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NORM OF COMPOSITION OPERATORS ON THE BLOCH SPACE

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The author estimates the semi-norm and norm of composition operators on the Block space; and obtains several necessary conditions for a composition operator to be isometric.

1. INTRODUCTION

Let $\mathbf{D}$ be the unit disk in the complex plane $\mathbb{C}$ and $H(\mathbf{D})$ be the set of all analytic functions on $\mathbf{D}$. A function $f \in H(\mathbf{D})$ is said to be a Bloch function if it satisfies

$$\|f\|_\beta = \sup_{z\in\mathbf{D}}(1-|z|^2)|f'(z)| < \infty.$$

The set of all Bloch functions is called the Bloch space, which becomes a Banach space [**1**] under the norm

$$\|f\|_\mathcal{B} = |f(0)| + \|f\|_\beta = |f(0)| + \sup_{z\in\mathbf{D}}(1-|z|^2)|f'(z)|.$$

Let φ be a holomorphic mapping of $\mathbf{D}$ into itself. It is known that φ induces a bounded composition operator $C_\varphi(f) = f\circ\varphi$ on the Bloch space ([**4**]). The semi-norm and norm of C_φ are defined as

$$\|C_\varphi\|_\beta = \sup_{\|f\|_\mathcal{B}=1}\|f\circ\varphi\|_\beta \quad \text{and} \quad \|C_\varphi\|_\mathcal{B} = \sup_{\|f\|_\mathcal{B}=1}\|f\circ\varphi\|_\mathcal{B}$$

respectively.

If $\|C_\varphi(f)\|_\mathcal{B} = \|f\|_\mathcal{B}$ for every $f\in\mathcal{B}$, then C_φ is called an isometric operator on the Bloch space.

On the Hardy space H^2, Littlewood [**2**] showed that

$$\|C_\varphi\|_{H^2} \leqslant \sqrt{\frac{1+|\varphi(0)|}{1-|\varphi(0)|}}.$$

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If φ is an inner function, Nordgren [3] showed that

$$\|C_\varphi\|_{H^2} = \sqrt{\frac{1+|\varphi(0)|}{1-|\varphi(0)|}}.$$

Moreover, C_φ is isometric if and only if φ is an inner function with $\varphi(0)=0$. If $\varphi(0)\neq 0$, Shapiro [5] showed that

$$\|C_\varphi\|_{H^2} = \sqrt{\frac{1+|\varphi(0)|}{1-|\varphi(0)|}}$$

if and only if φ is an inner function.

In this paper, we consider similar problems on the Bloch space. In Section 2, we estimate the semi-norm and norm of composition operators. In Section 3, we obtain several necessary conditions for a composition operator to be isometric. Finally, we propose a question.

2. Semi-norm and norm of Composition Operators

Denote

$$\tau_\varphi(z) = \frac{1-|z|^2}{1-|\varphi(z)|^2}|\varphi'(z)|, \quad \tau_\varphi^\infty = \sup_{z\in\mathbf{D}} \tau_\varphi(z).$$

By Schwarz-Pick Lemma, $\tau_\varphi(z) \leqslant 1$ and $\tau_\varphi^\infty \leqslant 1$.

Theorem 1. *Let φ be a holomorphic mapping of $\mathbf{D}$ into itself, then the semi-norm*

$$\|C_\varphi\|_\beta = \sup_{\|f\|_{\mathcal{B}}=1} (1-|z|^2)\left|f'(\varphi(z))\varphi'(z)\right| = \tau_\varphi^\infty.$$

Proof: By the definition,

$$\begin{aligned}\|C_\varphi\|_\beta &= \sup_{\|f\|_{\mathcal{B}}=1} (1-|z|^2)\left|f'(\varphi(z))\varphi'(z)\right| \\ &= \sup_{\|f\|_{\mathcal{B}}=1} \left(1-|\varphi(z)|^2\right)\left|f'(\varphi(z))\right| \cdot \frac{1-|z|^2}{1-|\varphi(z)|^2}|\varphi'(z)| \leqslant \tau_\varphi^\infty.\end{aligned}$$

