{-1} \cdot \|(Tf^{-N_1}(v))^s\| - \|A'\| \|(Tf^{-N_1}(v))^u\| \\ &> (b-k)\|(Tf^{-N_1}(v))^s\| , \end{aligned}$$

and

$$\begin{aligned} \frac{\|(Tf^{-N_1-1}(v))^u\|}{\|(Tf^{-N_1-1}(v))^s\|} &< \frac{\|T'^u\| \|(Tf^{-N_1}(v))^u\| + \|B'\| \|(Tf^{-N_1}(v))^s\|}{(b-k)\|(Tf^{-N_1}(v))^s\|} \\ &\leqq (b-k)^{-1}\left(a \frac{\|(Tf^{-N_1}(v))^u\|}{\|(Tf^{-N_1}(v))^s\|} + k\right) \\ &\leqq (b-k)^{-1}(a+k) < 1 . \end{aligned}$$

The last inequality holds since $b - k > 1$, $a + k < 1$.

By the induction on $\ell \geqq 1$, we have the following inequality

$$\|(Tf^{-N_1-\ell}(v))^s\| > (b-k)^\ell \|(Tf^{-N_1-\ell}(v))^u\|$$

for any $\ell \geqq 1$ and any non-zero $v \in T_z(\psi(B^s \times y))$, where $y \in B^u$, $z \in W \cap \psi(B^u \times y)$.

Now taking $N > N_1$ large enough, we have (2.6) for any $n \geqq N$.

Replacing Tf^{-1} (resp. E^u and E^s) by Tf (resp. E^s and E^u), we have (2.7) and (2.8) similarly.

§ 3. Construction of a map from a shift

First, we construct symbols, and using these symbols, we define a map from a shift to M. This map is used in our proof of the main theorem.

LEMMA 1. *There is a positive integer N_1 such that for any integer $N_0 \geqq N_1$, there are two embeddings*

$$\phi_i\colon (E^s(r) \times E^u(r_2),\ E^s(r) \times B_i^u) \longrightarrow E^s(r_2) \times E^u(r_2) \qquad (i = 0, 1)$$

$(0 < r < r_2 < r_1)$ of a pair of rectangles, where B_i^u is a u-dimensional disk contained in the interior of $E^u(r_2)$, satisfying the following 11 *conditions.*

(3.1) $f^{N_i}(\phi_i(E^s(r) \times B_i^u)) \subset \phi_j(E^s(r) \times E^u(r_2)) \quad (i, j = 0, 1)$.

(3.2) $f^{N_i}(\phi_i(E^s(r) \times \partial B_i^u)) \subset \phi_j(E^s(r) \times (E^u(r_2) - B_j^u)) \quad (i, j = 0, 1)$.

(3.3) $(f^{N_i})_*$: $H_{u-1}(\phi_i(E^s(r) \times \partial B_i^u)) \to H_{u-1}(\phi_j(E^s(r) \times (E^u(r_2) - B_j^u)))$ *is an isomorphism, where* $H_{u-1}(\ )$ *is the* $(u-1)$*-th homology group and* $(f^{N_i})_*$ *is the induced homomorphism of* f^{N_i} $(i, j = 0, 1)$.

(3.4) $\pi^s\phi_i(x \times B_i^u)$ *consists of a single point for* $x \in E^s(r)$, *and* $\pi^s\phi_i(E^s(r) \times B_i^u) = E^s(r) \quad (i = 0, 1)$.

(3.5) $\pi^u f^{N_i}\phi_i(E^s(r) \times y)$ *consists of a single point for* $y \in B_i^u \quad (i = 0, 1)$.

(3.6) $2\|v^s\| < \|v^u\|$ *for any non-zero* v *in* $T(f^{N_i}\phi_i(x \times B_i^u))$, $x \in E^s(r)$ $(i = 0, 1)$.

(3.7) $2\|v^u\| < \|v^s\|$ *for any non-zero* v *in* $T(\phi_i(E^s(r) \times y))$, $y \in B_i^u$ $(i = 0, 1)$.

(3.8) $\|(Tf^{N_i}(v))^u\| > 8\|v^u\|$ *for any non-zero* v *in* $T(\phi_i(x \times B_i^u))$, $x \in E^s(r)$ $(i = 0, 1)$.

(3.9) $8\|(Tf^{N_i}(v))^s\| < \|v^s\|$ *for any non-zero* v *in* $T(\phi_i(E^s(r) \times y))$, $y \in B_i^u$ $(i = 0, 1)$.

(3.10) *If we put* $A_0 = \phi_0(E^s(r) \times B_0^u)$ *and* $A_1 = \phi_1(E^s(r) \times B_1^u)$, *then* $A_0 \cap A_1 = \phi$.

(3.11) *There exists an integer* k $(0 \leqq k \leqq N_0 - 1)$ *such that* $f^k(A_1) \cap f^i(A_0) = \phi$ *for* $0 \leqq i \leqq N_0 - 1$ *and* $f^k(A_1) \cap f^i(A_1) = \phi$ *for* $0 \leqq i \neq k \leqq N_1 - 1$.

Proof. Since $z_1 \in W_{loc}^u(z_0) = E^u(r_1)$ and $z_1 \neq z_0$ by our assumption, there exists a positive number r_2 such that $z_1 \notin E^u(r_2)$, $r_2 < r_1$. Let r be a sufficiently small number such that $0 < r < r_2$, which we determine later.

