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A NOTE ON CENTRAL IDEMPOTENTS IN GROUP RINGS

by SONIA P. COELHO*

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Let G be a group and K a field. We denote by $\mathscr{U}(KG)$ the group of units of the group ring of G over K and for a group X we denote by $T(X)$ the set of torsion elements of G, i.e., the set of all elements of finite order.

In the study of group-theoretical properties of $\mathscr{U}(KG)$ it has been found that some conditions on this group lead to the fact that $T=T(G)$ is a group and every idempotent in KT is central in KG. For example, if K is a field of characteristic $p>0$, this will happen when G is infinite, non abelian and $\mathscr{U}(KG)$ is an FC-group ([**1**], Theorem A) or when G is nilpotent or FC, contains no p-elements and $T(\mathscr{U}(KG))$ forms a subgroup ([**4**], Theorem 4.1). This condition has also appeared in the description of the structure of some unit groups, as in [**5**], Lemma VI.3.22.

In this note, we shall study this condition and determine what it will imply about the structure of the original group and the given field. We will consider only groups whose torsion elements form a locally finite group. This is the case in all the situations which were mentioned above.

We shall assume that K is a field of characteristic $p>0$ and denote by $\mathscr{P}(K)$ the prime subfield of K. Also, A will be the set of all p'-elements of G and P the set of all p-elements of G.

We prove the following:

Theorem. *Let K be a field of characteristic $p>0$ and G a group such that the torsion T of G forms a locally finite subgroup. Then, every idempotent of KT is central in KG if and only if:*

(i) *A is an abelian subgroup of G.*

(ii) *if A is non central, then the algebraic closure Ω of $\mathscr{P}(K)$ in K is finite and for all $x\in G$ and all $t\in A$, there exists $r\in N$ such that $xtx^{-1}=t^{p^r}$. Furthermore, for each such r, we have that $(\Omega:\mathscr{P}(K))\mid r$,*

(iii) *P is a subgroup of G,*

(iv) *$T=P\times A$.*

Proof. *Necessity.* To establish (i), let $t\in G$ be a p'-element. Then, $e=o(t)^{-1}(1+t+\cdots+t^{o(t)-1})$ is a central idempotent. Given $x\in G$, we have that $xex^{-1}=e$. By

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considering the supports of the elements in both sides of the equation, we see that $xtx^{-1}=t^i$, for some natural number i. Therefore, we have that $\langle t\rangle \lhd G$.

Now, A is an abelian or Hamiltonian group, by [6], p. 160. But, if $A \supset K_8$, the quaternion group of order 8, then $p>2$ and $\mathscr{P}(K)K_8$ contains a matrix ring and hence a non-central idempotent, a contradiction.

For (ii), let L be a finite subfield of K and $t \in A$ a non-central element. We have that $L\langle t\rangle = \bigoplus_i K_i$, a direct sum of fields, such that at least one of them, say K_1, has the form $K_1 = L(\xi)$, where ξ is root of unity of order $o(t)$. Furthermore, the natural projection $L\langle t\rangle \to K_1$ sends t to ξ.

Given $x \in G$, we have that $xtx^{-1}=t^i$, for some i, as we have seen in the proof of (i). Hence, since every idempotent of KT is central, conjugation by x defines an automorphism ϕ of K_1. Consequently we see that $\phi(\xi)=\xi^i$.

Suppose now that we have a natural number r such that $xtx^{-1}=t^{p^r}$. Then, $\phi(\xi)=\xi^{p^r}$. Since every element in the copy of L contained in K_1 is fixed by ϕ, we have that $k^{p^r}=k$, for all $k \in L$. Then L is contained in a field with p^r elements, i.e., $(L:\mathscr{P}(K))\,|\,r$. Now, it is easy to conclude that $(\Omega:\mathscr{P}(K))\,|\,r$, for each such r.

Let us now show that such an r actually exists. As K_1 is finite, ϕ is a power of the Frobenius automorphism of K_1, say F, given by: $F(k)=k^p$, for all $k \in K_1$. If $\phi = F^r$, we have that $\phi(\xi)=\xi^{p^r}$, from which we conclude that $o(\xi)=o(t)$ divides p^r-i, that is, $p^r \equiv i (\operatorname{mod} o(t))$, so $xtx^{-1}=t^i=t^{p^r}$.

For (iii) and (iv), we observe first that every p-element commutes with every p'-element. If not, consider $\pi \in P$ and $t \in A$ such that $\pi t \pi^{-1} = t^{p^r} \neq t$.