On the other hand, let

$$F(z) = \frac{1}{2}\log\frac{1+z}{1-z}, \quad F(0)=0, \quad z\in\mathbf{D},$$

then $F\in\mathcal{B}$, $\|F\|_{\mathcal{B}}=1$ and $\left(1-|z|^2\right)|F'(z)|=1$ for $z\geqslant 0$. For any $\varepsilon>0$, there exists a $z_0\in\mathbf{D}$ such that $\tau_\varphi(z_0) > \tau_\varphi^\infty - \varepsilon$. Let $\varphi(z_0) = e^{i\theta}|\varphi(z_0)|$, $g(z) = F(e^{-i\theta}z)$, then

$$\begin{aligned}\|C_\varphi\|_\beta \geqslant \left\|C_\varphi(g)\right\|_\beta &\geqslant \sup_{z\in\mathbf{D}}\left(1-|\varphi(z)|^2\right)\left|F'\left(e^{-i\theta}\varphi(z)\right)\right| \cdot \frac{1-|z|^2}{1-|\varphi(z)|^2}|\varphi'(z)| \\ &\geqslant \left(1-|\varphi(z_0)|^2\right)\left|F'\left(e^{-i\theta}\varphi(z_0)\right)\right| \cdot \frac{1-|z_0|^2}{1-|\varphi(z_0)|^2}|\varphi'(z_0)| \\ &= \tau_\varphi(z_0) > \tau_\varphi^\infty - \varepsilon.\end{aligned}$$

For the arbitrariness of ε, we have $\|C_\varphi\|_{\mathcal{B}} \geqslant \tau_\varphi^\infty$. The proof is completed. $\square$

COROLLARY 1. *Let φ be a holomorphic mapping of* $\mathbf{D}$ *into itself, then*

$$\|C_\varphi\|_{\mathcal{B}} \geqslant \max\left\{1, \frac{1}{2}\log\frac{1+|\varphi(0)|}{1-|\varphi(0)|}\right\}$$

PROOF: As in the proof of theorem 1, let $\varphi(0) = e^{i\theta}|\varphi(0)|$,

$$F(z) = \frac{1}{2}\log\frac{1+z}{1-z} \quad F(0) = 0,$$

and $g(z) = F(e^{-i\theta}z)$. We have

$$\|C_\varphi\|_{\mathcal{B}} \geqslant \left\|g(\varphi)\right\|_{\mathcal{B}} \geqslant \left|g\Big((\varphi(0))\Big)\right| \geqslant \frac{1}{2}\log\frac{1+|\varphi(0)|}{1-|\varphi(0)|}.$$

Taking $f(z) = 1$, then $\|C_\varphi(f)\|_{\mathcal{B}} = 1$ and $\|C_\varphi\|_{\mathcal{B}} \geqslant 1$. The proof is completed. $\square$

LEMMA 1. *If $f \in \mathcal{B}$, then*

$$\left|f(z)\right| \leqslant \left|f(0)\right| + \frac{\|f\|_{\mathcal{B}} - |f(0)|}{2}\log\frac{1+|z|}{1-|z|}.$$

PROOF: By the definition,

$$\|f\|_{\mathcal{B}} = \left|f(0)\right| + \sup_{z\in\mathbf{D}}(1-|z|^2)\left|f'(z)\right|.$$

Hence

$$\left|f'(z)\right| \leqslant \frac{\|f\|_{\mathcal{B}} - |f(0)|}{1-|z|^2},$$

$$\begin{aligned}\left|f(z)\right| &= \left|f(0) + \int_0^z f'(z)\,dz\right| \leqslant \left|f(0)\right| + \int_0^{|z|}\left|f'(z)\right|d|z| \\ &\leqslant \left|f(0)\right| + \frac{\|f\|_{\mathcal{B}} - |f(0)|}{2}\log\frac{1+|z|}{1-|z|}.\end{aligned}$$

$\square$

THEOREM 2. *Let φ be a holomorphic mapping of* $\mathbf{D}$ *into itself, then*

$$\|C_\varphi\|_{\mathcal{B}} \leqslant \max\left\{1, \frac{1}{2}\log\frac{1+|\varphi(0)|}{1-|\varphi(0)|} + \tau_\varphi^\infty\right\}.$$

In particular, if $\varphi(0) = 0$, then $\|C_\varphi\|_{\mathcal{B}} = 1$.

