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PUSHFORWARD OF STRUCTURE SHEAF AND VIRTUAL GLOBAL GENERATION

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Abstract Let $f : X \longrightarrow Y$ be a generically smooth morphism between irreducible smooth projective curves over an algebraically closed field of arbitrary characteristic. We prove that the vector bundle $((f_*\mathcal{O}_X)/\mathcal{O}_Y)^*$ is virtually globally generated. Moreover, $((f_*\mathcal{O}_X)/\mathcal{O}_Y)^*$ is ample if and only if f is genuinely ramified.

Keywords: virtual global generation; genuinely ramified map; ampleness

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1. Introduction

Let X and Y be irreducible smooth projective curves over an algebraically closed field k – there is no assumption on the characteristic of k – and let $f : X \longrightarrow Y$ be a generically smooth morphism. Then, we have $\mathcal{O}_Y \subset f_*\mathcal{O}_X$. In [4] it was shown that the homomorphism of étale fundamental groups $f_* : \pi_1^{\mathrm{et}}(X) \longrightarrow \pi_1^{\mathrm{et}}(Y)$ induced by f is surjective if and only if $\mathcal{O}_Y$ is the unique maximal semistable subsheaf of $f_*\mathcal{O}_X$. We call f to be genuinely ramified if $\mathcal{O}_Y$ is the unique maximal semistable subsheaf of $f_*\mathcal{O}_X$. On the other hand, f is called primitive if the above homomorphism f_* of étale fundamental groups is surjective [5]. So f is genuinely ramified if and only if it is primitive.

The main result of [4] says the following: If $f : X \longrightarrow Y$ is genuinely ramified, and E is a stable vector bundle on Y, then f^*E is also stable. This was proved by investigating the quotient bundle $(f_*\mathcal{O}_X)/\mathcal{O}_Y$.

The dual vector bundle $((f_*\mathcal{O}_X)/\mathcal{O}_Y)^*$ is called the Tschirnhausen bundle for f (see [5]). The following is the main result of [5]: Let $f : X \longrightarrow Y$ be a general primitive degree r cover, where genus$(X) = g$ and genus$(Y) = h$, over an algebraically closed field of characteristic zero or greater than r. Then

(1) $((f_*\mathcal{O}_X)/\mathcal{O}_Y)^*$ is semistable if $h = 1$, and
(2) $((f_*\mathcal{O}_X)/\mathcal{O}_Y)^*$ is stable if $h \geq 2$.

Note that the above mentioned result of [4] can be reformulated as follows: Let $f : X \longrightarrow Y$ be a generically smooth morphism between irreducible smooth projective curves. Then f^*E is stable for every stable vector bundle E on Y if and only if:

$$\mu_{\min}(((f_*\mathcal{O}_X)/\mathcal{O}_Y)^*) > 0.$$

(Recall that $\mu_{\min}$ denotes the slope of the smallest quotient [9, p. 16, Definition 1.3.2].) See [5] for more on Tschirnhausen bundles.

A vector bundle on an irreducible smooth projective curve Z is called virtually globally generated if its pullback, under some surjective morphism to Z from some irreducible smooth projective curve, is generated by its global sections; see § 3.

We prove the following (see Theorem 3.3):

Let X and Y be irreducible smooth projective curves and:

$$f : X \longrightarrow Y,$$

a generically smooth morphism. Then $(f_*\mathcal{O}_X)^*$ is virtually globally generated.

Note that this implies that: $((f_*\mathcal{O}_X)/\mathcal{O}_Y)^*$ is virtually globally generated (see Corollary 3.5).

In Remark 3.6 it is shown that Corollary 3.5 fails in higher dimensions.

We prove the following (see Corollary 3.2):

Let $f : X \longrightarrow Y$ be a generically smooth morphism between two irreducible smooth projective curves. Then f is genuinely ramified if and only if $((f_*\mathcal{O}_X)/\mathcal{O}_Y)^*$ is ample.

It may be mentioned that the condition in Theorem 3.3 and Corollary 3.2 that f is generically smooth is essential. To give an example, take Y to be a smooth projective curve of genus at least two, and let $F_Y : Y \longrightarrow Y$, be the absolute Frobenius morphism of Y. Then $(F_{Y*}\mathcal{O}_Y))/\mathcal{O}_Y$ is in fact ample.

