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A new two-columns description for convective transport in stars

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Abstract. Assuming that the largest convective patterns generate the majority of convective transport, we devise a numerical scheme simplifying the convective velocity field using two parallel radial columns to represent up- and downstream flows. Horizontal exchange is described by fluid flow and radiation over the interface between those two columns. The main parameters of this convective description have a straightforward geometrical meaning, namely the diameter of the columns (representing the size of the convective cells) and the ratio of cross section between up- and downdrafts. For this geometrical setup, the equations of radiation hydrodynamics are solved time-dependently using an implicit scheme which has the advantage of being devoid of any time step limits. In order to demonstrate our approach, we present comparisons with detailed 2D hydrodynamics computations for the example of convection zones in Cepheids.

Keywords. hydrodynamics — Cepheids — convection — methods: numerical

1. Introduction

Current two- and three-dimensional hydrodynamics simulations of stellar photospheric convection are remarkably successful in reproducing the observed properties. In particular for the sun, comparisons with observational constraints from line profiles (e.g. Asplund *et al.*, 2000) and the solar granulation pattern (e.g. Stein & Nordlund, 1998, Wedemeyer *et al.*, 2004) show the high level of fidelity of multi-dimensional simulations. Indirectly, this also confirms various other (non-observed) properties of the numerical models such as temperature structure, convective flux, and convective velocities.

Motivated by this success of multidimensional hydrodynamics in modelling convective transport, we devised a numerical scheme which describes the circulating convective flow in the most simple manner with two parallel radial columns. Convective up- and downdraft motions are represented by radial fluid flow in these two columns while a horizontal component of the flow over the interface between these two columns closes the circulating motion. These two columns thereby do not stand for an individual convective cell but should be considered as a representation of all up- and downdraft motions.

This discretization scheme has three main parameters: First, the typical horizontal length scale D that can be interpreted as a diameter of the columns or as their distance from each other. Physically, this corresponds to the characteristic size of the flow patterns or eddies of the modelled convection.

Secondly, the parameter $cf1$ ('*cf*' for *column fraction*) describes the fraction of the sphere allocated to column 1. Accordingly, $cf2 = 1 - cf1$ is the relative cross section of column 2. This different size of the columns can be used to model convection zones with narrow down- and wide updrafts as observed, e.g., in the solar granulation.

Finally, a third constitutive parameter specifies details of the horizontal advection from one column to the other; but since this parameter not used in the examples below, we will not discuss it here further.

The big advantage of this two-columns discretization scheme is that its very simple setup allows for an efficient and straightforward application of the implicit solution method. This results in a code that is almost as fast as an implicit 1D-Code (as used, e.g., for stellar evolution or for stellar pulsations) while including the effects of convective transport by the simple, yet hydrodynamical consistent, circulating flow.

2. The discretization scheme

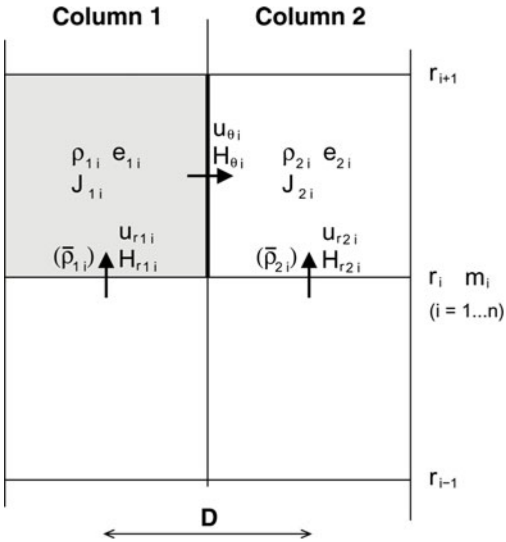

Figure 1. The two-columns discretization scheme. The primary variables (compare Table 1) are included in their staggered-mesh localization; D is the typical distance/diameter of the columns. Advection occurs, as indicated by arrows, in radial direction as well as over the interface between the two columns (thick line).

Figure 1 shows the setup of the two-columns discretization scheme. Note that, even though not drawn as such, all this takes place in spherical symmetry; in general, the two columns also do not have the same size. The area shaded in gray in Fig. 1 is the discretization volume for the scalar quantities ρ, e, J and the corresponding equations (see Table 1). Vector variables in radial (u_r, H_r, $\bar{\rho}$) and horizontal (u_θ, H_θ) direction are discretized in similar volumes located in staggered-mesh positions. For a detailed description of the discretization scheme see Stökl (2008).

