the number of entries following: that is, about 100 entries against about 60.

103	133	175	235	295	325	440	590	800
106	136	180	240	300	330	450	600	820
109	140	185	245	305	340	460	615	840
110	144	190	250	310	350	470	630	860
113	148	195	255	315	360	480	645	880
116	150	200	260	320	370	490	660	900
119	154	205	265	325	380	500	680	920
120	158	210	270	300	390	515	700	940
123	160	215	275	305	400	530	720	960
126	164	220	280	310	410	545	740	980
129	168	225	285	315	420	560	760	
130	170	230	290	320	430	575	780.	

In each case the smallest number of entries has not been taken but a series of numbers has been arranged so as to make the use of the proposed table rapid and simple. The smallest number of entries is given for the four figure table correct to ·0001 by the numbers

104	131	167	212	272	348	447	576	743
108	136	174	221	283	363	466	601	775
112	142	181	230	295	378	486	627	809
116	148	188	240	307	394	507	654	844
121	154	196	250	320	411	529	682	881
126	160	204	261	334	429	552	712	919
								959

or 55 entries in all; but it is clear that these entries are not so convenient as the somewhat larger number selected above.

On the Contact-Property of the Eleven-point Conic.

By R. E. Allardice, M.A.

It is sometimes the case that geometrical theorems, which are usually enunciated as properties of the triangle or the quadrilateral, may be stated more succinctly, and in a form that better suggests generalisation, as properties of the complete quadrilateral. Thus the theorem that is usually stated in the following form, "The three straight lines that join the middle points of opposite sides and the middle points of diagonals of a quadrilateral are concurrent and bisect one another at the point of concurrence," may be enunciated more simply as a property of the straight lines that join the middle points of the three pairs of opposite sides of a complete four-point; and the latter form of enunciation suggests how the theorem may be generalised by projection. So too the figure that consists of four points, any one of which is the orthocentre of the triangle formed by the other three, is better described as a complete four-point in which opposite sides intersect at right angles, (which may be called an orthic four-point).

This method of describing the property of the orthocentre suggests a way of stating the fundamental property of the nine-point circle, namely, as follows:—

In an orthic four-point, the middle points of the six sides and the three diagonal points are concyclic. It is now natural to enquire what the corresponding theorem is, when a quadrilateral of the most general kind is substituted for an orthic quadrilateral. It is well known that the nine-point circle is the locus of the centre of a conic that passes through the vertices of the orthic four-point; and in Taylor's *Ancient and Modern Geometry of Conics* (p. 283) it is proved that the locus of the centre of a conic that circumscribes a given four-point is a conic, (the eleven-point conic) which passes through the middle points of the six sides of the four-point, through the three diagonal points, and through two definite points at infinity which correspond to the circular points at infinity, in the case of the nine-point circle.

The following is an analytical proof of the analogue of the theorem that the nine-point circle touches the inscribed and escribed circles:—

Let ABCD (fig. 23) be the four-point; to show that the eleven-point conic of ABCD touches the similar conics inscribed in the triangles ABC, ABD, etc.

Take AC and BD as axes of co-ordinates; then the equations to AB, BC, CD, DA may be written

$$px + ry - 1 = 0, \qquad qx + ry - 1 = 0,$$
$$px + sy - 1 = 0, \qquad qx + sy - 1 = 0.$$

The equation to a conic through A, B, C, D, is

$$(px+sy-1)(qx+ry-1)+kxy=0\,;$$

that is

$$pqx^2+(pr+qs+k)xy+rsy^2-(p+q)x-(r+s)y+1=0.$$

The centre is given by

$$\left.\begin{aligned}2pqx+(pr+qs+k)y-(p+q)=0,\\(pr+qs)x+2rsy-(r-s)=0,\end{aligned}\right\}$$

that is

$$\left.\begin{aligned}2pqx+(pr+qs)y-(p+q)+ky=0,\\(pr+qs)x+2rsy-(r+s)\;+kx=0.\end{aligned}\right\}$$

Hence the locus of the centre is

$$x[2pqx+(pr+qs)y-(p+q)]-y[(pr+qs)x+2rsy-(r+s)]=0$$

that is,

$$2pqx^2-2rsy^2-\;(p+q)x+\;(r+s)y=0\,;$$

or $$4pqx^2-4rsy^2-2(p+q)x+2(r+s)y=0. \quad \ldots \quad (1)$$

This is the equation to the eleven-point conic.

The equation to a similar conic may be written

$$4pqx^2-4rsy^2+2lx+2my+2n=0. \quad \ldots \quad \ldots \quad (2)$$

The line of intersection of these two conics is

$$[l+(p+q)]x+[m-(r+s)]y+n=0.$$

To determine l, m, n we have the conditions that (2) touches $y=0$, $px+ry-1=0$ and $qx+ry-1=0$, (taking the conic to be inscribed in ABC).