2. Genuinely ramified maps, direct image and ampleness

The base field k is assumed to be algebraically closed. For a vector bundle E on an irreducible smooth projective curve X, if

$$E_1 \subset \cdots \subset E_{n-1} \subset E_n = E,$$

is the Harder–Narasimhan filtration of E, then define $\mu_{\max}(E) := \mu(E_1)$ and $\mu_{\min}(E) = \mu(E/E_{n-1})$ [9]. The subbundle $E_1 \subseteq E$ is called the maximal semistable subsheaf of E.

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Let X and Y be irreducible smooth projective curves and

$$f : X \longrightarrow Y, \tag{2.1}$$

a dominant generically smooth morphism. It is straight-forward to check that:

$$\mu_{\max}(f_*\mathcal{O}_X) = 0. \tag{2.2}$$

Indeed, $\mu_{\max}(f_*\mathcal{O}_X) \leq 0$ because degree$(\mathcal{O}_X) = 0$ [4, p. 12824, Lemma 2.2]. On the other hand, we have $\mathcal{O}_Y \subset f_*\mathcal{O}_X$, which implies that $\mu_{\max}(f_*\mathcal{O}_X) \geq 0$, and thus (2.2) holds.

The following proposition was proved in [4].

Proposition 2.1. ([4, p. 12828, Proposition 2.6] and [4, p. 12830, Lemma 3.1]). *The following five statements are equivalent:*

(1) *The maximal semistable subsheaf of $f_*\mathcal{O}_X$ is $\mathcal{O}_Y$.*
(2) $\dim H^0(X, f^*f_*\mathcal{O}_X) = 1$.
(3) *The fibre product $X \times_Y X$ is connected.*
(4) *The homomorphism of étale fundamental groups $f_* : \pi_1^{\text{et}}(X) \longrightarrow \pi_1^{\text{et}}(Y)$ induced by f is surjective.*
(5) *The map f does not factor through any nontrivial finite étale covering of Y.*

Any morphism f as in (2.1) is called *genuinely ramified* if the (equivalent) statements in Proposition 2.1 hold [4, p. 12828, Definition 2.5].

Proposition 2.2. *Let $f : X \longrightarrow Y$ be a genuinely ramified morphism of smooth projective curves. Then the vector bundle $((f_*\mathcal{O}_X)/\mathcal{O}_Y)^*$ is ample.*

Proof. Since f is genuinely ramified, from Proposition 2.1 it follows that:

$$\mu_{\max}((f_*\mathcal{O}_X)/\mathcal{O}_Y) < 0,$$

and hence we have:

$$\mu_{\min}(((f_*\mathcal{O}_X)/\mathcal{O}_Y)^*) = -\mu_{\max}((f_*\mathcal{O}_X)/\mathcal{O}_Y) > 0. \tag{2.3}$$

When the characteristic of k is zero, a vector bundle W on Y is ample if and only if the degree of every nonzero quotient of W is positive [7, p. 84, Theorem 2.4]. Therefore, from (2.3) we conclude that: $((f_*\mathcal{O}_X)/\mathcal{O}_Y)^*$ is ample, when the characteristic of k is zero. However, this characterization of ample bundles fails when the characteristic of k is positive (see [7, Section 3] for such examples).

We will inductive construct a sequence of vector bundles $\{V_i\}_{i\geq 0}$ on Y. First set $V_0 = \mathcal{O}_Y$. For any $i \geq 1$, let $V_i = f_*f^*V_{i-1}$. Since we have

$$\mathcal{O}_Y \subset V_1 = f_*f^*\mathcal{O}_Y = f_*\mathcal{O}_X,$$

it can be deduced that:

$$\mathcal{O}_Y \subset V_i, \tag{2.4}$$

for all $i \geq 0$. Indeed, this follows inductively, as the inclusion map $\mathcal{O}_Y \hookrightarrow V_j$ produces:

$$\mathcal{O}_Y \subset f_*\mathcal{O}_X = f_*f^*\mathcal{O}_Y \hookrightarrow f^*f_*V_j = V_{j+1}.$$

This proves (2.4) inductively.

Next we will show that the subsheaf $\mathcal{O}_Y$ in (2.4) is the maximal semistable subsheaf of V_i. This will also be proved using an inductive argument.