PROOF: Taking $f \in \mathcal{B}$ with $\|f\|_{\mathcal{B}} = 1$, we have

$$\begin{aligned}\left\|C_\varphi(f)\right\|_{\mathcal{B}} &= \left|f\big(\varphi(0)\big)\right| + \sup_{z\in\mathbf{D}}(1-|z|^2)\left|f'\big(\varphi(z)\big)\right|\left|\varphi'(z)\right| \\ &= \left|f\big(\varphi(0)\big)\right| + \sup_{z\in\mathbf{D}}\Big(1-\left|\varphi(z)\right|^2\Big)\left|f'\big(\varphi(z)\big)\right| \cdot \frac{1-|z|^2}{1-|\varphi(z)|^2}\left|\varphi'(z)\right| \\ &\leqslant \left|f\big(\varphi(0)\big)\right| + \Big(\|f\|_{\mathcal{B}} - \left|f(0)\right|\Big)\tau_\varphi^\infty \\ &= \left|f\big(\varphi(0)\big)\right| + \Big(1 - \left|f(0)\right|\Big)\tau_\varphi^\infty.\end{aligned}$$

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Using lemma 1,

$$\left|f(\varphi(0))\right| \leqslant |f(0)| + \frac{1-|f(0)|}{2}\log\frac{1+|\varphi(0)|}{1-|\varphi(0)|}.$$

Notice that $0 \leqslant |f(0)| \leqslant 1$, then

$$\begin{aligned}\|C_\varphi(f)\|_B &\leqslant |f(0)| + \big(1-|f(0)|\big)\Big(\frac{1}{2}\log\frac{1+|\varphi(0)|}{1-|\varphi(0)|} + \tau_\varphi^\infty\Big)\\ &\leqslant \max\Big\{1, \frac{1}{2}\log\frac{1+|\varphi(0)|}{1-|\varphi(0)|} + \tau_\varphi^\infty\Big\}.\end{aligned}$$

The proof is completed. □

To deal with τ_φ^∞, we need several concepts which can be found in [**6**]. Let $w \in T = \partial\mathbf{D}$ and $0 < \alpha < \pi/2$. The Stolz angle at w, denote by $S(w,\alpha)$, is the region between two straight lines in $\mathbf{D}$ that meet at w with angle 2α and are symmetric about the radius to w. For $f \in H(\mathbf{D})$, $w \in T$ and a constant $L \in \mathbb{C}$, then

$$\angle\lim_{z\to w} = L$$

means that $f(z) \to L$ as $z \to w$ through any Stolz angle at w. In this case, we say L is the angular limit of f at w. Suppose φ is a holomorphic mapping of $\mathbf{D}$ into itself and $w \in T$. If there exists a constant $\eta \in T$ such that

$$\angle\lim_{z\to w}\frac{\varphi(z)-\eta}{z-w}$$

exists finitely, then we say this limit is the angle derivative of φ at w and denote it by $\varphi'(w)$.

We need the following.

The Julia-Carathéodory Theorem. ([**6**].) *Suppose φ is a holomorphic self-map of $\mathbf{D}$, and $w \in T$. Then the following three statements are equivalent:*

(1) $\liminf\limits_{z\to w}\dfrac{1-|\varphi(z)|}{1-|z|} = \delta < \infty$;

(2) $\angle\lim\limits_{z\to w}\dfrac{\eta-\varphi(z)}{w-z}$ *exists for some* $(\eta \in T)$;

(3) $\angle\lim\limits_{z\to w}\varphi'(z)$ *exists, and* $\lim\limits_{z\to w}\angle\varphi(z) = \eta$.

Moreover, if one of the above conditions holds, then $\delta > 0$, the η in (1) and (2) are the same, and the limit in (2) coincides with that of the derivative in (3), with both equal $\overline{w}\eta\delta$.

Lemma 2. *Suppose φ is a holomorphic self-map of $\mathbf{D}$ and has an angle derivative at some point of $T = \partial\mathbf{D}$. Then*

$$\tau_\varphi^\infty = 1.$$

PROOF: By Theorems 1, 2 and the Schwarz-Pick lemma, we need only prove $\tau_\varphi^\infty \geqslant 1$. Assume that φ has an angle derivative μ at the point $w \in T$. Using the Julia-Carathéodory theorem for φ, then there exist $\delta > 0$ and $\eta \in T$ such that $\mu = \angle \lim_{z \to w} \varphi'(z) = \overline{w}\eta\delta$ and

$$\liminf_{z \to w} \frac{1 - |\varphi(z)|}{1 - |z|} = \delta.$$

Hence

$$\angle \lim_{z \to w} \tau_\varphi = \angle \lim_{z \to w} \frac{1 - |z|^2}{1 - |\varphi(z)|^2} |\phi'(z)| = 1$$

and $\|\tau_\varphi\|_\infty \geqslant 1$. The proof is completed. ∎

By Theorem 1, 2, Corollary of the Theorem 1 and the Lemma 2, we have following.