The radial distribution of the grid points r_i is determined by an adaptive grid equation (Dorfi & Drury, 1987). This adaptive grid equation is solved implicitly together with the physical equations and continuously adjusts the grid resolution according to the evolving physical structures. The adaptive grid also allows the usage of a Lagrangian outer boundary condition so that the grid can follow radius variations of the star, e.g. due to stellar pulsations or structural resettling caused by the onset of convective transport.

3. Method of solution

The system of discrete equations compiled in Table 1 is solved by an implicit solver. This method has the advantage of being not affected by the CFL (after Courant, Friedrichs & Lewy 1928) time step limit and in principle allows arbitrarily large time steps. The

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Table 1. The set of 16 primary variables and the corresponding discrete equations. A moment description for the radiation field (Mihalas & Mihalas, 1984) has been adopted for the equations of radiation hydrodynamics. The radially averaged densities $\overline{\rho}$ are used for assembling the radial momentum $\overline{\rho}u_r$; this formalism is necessary to implement the second order advection scheme for the momentum within the 5-point discretization stencil.

Variable	Description	Equation
r_i	Radius	Adaptive grid equation
m_i	Integrated mass	Poisson equation
$\rho_{1\,i}$, $\rho_{2\,i}$	Density	Equation of continuity
$\overline{\rho}_{1\,i}$, $\overline{\rho}_{2\,i}$	Averaged density	Radial averaging of ρ
$e_{1\,i}$, $e_{2\,i}$	Specific internal energy	Equation of energy
$u_{1\,i}$, $u_{2\,i}$	Radial velocity	Equation of motion, radial component
$u_{\theta\,i}$	Horizontal velocity	Equation of motion, horizontal component
$J_{1\,i}$, $J_{2\,i}$	0^{th} moment of intensity	Radiation energy equation
$H_{1\,i}$, $H_{2\,i}$	1^{st} moment of intensity, radial	Radiation flux Eq., radial component
$H_{\theta\,i}$	1^{st} moment of intensity, horizontal	Radiation flux Eq., horizontal component

Closures:
– tabulated equation of state (temperature, gas pressure), evaluated separately in each column: $T_1 = T(\rho_1, e_1)$, $T_2 = T(\rho_2, e_2)$, $P_1 = P(\rho_1, e_1)$, $P_2 = P(\rho_2, e_2)$
– tabulated opacities (Rosseland mean), evaluated separately in each column: $\kappa_1 = \kappa(\rho_1, e_1)$, $\kappa_2 = \kappa(\rho_2, e_2)$
– closure of radiation moments with an Eddington factor $f_{\mathrm{edd}} = \mathbf{K}/J = 1/3$

inclusion of elliptical parts into the system of physical equations (Poisson and grid equation) is also only possible with an implicit scheme.

As the system of equations contains nonlinear terms, a Newton-Raphson iteration is used where each iteration step involves the inversion of the Jacobi matrix. According to the number of equations, the Jacobian consists of 16×16 sub-matrices which form a pentadiagonal banded structure reflecting the discretization on a 5-point stencil. The inversion of the Jacobi matrix uses the customary approach of elimination of the two lower sub-diagonals and subsequent back-substitution of the resulting upper triangular matrix.

Usually, the models have 500 radial grid points. Due to the adaptive grid, the majority of them clusters around the steep gradients in the photosphere and in the convective region.

4. Stationary solutions

The stationary solution for the convective velocity field is obtained by computing the temporal evolution starting from a hydrostatic, purely radiative initial model. In order to make the convective circulation go in the intended sense of rotation (i.e. wide up-, and narrow downdrafts) small radial velocity perturbations ($u \leqslant 1\,\mathrm{m/sec}$) are applied to the hydrostatic model using the Schwarzschild convection criterion as a guide. Starting from these perturbations, the convective velocities develop quickly and after a dynamic phase of growth which lasts about a thermal time scale of the involved part of the envelope, the convection approaches the stationary solution. This evolution typically requires about 1000 time steps that increase from a few seconds at the start up to 10^{10} sec for the stationary solution.

The ultimate aim of our work is the investigation of the interaction of convection and pulsation in Cepheids. Hence, as a first step, we computed stationary solutions for Cepheid convection zones.

Figure 2 illustrates the results for a rather typical Cepheid with $T_{\rm eff} = 5400\,K$, $L = 10^3\,L_\odot$, and $M = 4.75\,M_\odot$. The geometrical parameters are $D = 20\,H_{p0}$ and $cf1 = 0.8$. According to the (arbitrary) convention of using column 1 for up-, and column 2 for downdrafts, the latter corresponds to a downstream cross section of 20% of the sphere. The typical horizontal extension D of the columns is specified relative to the characteristic photospheric pressure scale height, $H_{p0} = \mathcal{R}T_{\rm eff}/g$. This formalism is motivated by Freytag *et al.* (1997) where a horizontal scale of photospheric convection of about $10\,H_{p0}$ was found to be a good estimate for a broad range of stellar parameters.