First, $\mathcal{O}_Y$ is obviously the maximal semistable subsheaf of V_0. Next, from Proposition 2.1 we know that $\mathcal{O}_Y$ is the maximal semistable subsheaf of V_1 (recall that f is genuinely ramified). Let

$$\mathcal{O}_Y = E^1_1 \subset E^1_2 \subset \cdots \subset E^1_{n_1-1} \subset E^1_{n_1} = V_1,$$

be the Harder–Narasimhan filtration of V_1. Since f^*W is semistable if W is so (see [4, pp. 12823–12824, Remark 2.1]), we conclude that:

$$\mathcal{O}_X = f^*E^1_1 \subset \cdots \subset f^*E^1_{n_1-1} \subset f^*E^1_{n_1} = f^*V_1, \tag{2.5}$$

is the Harder–Narasimhan filtration of f^*V_1.

For any vector bundle B on X, we have $\mu_{\max}(f_*B) \leq \mu_{\max}(B)/\mathrm{degree}(f)$ [4, Lemma 2.2, p. 12824]. In view of the Harder–Narasimhan filtration in (2.5), this implies that:

$$\mu_{\max}((f_*f^*E^1_{j+1})/(f_*f^*E^1_j)) < 0,$$

for all $1 \leq j \leq n_1 - 1$, because $\mu_{\max}((f^*E^1_{j+1})/(f^*E^1_j)) < 0$. Also, as noted before, the maximal semistable subsheaf of $f_*\mathcal{O}_X$ is $\mathcal{O}_Y$. Combining these we conclude that $\mathcal{O}_Y$ is the maximal semistable subsheaf of $f_*f^*V_1 = V_2$.

The above argument works inductively. To explain this, let

$$\mathcal{O}_Y = E^\ell_1 \subset E^\ell_2 \subset \cdots \subset E^\ell_{n_\ell-1} \subset E^\ell_{n_\ell} = V_\ell,$$

be the Harder–Narasimhan filtration of V_ℓ. As before, we have:

$$\mu_{\max}((f_*f^*E^\ell_{j+1})/(f_*f^*E^\ell_j)) < 0,$$

for all $1 \leq j \leq n_\ell - 1$, because $\mu_{\max}((f^*E^\ell_{j+1})/(f^*E^\ell_j)) < 0$. Using this together with the fact that the maximal semistable subsheaf of $f_*\mathcal{O}_X$ is $\mathcal{O}_Y$ we conclude that $\mathcal{O}_Y$ is the maximal semistable subsheaf of $f_*f^*V_\ell = V_{\ell+1}$.

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The projection formula (see [8, p. 124, Ch. II, Ex. 5.1(d)], [11]) gives that $V_{i+1} = f_*f^*V_i = V_i \otimes (f_*\mathcal{O}_X)$ for all $i \geq 1$. This implies that:

$$V_i = (f_*\mathcal{O}_X)^{\otimes i} = V_1^{\otimes i}, \tag{2.6}$$

for all $i \geq 1$.

Now we assume that the characteristic of k is positive (recall that the proposition was proved when the characteristic of k is zero). Let p be the characteristic of k. Let

$$F_Y : Y \longrightarrow Y$$

be the absolute Frobenius morphism of Y. For any vector bundle W on Y, we have the inclusion:

$$F_Y^*W \subset W^{\otimes p},$$

it is constructed using the map $W \longrightarrow W^{\otimes p}$ defined by $v \longmapsto v^{\otimes p}$. Therefore, from (2.6) we have

$$(F_Y^n)^*V_1 \subset (V_1)^{\otimes np} = V_{np} \tag{2.7}$$

for all $n \geq 1$. Since $\mathcal{O}_Y$ in (2.4) is the maximal semistable subsheaf of V_i, from (2.7) we have:

$$(F_Y^n)^*(V_1/\mathcal{O}_Y) = ((F_Y^n)^*V_1)/\mathcal{O}_Y \subset V_{np}/\mathcal{O}_Y,$$

and

$$\mu_{\max}((F_Y^n)^*(V_1/\mathcal{O}_Y)) < 0, \tag{2.8}$$

because $\mu_{\max}(V_{np}/\mathcal{O}_Y) < 0$.

From (2.8) it follows that:

$$\mu_{\min}((F_Y^n)^*(V_1/\mathcal{O}_Y)^*) = -\mu_{\max}((F_Y^n)^*(V_1/\mathcal{O}_Y)) > 0,$$

for all $n \geq 1$. This implies that $(V_1/\mathcal{O}_Y)^* = ((f_*\mathcal{O}_X)/\mathcal{O}_Y)^*$ is ample [1, p. 542, Theorem 2.2]. □

3. Virtual global generation

Let E be a vector bundle on an irreducible smooth projective curve Z. It will be called *virtually globally generated* if there is a finite surjective morphism:

$$\phi : M \longrightarrow Z,$$

from an irreducible smooth projective curve M such that ϕ^*E is generated by its global sections. The vector bundle E is called *étale trivializable* if there is a pair (M, ϕ) as above such that ϕ is étale and ϕ^*E is trivializable.