Since $f^{-1}|E^u(r_1)$ is a contraction, there is a positive integer L such that $f^{-L}(z_1) \in E^u(r_2)$.

Let j: $E^s(r) \times E^u(r) \to E^s(r_2) \times E^u(r_2)$ be the inclusion map. Let $N_2 = N(j, 1/2, 8)$ be the integer given in the main lemma. By the λ-lemma, the transversality condition of the main theorem, and the fact that $f|E^u(r_1)$ is an expansion, there are positive number $N_3 > N_2$ and an u-dimensional disk D^u in $E^u(r_2)$ satisfying the following conditions.

(1) D^u is a neighborhood of $f^{-L}(z_1)$ in $E^u(r_2)$.

(2) $f^{N_3}(f^{-L}(z_1)) \in E^s(r)$ and $f^{N_3}(D^u) \subset E^s(r) \times E^u(r_2)$.

(3) $f^{N_3}(\partial D^u) \subset E^s(r) \times (E^u(r_2) - E^u(r))$.

(4) $8\|v^s\| < \|v^u\|$ for any non-zero $v \in T(f^{N_3}(D^u))$.

(5) $D^u \cap E^u(r) = \phi$.

(6) There exists a positive number r_3 such that $f^L(D^u) \subset \operatorname{Int} E^u(r_3) - f^{-1}(E^u(r_3))$, and $f^{L+i}(D^u) \cap f^L(D^u) = \phi$ for $1 \leqq i \leqq N_3 - L$.

From conditions (3) and (4), $\pi^u | f^{N_3}(D^u)$: $f^{N_3}(D^u) \to E^u(r_2)$ is a diffeomorphism onto its image, and its image contains $E^u(r)$. And there is a neighborhood W of D^u in $E^s(r_2) \times E^u(r_2)$ such that

(7) $\pi^u \circ (f^{N_3}| W)$ is a submersion,

(8) $f^L(W) \cap E^s(r) \times E^u(r) = \phi$,

(9) $f^L(W) \subset E^s(\delta) \times E^u(r_3) - f^{-1}(E^s(\delta) \times E^u(r_3))$ for some $\delta > 0$, and

(10) $f^{L+i}(W) \cap f^L(W) = \phi$ for $1 \leqq i \leqq N_3 - L$.

Therefore, there are positive number δ' and an embedding ψ: $E^s(\delta') \times D^u \to E^s(r_2) \times E^u(r_2)$ satisfying the following conditions.

(11) $\psi | D^u$ is the inclusion map, and $\psi(E^s(\delta') \times D^u) \subset W$.

(12) $\pi^u \circ f^{N_3}(\psi(E^s(\delta') \times y))$ consists of a single point for $y \in D^u$.

(13) $\pi^s(\psi(x \times D^u)) = x$ for $x \in E^s(\delta')$.

(14) $\psi(E^s(\delta') \times D^u) \cap E^s(r) \times E^u(r) = \phi$.

If we take $\delta' > 0$ small enough, then we have the following conditions.

(15) $f^{N_3}(\psi(E^s(\delta') \times D^u)) \subset E^s(r) \times E^u(r_2)$

(16) $f^{N_3}(\psi(E^s(\delta') \times \partial D^u)) \subset E^s(r) \times (E^u(r_2) - E^u(r))$.

(17) $4\|v^s\| < \|v^u\|$ for any non-zero v in $T(f^{N_3}(\psi(x \times D^u)))$, $x \in E^s(\delta')$.

By (17) we can take a small number ρ $(0 < \rho < 1/2)$ such that

(18) $2\|(Tf^{N_3}(v))^s\| < \|(Tf^{N_3}(v))^u\|$ if $v \in T(\psi(E^s(\delta') \times D^u))$ and $\|v^s\| < \rho\|v^u\|$.

By a similar argument, we can take a large number $L_1 > 0$ such that

(19) $\|(Tf^{N_3}(v))^u\| > L_1^{-1}\|v^u\|$ for any non-zero $v \in T(\psi(E^s(\delta') \times D^u)$ with $\|v^s\| < \rho\|v^u\|$, and

(20) $\|(Tf^{N_3}(v))^s\| < L_1\|v^s\|$ for any non-zero $v \in T(\psi(E^s(\delta') \times y))$, $y \in D^u$.

Let $N_4 = N(\psi, \rho, 8L_1)$ be the positive integer given in the main lemma, and $N_1 = N_3 + N_4$. Let ϕ_1: $E^s(r) \times D^u \to E^s(r) \times E^u(r)$ be defined by $f^{-N_4}\phi(\psi, \rho, 8L_1, N_4)$, where $\phi(\psi, \rho, 8L_1, N_4)$ is given in the main lemma.

Let $N_0 \geqq N_1$ be a given integer. Then, $N_0 \geqq N_1 > N_3 > N_2$. Therefore, we have an embedding $\phi(j, 1/2, 8, N_0)$: $E^s(r) \times E^u(r) \to E^s(r_2) \times E^u(r_2)$ by the main lemma. Now, define ϕ_0: $E^s(r) \times E^u(r) \to E^s(r) \times E^u(r)$ by $\phi_0 = f^{-N_0} \circ \phi(j, 1/2, 8, N_0)$.

https://doi.org/10.1017/S0027763000019292 Published online by Cambridge University Press

Put $B_0^u = E^u(r)$ and $B_1^u = D^u$. Then, by our construction, (3.4) and (3.5) are satisfied. By the definition of ϕ_0, ϕ_1 and (18), we see (3.6) holds. Also, by the definition of ϕ_0, ϕ_1 and the fact $\rho < 1/2$, (3.7) holds. (3.8) is a consequence of the definition of ϕ_0, ϕ_1, and (19). Also, (3.9) is a consequence of the definition of ϕ_0, ϕ_1 and (20).

