Canad. J. Math. Vol. **68** (5), 2016 pp. 999–1022
http://dx.doi.org/10.4153/CJM-2015-017-4

Quotients of $A_2 * T_2$

Masaki Izumi, Scott Morrison, and David Penneys

Abstract. We study unitary quotients of the free product unitary pivotal category $A_2 * T_2$. We show that such quotients are parametrized by an integer $n \geq 1$ and an $2n$-th root of unity ω. We show that for $n = 1, 2, 3$, there is exactly one quotient and $\omega = 1$. For $4 \leq n \leq 10$, we show that there are no such quotients. Our methods also apply to quotients of $T_2 * T_2$, where we have a similar result.

The essence of our method is a consistency check on jellyfish relations. While we only treat the specific cases of $A_2 * T_2$ and $T_2 * T_2$, we anticipate that our technique can be extended to a general method for proving the nonexistence of planar algebras with a specified principal graph.

During the preparation of this manuscript, we learnt of Liu's independent result on composites of A_3 and A_4 subfactor planar algebras (arxiv:1308.5691). In 1994, Bisch–Haagerup showed that the principal graph of a composite of A_3 and A_4 must fit into a certain family, and Liu has classified all such subfactor planar algebras. We explain the connection between the quotient categories and the corresponding composite subfactor planar algebras. As a corollary of Liu's result, there are no such quotient categories for $n \geq 4$.

This is an abridged version of arxiv:1308.5723.

1 Introduction

In [BJ97], Bisch–Jones defined the free product of two Temperley–Lieb subfactor planar algebras as a certain planar subalgebra of the tensor product subfactor planar algebra. Given two subfactor planar algebras $\mathcal{P}_\bullet$ and $\mathcal{Q}_\bullet$, their tensor product is given by $(\mathcal{P} \otimes \mathcal{Q})_{n,\pm} = \mathcal{P}_{n,\pm} \otimes \mathcal{Q}_{n,\pm}$, which is again a subfactor planar algebra, with tangles acting diagonally on each tensor factor. In an unpublished article, Bisch–Jones defined the free product $(\mathcal{P} * \mathcal{Q})_\bullet$ to be the planar subalgebra of $(\mathcal{P} \otimes \mathcal{Q})_\bullet$ consisting of up-to-isotopy non-crossing diagrams. See also [BL10, Section 8].

There is an analogous definition for pivotal tensor categories. Given two such categories $\mathcal{C}$ and $\mathcal{D}$, we can define the tensor product $\mathcal{C} \boxtimes \mathcal{D}$, which is again a pivotal tensor category. The objects are words of objects in $\mathcal{C}$ and $\mathcal{D}$, and the morphism spaces are just the tensor products of the morphism spaces for the subwords consisting of letters from $\mathcal{C}$ and from $\mathcal{D}$. One can represent these diagrammatically as superimposed diagrams from $\mathcal{C}$ and $\mathcal{D}$, as if on two stacked panes of glass. The free product $\mathcal{C} * \mathcal{D}$ is the pivotal subcategory of $\mathcal{C} \boxtimes \mathcal{D}$ containing only morphisms that can be presented

Received by the editors June 11, 2014; revised March 2, 2015.
Published electronically June 28, 2016.
Author M. I. was supported by JSPS, the Grant-in-Aid for Scientific Research (B) 22340032. Author S. M. was supported by an Australian Research Council Discovery Early Career Researcher Award, DE120100232 and Discovery Project 'Subfactors and symmetries' DP140100732. Author D. P. was supported in part by the Natural Sciences and Engineering Research Council of Canada. Authors S. M. and D. P. were both supported by DOD-DARPA grant HR0011-12-1-0009.
AMS subject classification: **46L37**.
Keywords: pivotal category, free product, quotient, subfactor, intermediate subfactor.

diagrammatically as non-crossing diagrams. For more details, see [IMP13]. Note that if $\mathcal{C}$ and $\mathcal{D}$ are spherical (resp. unitary), then both $\mathcal{C} \boxtimes \mathcal{D}$ and $\mathcal{C} * \mathcal{D}$ are.

(a) a typical $\otimes$-product morphism (b) a typical free product morphism

One can think of the tensor product $\mathcal{C} \boxtimes \mathcal{D}$ as a quotient of the free product $\mathcal{C} * \mathcal{D}$; namely, there is a faithful, dominant functor $\mathcal{C} * \mathcal{D} \to \mathcal{C} \boxtimes \mathcal{D}$ (which is not full). We can obtain $\mathcal{C} \boxtimes \mathcal{D}$ from $\mathcal{C} * \mathcal{D}$ by adding natural symmetric isomorphisms $cd \to dc$ for all $c \in \mathcal{C}$ and $d \in \mathcal{D}$. Hence, we define a quotient of $\mathcal{C} * \mathcal{D}$ to be a pivotal tensor category $\mathcal{E}$ together with a faithful dominant functor $F : \mathcal{C} * \mathcal{D} \to \mathcal{E}$.

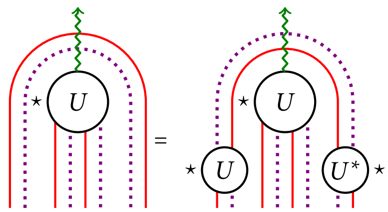
The easiest example is the case where $\mathcal{C} = \mathcal{D} = A_2$, the unitary quotient of Temperley–Lieb where the loop parameter is $\delta = 1$. In this case, $A_2 \cong \mathrm{Vec}(\mathbb{Z}/2\mathbb{Z})$, the free product $A_2 * A_2 \cong \mathrm{Vec}(\mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z})$, and $A_2 \boxtimes A_2 \cong \mathrm{Vec}(\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z})$. Since $\mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z} \cong D_\infty$, the infinite dihedral group, and all (nontrivial) quotients of D_∞ are of the form D_{2n} for some $n \geq 2$, we see all quotients of $A_2 * A_2$ are of the form $\mathrm{Vec}(D_{2n}, \omega)$ for some $\omega \in H^3(D_{2n}, \mathbb{C}^\times)$, *i.e.,* they are classified by an $n \geq 2$ and some cohomological data. In subfactor theory, this corresponds to the $A_{2n-1}^{(1)}$ subfactors [Pop94], which are closely related to the $D_{n+2}^{(1)}$ subfactors [IK93]. See [IMP13] for more details.

In this article, we study quotients of $A_2 * T_2$, where $T_2 = \frac{1}{2} A_4$ is the even half of the unitary quotient of Temperley–Lieb where $\delta = \tau = \frac{1}{2}(1 + \sqrt{5})$. Sometimes T_2 is also called the "golden" or "Fibonacci" category. Now $A_2 * T_2$ is no longer pointed, but we still find that there is at most one quotient for each $n \in \mathbb{N}$ and a $2n$-th root of unity ω. This root of unity is analogous to the cohomological data in the previous case.

We show that such a quotient has a unitary generator U with $4n$ strings that is a rotational eigenvector with eigenvalue ω_U, a $2n$-th root of unity. We then derive jellyfish relations for our generator, and these relations are sufficient to evaluate all closed diagrams, which determines the quotient, which we name $\mathcal{AT}_{n,\omega_U}$, provided it exists. Our main theorem is as follows.

Theorem 1.1 *For $n \in \{1, 2, 3\}$, $\mathcal{AT}_{n,\omega_U}$ exists if and only if $\omega_U = 1$. For $4 \leq n \leq 10$, $\mathcal{AT}_{n,\omega_U}$ does not exist.*

We prove this theorem using a variation of Bigelow's jellyfish algorithm [Big10, BMPS12]. We take the diagram below and apply jellyfish relations in two different ways.



We either pull U through a bundle of strings, or first create a cancelling pair UU^* on those strings, and then pull U between them. (In fact, the purple dotted string here is a bundle of $2n-1$ strings.) When $n \in \{1,2,3\}$, this calculation shows $\omega_U = 1$. When $4 \le n \le 10$, we obtain different answers, so there are no such quotients of $A_2 * T_2$.

These quotients of $A_2 * T_2$ are related to quotients of Bisch–Jones' Fuss–Catalan subfactor planar algebra $A_3 * A_4$ [BJ97]. Starting with a composed inclusion of subfactors $N \subset P \subset M$ where $N \subset P$ is A_4 and $P \subset M$ is A_3, the planar algebra $\mathcal{P}_\bullet$ of $N \subset M$ contains the $A_3 * A_4$ Fuss–Catalan subfactor planar algebra $\mathcal{FC}_\bullet$. We call such a $\mathcal{P}_\bullet$ a *composite $A_3 - A_4$ subfactor planar algebra.* In this case, the even half $\frac{1}{2}\mathcal{P}_+$ of $N - N$ bimodules is a quotient of $A_2 * T_2$, so it must be of the form $\mathcal{AT}_{n,\omega_U}$. These parameters n, ω_U also appear naturally from $\mathcal{P}_\bullet$. In 1994, Bisch–Haagerup found the list of possible candidates for the principal graphs for composite $A_3 - A_4$ subfactor planar algebras (see Section 3).

Simultaneously and independently, Liu proved the complete theorem for these composite subfactor planar algebras. (In fact, the articles [Liu13, IMP13] were coordinated to appear on the `arXiv` the same day.)

Theorem 1.2 ([Liu13]) *For $n \in \{1,2,3,\infty\}$, there is a unique $A_3 - A_4$ composite subfactor planar algebra. For $4 \le n < \infty$, no such composite exists.*

Since the even half $\frac{1}{2}\mathcal{P}_+$ must be of the form $\mathcal{AT}_{n,\omega_U}$, the weaker category result implies a weaker subfactor result. In light of Liu's stronger theorem, we would like the stronger subfactor result to imply a stronger category result. Luckily, we have the following theorem, which we prove in Section 3 by finding a Frobenius algebra object in $\mathcal{AT}_{n,\omega_U}$ for every n and ω_U.