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If degree$(E) < 0$, then E is not virtually globally generated. More generally, E is not virtually globally generated if it admits a quotient of negative degree. To give a nontrivial example of vector bundle which is not virtually globally generated, let Z be a compact connected Riemann surface of genus g, with $g \geq 2$. Note that the free group of g generators is a quotient of $\pi_1(Z)$. To see this, express $\pi_1(Z)$ as the quotient of the free group, with generators $a_1, \cdots, a_g, b_1, \cdots, b_g$, by the single relation $\prod_{i=1}^{g}[a_i, b_i] = 1$. Then the quotient of $\pi_1(Z)$ by the normal subgroup generated by $b_1, \cdots, b_g$ is the free group generated by $a_1, \cdots, a_g$. Therefore, there is homomorphism:

$$\rho : \pi_1(Z) \longrightarrow \mathrm{U}(r),$$

where $\mathrm{U}(r)$ is the group of $r \times r$ unitary matrices, such that $\rho(\pi_1(Z))$ is a dense subgroup of $\mathrm{U}(r)$ (the subgroup of $\mathrm{U}(r)$ generated by two general elements of it is dense in $\mathrm{U}(r)$). Let E denote the flat unitary vector bundle on Z given by ρ. This vector bundle E is stable of degree zero [10]. Let M be a compact connected Riemann surface and

$$\phi : M \longrightarrow Z,$$

a surjective holomorphic map. Since the image of the induced homomorphism:

$$\phi_* : \pi_1(M) \longrightarrow \pi_1(Z),$$

is a subgroup of $\pi_1(Z)$ of finite index, the image of the following composition of homomorphisms:

$$\pi_1(M) \xrightarrow{\phi_*} \pi_1(Z) \xrightarrow{\rho} \mathrm{U}(r),$$

is a dense subgroup of $\mathrm{U}(r)$. This implies that ϕ^*E is a stable vector bundle of degree zero [10]. In particular, we have

$$H^0(M, \phi^*E) = 0.$$

Hence E is not virtually globally generated.

Theorem 3.1. *Let X and Y be irreducible smooth projective curves over k and*

$$f : X \longrightarrow Y,$$

a generically smooth morphism. Then $f_*\mathcal{O}_X$ fits in a short exact sequence of vector bundles on Y:

$$0 \longrightarrow E \longrightarrow f_*\mathcal{O}_X \longrightarrow V \longrightarrow 0,$$

where E is étale trivializable and V^* is ample.

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Proof. Let

$$S^f \subset f_*\mathcal{O}_X, \tag{3.1}$$

be the maximal semistable subbundle. From (2.2) we know that $\text{degree}(S^f) = 0$.

The algebra structure of $\mathcal{O}_X$ produces an algebra structure on the direct image $f_*\mathcal{O}_X$. The subsheaf S^f in (3.1) is a subalgebra. Moreover, there is an étale covering $g : Z \longrightarrow Y$ such that:

- f factors through g, meaning there is a morphism:

$$h : X \longrightarrow Z \tag{3.2}$$

  such that $g \circ h = f$, and
- the subsheaf $g_*\mathcal{O}_Z \subset f_*\mathcal{O}_X$ coincides with S^f.

(See the proof of [4, p. 12828, Proposition 2.6] and [4, p. 12829, (2.13)].) Moreover, the map h in (3.2) is genuinely ramified [4, p. 12829, Corollary 2.7].

Consider the short exact sequence of vector bundles on Y:

$$0 \longrightarrow S^f \longrightarrow f_*\mathcal{O}_X \longrightarrow Q := (f_*\mathcal{O}_X)/S^f \longrightarrow 0. \tag{3.3}$$

The pullback g^*Q, where Q is the vector bundle in (3.3), is identified with $(h_*\mathcal{O}_X)/\mathcal{O}_Z$, where h is the map in (3.2). From Proposition 2.2 we know that $((h_*\mathcal{O}_X)/\mathcal{O}_Z)^*$ is ample, Since $((h_*\mathcal{O}_X)/\mathcal{O}_Z)^* = g^*Q^*$, this implies that Q^* in (3.3) is ample (see [6, p. 73, Proposition 4.3]).