Now, we can extend ϕ_i $(i = 0, 1)$ to an embedding ϕ_i: $E^s(r) \times E^u(r_2) \to E^s(r) \times E^u(r_2)$ such that it satisfies conditions (3.1), (3.2), and $\phi_i(\partial E^s(r) \times E^u(r_2)) \subset \partial E^s(r) \times E^u(r_2)$. Then, (3.3) is a consequence of the fact that $\pi^u | f^{N_3}(D^u)$ is a diffeomorphism and its image contains $E^u(r)$.

By the definition of ϕ_1 and (14), $f^{N_4}\phi_1(E^s(r) \times B_1^u) \cap E^s(r) \times E^u(r) = \phi$. Also, it follows from the definition that $f^{N_4}\phi_0(E^s(r) \times B_0^u) = f^{-(N_0-N_4)}\phi(j, 1/2, 8, N_0)(E^s(r) \times B_0^u) \subset f^{-(N_0-N_4)}(E^s(r) \times E^u(r)) \subset E^s(r) \times E^u(r)$, since $N_0 - N_4 \geqq N_1 - N_4 = N_3 > N_2 = N(j, 1/2, 8)$. Therefore, $f^{N_4}\phi_1(E^s(r) \times B_1^u) \cap f^{N_4}\phi_0(E^s(r) \times B_0^u) = \phi$. This proves (3.10).

Finally, put $k = N_4 + L$. Then, $0 \leqq k < N_4 + N_3 = N_1 \leqq N_0$. By (10) and our construction, $f^k(A_1) \cap f^i(A_1) \subset f^L(W) \cap f^{i-k+L}(W) = \phi$ for $k < i < N_0$. Thus, $f^k(A_1) \cap f^i(A_1) = \phi$ for $k < i < N_1$. Also, if $0 \leqq i < k$, $f^k(A_1) \cap f^i(A_1) = \phi$ by (9). Finally, if $0 \leqq i < N_0$, $f^i(A_0) \subset E^s(r) \times E^u(r)$. Therefore, by the fact that $f^k(A_1) \subset f^L(W)$ and (8), we have $f^k(A_1) \cap f^i(A_0) = \phi$ for $0 \leqq i < N_0$. This completes the proof.

Let $\sum = \{A_0, A_1\}^Z$ be a two-sided shift on two symbols A_0 and A_1. By definition, an element of Σ is a bisequence $\boldsymbol{a} = (a_i)_{i \in Z}$ such that $a_i = A_0$ or A_1 for each $i \in Z$, where Z is the set of integers. The metric d on Σ is defined by $d(\boldsymbol{a}, \boldsymbol{b}) = \sum_{i \in Z} 1/2^{|i|} d(a_i, b_i)$, where $\boldsymbol{a} = (a_i)$, $\boldsymbol{b} = (b_i)$, $d(a_i, b_i) = 0$ if $a_i = b_i$, and $d(a_i, b_i) = 1$ if $a_i \neq b_i$. By this metric Σ is a compact metric space.

Now, define integers $k(\boldsymbol{a}, i)$ and $n(\boldsymbol{a}, i)$ for $\boldsymbol{a} = (a_i) \in \Sigma$ as follows.

$$k(\boldsymbol{a}, i) = \begin{cases} N_0 & \text{if } a_i = A_0 \\ N_1 & \text{if } a_i = A_1 \end{cases}$$

$$n(\boldsymbol{a}, i) = \begin{cases} \displaystyle\sum_{j=0}^{i-1} k(\boldsymbol{a}, j) & \text{if } i \geqq 0 \\ \displaystyle -\sum_{j=-1}^{i} k(\boldsymbol{a}, j) & \text{if } i < 0 \end{cases}$$

Define a subset $F^{-i}(\boldsymbol{a})$ of M as follows,

$$F^{-i}(\boldsymbol{a}) = \begin{cases} (f^{k(\boldsymbol{a},0)}|a_0)^{-1} \circ \cdots \circ (f^{k(\boldsymbol{a},i-1)}|a_{i-1})^{-1}(a_i) & \text{if } i > 0 \\ a_0 & \text{if } i = 0 \\ f^{k(\boldsymbol{a},-1)}(\cdots f^{k(\boldsymbol{a},i+1)}(f^{k(\boldsymbol{a},i)}(a_i) \cap a_{i+1}) \cap \cdots) \cap a_{-1}) \cap a_0 & \text{if } i < 0. \end{cases}$$

Then for ℓ, $m > 0$, $\bigcap_{-\ell \leqq i \leqq m} F^{-i}(\boldsymbol{a}) \ni x$ if and only if $f^{n(a,i)}(x) \in a_i$ for $0 \leqq i \leqq m$, and there exist $y_i \in a_i$ with $f^{k(a,i)}(y_i) = y_{i+1}$ for $-\ell \leqq i < 0$ and $y_0 = x$.