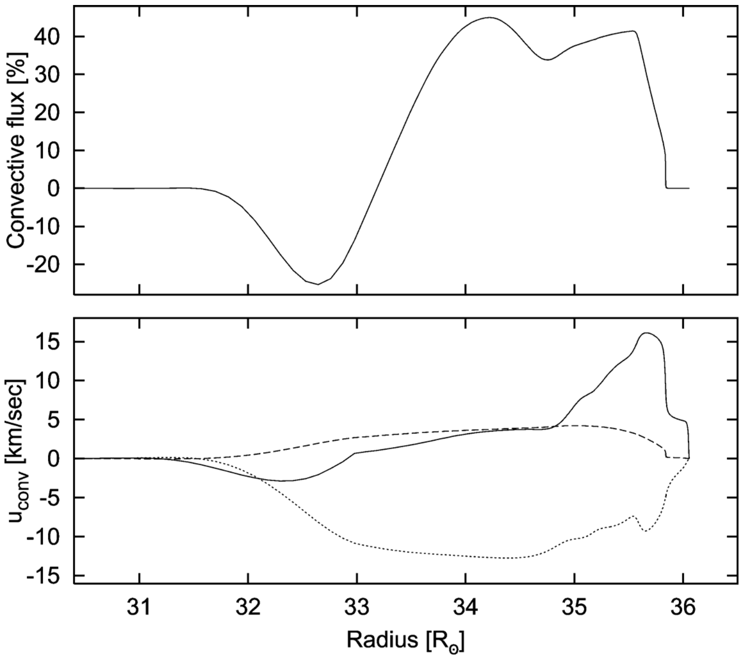

Figure 2. Stationary solution for a Cepheid convection zone: The upper panel shows the convective transport in units of the total luminosity. The convective velocities are given in the lower panel: updraft (dashed line), downdraft (dotted line) and horizontal (solid line). A positive sign of the horizontal velocity corresponds to a flow from column 1 to column 2, i.e. from updraft to downdraft. The figure focuses only on the outer convective region, the model actually extends down to about $3.6\,R_\odot$.

5. Comparison with 2D-computations

In order to verify the convection zones computed with the two-columns scheme, Fig. 3 compares them with results from 2D hydrodynamics carried out with $\mathrm{CO^5BOLD}$ (Freytag, 2008). The figure gives examples for convection zones in cool and hot Cepheids which are qualitatively quite different:

For lower effective temperatures, a deep convective region contains both the H and He II ionization zones. In that case, convection carries a substantial fraction of the energy flux and the extended downdrafts lead to a pronounced overshoot at the lower boundary.

For hotter stars, where radiative transport is more effective, convection becomes less vigorous and is thus no longer able to bridge the gap between the two ionization zones; hence, there only remains a thin convective shell at the H ionization zone. The He II

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ionization zone only appears as a very slight bump in the convective flux. However, even though not visible in Fig. 3, there is still a substantial, extended convective velocity field.

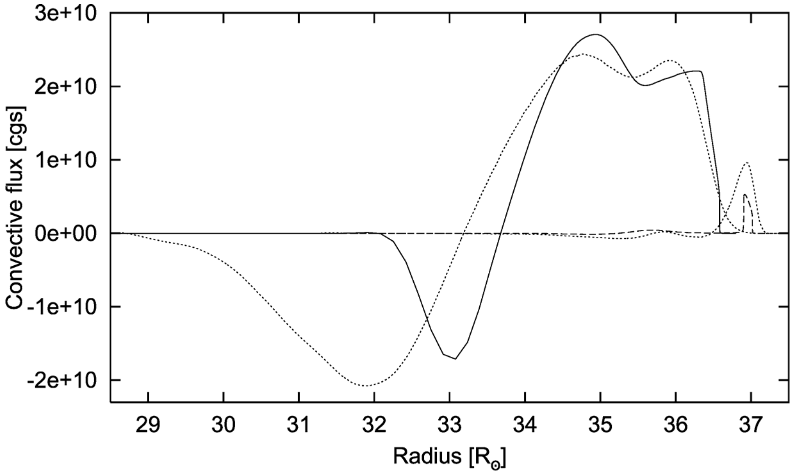

Figure 3. Comparison of the two-columns scheme with 2D hydrodynamics. Shown are convective fluxes computed with the two-columns code for Cepheids with effective temperatures of 5200 K (solid line) and 5800 K (dashed line) as well as the corresponding results obtained with CO^{5}BOLD (both dotted lines).