Theorem 1.3 *There is a bijective correspondence between quotients of $A_2 * T_2$ and composite $A_3 - A_4$ subfactor planar algebras.*

Hence, using Liu's theorem together with the above result yields the following corollary.

Corollary 1.4 *For $4 \le n < \infty$, $\mathcal{AT}_{n,\omega_U}$ does not exist.*

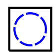
The methods of this article and Liu's article [Liu13] also apply to quotients of $T_2 * T_2$ with little alteration, where similar results are proved. Again, we obtain a partial classification for quotient categories, and Liu obtains a full classification for composite subfactor planar algebras. These two viewpoints are really the same by looking at even

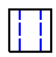
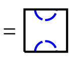
https://doi.org/10.4153/CJM-2015-017-4 Published online by Cambridge University Press

halves and Frobenius algebras, so we get a full classification of composite categories from Liu's stronger classification.

2 Quotients of $A_2 * T_2$

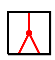
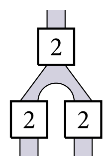
We recall that the pivotal category A_2 has two simple objects 1 and θ, which are both symmetrically self-dual. The category A_2 has no generators as a pivotal category (meaning all morphisms are tensor generated by identities and (co)evaluations) and the relations

(A1)  = 1,

(A2)  = .

Here, we represent θ by a dashed blue strand. The first relation simply says $\dim\theta = 1$. In a unitary category, the second relation follows from the first by calculating the norm of the difference of the two terms.

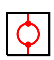
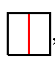
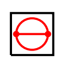
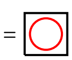
Recall that T_2, the even half of A_4 (A_4 is Temperley–Lieb with $\delta = \tau = \frac{1+\sqrt{5}}{2}$), has two simple objects $1, \rho$ where $\rho \otimes \rho \cong 1 \oplus \rho$. We denote ρ by a solid red strand, and we write a trivalent vertex for the intertwiner $\rho \otimes \rho \to \rho$ given by

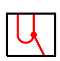
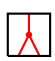
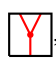
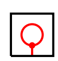
 $= \left(\frac{[2]}{[3]-1}\right)^{1/2}$  ,

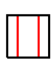
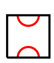
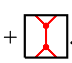
where we just write 2 for $f^{(2)}$, and $[2] = [3] = \tau$.

The following proposition is straightforward from the definition of the trivalent vertex.

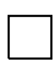
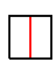
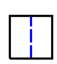
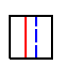
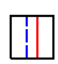
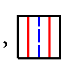
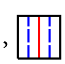
Proposition 2.1 *We have the following skein relations in T_2:*

 = ,  =  $= \tau$,

 = * = ,  = 0,

(T1)  $= \frac{1}{\tau}$  + .

The distinct simple objects of $A_2 * T_2$ are all alternating words in ρ, θ. Hence representatives of the isomorphism classes of simples are given by

, , , , , , , . . .

$1 \quad \rho \quad \theta \quad \rho\theta \quad \theta\rho \quad \rho\theta\rho \quad \theta\rho\theta$

2.1 Finding Generators

We now show all unitary quotients of $A_2 * T_2$ are parametrized by an $n \in \mathbb{N}$ and an n-th root of unity ω. Suppose we are working in some unitary quotient of $A_2 * T_2$.

https://doi.org/10.4153/CJM-2015-017-4 Published online by Cambridge University Press

The following proposition, with the same proof, was known to Bisch and Haagerup, and follows by a simple Frobenius reciprocity argument.

Proposition 2.2 *Suppose that $(\rho\theta)^k \ncong (\theta\rho)^k$ for some $k \geq 1$. Then all alternating words in ρ, θ with length less than or equal to $2k+2$ give distinct simple objects, except that $(\rho\theta)^{k+1}$ may not be distinct from $(\theta\rho)^{k+1}$.*

Proof We induct on k. If $k = 1$, then it is a straightforward calculation using Frobenius reciprocity (which holds in the unitary quotient) to show that 1, ρ, θ, $\rho\theta$, $\theta\rho$, $\rho\theta\rho$, $\theta\rho\theta$, $\rho\theta\rho\theta$ are distinct and simple.

For example, once one shows $\rho\theta$ and $\theta\rho$ are irreducible, we have

$$\langle \rho\theta\rho\theta, \rho\theta\rho\theta \rangle = \langle \rho\theta\rho, \rho\theta\rho \rangle = \langle \rho\theta, \rho\theta \rangle + \langle \rho\theta\rho, \rho\theta \rangle = 1 + \langle \theta\rho, \rho\theta \rangle + \langle \theta\rho, \theta \rangle = 1.$$

A similar calculation shows that $\theta\rho\theta\rho$ is simple, but note that we cannot yet compute $\langle \rho\theta\rho\theta, \theta\rho\theta\rho \rangle$.

Suppose the result holds true for $k > 1$, and suppose we also know that $(\rho\theta)^{k+1} \neq (\theta\rho)^{k+1}$. Then we calculate:

$$\begin{aligned}
\left\langle (\rho\theta)^k\rho, (\rho\theta)^k \right\rangle &= \left\langle (\theta\rho)^k, (1 \oplus \rho)(\theta\rho)^{k-1}\theta \right\rangle \\
&= \left\langle (\theta\rho)^k, (\theta\rho)^{k-1}\theta \right\rangle + \left\langle (\theta\rho)^k, (\rho\theta)^k \right\rangle = 0 \\
\left\langle (\rho\theta)^k\rho, (\rho\theta)^k\rho \right\rangle &= \left\langle (\rho\theta)^k(1 \oplus \rho), (\rho\theta)^k \right\rangle \\
&= \left\langle (\rho\theta)^k, (\rho\theta)^k \right\rangle + \left\langle (\rho\theta)^k\rho, (\rho\theta)^k \right\rangle = 1 \\
\left\langle (\theta\rho)^k\theta, (\theta\rho)^k \right\rangle &= \left\langle (\rho\theta)^k, (\rho\theta)^{k-1}\theta \right\rangle = 0 \\
\left\langle (\theta\rho)^k\theta, (\theta\rho)^k\theta \right\rangle &= \langle (\theta\rho)^k, (\theta\rho)^k \rangle = 1
\end{aligned}$$

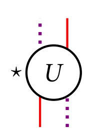
(simples with different dimensions)

$$\begin{aligned}
\left\langle (\rho\theta)^k\rho, (\theta\rho)^k\theta \right\rangle &= 0 \\
\left\langle (\rho\theta)^{k+1}, (\rho\theta)^k\rho \right\rangle &= \left\langle (\theta\rho)^k\theta, (1 \oplus \rho)(\theta\rho)^k \right\rangle \\
&= \left\langle (\theta\rho)^k\theta, (\theta\rho)^k \right\rangle + \left\langle (\theta\rho)^k\theta, (\rho\theta)^k\rho \right\rangle = 0
\end{aligned}$$

and so forth. We see that $(\rho\theta)^k\rho, (\theta\rho)^k\theta, (\rho\theta)^{k+1}, (\theta\rho)^{k+1}$ are all distinct and simple, except that possibly $(\rho\theta)^{k+1} \cong (\theta\rho)^{k+1}$. ■

Corollary 2.3 *Either there is an $n \in \mathbb{N}$ such that $(\rho\theta)^n \cong (\theta\rho)^n$, or there is no such n. In either case, we know all the distinct simples generated by θ, ρ.*

If there is an $n \in \mathbb{N}$ as in Corollary 2.3, there is a unitary isomorphism $U\colon (\rho\theta)^n \to (\theta\rho)^n$. Let $\zeta = (\theta\rho)^{n-1}\theta$. We denote ζ by a dotted purple strand, and we denote the isomorphism U by

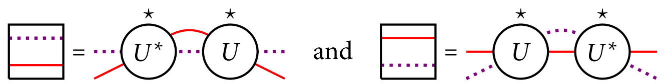


.

https://doi.org/10.4153/CJM-2015-017-4 Published online by Cambridge University Press

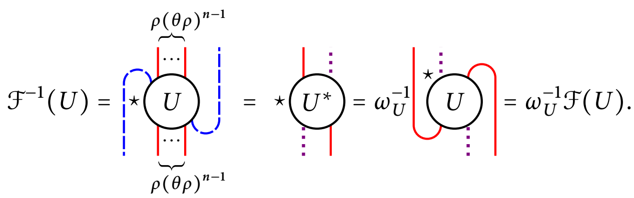
Since $U^*U = \mathbf{1}_{\zeta\rho}$ and $UU^* = \mathbf{1}_{\rho\zeta}$, we immediately obtain

(AT1)



We can normalize by a phase so that for some $2n$-th root of unity ω_U,

(AT2)

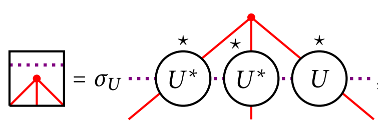


Remark 2.4 Not assuming unitarity, there are exactly 2 ways to put a $*$-structure on the quotient. However, we only consider the unitary case.

2.2 Jellyfish Relations

We now derive jellyfish relations for our generators U and U^*. Recall that a diagram is in jellyfish form if all the appearances of U or U^* are at the top of the diagram. A jellyfish relation is a relation which rewrites a diagram with a strand above a generator U or U^* as a linear combination of diagrams in jellyfish form. The existence of a complete set of jellyfish relations implies that we can rewrite an arbitrary diagram in jellyfish form.

Lemma 2.5



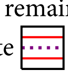
where σ_U is a choice of square root of ω_U.