PROPOSITION 1. (a) *$\bigcap_{i \in Z} F^{-i}(\boldsymbol{a})$ consists of a single point of M for each $\boldsymbol{a} \in \Sigma$.*

(b) *A map $p\colon \Sigma \to M$ defined by $p(a) = \bigcap_{i \in Z} F^{-i}(\boldsymbol{a})$ $(\boldsymbol{a} \in \Sigma)$ is continuous.*

(c) *If there exists an integer $i \geqq 0$ such that $a_i \neq b_i$ for $\boldsymbol{a} = (a_i)$, $\boldsymbol{b} = (b_i) \in \Sigma$, then $p(\boldsymbol{a}) \neq p(\boldsymbol{b})$.*

(d) *If $N_0 = N_1$, then $p \circ \sigma = f^{N_0} \circ p$, where $\sigma\colon \Sigma \to \Sigma$ is the shift map defined by $\sigma(\boldsymbol{a}) = (\boldsymbol{b})$, $b_i = a_{i+1}$ for $i \in Z$.*

Proof. It follows from (3.1)~(3.3) that $\bigcap_{-\ell \leqq i \leqq 0} F^{-i}(\boldsymbol{a}) \neq \phi$ for any $\boldsymbol{a} \in \Sigma$ and positive integer ℓ (cf. Kurata [2], 50–51). Let m be any positive integer. For $\boldsymbol{a} = (a_i) \in \Sigma$, we define $\boldsymbol{b} = (b_i)$ by $b_i = a_{i+m}$, $i \in Z$. Then, if x is a point of $\bigcap_{-(\ell+m) \leqq i \leqq 0} F^{-i}(\boldsymbol{b})(f^{k(a,0)}|a_0)^{-1} \circ \cdots \circ (f^{k(a,m-1)}|a_{m-1})^{-1}(x) \subset \bigcap_{-\ell \leqq i \leqq m} F^{-i}(\boldsymbol{a})$. This implies that $\bigcap_{-\ell \leqq i \leqq m} F^{-i}(\boldsymbol{a}) \neq \phi$ for any positive integers ℓ and m. Since $\bigcap_{-\ell \leqq i \leqq m} F^{-i}(\boldsymbol{a})$ is compact, $\bigcap_{i \in Z} F^{-i}(\boldsymbol{a}) \neq \phi$ for each $\boldsymbol{a} \in \Sigma$.

In order to prove (a), we shall prove the following assertion $(*)$ by induction.

$(*)$ Let x be a point of $\bigcap_{i \in Z} F^{-i}(\boldsymbol{a})$. Then, the following formula holds for any integer $\ell \geqq 0$.

$$\bigcap_{-\ell \leqq i \leqq \ell} F^{-i}(\boldsymbol{a}) \subset E^s(\pi^s(x); 2r(\tfrac{1}{2})^\ell) \times E^u(\pi^u(x); 2r(\tfrac{1}{2})^\ell),$$

where $E^\sigma(y; r')$ denotes the σ-dimensional disk in $E^\sigma(r_1)$ with the center y and the radius r' $(\sigma = s, u)$.

If $\ell = 0$, $(*)$ follows from the fact $A_i \subset E^s(r) \times E^u(r)$ $(i = 0, 1)$. Now, assume that $(*)$ holds for some $\ell \geqq 0$.

For $\boldsymbol{a} = (a_i) \in \Sigma$, let $\boldsymbol{b} = (b_i)$ (resp. $\boldsymbol{c} = (c_i)$) be an element of Σ given by $b_i = a_{i+1}$ (resp. $c_1 = a_{i-1}$), $i \in Z$. Then,

(21) $\bigcap_{|i| \leqq \ell+1} F^{-i}(\boldsymbol{a}) \subset (f|a_0)^{-k(a,0)}(\bigcap_{|i| \leqq \ell} F^{-i}(\boldsymbol{b})) \cap f^{k(a,-1)}(\bigcap_{|i| \leqq \ell} F^{-i}(\boldsymbol{c}))$.

Let $x \in \bigcap_{i \in Z} F^{-i}(\boldsymbol{a})$, $y \in \bigcap_{i \in Z} F^{-i}(\boldsymbol{b})$, $w \in \bigcap_{i \in Z} F^{-i}(\boldsymbol{c})$ be such that $f^{k(a,-1)}(w) = x$, $f^{k(a,0)}(x) = y$.

Define $j = 0$ if $c_0 = A_0$ and $j = 1$ if $c_0 = A_1$. Then, there is a point $w' \in E^s(r)$ such that $\phi_j(w' \times B_j^u) \ni w$ since $w \in c_0$. For $y' \in \phi_j(w' \times B_j^u) \cap \phi_j(E^s(r) \times t')$, put $C(y') = \{z;\, d(\pi^s(z), \pi^s(y')) < \varepsilon,\ z \in \phi_j(E^s(r) \times t')\}$, and for $y'' \in f^{k(a,-1)}\phi_j(E^s(r) \times t'')$, $t'' \in B_j^n$, put $C'(y'') = \{z;\, d(z, y'') < \varepsilon/8,\ z \in f^{k(a,-1)}\phi_j(E^s(r)$

$\times t'')\}$, where $\varepsilon = 2r(1/2)^{\ell}$. Note that $\pi^s\phi_j(x' \times B^u_j) = x'$ and $\pi^s(y') = \pi^s(w) = w'$. Then, by using the inductive hypothesis, we have the following.