Assume

$$4pqx^2-4rsy^2+2lx+2my+2n\equiv ax^2+2hxy+by^2+2gx+2fy+c\,;$$

then, $\quad a=4pq,\ b=-4rs,\ c=2n,\ f=m,\ g=l,\ h=0.$

Hence,
$$\begin{aligned}A&=-\;8rsn-m^2, & F&=-4pqm,\\ B&=\;\;8pqn-l^2, & G&=\;\;4rsl,\\ C&=-16pqrs, & H&=\;\;lm.\end{aligned}$$

The condition that (2) touch $\lambda x+\mu y+\nu=0$ is

$$A\lambda^2+B\mu^2+C\nu^2+2F\mu\nu+2G\nu\lambda+2H\lambda\mu=0.$$

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For the three sides of ABC, we have

1° $\lambda = \nu = 0, \mu = 1,$ $\quad\therefore\ B = 0$;

2° $\lambda = p, \quad \mu = r, \ \nu = -1, \quad \therefore\ Ap^2 + C - 2Fr - 2Gp + 2Hpr = 0$;

3° $\lambda = q, \quad \mu = r, \ \nu = -1, \quad \therefore\ Aq^2 + C - 2Fr - 2Gq + 2Hqr = 0.$

Taking (2) and (3) we have

$$A(p^2 - q^2) - 2G(p-q) + 2Hr(p-q) = 0 \text{ ; and}$$
$$A(p^2q - pq^2) - C(p-q) + 2Fr(p-q) = 0.$$

Hence $\quad A(p-q) - 2G + 2Hr = 0$; and
$Apq - C + 2Fr = 0$; and also $B = 0$.

These conditions reduce to

$$l^2 - 8pqn = 0, -(m^2 + 8rsn) + 16rs - 8mr = 0$$
$$\text{and } 8s(p+q) - 4m(p+q) + 4sl - lm = 0. \quad \ldots \quad \ldots \quad (3)$$

Now we have to show that

$(l+p+q)x + (m-r-s)y + n = 0$ touches the conic (1).

Assume

$$4pqx^2 - 4rsy^2 - 2(p+q)x + 2(r+s)y$$
$$\equiv a'x^2 + 2h'xy + b'y^2 + 2g'x + 2f'y + c' \text{ ;}$$

then

$$A' = -(r+s)^2, \quad B' = -(p+q)^2, \quad C' = -16pqrs,$$
$$F' = -4pq(r+s), \ G' = -4rs(p+q), \ H' = -(p+q)(r+s).$$

Hence we have to show that

$$K \equiv -(r+s)^2(l+p+q)^2 - (p+q)^2(m-r-s)^2 - 8pq(r+s)(m-r-s)n$$
$$-16pqrsn^2 - 8rs(p+q)(l+p+q)n - 2(p+q)(r+s)(l+p+q)(m-r-s) = 0.$$

Now

$$K \equiv -(r+s)^2[l^2 + 2l(p+q) + (p+q)^2] - (p+q)^2[m^2 - 2m(r+s) + (r+s)^2]$$
$$-16pqrsn^2 - 8pqmn(r+s) + 8pqn(r+s)^2 - 8rsln(p+q) - 8rsn(p+q)^2$$
$$-2(p+q)(r+s)[lm - l(r+s) + m(p+q) - (p+q)(r+s)]$$
$$\equiv -m^2(p+q)^2 - 16pqrsn^2 - 8pqmn(r+s) - 8rsln(p+q)$$
$$-8rsn(p+q)^2 - 2lm(p+q)(r+s)$$

Now $\quad (p+q)^2(8rsn + m^2) = (p+q)(2lmr - 8rls), \quad$ [from (3)],

Hence

$$K \equiv (p+q)(8rls - 2lmr) - 16pqrsn^2 - 8pqmn(r+s) - 8rsln(p+q)$$
$$-2lmr(p+q) - 2lms(p+q)$$

Again, $\quad 8rls(p+q) = 4rlm(p+q) - 4rsl^2 - rl^2m$
$$= 4rlm(p+q) - 32pqnrs - 8pqnmr$$

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$$\therefore \mathrm{K} \equiv 4rlm(p+q) - 32pqnrs - 8pqmnr - 2lmr(p+q) - 16pqrsn^2$$
$$- 8pqmn(r+s) - 8rsln(p+q) - 2lmr(p+q) - 2lms(p+q).$$
$$\equiv 2s[-16pqnr - 8pqrn^2 - 4pqmn - (p+q)(rln+lm)]$$
$$\equiv s[-4l^2r - 2l^2rn - l^2m - 2l(p+q)(rn+m)]$$
$$\equiv -ls[4lr + 2lrn + lm + 2(p+q)(rn+m)].$$

Now if we eliminate s between the equations (3) we get

$$4lr + 2lrn + lm + 2(p+q)(rn+m) = 0\ ;\ \therefore \mathrm{K} = 0.$$

It may easily be proved that the eleven-point conic passes through the following eleven points :—the vertices of the diagonal triangle of ABCD, the middle points of the six sides of ABCD and two definite points at infinity, namely, the centres of the two parabolas that pass through A, B, C, D ; and by means of the equation to the eleven-point conic given above, various particular cases may be considered.

[*Note.*—Since writing the above, I have found references to the following papers by Professor Beltrami on the eleven-point conic :— *Intorno alle coniche di nove punti e ad alcune questioni che ne dipendono*, Bologna, Mem. Accad., II., 1862, pp. 361-365, and Bologna, Rendiconto 1862-63, pp. 82-85; *Sulle coniche di nove punti*, Napoli, Giorn. di Mat., I., 1863, pp. 109-118 ; *Estenzione allo spazio di tre dimensioni dei teoremi, relativi alle coniche di nove punti*, Napoli, Giorn. di Mat., I., 1863, pp. 208-217, 354-360.]

Deduction of the Thermodynamical Relations.

By William Peddie, D.Sc.

In his *Theory of Heat*, Clerk-Maxwell gives a very elegant and simple geometrical proof of the four thermodynamical relations, and points out that his construction shows that the truth of any one is a necessary consequence of the truth of any other. The usual analytical proof of these relations can be made as simple as Maxwell's geometrical proof, and the fact that any one is a necessary consequence of any other becomes evident when it is considered that they are all deduced by a common process from identical transformations of one equation. The following method of proof seems to me to bring out more directly their necessary interdependence.

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