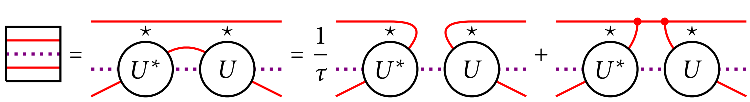
Remark 2.6 Note that switching the sign of U switches the sign of σ_U.

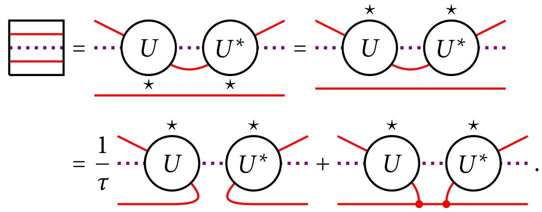
Proof Recall that ζ and $\zeta\rho$ are distinct simple objects by Proposition 2.2, so

$$\langle \zeta\rho, \zeta\rho^2 \rangle = \langle \zeta\rho, \zeta \rangle + \langle \zeta\rho, \zeta\rho \rangle = 1.$$

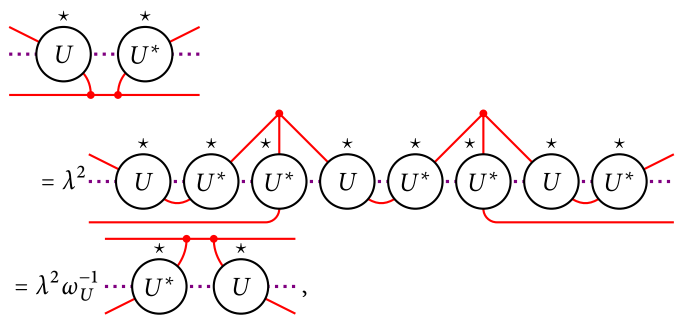
Thus, both diagrams live in the same 1-dimensional morphism space. Since the diagrams have the same nonzero norm $\tau \dim(\zeta)$ (using equation (AT1)), there is a unimodular scalar $\lambda \in \mathbb{T}$ such that the diagram on the left is equal to λ times the diagram on the right. It remains to determine the scalar λ.

We can write  in two different ways:



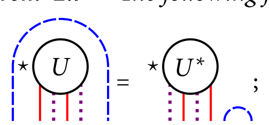


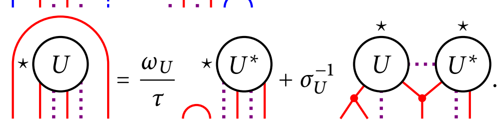
On the right-hand sides, the first terms are equal, so the second terms must be equal as well. We have



so $\lambda^2 = \omega_U$. ∎

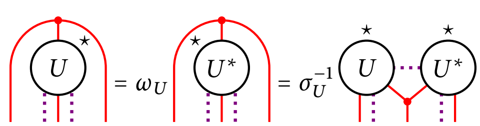
Theorem 2.7 *The following jellyfish relations hold for U:*

(i) 

(ii) 

Proof (i) follows from relations (A2) and (AT2). In more detail, the rightmost ζ strand below the U on the left-hand side of (i) is a bundle of strands $(\theta\rho)^{n-1}\theta$, so we can use Relation (A2) to reconnect the θ strands. Now using relation (AT2) gives the right-hand side.

To prove (ii), we first note that the intermediate result



follows from Lemma 2.5 by multiplying on the left by U and on the right by U^*. (ii) now follows from this intermediate result by the relations in Proposition 2.1. ∎

Remark 2.8 We get jellyfish relations for U^* by taking adjoints of the above relations.

https://doi.org/10.4153/CJM-2015-017-4 Published online by Cambridge University Press

Theorem 2.9 *Relations (A1)–(A2), the relations in Proposition 2.1, relations (AT1)–(AT2), and the relations in Theorem 2.7 are sufficient to evaluate all closed diagrams.*

Proof We may assume our diagram is in jellyfish form by Theorem 2.7 and Remark 2.8.

Suppose we have such a diagram D. We show that we can evaluate D by induction on the number of generators U, U^* in the diagram.

$k = 0$: If there are no U's or U^*'s, we can evaluate D using Relation (A1) and Proposition 2.1.

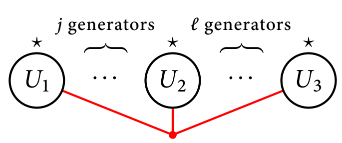
$k = 1$: If there is only one U or U^* in D, all θ strings connect back to the generator, so somewhere there is an innermost θ cap. Inside this cap is a red diagram with only one boundary point, which must be zero by Proposition 2.1.

$k \geq 2$: Suppose there are $k \geq 2$ generators U or U^*, and suppose that we can evaluate all closed diagrams D with $k-1$ generators.

If there are no trivalent vertices, the usual argument in the jellyfish algorithm applies [BMPS12], so there must be two generators connected by at least n strings along the boundary. We can then use relation (AT1) to obtain a diagram with $k-2$ generators. We are finished by the induction hypothesis.

Suppose now that there are trivalent vertices. Using relation (T1) in Proposition 2.1, which does not increase the number of generators, we can assume that no two trivalent vertices of D are connected. Hence each string connected to a trivalent vertex connects to a generator. If there is a vertex connected by two ρ strings to a generator U', then between those ρ strings there is an innermost θ cap. The argument from the $k = 1$ case shows that $D = 0$.

Now we can assume each vertex attaches to 3 distinct generators. Isotope D so that all trivalent vertices have strings emanating from the top, and these strings travel upward and attach to U's or U^*'s with no critical points (this amounts to picking a linear ordering of the generators rather than the cyclic ordering afforded by jellyfish form)



(Here, U_1, U_2, and U_3 are either U or U^*.) Hence each trivalent vertex bounds two inner regions in the diagram. Pick an innermost trivalent vertex v, *i.e.,* a trivalent vertex for which these two inner regions contain no other trivalent vertices. Let U_1, U_2, U_3 be the distinct generators attached to v as in the diagram above. Let j be the number of distinct generators between U_1 and U_2, and let ℓ be the number of distinct generators between U_2 and U_3.

If j and ℓ are both zero, then U_2 is connected to either U_1 or U_3 by at least n strings, and we can use relation (AT1) to reduce the number of generators by 2. If $j > 0$, looking at the region above our innermost trivalent vertex v and between U_1 and U_2, we see have a polygonal region whose vertices are the copies of U or U^*, with some number of diagonals. There are two strings connecting v to U_1 and U_2, we consider these as a single distinguished edge of the polygon. Now, the usual jellyfish argument proceeds by showing that every polygon with diagonals has a vertex with no incident diagonals. If we were assured only one such vertex, it may be the case that this vertex is U_1 or U_2, and we would get stuck at this point. Luckily, we have the following lemma.

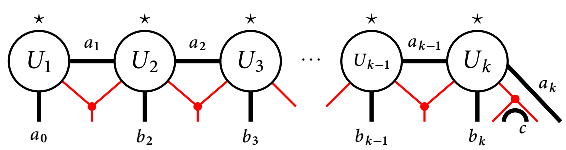
Lemma 2.10 *Every polygon with four or more sides, with certain diagonals drawn, has a pair of nonadjacent vertices with no incident diagonals.*

Hence, at least one of the j generators strictly between U_1 and U_2 is connected to one its neighbors by at least n strands. Hence, we can reduce the diagram using relation (AT1), leaving it in jellyfish form, and we are finished by the induction hypothesis. ∎

Definition 2.11 For $1 \le n < \infty$, let $\mathcal{AT}_{n,\omega_U}$ be the unitary quotient of $A_2 * T_2$ generated by U satisfying relation (A1), the relations of Proposition 2.1, and relations (AT1)–(AT2), provided that it exists. Note that $\mathcal{AT}_{1,1}$ is $A_2 \boxtimes T_2$.

2.3 A Basis for Jellyfish Calculations

Proposition 2.12 *Consider all diagrams of the form*



where the labels on the thick strings indicate the total number of ρ and θ strings in the bundle, satisfying the following criteria:

- *k is even;*
- *the U_i's alternate between U and U^*, but U_1 is not necessarily U;*
- *$a_0 \ge 2n$ and $0 \le a_k \le 2n-2$;*
- *$1 \le c \le 2n-1$ is odd, and the bundle is of the form $\theta(\rho\theta)^j$ for some j;*
- *for $2 \le i \le k$, $1 \le b_i \le 2n-1$ and b_i is odd;*
- *for $i = 1,\dots,k-1$, $1 \le a_i \le 2n-1$ and $a_i - a_{i\pm1}$ is odd (the parity of the a_is alternates).*

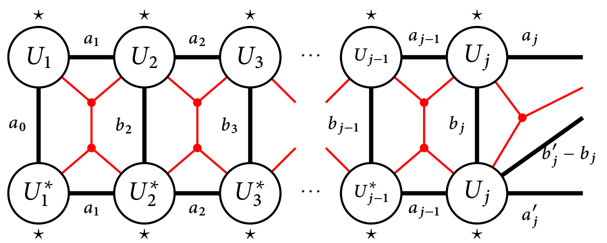
Then each such diagram has nonzero norm squared, and distinct diagrams with the same number of strings attached to the external boundary are orthogonal.

Proof A simple calculation shows the norm squared of each diagram is a power of τ, which is nonzero.