(22) $\bigcap_{|i|\leqq\ell} F^{-i}(c) \subset E^s(\pi^s(w); \varepsilon) \times E^u(\pi^u(w); \varepsilon) \subset (\pi^s|c_0)^{-1}(E^s(\pi^s(w); \varepsilon))$
$= \bigcup\{C(y'); y' \in \phi_j(w' \times B^u_j)\}$.

By (3.9), the following inequality holds.

$$d(\pi^s \circ f^{k(a,-1)}(z), \pi^s \circ f^{k(a,-1)}(y')) < \tfrac{1}{8} d(\pi^s(z), \pi^s(y')),$$
$$\text{for } z,\ y' \in \phi_j(E^s(r) \times t)\,.$$

Since $f^{k(a,-1)}\phi_j(E^s(r) \times t) \subset E^s(r) \times t'$ by (3.5), it follows that

$$d(\pi^s \circ f^{k(a,-1)}(z), \pi^s \circ f^{k(a,-1)}(y')) = d(f^{k(a,-1)}(z), f^{k(a,-1)}(y'))\,.$$

Therefore, $f^{k(a,-1)}(C(y')) \subset C'(f^{k(a,-1)}(y'))$. This and (22) imply the following.

(23) $f^{(a,-1)}(\bigcap_{|i|\leqq\ell} F^{-i}(c)) \subset \bigcup\{C'(y''); y'' \in f^{k(a,-1)}(\phi_j(w' \times B^u_j))\}$.

Define $h = 0$ if $a_0 = A_0$ and $h = 1$ if $a_0 = A_1$. Let $z' \in B^u_h$ be a point such that $x \in \phi_h(E^s(r) \times z') \subset a_0$. For $x' \in \phi_h(E^s(r) \times z')$, put

$$B'(x') = \{z; d(\pi^u(z), \pi^u f^{k(a,0)}(x')) < \varepsilon,\ z \in f^{k(a,0)}\phi_h(t \times B^u_h),\ x' \in \phi_h(t \times B^u_h)\}$$

and

$$B'(x') = \{z; d(z, x') < \varepsilon/8,\ z \in \phi_h(t \times B^u_h),\ x' \in \phi_h(t \times B^u_h)\}.$$

Then, by using the inductive hypothesis, we have the following.

$$\begin{aligned}\bigcap_{|i|\leqq\ell} F^{-i}(b) \cap f^{k(a,0)}(a_0) &\subset E^s(\pi^s(y); \varepsilon) \times E^u(\pi^u(y); \varepsilon) \cap f^{k(a,0)}(a_0)\\ &\subset (\pi^u)^{-1}E^u(\pi^u(y); \varepsilon) \cap f^{k(a,0)}\phi_h(E^s(r) \times B^u_h)\\ &= \bigcup_{i \in E^s(r)} \{(\pi^u)^{-1}E^u(\pi^u(y); \varepsilon) \cap f^{k(a,0)}\phi_h(t \times B^u_h)\}\,.\end{aligned}$$

Note that $(\pi^u)^{-1}(\pi^u(y)) \cap f^{k(a,0)}(a_0) = f^{k(a,0)}\phi_h(E^s(r) \times z')$ and $\pi^u f^{k(a,0)}(x') = \pi^u(y)$. Then, we have

$$\bigcap_{|i|\leqq\ell} F^{-i}(b) \cap f^{k(a,0)}(a_0) \subset \bigcup\ \{B(x'); x' \in \phi_h(E^s(r) \times z')\}\,.$$

By (3.8) and the definition of $B'(x')$ we have the following.

(24) $(f^{k(a,0)}|a_0)^{-1}(\bigcap_{|i|\leqq\ell} F^{-i}(b)) \subset \bigcup\{B'(x'); x' \in \phi_h(E^s(r) \times z')\}$

Combining (21), (23) and (24), we have

(25) $\bigcap_{|i|\leqq\ell+1} F^{-i}(a) \subset \{(\bigcup B'(x'); x' \in \phi_h(E^s(r) \times z')\})$
$\cap\ (\bigcup\{C'(y''); y'' \in f^{k(a,-1)}\phi_h(w' \times B^u_h)\}).$

By definition and (3.5), if $y'' \in f^{k(a,-1)}\phi_h(w' \times B^u_h)$, $d(z', y'') = d(\pi^s(z'), \pi^s(y'')) < \varepsilon/8$ for $z' \in C'(y'')$ and $\pi^u(z') = \pi^u(y'')$. Similarly, by (3.4) and definition, if $x' \in \phi_h(E^s(r) \times z')$, $d(z, x') = d(\pi^u(z), \pi^u(x')) < \varepsilon/8$ for $z' \in C'(y'')$

and $\pi^s(z) = \pi^s(x')$. These facts and (3.6), (3.7) imply that

(26) $B'(x') \cap C'(y'') \subset E^s(\pi^s(x); \varepsilon/2) \times E^u(\pi^u(x); \varepsilon/2)$ for any $x' \in \phi_h(E^s(r) \times z')$ and $y'' \in f^{k(a,-1)}\phi_h(w' \times B^u_h)$.

Therefore, (25) and (26) imply that

$$\bigcap_{|i| \leqq \ell+1} F^{-i}(\boldsymbol{a}) \subset E^s\left(\pi^s(x);\ \frac{\varepsilon}{2}\right) \times E^u\left(\pi^u(x);\ \frac{\varepsilon}{2}\right).$$

Thus $(*)$ holds for $\ell + 1$, and this proves $(*)$ for any $\ell \geqq 0$.