The group ring $\mathscr{P}(K)\langle \pi, t\rangle$ is finite and, if J denotes its Jacobson radical, we have that $(\mathscr{P}(K)\langle \pi, t\rangle)/J$ is semi-simple artinian. If it were abelian, then, denoting by $\bar{x}$ the image of an element $x \in \mathscr{P}(K)\langle \pi, t\rangle$ under the natural projection $\mathscr{P}(K)\langle \pi, t\rangle \to (\mathscr{P}(K)\langle \pi, t\rangle)/J$, we should have: $\bar{\pi}\bar{t}\bar{\pi}^{-1}=\bar{t}^{p^r}=\bar{t}$ or $\bar{t}^{p^r-1}=1$.

But in this case $t^{p^r-1}-1$ belongs to J, which is nilpotent. Hence, it is easy to see that t^{p^r-1} is a p-element. As it is also a p'-element, we have: $t^{p^r-1}=1$, or $t^{p^r}=t$, a contradiction.

Then, $(\mathscr{P}(K)\langle \pi, t\rangle)/J$ contains a matrix ring, hence a non-central idempotent. As J is nilpotent, $\mathscr{P}(K)\langle \pi, t\rangle$ itself contains a non-central idempotent ([3], Lemma 2.3.7), a contradiction.

As T is locally finite, it is easy to see now that P is a subgroup of G and $T = P \times A$, which proves (iii) and (iv).

Sufficiency. We now suppose that conditions (i) to (iv) hold, and set e an idempotent of KT. Writing e in the form:

$$e = \sum_{i,j} \alpha_{ij} g_i h_j,$$

with $\alpha_{ij} \in K$, $g_i \in P$, $h_j \in A$, we see that

$$f = \sum_j \left(\sum_i \alpha_{ij} \right) h_j$$

is an idempotent of KA, and $e-f=\delta\in\Delta(T,P)$, the kernel of the natural epimorphism $KT\rightarrow KA$.

As T is locally finite, by considering the subgroup generated by the g_i's and the h_j's we can conclude that δ is nilpotent. Furthermore, it is easy to see that δ commutes with f.

If $\delta^{p^n}=0$, we have that $e^{p^n}=f^{p^n}+\delta^{p^n}=f^{p^n}$, that is, $e=e^{p^n}=f^{p^n}=f$. Then, e belongs to KA. Furthermore, by considering the subgroup generated by the support of e, we may suppose that A is finite.

If A is central, then e is central. If not, take $x\in G$. If $A=\langle t_1\rangle\times\langle t_2\rangle\times\cdots\times\langle t_s\rangle$, a direct product of cyclic groups, we claim that we can choose a natural u such that $xt_ix^{-1}=t_i^{p^u}$, for all i. In fact, if

$$xt_ix^{-1}=t_i^{p^{u_i}}, 1\leqq i\leqq s,$$

and

$$xt_1t_2\ldots t_sx^{-1}=(t_1t_2\ldots t_s)^{p^u},$$

we have:

$$t_1^{p^{u_1}}t_2^{p^{u_2}}\ldots t_s^{p^{u_s}}=t_1^{p^u}t_2^{p^u}\ldots t_s^{p^u}.$$

Therefore, $t_1^{p^{u_i}}=t_i^{p^u}$, and $xt_ix^{-1}=t_i^{p^u}$, for all i.

Now, e can be written in the form: $e=f(t_1,\ldots,t_s)$ where $f(X_1,\ldots,X_s)$ is a polynomial in the commuting variables $X_1,\ldots,X_s$, with coefficients in the algebraic closure Ω of $\mathscr{P}(K)$ in K. In fact, e is actually an idempotent of KA, with A finite abelian, hence e is a sum of primitive idempotents of ΩA ([2], Theorem 2.12, p. 19).

Conjugating e by x, we obtain $xex^{-1}=f(t_1^{p^u},t_2^{p^u},\ldots,t_s^{p^u})$. But from the hypothesis (ii) every $k\in\Omega$ satisfies $k^{p^u}=k$, hence this is true for the coefficients of f. Then

$$xex^{-1}=f(t_1^{p^u},t_2^{p^u},\ldots,t_s^{p^u})=(f(t_1,\ldots,t_s))^{p^u}=e^{p^u}=e,$$

as we wanted to show.

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Instituto de Matemática e Estatística
Universidade de São Paulo
C.P. 20.570 (Ag. Iguatemi)
01000—S. Paulo
Brazil