Since Q^* is ample, in view of (3.3), it suffices to prove that S^f is a finite vector bundle.

Fix an étale Galois covering $\varphi : M \longrightarrow Y$ that dominates g. In other words, there is a morphism:

$$\beta : M \longrightarrow Z$$

such that $g \circ \beta = \varphi$. Since φ is an étale Galois covering, the vector bundle $\varphi^*\varphi_*\mathcal{O}_M$ is trivializable. On the other hand,

$$S^f = g_*\mathcal{O}_Z \subset \varphi_*\mathcal{O}_M,$$

and S^f is a subbundle of $\varphi_*\mathcal{O}_M$. Consider the subbundle

$$\varphi^*S^f \subset \varphi^*\varphi_*\mathcal{O}_M. \tag{3.4}$$

We have $\text{degree}(\varphi^*S^f) = 0$, because $\text{degree}(S^f) = 0$, and we also know that $\varphi^*\varphi_*\mathcal{O}_M$ is trivializable. Consequently, the subbundle φ^*S^f in (3.4) is also trivializable. Hence S^f is étale trivializable. □

Corollary 3.2. *Let $f : X \longrightarrow Y$ be a generically smooth morphism between two irreducible smooth projective curves. Then f is genuinely ramified if and only if $((f_*\mathcal{O}_X)/\mathcal{O}_Y)^*$ is ample.*

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Proof. In view of Proposition 2.2 it suffices to show that $((f_*\mathcal{O}_X)/\mathcal{O}_Y)^*$ is not ample if f is not genuinely ramified. If f is not genuinely ramified, then $\mathrm{rank}(S^f) \geq 2$ (see (3.1)). Hence $(S^f/\mathcal{O}_Y)^*$ is a quotient of $((f_*\mathcal{O}_X)/\mathcal{O}_Y)^*$ (see (3.3)). But $\mathrm{degree}((S^f/\mathcal{O}_Y)^*) = 0$ because $\mathrm{degree}((S^f) = 0$. Now $((f_*\mathcal{O}_X)/\mathcal{O}_Y)^*$ is not ample because its quotient $(S^f/\mathcal{O}_Y)^*$ is not ample. □

Theorem 3.3. *Let X and Y be irreducible smooth projective curves and*

$$f : X \longrightarrow Y,$$

a generically smooth morphism. Then $(f_*\mathcal{O}_X)^*$ is virtually globally generated.

Proof. First assume that the characteristic of k is zero. We will show that the short exact sequence in (3.3) splits. First, the inclusion map $\mathcal{O}_Z \hookrightarrow h_*\mathcal{O}_X$ splits naturally, where h is the map in (3.2); in other words,

$$h_*\mathcal{O}_X = \mathcal{O}_Z \oplus F\,;$$

the fibre of F over any $z \in Z$ is the space of functions on $h^{-1}(z)$ whose sum is zero. Now we have

$$f_*\mathcal{O}_X = g_*h_*\mathcal{O}_X = g_*(\mathcal{O}_Z \oplus F) = (g_*\mathcal{O}_Z) \oplus g_*F = S^f \oplus g_*F. \tag{3.5}$$

From (3.3) and (3.5) it follows that the vector bundle g_*F is isomorphic to Q. Therefore, from (3.5) we have

$$(f_*\mathcal{O}_X)^* = (S^f)^* \oplus Q^*. \tag{3.6}$$

Now $(S^f)^*$ is virtually globally generated because S^f is étale trivializable, and Q^* is virtually globally generated because Q^* is ample by Theorem 3.1 (see [3, p. 46, Theorem 3.6]). Therefore, from (3.6) it follows that $(f_*\mathcal{O}_X)^*$ is virtually globally generated.