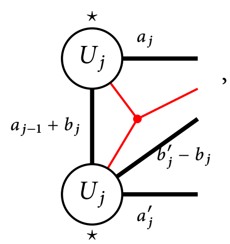
Suppose now that we have two distinct diagrams with the same number of external strings. Let the first diagram have constants $(a_0,\dots a_k, b_2,\dots,b_k,c)$, and let the second diagram have constants $(a'_0,\dots a'_\ell, b'_2,\dots,b'_\ell,c')$. Without loss of generality, assume $k \le \ell$. We cannot have $a_i = a'_i$ for all $i = 0,\dots,k$, since the b_is are determined by the a_i's, and then c, c' are determined by the a_i's and the number of external boundary points.

https://doi.org/10.4153/CJM-2015-017-4 Published online by Cambridge University Press

Let j be minimal such that $a_j \neq a'_j$, which implies that $b_j \neq b'_j$. If $a_j > a'_j$, then $b_j < b'_j$. Taking the inner product, we get the following sub-diagram:

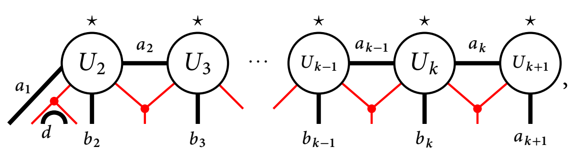


We can iteratively cancel the first $j-1$ pairs of generators U_i, U_i^* counting from the left, since $a_0 \geq 2n$, and $a_{i-1} + b_i \geq 2n$ for all $i = 2, \dots, j-1$. We then get some power of τ times



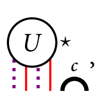
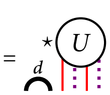
which is zero since $a_{j-1} + b_j \geq 2n$. Similarly, we get zero if $a'_j > a_j$. ■

Corollary 2.13 *The conclusion of Proposition 2.12 holds for diagrams of the form*



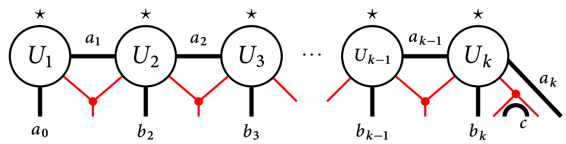
where the a_is and b_is satisfy the same criteria, but instead of a_0 and c, we have $a_{k+1} \geq 2n$ and $0 \leq d \leq 2n-2$, where the d bundle is of the form $(\rho\theta)^j$.

Notation 2.14 We use the following notation for one-car trains:

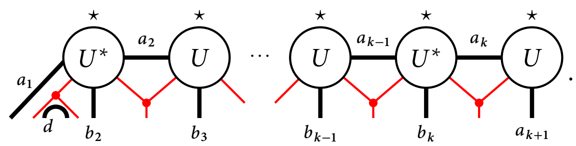
$\mathrm{train}\big[\{\cdot\}, \mathrm{wheel}[c]\big] =$  , $\mathrm{train}\big[\mathrm{wheel}[d], \{\cdot\}\big] =$  .

We use the following notation for diagrams in the form of Proposition 2.12:

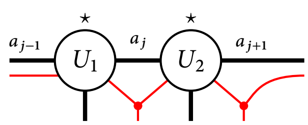
$$\mathrm{train}\big[a_0, a_1, \dots, a_{k-1}, a_k, \mathrm{wheel}[c]\big] =$$



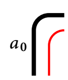
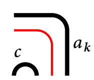
$$\mathrm{train}[\mathrm{wheel}[d], a_1, a_2 \dots, a_k, a_{k+1}] =$$



We now omit the b_is, since they can be recovered from the a_is. These diagrams can be thought of as products of two-car trains

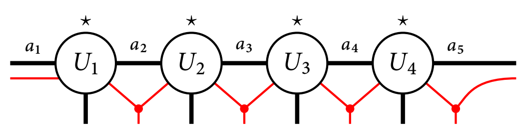
(2.1) $$\mathrm{twocar}[a_{j-1}, a_j, a_{j+1}] =$$ 

with a caboose and an engine:

$\mathrm{caboose}[a_0] =$  $\qquad \mathrm{engine}[a_k, c] =$  ,

with the convention that the last twocar in the train must have all its strings connected to the caboose (and no strings going downward). We multiply the train parts by concatenating horizontally:

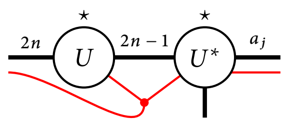
$$\mathrm{twocar}[a_1, a_2, a_3]\,\mathrm{twocar}[a_3, a_4, a_5] =$$

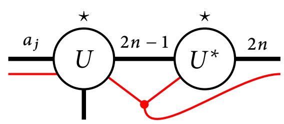


To simplify future calculation, we will also allow the external a_is in our two car trains to surpass $2n$, *i.e.*, for the two car train in Equation (2.1), if $a_j < 2n-2$, then a_{j-1} or a_{j+1} may be more than $2n-1$. If $a_j = 2n-1$, we have the following two-car

https://doi.org/10.4153/CJM-2015-017-4 Published online by Cambridge University Press

trains for which one of $a_{j\pm1}$ is $2n$:

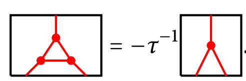
twocar$[2n, 2n-1, a_j]$ =  where a_j is even,

twocar$[a_j, 2n-1, 2n]$ =  where a_j is even.

When we use the two-car trains with $2n$ horizontal strands, we have the following multiplication rules for contracting the $2n$ strands:

$$\mathrm{twocar}[x_1,x_2,2n]\,\mathrm{twocar}[2n,x_3,x_4] = \begin{cases} 0 & \text{if } x_2, x_3 < 2n-1, \\ \mathrm{twocar}[x_1,x_3,x_4] & \text{if } x_2 = 2n-1 \text{ and } x_3 < 2n-1, \\ \mathrm{twocar}[x_1,x_2,x_4] & \text{if } x_2 < 2n-1 \text{ and } x_3 = 2n-1, \\ \frac{-1}{\tau}\,\mathrm{twocar}[x_1,2n-1,x_4] & \text{if } x_2, x_3 = 2n-1. \end{cases}$$

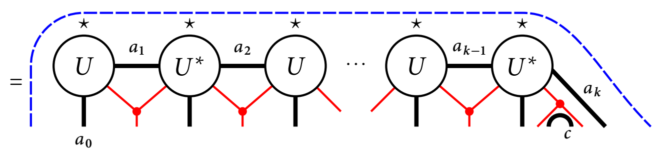
The last identity follows from the fact that

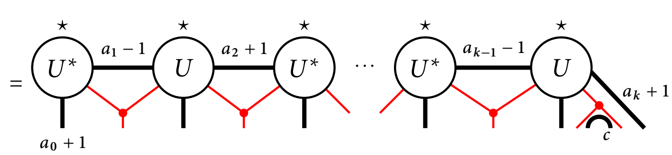


We now describe the nice jellyfish relations for these two car trains. The proofs of the following three lemmas are straightforward applications of Theorem 2.7 and Remark 2.8.

Lemma 2.15 *The diagrams in the form of Proposition 2.12 satisfy the following jellyfish relation for θ strings, where we assume the a_{2j}s are even and the a_{2j+1}'s are odd:*

$\theta(\mathrm{train}[a_0, a_1, \dots, a_{k-1}, a_k, \mathrm{wheel}[c]])$

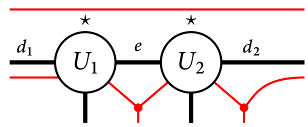




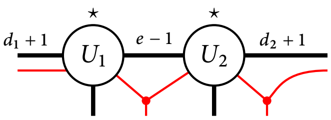
$= \mathrm{train}[a_0+1, a_1-1, \dots, a_{k-1}-1, a_k+1, \mathrm{wheel}[c]]$.

https://doi.org/10.4153/CJM-2015-017-4 Published online by Cambridge University Press

Lemma 2.16 *When d_1, d_2 are odd and $e \le 2n-2$ is even,*

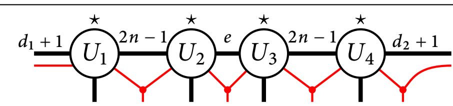
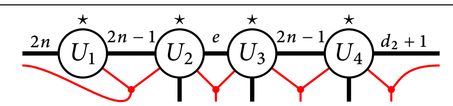
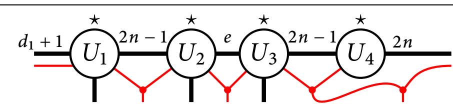
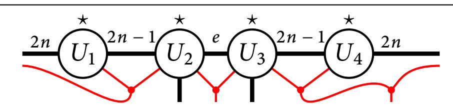


$$= \frac{1}{\tau}$$



$$+ X,$$

where X is the diagram on the right in the table below according to the values of d_1, d_2.

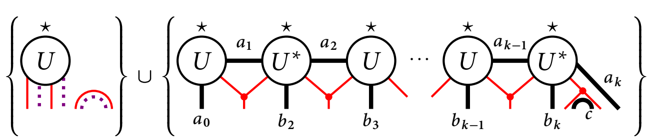
d_1, d_2	X
$d_1, d_2 < 2n-1$	
$d_2 < 2n-1 \le d_1$	
$d_1 < 2n-1 \le d_2$	
$2n-1 \le d_1, d_2$	

2.4 Existence Results for $\mathcal{AT}_{n,\omega_U}$

We now use our jellyfish relations to prove our existence results for the $\mathcal{AT}_{n,\omega_U}$. We can prove these results thanks to the train bases afforded by Proposition 2.12.

Definition 2.17 Consider the set of diagrams of the form of Proposition 2.12 with $8n$ external boundary points, together with the one car train below:

$B_r =$

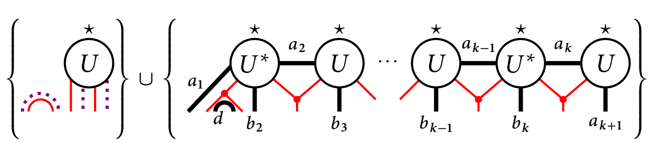


$\subset P_{(\rho\zeta)^3\zeta\rho}$.

Then B_r is orthogonal and linearly independent. We call B_r the *right train basis*.