It is clear that $(*)$ implies (a).

Now, we reformulate $(*)$ as follows.

$$(*)' \qquad \bigcap_{-\ell \leqq i \leqq \ell} F^{-i}(\boldsymbol{a}) \subset E^s\left(\pi^s p(\boldsymbol{a});\ 2r\left(\frac{1}{2}\right)^{\ell}\right) \times E^u\left(\pi^u p(\boldsymbol{a});\ 2r\left(\frac{1}{2}\right)^{\ell}\right),$$

$$\text{where } p(\boldsymbol{a}) = \bigcap_{i \in \boldsymbol{Z}} F^{-i}(\boldsymbol{a}),\ \boldsymbol{a} \in \Sigma, \text{ and } \ell \geqq 0\,.$$

From this formula and the definition of the metric on Σ, continuity of $p: \Sigma \to M$ is easily proved.

By the definition of p, it is clear that $f^{n(\boldsymbol{a},i)}(p(\boldsymbol{a})) \in a_i$ for any $i \geqq 0$. Let $\boldsymbol{a} = (a_i)$, $\boldsymbol{b} = (b_i)$ be elements of Σ such that $a_i \neq b_i$ for some $i \geqq 0$. Then, it is clear that $f^{n(\boldsymbol{a},i)}(p(\boldsymbol{a})) \neq f^{n(\boldsymbol{b},i)}(p(\boldsymbol{b}))$ since $A_0 \cap A_1 = \phi$. Thus, $p(\boldsymbol{a}) \neq p(\boldsymbol{b})$.

Finally, if $N_0 = N_1$, then it is clear from the definition that $p \circ \sigma = f^{N_0} \circ \sigma$. This completes the proof.

§4. Proof of the Main Theorem

In this section, we give a proof of the main theorem.

Let $N = 2N_1$. Let p be an integer greater than N. Put $N_0 = p - N_1$. Then $N_0 \geqq N_1$ and $p = N_0 + N_1$. We apply Proposition 1 in this case.

Define $\boldsymbol{a} = (a_i) \in \Sigma$ by $a_{2i} = A_0$ and $a_{2i+1} = A_1$ for $i \in \boldsymbol{Z}$. Put $x = p(\boldsymbol{a}) = \bigcap F^{-i}(\boldsymbol{a})$. Then, $f^{n(\boldsymbol{a},i)}(x) \in a_i$ for $i \geqq 0$, and there are $x_j \in a_j$ such that $f^{k(\boldsymbol{a},j)}(x_j) = x_{j+1}$ for $j < 0$ and $x_0 = x$. Put $y = f^p(x) = f^{N_0+N_1}(x) = f^{n(\boldsymbol{a},2)}(x)$. Then, $f^{n(\boldsymbol{a},i)}(y) = f^{n(\boldsymbol{a},i)+n(\boldsymbol{a},2)}(x) = f^{n(\boldsymbol{a},i+2)}(x)$. Hence $f^{n(\boldsymbol{a},i)}(y) \in a_{i+2} = a_i$ for $i \geqq 0$. Put $y_j = f^{n(\boldsymbol{a},2)}(x_j)$ for $j < 0$. Then, $y_j \in a_{j+2} = a_j$ and $f^{k(\boldsymbol{a},j)}(y_j) = f^{n(\boldsymbol{a},2)}(x_{j+1}) = y_{j+1}$ for $j < 0$ and $y_0 = y$. Therefore, $y \in \bigcap_{i \in \boldsymbol{Z}} F^{-i}(\boldsymbol{a}) = p(\boldsymbol{a})$ Thus, $y = p(\boldsymbol{a}) = x$. This proves $f^p(x) = x$.

Next, we shall prove that p is the minimal period of x.

By definition, $x \in A_0$, $f^{N_0}(x) \in A_1$, and $A_0 \cap A_1 = \phi$. Since $z_0 \in A_0$ is a fixed point of f, $x \neq z_0$.

https://doi.org/10.1017/S0027763000019292 Published online by Cambridge University Press

Since $N_0 \geqq N_1$ and $p = N_1 + N_0$, $N_0 \geqq p/2$. By the similar argument as in the proof of the main lemma, $f^i(x') \in E^s(r_1) \times E^u(r_1)$ for $0 \leqq i \leqq N_0$ if $x' \in A_0$. Let d be the minimal period of x.

If $d < p$, then $d \leqq p/2$ since d is a divisor of p. Therefore, $d \leqq N_0$. This implies that $f^i(x) \in E^s(r_2) \times E^u(r_2)$ for $0 \leqq i \leqq d$ and $f^d(x) = x$. But this is impossible by Hartman's theorem [1], since $x \neq z_0$. (We replace r for a smaller value if necessary to apply Hartman's theorem.) This completes the proof of (a) of the main theorem.

Next, we prove (b) of the main theorem. In this case, we take $N_0 = N_1$. Then, combining Proposition 1 and Lemma 1, we have the following proposition.

PROPOSITION 2. *Under the assumption of the main theorem, we have the following properties.*

(4.1) $p\colon \Sigma \longrightarrow M$ *is a continuous map, and* $p(\boldsymbol{a}) \neq p(\boldsymbol{b})$ *if* $a_i \neq b_i$ *for some* $i \geqq 0$, *where* $\boldsymbol{a} = (a_i)$, $\boldsymbol{b} = (b_i) \in \Sigma$.