Next assume that the characteristic of k is positive. As before,

$$F_Y : Y \longrightarrow Y$$

is the absolute Frobenius morphism of Y. Consider the exact sequence in (3.3); recall that S^f is the maximal semistable subsheaf of $f_*\mathcal{O}_X$. Therefore, there is an integer n_0 such that for all $n \geq n_0$, we have

$$(F_Y^n)^*f_*\mathcal{O}_X = (F_Y^n)^*S^f \oplus (F_Y^n)^*Q$$

[2, p. 356, Proposition 2.1]. Therefore,

$$(F_Y^n)^*(f_*\mathcal{O}_X)^* = (F_Y^n)^*(S^f)^* \oplus (F_Y^n)^*Q^*. \tag{3.7}$$

Now $(F_Y^n)^*(S^f)^*$ is virtually globally generated because S^f is étale trivializable and the Frobenius morphism commutes with étale morphisms. Also, Q^* is virtually globally generated because Q^* is ample by Theorem 3.1 (see [2, p. 357, Theorem 2.2]). Therefore, from (3.7) it follows that $(f_*\mathcal{O}_X)^*$ is virtually globally generated. □

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Corollary 3.4. *Let X and Y be irreducible smooth projective curves and*

$$f : X \longrightarrow Y,$$

a generically smooth morphism. Then the following statements hold:

- *If the characteristic of k is zero, then*

$$(f_*\mathcal{O}_X)^* = E \oplus A,$$

  where E is étale trivializable and A is ample.
- *If the characteristic of k is positive, then there is an integer n such that:*

$$(F_Y^n)^*(f_*\mathcal{O}_X)^* = E \oplus A,$$

  where E is étale trivializable and A is ample.

Proof. In view of Theorem 3.3, this follows immediately from [3, p. 40, Theorem 1.1]. □

Corollary 3.5. *Let X and Y be irreducible smooth projective curves and*

$$f : X \longrightarrow Y,$$

a generically smooth morphism. Then $((f_*\mathcal{O}_X)/\mathcal{O}_Y)^*$ is virtually globally generated.

Proof. From Theorem 3.3 we know that there is a finite surjective map:

$$\phi : M \longrightarrow Y,$$

such that $\phi^*(f_*\mathcal{O}_X)^*$ is generated by its global sections. We have the short exact sequence of vector bundles on M:

$$0 \longrightarrow \phi^*((f_*\mathcal{O}_X)/\mathcal{O}_Y)^* \longrightarrow \phi^*(f_*\mathcal{O}_X)^* \longrightarrow \phi^*(\mathcal{O}_Y)^* = \mathcal{O}_M \longrightarrow 0. \tag{3.8}$$

Since $\phi^*(f_*\mathcal{O}_X)^*$ is generated by its global sections, it has a section that projects to a nonzero section of $\mathcal{O}_M$. Choosing such a section we obtain a splitting of (3.8). Since $\phi^*(f_*\mathcal{O}_X)^*$ is generated by its global sections, its direct summand $\phi^*((f_*\mathcal{O}_X)/\mathcal{O}_Y)^*$ is also generated by its global sections. □

Remark 3.6. Corollary 3.5 is not valid in higher dimensions. To give an example, let X denote $\mathbb{CP}^2$ blown up at the point $(1, 0, 0)$. The involution of $\mathbb{CP}^2$ defined by

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$(x, y, z) \longmapsto (x, -y, -z)$ lifts to X; let

$$\tau : X \longrightarrow X,$$

be this lifted involution. Set $Y := X/(\mathbb{Z}/2\mathbb{Z})$ to be the quotient of X for the action of $\mathbb{Z}/2\mathbb{Z}$ given by τ. Let

$$f : X \longrightarrow X/(\mathbb{Z}/2\mathbb{Z}) = Y,$$

be the quotient map. Then the line bundle $((f_*\mathcal{O}_X)/\mathcal{O}_Y)^*$ is not virtually globally generated. To see this, first note that the line bundle $f^*(((f_*\mathcal{O}_X)/\mathcal{O}_Y)^*)$ is virtually globally generated if $((f_*\mathcal{O}_X)/\mathcal{O}_Y)^*$ is virtually globally generated. But

$$f^*(((f_*\mathcal{O}_X)/\mathcal{O}_Y)^*) = \mathcal{O}_X(D_e + D_\infty),$$

where $D_e \subset X$ is the exceptional divisor and $D_\infty \subset X$ is the inverse image of

$$\{(0, y, z) \in \mathbb{CP}^2 \mid y, z \in \mathbb{C}\} \subset \mathbb{CP}^2.$$

It is easy to see that $\mathcal{O}_X(D_e + D_\infty)$ is not virtually globally generated. Indeed, if

$$\varpi : Z \longrightarrow X$$

is a finite surjective proper map, then every section of $\varpi^*\mathcal{O}_X(D_e + D_\infty)$ vanishes on $\varpi^{-1}(D_e)$.

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