Similarly, we define the *left train basis* by

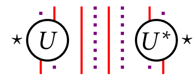
$B_\ell =$



$\subset P_{\zeta\rho(\rho\zeta)^3}$.

https://doi.org/10.4153/CJM-2015-017-4 Published online by Cambridge University Press

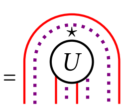
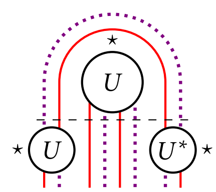
Given an element in B_ℓ, attaching



underneath yields an element of B_r.

Theorem 2.18 *For $n = 1, 2,$ or 3, $\mathcal{AT}_{n,\omega_U}$ does not exist if $\omega_U \neq 1$.*

Proof Suppose $\mathcal{AT}_{n,\omega_U}$ exists. We compute $\rho\zeta(U)$ in two ways:

(2.2) $\rho\zeta(U) =$  $=$  $= U(\zeta\rho(U))U^*.$

Note that by Lemmas 2.15 and 2.16, applying the ρ and θ strings in the order on the left always gives us a linear combination of elements from B_r, the right train basis. Similarly, applying the ρ and θ strings in the order on the right always gives us a linear combination of elements from B_ℓ, the left train basis. Then conjugating by U as in equation (2.2), we get back some linear combination of elements in B_r.

We can calculate the coefficients in the linear combinations for $\rho\zeta(U)$ and $U(\zeta\rho(U))U^*$ after applying the jellyfish relations. These are displayed in full in Appendix A of the unabridged version [IMP13] of this paper, as Figures 2, 3, and 4.

These coefficients can agree only if $\sigma_U = \sigma_U^{-1}$, and hence $\omega_U = 1$. ■

Theorem 2.19 *$\mathcal{AT}_{n,\omega_U}$ does not exist for $4 \leq n \leq 10$.*

Proof The technique is the same as the proof of Theorem 2.18. We compute $\rho\zeta(U)$ in two different ways as in Equation (2.2), and we get different linear combinations. Hence by Proposition 2.12, $\mathcal{AT}_{n,\omega_U} = 0$ for $4 \leq n \leq 10$.

The coefficients for $n = 4$ also appear in [IMP13, Fig. 5, Appendix A]. Similar computations for $5 \leq n \leq 10$ using the same code yield different linear combinations. ■

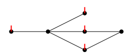
Conjecture 2.20 *The technique used for Theorems 2.18 and 2.19 should show that*

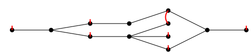
(i) *$\mathcal{AT}_{n,\omega_U}$ exists only if $\omega_U = 1$ for all $1 \leq n < \infty$, and*
(ii) *$\mathcal{AT}_{n,\omega_U}$ does not exist for all $4 \leq n < \infty$.*

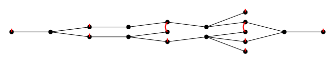
https://doi.org/10.4153/CJM-2015-017-4 Published online by Cambridge University Press

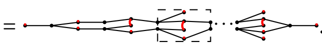
3 Application to Subfactors at Index $3+\sqrt{5}$

We now connect the categories $\mathcal{AT}_{n,\omega_U}$ to index $3+\sqrt{5}$ subfactors. In 1994, Bisch–Haagerup found a sequence of possible principal graphs converging to the Fuss–Catalan principal graph at index $3+\sqrt{5}$:

$\mathcal{BHF}_1 =$  ,

$\mathcal{BHF}_2 =$  ,

$\mathcal{BHF}_3 =$  ,

$\mathcal{BHF}_n =$  .

For $n \geq 4$, the dashed section appears a total of $n-3$ times in $\mathcal{BHF}_n$.

The main result of this section is the following theorem.

Theorem 3.1 *A subfactor with principal graph $\mathcal{BHF}_n$ exists if and only if $\mathcal{AT}_{n,\omega_U}$ exists for some $\omega_U^{2n} = 1$.*

Proof Existence of the subfactor implies existence of such a fusion category by Theorem 3.8. The converse follows from Theorem 3.10. ∎

Corollary 3.2 *A unique subfactor exists with principal graphs $\mathcal{BHF}_n$ for $n = 1, 2, 3$. No subfactor exists with principal graph $\mathcal{BHF}_n$ for $4 \leq n \leq 10$.*

Proof By Theorem 3.1, uniqueness and nonexistence follow from Theorems 2.18 and 2.19, respectively. Subsection 3.3 shows existence for $n = 1, 2, 3$. ∎

Independently, and by a different method, Liu [Liu13] showed that no subfactor with principal graph $\mathcal{BHF}_n$ exists for any $n \geq 4$. Liu's result together with Theorem 3.1 shows that $\mathcal{AT}_{n,\omega_U}$ does not exist for any $n \geq 4$.

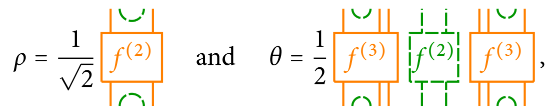
3.1 From Subfactors to Quotients of $A_2 * T_2$

Let $N \subset M$ be a 1-supertransitive subfactor at index $3+\sqrt{5}$ with intermediate subfactor P.

By taking duals, we can assume that $[M{:}P] = 2$ and $[P{:}N] = \tau^2 = \frac{3+\sqrt{5}}{2}$. Denote the planar algebra for $N \subset M$ by $\mathcal{P}_\bullet$ and the principal even half of $N-N$ bimodules by $\frac{1}{2}\mathcal{P}_+$. In this subsection, we show that $\frac{1}{2}\mathcal{P}_+$ must be a quotient of $A_2 * T_2$.

By [BJ97], $\mathcal{P}_\bullet$ has a Fuss–Catalan planar subalgebra $\mathcal{FC}_\bullet$. Recall that $\mathcal{FC}_{j,+}$ consists of all $A_3 * A_4$ diagrams with boundaries of the form $ab(baab)^{j-1}ba$, where a and b are the usual generators of A_3 and A_4. Similarly, $\mathcal{FC}_{j,+}$ of those diagrams with boundary $ba(abba)^{j-1}ab$. In the diagrams below we represent a by a dashed green string and b by a solid orange string.

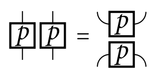
Definition 3.3 Define projections ρ and θ in $\mathcal{FC}_+ \subset \mathcal{P}_+$ by



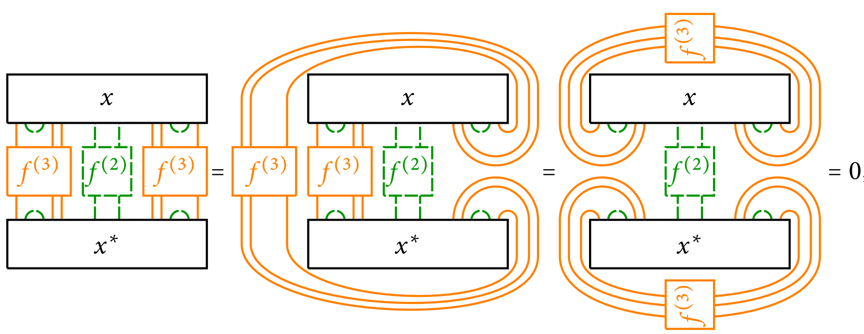
These correspond to $N - N$ bimodules, and so to (possibly a collection of) vertices on Γ_+. Clearly, ρ is a minimal projection of trace τ, and $\rho^2 \cong 1 \oplus \rho$.

Lemma 3.4 *The projection θ satisfies $\theta \otimes \theta \cong 1$ and $\theta \not\cong 1$.*

Proof First, $\theta^2 \cong 1$, since and $f^{(2)}$ have dimension 1. This follows from the fact that if p is a trace 1 symmetrically self-dual projection in a factor planar algebra [BHP12], then



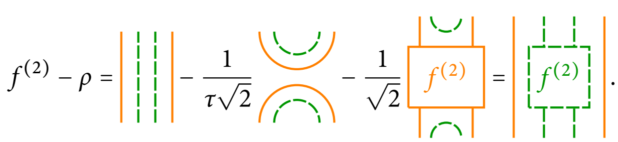
(take the norm squared of the difference). Now let x be an intertwiner from θ to the empty diagram. Then by sphericality, and the fact that $\dim(f^{(3)}) = 1$, we have



since any intertwiner from $f^{(2)}$ to the empty diagram must be zero. ∎

Proposition 3.5 *In $FC_\bullet$, $f^{(2)} \cong \rho \oplus \rho\theta\rho$.*

Proof We have



It is then easy to see that $f^{(2)} \cong \rho \oplus \rho\theta\rho$, since $f^{(1)} \cong f^{(2)} f^{(3)} \cong f^{(3)} f^{(2)}$. ∎

Corollary 3.6 *The even half $\frac{1}{2}\mathcal{P}_+$ of $P_\bullet$ is generated by ρ and θ. Hence $\frac{1}{2}\mathcal{P}_+$ is either $A_2 * T_2$ or $\mathcal{AT}_{n,\omega_U}$ for some $1 \le n < \infty$ and some $2n$-th root of unity ω_U.*

Proof Note that all of the $N - N$ bimodules are summands of a tensor power of $f^{(2)} \cong \rho \oplus \rho\theta\rho$, and thus every $N - N$ bimodule is a summand of some alternating

https://doi.org/10.4153/CJM-2015-017-4 Published online by Cambridge University Press

word in ρ, θ. Hence, by sending $\theta \in A_2$ to $\theta \in \frac{1}{2}\mathcal{P}_+$ (defined above) and $\rho \in T_2$ to $\rho \in \frac{1}{2}\mathcal{P}_+$, we get a dominant functor $F\colon A_2 * T_2 \to \frac{1}{2}\mathcal{P}_+$. This functor is faithful, because $A_2 * T_2 \cong \frac{1}{2}\mathcal{FC}_+$ and $\mathcal{FC}_\bullet$ is a planar subalgebra of $\mathcal{P}_\bullet$. Thus, $\frac{1}{2}\mathcal{P}_+$ is a quotient of $A_2 * T_2$. ∎

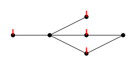
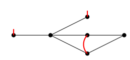
We now show that any subfactor with principal graph $\mathcal{BHF}_n$ must have an intermediate subfactor, and thus its even half must be $\mathcal{AT}_{n,\omega_U}$ for some $2n$-th root of unity ω_U.