COROLLARY 2. *Suppose that φ is a holomorphic self-map of $\mathbf{D}$ and has an angle derivative at some point of $T = \partial\mathbf{D}$. Then*

$$\|C_\varphi\|_\beta = 1,$$

$$\max\left\{1, \frac{1}{2}\log\frac{1 + |\varphi(0)|}{1 - |\varphi(0)|}\right\} \leqslant \|C_\varphi\|_B \leqslant 1 + \frac{1}{2}\log\frac{1 + |\varphi(0)|}{1 - |\varphi(0)|}.$$

Moveover, for any $a \in [0, 1)$, the left equality of the above ineqality can hols for $\varphi(z) = a$ and the right for $\varphi(z) = (a - z)/(1 - \overline{a}z)$.

3. ISOMETRIC COMPOSITION OPERATORS

LEMMA 3. *If the composition operator C_φ is isometric on the Bloch space then $\varphi(0) = 0$ and $\|\varphi\|_B = 1$.*

PROOF: Let $\varphi(0) = a$ and

$$\psi(z) = \frac{a - z}{1 - \overline{a}z}.$$

By the Schwarz-Pick lemma, we have

$$\|\psi\|_B = |a| + \|\psi\|_\beta = |a| + 1$$

and

$$\left\|C_\varphi(\psi)\right\|_B = \left|\psi\big(\varphi(0)\big)\right| + \sup_{z \in D}\left(1 - |\varphi(z)|^2\right)\left|\psi'\big(\varphi(z)\big)\right|\frac{1 - |z|^2}{1 - |\varphi(z)|^2}|\varphi'(z)| \leqslant 1.$$

If C_φ is isometric, then $a = 0$ and $\|\varphi\|_B = 1$. The lemma is proved. ∎

THEOREM 3. *Suppose that the composition operator C_φ is isometric on the Bloch space. Then*

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(1) *φ' is bounded on* $\mathbf{D} \Longrightarrow \varphi(z) = e^{i\theta}z$.

(2) *φ is univalent on* $\mathbf{D} \Longrightarrow \varphi(z) = e^{i\theta}z$.

(3) *If $\varphi(z) \neq e^{i\theta}z$, then for every $a \in \mathbf{D}$ there exists a sequence $\{z_n\}$ such that $|z_n| \to 1$ and $\varphi(z_n) \to a$.*

PROOF: (1) For the isometry of C_φ, we know $\|\varphi\|_B = 1$ and $\varphi(0) = 0$ by Lemma 3, hence

$$\sup_{z\in\mathbf{D}}\left(1 - |\varphi(z)|^2\right) \cdot \frac{1 - |z|^2}{1 - |\varphi(z)|^2}|\varphi'(z)| = 1.$$

If $\varphi(z) \neq e^{i\theta}z$, by the Schwarz-pick lemma, we have $\tau_\varphi(z) < 1$ for $z \in \mathbf{D}$. Then there exists a sequence $\{z_n\}$ such that $|z_n| \to 1$, $|\varphi(z_n)| \to 0$ and $\tau_\varphi(z_n) \to 1$. Hence

$$|\varphi'(z_n)| > \frac{1 - |\varphi(z_n)|^2}{2(1 - |z_n|^2)} \to \infty$$

for the sufficient large n. This contradicts the fact that that φ' is bounded on $\mathbf{D}$.

(2) Assuming that φ is univalent, we know $\varphi(0) = 0$ by lemma 3. Then there exists a $\varepsilon > 0$ such that $|\varphi(z_n)| > \varepsilon$ for any sequence $|z_n| \to 1$ with the sufficient large n. If $\varphi(z) \neq e^{i\theta}z$, by the proof of (1), we have a contradiction.

(3) For any $a \in \mathbf{D}$, we take $\psi(z) = (a - z)/(1 - \bar{a}z)$. By the Schwarz-Pick lemma and the lemma 3, we have

$$|a| + \sup_{z\in\mathbf{D}}\left(1 - |\varphi(z)|^2\right)\left|\psi'(\varphi(z))\right| \cdot \frac{1 - |z|^2}{1 - |\varphi(z)|^2}|\varphi'(z)| = |a| + 1,$$

$$\sup_{z\in\mathbf{D}}\left(1 - \left|\psi(\varphi(z))\right|^2\right) \cdot \frac{1 - |z|^2}{1 - |\varphi(z)|^2}|\varphi'(z)| = 1.$$

Then there exists a sequence $\{z_n\}$ such that $|z_n| \to 1$ and $\left|\psi(\varphi(z_n))\right| \to 0$, so $\varphi(z_n) \to a$.

The proof is completed. ∎

This theorem suggests the following question.

QUESTION. Does there exist a function φ other than $e^{i\theta}z$ such that C_φ is isometric on the Bloch space?

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