The comparison of the two-columns scheme with 2D hydrodynamics confirms that the two-columns scheme is able to reproduce many properties of the convection zone:

- An extended lower overshoot.
- The efficiency of convective transport, also in the overshoot region.
- Two distinctive bumps in the convective flux correlated to H and He II ionization.
- An inward flux of kinetic energy (not illustrated).
- The magnitude of convective velocities (not illustrated).
- The transition with increasing effective temperature from an extended convection zone embracing both the H and He II ionization zones to a thin, inefficient convective shell.

Most differences in the results between the two-columns scheme and the 2D simulations, in particular the different depth of the overshoot, are probably due to the simplified geometrical picture of the two columns scheme. An other prominent effect are the considerably smeared-out curves for the 2D results (see upper boundaries of the convective regions) that are caused by the spatial and temporal averaging necessary for extracting such integral quantities from the 2D simulations. Generally, the results for the convection zones in Cepheids agree well with those found for the similar A-type stars (Steffen *et al.*, 2005).

The main advantage of the two-columns model over multi-dimensional hydrodynamics are the large time steps possible with the implicit solution method. The two-columns code requires about one minute CPU-time to follow the temporal evolution of the convection zone up to the stationary solution. In contrast, the 2D simulations of the Cepheid convection zones with CO^{5}BOLD took up to six months to reach sufficiently stationary states.

6. Summary

The two-columns scheme allows for a non-local, time-dependent description of convection and the main parameters of the scheme have a straightforward geometrical meaning. Moreover, the convective solutions are qualitatively quite robust and do not vary drastically within the reasonable parameter range (Stökl, 2008).

Due to the implicit solution method, very large time steps are possible which make the method in many applications much faster than multi-D hydrodynamics. The simplified convective circulation of the two-columns scheme has stationary solutions. Hence, this method can be applied to problems where long time series are required, e.g. stellar pulsation or stellar evolution, without the necessity of resolving the dynamical time scale of convective turbulence.

Comparisons with 2D hydrodynamics proved that the scheme is able to reproduce many important properties of convection such as convective flux, overshoot, kinetic energy flux, and the range of convective velocities. First time dependent Cepheid pulsations computed with the two-columns scheme have also already demonstrated the feasibility of our approach to simulate simultaneously convection and pulsations.

Despite these achievements, the two-columns model is basically still a parameter-depended description. The good quantitative agreement between two-columns and 2D is in part also a result of suitable values for the free-parameters. However, the adopted parameters $cf1 = 0.8$ and $D = 20\,H_{p0}$ are by no means the outcome of an extensive parameter tuning.

An other limit of the scheme is the very coarse spatial resolution in horizontal direction and the simplistic description of the (vertical and horizontal) spectrum of convective velocities. So while the method is able to reproduce the main macroscopic properties of convection through the simple circulation fluid flow, one should not expect a correct description of more subtle details or turbulence effects from such a simple scheme.

Acknowledgements

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Discussion

Kupka: The type of model for convection you are using here is known in meteorology under the name of mass-flux model. If one ties both velocity and temperature to the same column (up≡hot, cold≡down), sign symmetries in some third (and fourth) order correlations are violated and one runs into troubles with that particularly in overshooting zones. Secondly, I wonder about your choice of the column fraction value. There are problems known to exist for A-stars where the simulations are known to not recover the observed, blue-wards curved line profiles. For line profiles of Cepheids this non-solar-like shape is even more prominent. I'd thus not rely on the simulations as a valid test of models until they themselves have been probed with observational data.

Stoekl: (1) I'm not familiar with these meteorology models – however, my implementation does not couple the sign of velocity and temperature variations. I can not argue about 3^{rd} & 4^{rd} order moments as I've never translated the 2-columns scheme into that formulism. (2) The value for the fraction in cross section for up- and downdrafts is left as a free parameter. Therefore, when observations – of line profiles or of whatever – can provide constrains, this is a very welcome input. Despite the mentioned problems of 2D (and 3D) hydro simulations, I think that the qualitatively agreement of 2 columns & 2D simulations is a promising result.

Woitke: What is the difference between your "2 column" approach and a regular radiation-hydro code with an extremely coarse resolution like 500×2?

Stoekl: There is none. – in principle – the geometric interpretation of the 2-columns-scheme is not that of a usual 2D–grid.