The following lemma is well known to experts. (In fact, a much stronger version is true, but we do not need it here.)

Lemma 3.7 *Suppose the subfactor $N \subset M$ has planar algebra $\mathcal{P}_\bullet$ and principal graph Γ_+. Suppose Γ_+ is 1 supertransitive, has depth greater than 2, and has exactly one univalent (self-dual) vertex β at depth 2. Then $e_1 + \beta$ is a biprojection* [Bis94, Lan02]*, so there is an intermediate subfactor $N \subset P \subset M$ where $[P : N] = 2$.*

Theorem 3.8 *If the principal graph of $N \subset M$ is $\mathcal{BHF}_n$, then there is an intermediate subfactor $N \subset P \subset M$ such that $[M : P] = 2$ and $[P : N] = \frac{3+\sqrt{5}}{2}$. Hence, the even half of $N \subset M$ is necessarily $\mathcal{AT}_{n,\omega_U}$ for some $\omega_U^{2n} = 1$.*

Proof If $n = 1$, then the dual graph Γ_- must be one of

 or .

(In fact, the dual graph cannot be the second graph above, since the dual even half must also be $\mathcal{AT}_{2,\omega_U}$, which only exists if $\omega_U = 1$ by Theorem 2.18.) If $n \geq 2$, since $\mathcal{BHF}_n$ starts with a triple point, the dual graph Γ_- also starts with a triple point, and by Ocneanu's triple point obstruction [Haa94], Γ_- has a univalent vertex at depth 2. Thus for any n, applying Lemma 3.7 to the dual subfactor yields an intermediate subfactor with the desired indices.

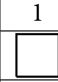
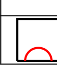
Now that we know there is an intermediate subfactor, Corollary 3.6 implies that the even half of $N \subset M$ must be $\mathcal{AT}_{k,\omega_U}$ for some $2k$-th root of unity ω_U. By Proposition 2.2, it suffices to count the even vertices of $\mathcal{BHF}_n$ to see that $k = n$. ∎

3.2 From Quotients of $A_2 * T_2$ to Subfactors

Proposition 3.9 *In $A_2 * T_2$, $A = 1 \oplus \rho \oplus \rho\theta\rho$ is a Frobenius algebra object with Frobenius subalgebra object $B = 1 \oplus \rho$.*

Proof First, it is well known that B is an algebra object, but we provide a proof as a warmup to showing that A is an algebra. We need to specify the map $B \otimes B \to B$, which can be thought of as 8 maps between the summands. Since the map must be unital and rotationally invariant, we only have one unknown parameter:

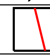
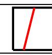
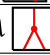
maps from $X \otimes Y \to 1$:

$\otimes$	1	ρ
1		0
ρ	0	

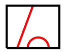
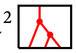
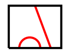
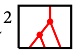
,

https://doi.org/10.4153/CJM-2015-017-4 Published online by Cambridge University Press

maps from $X \otimes Y \to \rho$:

$\otimes$	1	ρ
1	0	
ρ		λ 

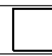
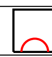
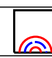
for some constant $\lambda \in \mathbb{C}$. Checking associativity amounts to checking associativity of $\rho \otimes \rho \otimes \rho \to \rho \otimes \rho \to 1$, which yields the following equation:

 $+ \lambda^2$  $=$  $+ \lambda^2$  .

This equation is satisfied whenever $\lambda = \pm\tau^{-1/2}$ by relation (T1).

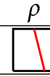
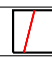
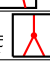
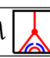
We now specify the map $A \otimes A \to A$ for $A = 1 \oplus \rho \oplus \rho\theta\rho$ by specifying maps between the summands as before. We already know one constraint if B is a subalgebra.

maps from $X \otimes Y \to 1$:

$\otimes$	1	ρ	$\rho\theta\rho$
1		0	0
ρ	0		0
$\rho\theta\rho$	0	0	

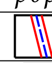
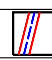
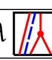
,

maps from $X \otimes Y \to \rho$:

$\otimes$	1	ρ	$\rho\theta\rho$
1	0		0
ρ		$\frac{\pm 1}{\sqrt{\tau}}$ 	0
$\rho\theta\rho$	0	0	λ 

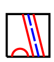
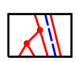
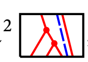
,

maps from $X \otimes Y \to \rho\theta\rho$:

$\otimes$	1	ρ	$\rho\theta\rho$
1	0	0	
ρ	0	0	λ 
$\rho\theta\rho$		λ 	0

.

We get the following constraint

 $\pm \frac{\lambda}{\sqrt{\tau}}$  $= \lambda^2$  ,

which is satisfied if $\lambda = \mp\sqrt{\tau}$. We leave it to the reader that this restriction is sufficient for the map $A \otimes A \to A$ to be associative. ∎

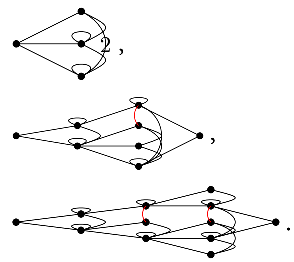
Theorem 3.10 *Suppose $\mathcal{AT}_{n,\omega_U}$ exists for some $\omega_U^{2n} = 1$. Then there are* II_1*-factors $N \subset P \subset M$ where $[M{:}P] = 2$ and $[P{:}N] = \frac{3+\sqrt{5}}{2}$ such that the even half of $N \subset M$ is $\mathcal{AT}_{n,\omega_U}$, and the principal graph of $N \subset M$ is $\mathcal{BHF}_n$.*

Proof By Proposition 3.9, $1 \oplus \rho \oplus \rho\theta\rho$ is a Frobenius algebra object with subalgebra $1 \oplus \rho$ in $A_2 * T_2$, and thus they are also algebra objects in $\mathcal{AT}_{n,\omega_U}$. Now the usual construction (see Remark 3.11) provides a subfactor $N \subset M$ with $f^{(2)} = \rho \oplus \rho\theta\rho$. A straightforward calculation shows that the fusion graph of $\rho \oplus \rho\theta\rho$ in $\mathcal{AT}_{n,\omega_U}$ is the same as the even part of $\mathcal{BHF}_n$. We give the fusion graph for $n = 1, 2, 3$ below.

$n = 1$

$n = 2$

$n = 3$



Finally, we can show that $\mathcal{BHF}_n$ is the unique principal graph with this even part. First, we note that $N \subset M$ is irreducible. Since $[M : P] = 2$ and $[P : N] = \frac{3+\sqrt{5}}{2}$, $\dim(f^{(1)}) = \sqrt{2}\tau$, which cannot be written as the sum of two numbers from the set $\{2\cos(\pi/k) | k \geq 3\}$. Next, since $f^{(1)}$ is simple and $f^{(1)} \otimes f^{(1)} \cong 1 \oplus f^{(2)}$, the number of self-loops on a vertex in the even principal graph is exactly one more than the valence of that vertex in the principal graph. This condition uniquely determines the number of vertices at each odd depth, and their connectivity to the vertices at even depths. ■

Remark 3.11 It is well known that Frobenius algebra objects in unitary fusion categories correspond to finite depth subfactor planar algebras. See [GS12, Section 2] for a good background and a dictionary between the two viewpoints.

3.3 Existence of $\mathcal{AT}_{n,1}$ for $n \in \{1, 2, 3\}$

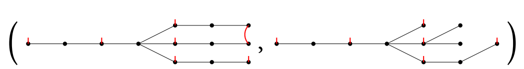
We now show that $\mathcal{AT}_{n,1}$ exists for $n \in \{1, 2, 3\}$. Hence by Theorems 2.18 and 3.10, there is a unique hyperfinite subfactor whose principal graph is $\mathcal{BHF}_n$ for $n = 1, 2,$ and 3.

First, $\mathcal{AT}_{1,1} = A_2 \boxtimes T_2$, which exists. Second, Bisch and Haagerup showed that if $N_0 \subset N_1$ is the hyperfinite A_4 subfactor, and $\beta \in \mathrm{Out}(N_1 \otimes N_1)$ is the flip automorphism, then both the $N - N$ and $P - P$ bimodules associated with composed inclusion

$$N = N_0 \otimes N_1 \subset P = N_1 \otimes N_1 \subset M = (N_1 \otimes N_1) \rtimes \langle \beta \rangle$$

are equivalent to $\mathcal{AT}_{2,1}$.

We now construct $\mathcal{BHF}_3$ as an intermediate subfactor of a reduced subfactor of the $3^{\mathbb{Z}/4}$ subfactor with principal graphs



which was constructed in unpublished work of Izumi and also in [PP13].