(4.2) $f^{N_0} \circ p = p \circ \sigma$.

(4.3) $A_0 \cap A_1 = \phi$.

(4.4) *There is an integer* k $(0 \leqq k \leqq N_0 - 1)$ *such that* $f^k(A_1) \cap f^i(A_0) = \phi$ *for* $0 \leqq i \leqq N_0 - 1$, *and* $f^k(A_1) \cap f^i(A_1) = \phi$ *for* $0 \leqq i \neq k \leqq N_0 - 1$.

Now, we can prove the conclusion (b) of the main theorem using Proposition 2. Our proof is similar to the one in Li and Yorke [3] and Marotto [4].

For $\boldsymbol{a} = (a_i) \in \Sigma$, let $R(\boldsymbol{a}, n)$ be the number of a_i's which is equal to A_0 for $0 \leqq i \leqq n$. For each $w \in (0, 1)$, choose an element $\boldsymbol{a}^w = (a_i^w) \in \Sigma$ satisfying the following conditions.

(1) If $a_i^w = A_0$, then $i = k^2$ for some $k \in \boldsymbol{Z}$.

(2) $\lim_{n\to\infty} R(\boldsymbol{a}^w, n^2)/n = w$.

Put $h = f^{N_0}$, and put $S_0 = \{h^k(p(\boldsymbol{a}^w)); w \in (0, 1), h \geqq 0\}$. Then, $h(S_0) \subset S_0$ by definition. Also, by (4.2), $S_0 = p\{\sigma^k(\boldsymbol{a}^w); w \in (0, 1), k \geqq 0\} \supset p\{\boldsymbol{a}^w; w \in (0, 1)\}$. By (1), (2), and (4.1), it is clear that $p\{\boldsymbol{a}^w; w \in (0, 1)\}$ is uncountable. Hence S_0 is an uncountable subset of M.

Suppose that $x = h^n(p(\boldsymbol{a}^w)) = p(\sigma^n(\boldsymbol{a}^w)) \in S_0$ be a periodic point of h. Since $h^\ell(x) = h^{\ell+n}(p(\boldsymbol{a}^w)) = p(\sigma^{\ell+n}(\boldsymbol{a}^w))$, $h^\ell(x) \in a_{\ell+n}^w$ for any $i \geqq 0$ by the definition of p and σ. Therefore, by (4.3) a_i^w must be periodic in i for sufficiently large i. But this is impossible by (1) and (2). Thus S_0 does not contain any periodic point of h.

Suppose that $x = h^n(p(\boldsymbol{a}^w))$, $y = h^m(p(\boldsymbol{a}^{w'})) \in S_0$, and $x \neq y$. Then, $h^i(x) \in a^w_{i+n}$ and $h^i(y) \in a^{w'}_{i+m}$. By (1) and (2), there exists an infinite number of i's such that $a^w_{n+i} \neq a^{w'}_{m+i}$. Since $A_0 \cap A_1 = \phi$, and A_0 and A_1 are compact, the distance L of A_0 and A_1 is positive. Therefore, there are infinite number of i's such that $d(h^i(x), h^i(y)) \geqq L > 0$.

Thus, we have the following

(3) $\limsup_{i\to\infty} d(h^i(x), h^i(y)) \geqq L > 0$ for x, $y \in S_0$ $(x \neq y)$.

By a similar argument as above, we can prove that the above (3) holds for any $x \in S_0$ and any periodic point $y \in p(\Sigma)$ of h.

Let y be a periodic point of h in $M - p(\Sigma)$. Then, the positive orbit $\mathrm{orb}^+_h(y) = \{h^i(y); i \geqq 0\}$ of y under h is a finite set disjoint from the compact set $p(\Sigma)$ which contains S_0. Therefore, the following (4) holds.

(4) $\limsup_{i\to\infty} d(h^i(x), h^i(y)) > 0$ for $x \in S_0$ and a periodic point y of h.

Put $S = \{f^k(x); x \in S_0, k \geqq 0\}$. Then, $S \supset S_0$ and $f(S) \subset S$ by definition. Therefore, S is an uncountable set.

If $y \in S$ is a periodic point of f, then there is an integer $i \geqq 0$ such that $f^i(y) \in S_0$ and $f^i(y)$ is a periodic point of $h = f^{N_0}$. But S_0 does not contain any periodic points of h as stated before. Therefore, S does not contain any periodic points of f.

Let $\tilde{\Sigma} = \{A_0, f(A_0), \cdots, f^{N_0-1}(A_0), A_1, f(A_1), \cdots, f^{N_0-1}(A_1)\}^Z$ be the two-sided shift on $2N_0$ symbols, and let Σ_1 be a subshift of finite type with the following transition matrix.

$$
\begin{array}{c c}
 & \overbrace{\qquad\qquad\quad}^{N_0}\ \overbrace{\qquad\qquad\quad}^{N_0} \\
\begin{array}{c} N_0\left\{\begin{array}{c} \\ \\ \\ \\ \end{array}\right. \\ N_0\left\{\begin{array}{c} \\ \\ \\ \\ \end{array}\right. \end{array} &
\left(\begin{array}{cccc:cccc}
0 & 1 & & & & & & \\
 & \ddots & \ddots & & & & & \\
 & & \ddots & 1 & & & & \\
1 & & & 0 & 1 & & & \\
\hdashline
 & & & & 0 & 1 & & \\
 & & & & & \ddots & \ddots & \\
 & & & & & & \ddots & 1 \\
1 & & & & 1 & & & 0
\end{array}\right)
\end{array}
$$

Then, by definition an element of Σ_1 is a bisequence of symbols consisting of the blocks of symbols of the forms A_0, $f(A_0), \cdots, f^{N_0-1}(A_0)$ and A_1, $f(A_1), \cdots, f^{N_0-1}(A_1)$.