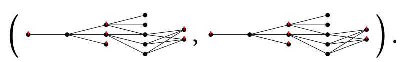
We will denote the even bimodules on the dual principal graph lexicographically left to right and bottom to top by $1, \kappa, \beta\kappa, \chi, \sigma, \beta$. Using the `FusionAtlas` program `FindFusionRules`, we see that the dual principal even half ${}_M\mathrm{Mod}_M$ of $3^{\mathbb{Z}/4}$ has the following fusion rules

$\otimes$	κ	$\beta\kappa$	χ	σ	β
κ	$1{+}\kappa{+}\chi{+}\sigma{+}\beta\kappa$	$\kappa{+}\chi{+}\sigma{+}\beta{+}\beta\kappa$	$\kappa{+}2\chi{+}\sigma{+}\beta\kappa$	$\kappa{+}\chi{+}\beta\kappa$	$\beta\kappa$
$\beta\kappa$	$\kappa{+}\chi{+}\sigma{+}\beta{+}\beta\kappa$	$1{+}\kappa{+}\chi{+}\sigma{+}\beta\kappa$	$\kappa{+}2\chi{+}\sigma{+}\beta\kappa$	$\kappa{+}\chi{+}\beta\kappa$	κ
χ	$\kappa{+}2\chi{+}\sigma{+}\beta\kappa$	$\kappa{+}2\chi{+}\sigma{+}\beta\kappa$	$1{+}2\kappa{+}\chi{+}\sigma{+}\beta{+}2\beta\kappa$	$\kappa{+}\chi{+}\sigma{+}\beta\kappa$	χ
σ	$\kappa{+}\chi{+}\beta\kappa$	$\kappa{+}\chi{+}\beta\kappa$	$\kappa{+}\chi{+}\sigma{+}\beta\kappa$	$1{+}\chi{+}\sigma{+}\beta$	σ
β	$\beta\kappa$	κ	χ	σ	1

The Frobenius–Perron dimensions of the $M-M$ bimodules are as follows:

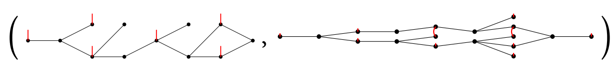
$$\dim(1)=\dim(\beta)=1, \qquad \dim(\kappa)=\dim(\beta\kappa)=2+\sqrt{5},$$
$$\dim(\chi)=2\tau^2=3+\sqrt{5}, \qquad \dim(\sigma)=2\tau=1+\sqrt{5}.$$

Using the `FusionAtlas` program `ExtractPairOfBigraphsWithDuals`, we compute the principal graphs of the reduced subfactor at σ to be



Here, the reduced subfactor is $M\subset Q$, where Q is the commutant of the right M-action on σ. By Lemma 3.7 applied to the dual graph, there is an intermediate subfactor $M\subset P\subset Q$ such that $[Q:P]=2$. By Goldman's Theorem [Gol59], we have $Q\cong P\rtimes\mathbb{Z}/2$ and ${}_PQ_P\cong 1_P\oplus\alpha$ for some α with dimension 1.

Theorem 3.12 *The principal graphs of $M\subset P$ are*



Hence, a subfactor with principal graph $\mathcal{BHF}_3$ exists, and $\mathcal{AT}_{3,1}$ exists.

Proof We factor ${}_MQ_Q\cong{}_MP\otimes_P Q_Q$, and for notational convenience we write $\xi={}_MP_P$, so ${}_MP_M=\xi\overline{\xi}$. Since $\sigma\overline{\sigma}\cong{}_MQ_M$, we have

$$1\oplus\beta\oplus\chi\oplus\sigma=\sigma\overline{\sigma}={}_MQ_M={}_MQ\otimes_Q Q_M=\xi(Q\otimes_Q Q)\overline{\xi}=\xi(1_P\oplus\alpha)\overline{\xi}=\xi\overline{\xi}\oplus\xi\alpha\overline{\xi}.$$

We also know $\xi\overline{\xi}$ has dimension $2\tau^2=3+\sqrt{5}$ and is not irreducible, since it contains a copy of the trivial. Hence by the Frobenius–Perron dimensions listed above, we must have $\xi\overline{\xi}=1\oplus\beta\oplus\sigma$. We immediately see that the even half of $M\subset P$ is the even half of $3^{\mathbb{Z}/4}$.

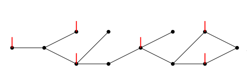
We continue computing the principal graph. We have

$$\langle\sigma\xi,\sigma\xi\rangle=\langle\sigma^2,\xi\overline{\xi}\rangle=\langle 1\oplus\sigma\oplus\beta\oplus\chi,1\oplus\sigma\oplus\beta\rangle=3,$$

so $\sigma\xi$ breaks up into 3 distinct irreducibles $\sigma\xi=\xi\oplus\nu\oplus\mu$. Moreover,

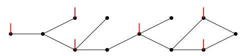
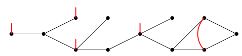
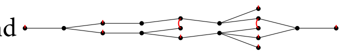
$$\sigma\xi\overline{\xi}=\sigma(1\oplus\sigma\oplus\beta)=1\oplus 3\sigma\oplus\beta\oplus\chi,$$

so without loss of generality, ν is a univalent vertex, and μ connects to only σ and χ. As before, $\langle\chi\xi,\chi\xi\rangle=3$, and $\chi(1\oplus\sigma\oplus\beta)=\sigma\oplus 3\chi\oplus\kappa\oplus\beta\kappa$. Since $\kappa,\beta\kappa$ are self-dual, the principal graph is



https://doi.org/10.4153/CJM-2015-017-4 Published online by Cambridge University Press

Using the FusionAtlas program FindGraphPartners, the only possible dual graphs are

 ,  , and  ,

and Ocneanu's triple point obstruction [Haa94, MPPS12] implies that the third graph must be the dual graph. ∎

4 Quotients of $T_2 * T_2$

Our method also applies to composites of two copies of T_2 with little alteration. The interested reader can see more details at [IMP13]. As the techniques are highly similar to those of Section 2, we merely state our results and their connections to subfactor theory.

Suppose we have two copies of T_2 generated by objects ρ, μ, together with intertwiners $\rho \otimes \rho \to \rho$ and $\mu \otimes \mu \to \mu$, both satisfying the relations in Proposition 2.1.

As before, we see nontrivial unitary quotients of $T_2 * T_2$ are parametrized by an n such that the alternating words in ρ, μ and μ, ρ of length n are isomorphic, and an n-th root of unity ω_U, resulting in a unitary isomorphism $U: (\rho\mu\cdots) \to (\mu\rho\cdots)$ satisfying similar relations.

Arguing as in Theorem 2.9, similar relations as before are sufficient to evaluate all closed diagrams, and thus we give the following definition.

Definition 4.1 For $2 \leq n < \infty$, let $\mathcal{TT}_{n,\omega_U}$ be the unitary quotient of $T_2 * T_2$ generated by U, provided that it exists. Note that $\mathcal{TT}_{2,1}$ is $T_2 \boxtimes T_2$.

We get a similar basis for our jellyfish calculations, and we can use a similar twocar formalism. Our uniqueness and non-existence proofs are also similar. As in equation (2.2), we evaluate a similar diagram in two different ways. Using very similar Mathematica code, we see that for $n = 2, 3$, we must have $\omega_U = 1$, but $n = 4, \dots, 10$ are not possible.

Hence we have the following theorems. Again, for more details, see [IMP13].

Theorem 4.2 *For* $n = 2, 3$, $\mathcal{TT}_{n,\omega_U}$ *exists only if* $\omega_U = 1$.

Theorem 4.3 *For* $4 \leq n \leq 10$, $\mathcal{TT}_{n,\omega_U}$ *does not exist.*

Conjecture 4.4 *The technique used for Theorems 4.2 and 4.3 should show that*

(i) $\mathcal{TT}_{n,\omega_U}$ *exists only if* $\omega_U = 1$ *for all* $2 \leq n < \infty$, *and*
(ii) $\mathcal{TT}_{n,\omega_U}$ *does not exist for all* $4 \leq n < \infty$.

4.1 Application to Subfactors

The techniques of Section 3 can be used to prove the following theorem.

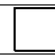
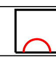
Theorem 4.5 *Any* $A_4 - A_4$ *composite subfactor has (dual) even half* $T_2 * T_2$ *or* $\mathcal{TT}_{n,\omega_U}$ *for some n-th root of unity* ω_U.

Conversely, in $T_2 * T_2$, $1 \oplus \rho \oplus \rho\mu\rho$ is a Frobenius algebra object with subalgebra $1 \oplus \rho$. Thus, if $\mathcal{TT}_{n,\omega_U}$ exists, then there is an $A_4 - A_4$ composite subfactor with (dual) even half $\mathcal{TT}_{n,\omega_U}$.

Proof If we have a composite subfactor $N \subset P \subset M$, where $N \subset P$ and $P \subset M$ are A_4 subfactors, we can define ρ, μ analogously to Definition 3.3. Thus, $\rho, \mu \not\cong 1$ are irreducible, satisfying $\rho^2 \cong 1 \oplus \rho$ and $\mu^2 \cong 1 \oplus \mu$ by the same proof as in Lemma 3.4. Again, we have $f^{(2)} \cong \rho \oplus \rho\mu\rho$, so the $M - M$ bimodules are generated by ρ, μ, and the even half is either $T_2 * T_2$ or $\mathcal{TT}_{n,\omega_U}$ for some ω_U.

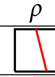
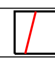
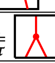
For the converse, the algebra map is given by

maps from $X \otimes Y \to 1$:

$\otimes$	1	ρ	$\rho\mu\rho$
1		0	0
ρ	0		0
$\rho\mu\rho$	0	0	

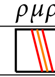
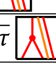
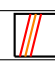
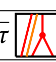
,

maps from $X \otimes Y \to \rho$:

$\otimes$	1	ρ	$\rho\mu\rho$
1	0		0
ρ		$\frac{\pm 1}{\sqrt{\tau}}$ 	0
$\rho\mu\rho$	0	0	$\mp\sqrt{\tau}$ 

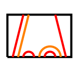
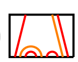
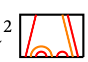
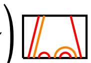
,

maps from $X \otimes Y \to \rho\mu\rho$:

$\otimes$	1	ρ	$\rho\mu\rho$
1	0	0	
ρ	0	0	$\mp\sqrt{\tau}$ 
$\rho\mu\rho$		$\mp\sqrt{\tau}$ 	λ 

.