Define a map p_1: $\Sigma_1 \to M$ by the following manner. For $\boldsymbol{a} = (\alpha_i) \in \Sigma_1$,

$p_1(\boldsymbol{\alpha})$ is a point $x \in M$ satisfying the following two conditions.

(5) $f^i(x) \in \alpha_i$ for $i \geqq 0$,

(6) For each $i < 0$, there is a point $y_i \in \alpha_i$ such that $f(y_{i-1}) = y_i$ and $y_0 = x$.

By Proposition 1 (a), there is a unique point $x \in M$ satisfying (5) and (6). Similarly to Proposition 1, we have the following proposition.

PROPOSITION 3. (a) *p_1: $\Sigma_1 \to M$ is a continuous map.*

(b) *If there exists an integer $i \geqq N_0$ such that $\alpha_i \neq \beta_i$ for $\boldsymbol{\alpha} = (\alpha_i)$, $\boldsymbol{\beta} = (\beta_i) \in \Sigma_1$, then $p_1(\boldsymbol{\alpha}) \neq p_1(\boldsymbol{\beta})$.*

(c) *$p_1 \circ \sigma_1 = f \circ p_1$, where σ_1: $\Sigma_1 \to \Sigma_1$ is the shift map of Σ_1.*

(d) *For each $\boldsymbol{a} = (a_i) \in \Sigma$, define $t(\boldsymbol{a}) = (\boldsymbol{\alpha})$ be an element of Σ_1 such that $\alpha_{N_0 i + j} = f^j(a_i)$ for $0 \leqq j \leqq N_0 - 1$, and $i \in \boldsymbol{Z}$. Then, t: $\Sigma \to \Sigma_1$ is continuous, $p_1 \circ t = p$, and $\sigma_1^{N_0} \circ t = t \circ \sigma$.*

Now, we come back to the proof of the theorem. By Proposition 2, Proposition 3, and the definition of S, $S = \{p_1 \circ \sigma_1^k \circ t(\boldsymbol{a}^w); w \in (0, 1), k \geqq 0\}$. By a similar argument using the properties (4.2) and (5), we can prove the conclusions (iii) and (iv) of (b) of the main theorem.

Finally, let $x = p \circ \sigma^n(\boldsymbol{a}^w)$, $y = p \circ \sigma^m(\boldsymbol{a}^{w'}) \in S_0$. Suppose that $m \geqq n$. Then, $m + i$ and $n + i$ cannot be squares of some integers for $i = k^2 - n + s$, $1 \leqq s \leqq 2k - (m - n)$. By the condition (1), we have $a^w_{n+i} = a^{w'}_{m+i}$ for $i = k^2 = n + s$, $1 \leqq s \leqq 2k - (m - n)$. If we tend k to infinity, we have the following.

$$\liminf_{i \to \infty} d(\sigma^{n+i}(\boldsymbol{a}^w), \sigma^{m+1}(\boldsymbol{a}^{w'})) = 0 .$$

Since p: $\Sigma \to M$ is uniformly continuous,

$$\liminf_{i \to \infty} d(p \circ \sigma_i(\boldsymbol{a}^w), p \circ \sigma^i(\boldsymbol{a}^{w'})) = \liminf_{i \to \infty} d(h^i(x), h^i(y)) = 0 .$$

Since $h = f^{N_0}$,

$$\liminf_{i \to \infty} d(f^i(x), f^i(y)) \leqq \liminf_{i \to \infty} \, d(h^i(x), h^i(y)).$$

Thus

$$\liminf_{i \to \infty} d(f^i(x), f^i(y)) = 0 \qquad \text{for } x, \ y \in S_0 .$$

This completes the proof.

Remark 6. If we replace the condition (1) by the following condition (1)′ if $a^w_i = A_1$, then $i = k^2$ some $k \in \boldsymbol{Z}$, and if we construct the set S

similarly, then we can prove the same conclusion (b). Furthermore, (v) holds for $x, y \in S$.

References

[1] P. Hartman, Ordinary differential equations, John Wiley and Sons, Inc. (1964), p. 245, Lemma 8.1.
[2] M. Kurata, Hartman's theorem for hyperbolic sets, Nagoya Math. J., **67** (1977), 41–52.
[3] T. Y. Li and J. A. Yorke, Period three implies chaos, Amer. Math. Monthly, **82** (1975), 985–992.
[4] F. R. Marotto, Snap-back repellers imply chaos in $\boldsymbol{R}^n$, J. Math. Analysis and Appl., **63** (1978), 199–223.
[5] J. Palis, On Morse-Smale dynamical systems, Topology, **8** (1969), 385–404.
[6] S. Smale, Diffeomorphisms with many periodic points, Differential and Combinatorial Topology, Princeton University Press (1965), 63–80.

Department of Mathematics
College of General Education
Nagoya University

Department of Mathematics
Nagoya Institute of Technology