We analyze maps $\rho\mu\rho \otimes \rho\mu\rho \otimes \rho\mu\rho \to \rho\mu\rho$ to get the restriction

λ^2  $+ \left(\frac{-\lambda^2}{\tau} + \tau\right)$  $= \lambda^2$  $+ \left(\frac{-\lambda^2}{\tau} + \tau\right)$  ,

i.e., $\lambda^2 = 1$. ∎

Corollary 4.6 *There is a unique $A_4 - A_4$ composite subfactor for $n = 2, 3$. For $n = 4, \dots, 10$ there is no such composite subfactor.*

Proof By Theorem 4.5, uniqueness and nonexistence follow from Theorems 4.2 and 4.3 respectively. Existence for $n = 2, 3$ is proved below. ∎

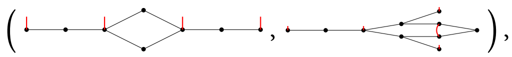
Liu's method also applies to the $A_4 - A_4$ composite subfactors, and he shows that no subfactor with even half $\mathcal{TT}_{n,\omega_U}$ exists for any $n \geq 4$ [Liu13]. His result together with Theorem 4.5 shows that $\mathcal{TT}_{n,\omega_U}$ does not exist for any $n \geq 4$.

Existence of $\mathcal{TT}_{2,1}$ and $\mathcal{TT}_{3,1}$.

Clearly $\mathcal{TT}_{2,1} = T_2 \boxtimes T_2$ exists. We can construct $\mathcal{TT}_{3,1}$ from the 2D2 subfactor with

https://doi.org/10.4153/CJM-2015-017-4 Published online by Cambridge University Press

principal graphs



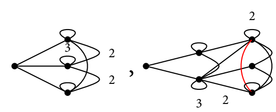
which is constructed in unpublished work of Izumi, and also in [MP14]. First, naming the even bimodules on the dual graph $1, f^{(2)}, \rho, \sigma, \overline{\sigma}, \mu$ lexicographically from left to right, bottom to top, the fusion rules are

$\otimes$	$f^{(2)}$	ρ	σ	$\overline{\sigma}$	μ
$f^{(2)}$	$1{+}2f^{(2)}{+}\sigma{+}\overline{\sigma}{+}\rho{+}\mu$	$f^{(2)}{+}\overline{\sigma}$	$f^{(2)}{+}\sigma{+}\overline{\sigma}{+}\mu$	$f^{(2)}{+}\sigma{+}\overline{\sigma}{+}\rho$	$f^{(2)}{+}\sigma$
ρ	$f^{(2)}{+}\sigma$	$1{+}\rho$	$f^{(2)}$	$\overline{\sigma}{+}\mu$	$\overline{\sigma}$
σ	$f^{(2)}{+}\sigma{+}\overline{\sigma}{+}\rho$	$\sigma{+}\mu$	$f^{(2)}{+}\overline{\sigma}$	$1{+}f^{(2)}{+}\mu$	$f^{(2)}$
$\overline{\sigma}$	$f^{(2)}{+}\sigma{+}\overline{\sigma}{+}\mu$	$f^{(2)}$	$1{+}f^{(2)}{+}\rho$	$f^{(2)}{+}\sigma$	$\overline{\sigma}{+}\rho$
μ	$f^{(2)}{+}\overline{\sigma}$	σ	$\sigma{+}\rho$	$f^{(2)}$	$1{+}\mu$

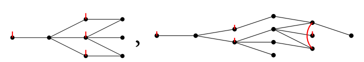
.

We see that ρ and μ give two copies of A_4, and they satisfy the relation $\rho\mu\rho \cong f^{(2)} \cong \mu\rho\mu$, but $\rho\mu \ncong \mu\rho$. Hence $\mathcal{TT}_{3,1}$ exists.

$A_4 - A_4$ composite principal graphs.
It is possible to determine the principal graph as in Theorem 3.10. For $n = 2, 3$, the fusion graphs for the $N - N$ bimodules with respect to $\rho \oplus \rho\mu\rho$ are given by



resulting in the following principal graphs for $n = 2, 3$ respectively:



The first is the tensor product $A_4 \otimes A_4$.

Liu pointed out to us that these two $A_4 - A_4$ composite subfactors are also the reduced subfactors of $\mathcal{BHF}_2$ and $\mathcal{BHF}_3$ at $\rho\theta\rho$.

Acknowledgements We thank Vaughan Jones, Dietmar Bisch, and Uffe Haagerup for many helpful conversations about the "fish" subfactors. We also thank Zhengwei Liu for explaining his perspective on the problem and sharing his excellent results with us prior to publication. We'd like to thank the referees for helpful comments.

We thank the Mathematical Sciences Institute at the Australian National University for hosting all the authors while undertaking this research. Scott Morrison would like to thank the Erwin Schrödinger Institute and its 2014 programme on "Modern Trends in Topological Quantum Field Theory" for their hospitality. All the authors would like to thank the Banff International Research Station for hosting the 2014 workshop on Subfactors and Fusion Categories.

References

[BHP12] A. Brothier, M. Hartglass, and D. Penneys, *Rigid C^*-tensor categories of bimodules over interpolated free group factors.* J. Math. Phys. **53**(2012), no. 12, 123525. http://dx.doi.org/10.1063/1.4769178

[Big10] S. Bigelow, *Skein theory for the ADE planar algebras.* J. Pure Appl. Algebra **214**(2010), no. 5, 658–666. http://dx.doi.org/10.1016/j.jpaa.2009.07.010

[Bis94] D. Bisch, *A note on intermediate subfactors.* Pacific J. Math. **163**(1994), no. 2, 201–216. http://dx.doi.org/10.2140/pjm.1994.163.201

[BJ97] ———, *Algebras associated to intermediate subfactors.* Invent. Math. **128**(1997), 89–157. http://dx.doi.org/10.1007/s002220050137

[BL10] B. Bhattacharyya and Z. Landau, *Intermediate standard invariants and intermediate planar algebras.* 2010. http://www.eecs.berkeley.edu/~landau/publications.htm

[BMPS12] S. Bigelow, S. Morrison, E. Peters, and N. Snyder, *Constructing the extended Haagerup planar algebra.* Acta Math. **209**(2012), no. 1, 29–82. http://dx.doi.org/10.1007/s11511-012-0081-7

[Gol59] M. Goldman, *On subfactors of factors of type* II_1. Michigan Math. J. **6**(1959), 167–172. http://dx.doi.org/10.1307/mmj/1028998188

[GS12] P. Grossman and N. Snyder, *Quantum subgroups of the Haagerup fusion categories*. Comm. Math. Phys. **311**(2012), no. 3, 617–643. http://dx.doi.org/10.1007/s00220-012-1427-x

[Haa94] U. Haagerup, *Principal graphs of subfactors in the index range* $4 < [M : N] < 3 + \sqrt{2}$. In: Subfactors (Kyuzeso, 1993), World Sci. Publ., River Edge, NJ, 1994, pp. 1–38.

[IK93] M. Izumi and Y. Kawahigashi, *Classification of subfactors with the principal graph* $D_n^{(1)}$. J. Funct. Anal. **112**(1993), no. 2, 257–286. http://dx.doi.org/10.1006/jfan.1993.1033

[IMP13] M. Izumi, S. Morrison, and D. Penneys, *Quotients of* $A_2 * T_2$. Canad. J. Math., to appear; extended version available as *Fusion categories between* $\mathcal{C} \boxtimes \mathcal{D}$ *and* $\mathcal{C} * \mathcal{D}$. arxiv:1308.5723.

[Lan02] Z. A. Landau, *Exchange relation planar algebras.* In: Proceedings of the Conference on Geometric and Combinatorial Group Theory, Part II (Haifa, 2000), Geom. Dedicata **95**(2002), 183–214. http://dx.doi.org/10.1023/A:1021296230310

[Liu13] Z. Liu, *Composed inclusions of* A_3 *and* A_4 *subfactors.* Adv. Math. **279**(2015), 307–371. http://dx.doi.org/10.1016/j.aim.2015.03.017

[MP14] S. Morrison and D. Penneys, 2-*supertransitive subfactors at index* $3 + \sqrt{5}$. J. Funct. Anal. **269**(2015), no. 9, 2845–2870. http://dx.doi.org/10.1016/j.jfa.2015.06.023

[MPPS12] S. Morrison, D. Penneys, E. Peters, and N. Snyder, *Subfactors of index less than 5, Part 2: Triple points.* Internat. J. Math. **23**(2012), 1250016, 33. http://dx.doi.org/10.1142/S0129167X11007586

[Pop94] S. Popa, *Classification of amenable subfactors of type II.* Acta Math. **172**(1994), no. 2, 163–255. http://dx.doi.org/10.1007/BF02392646

[PP13] D. Penneys and E. Peters, *Calculating two-strand jellyfish relations.* Pacific J. Math. **277**(2015), no. 2, 463–510. http://dx.doi.org/10.2140/pjm.2015.277.463

Department of Mathematics, Graduate School of Science, Kyoto University, Kyoto 606-8502, Japan
e-mail: izumi@math.kyoto-u.ac.jp

Mathematical Sciences Institute, Australian National University, Canberra, Australia
e-mail: scott.morrison@anu.edu.au

Department of Mathematics, University of California, Los Angeles, Los Angeles, CA, 90095-1555, USA
e-mail: dpenneys@math.ucla.edu