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A Taylor–Wiles System for Quaternionic Hecke Algebras

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Abstract. Let $\ell > 3$ be a prime. Fix a regular character χ of $\mathbf{F}_{\ell^2}^{\times}$ of order $\leqslant \ell - 1$, and an integer M prime to ℓ. Let $f \in S_2(\Gamma_0(M\ell^2))$ be a newform which is supercuspidal of type χ at ℓ. For an indefinite quaternion algebra over $\mathbf{Q}$ of discriminant dividing the level of f, there is a local quaternionic Hecke algebra $\mathbf{T}$ of type χ associated to f. The algebra $\mathbf{T}$ acts on a quaternionic cohomological module $\mathbf{M}$. We construct a Taylor–Wiles system for $\mathbf{M}$, and prove that $\mathbf{T}$ is the universal object for a deformation problem (of type χ at ℓ and semi-stable outside) of the Galois representation $\bar{\rho}_f$ over $\bar{\mathbf{F}}_\ell$ associated to f; that $\mathbf{T}$ is complete intersection and that the module $\mathbf{M}$ is free of rank 2 over $\mathbf{T}$. We deduce a relation between the quaternionic congruence ideal of type χ for f and the classical one.

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Introduction

The fact that certain Hecke algebras are complete intersections and universal deformation rings is a fundamental ingredient in Wiles' proof of the modularity of semistable elliptic curves over $\mathbf{Q}$ [36, 39].

Taylor and Wiles' original construction makes use of the so-called 'multiplicity one' result for the ℓ-adic cohomology of the modular curve: namely the fact that this cohomology is free of rank 2 over the Hecke algebra when localized at certain maximal ideals. This result generalizes a theorem of Mazur [21]. Its proof is based on the q-expansion principle for classical modular forms. The Gorenstein property for the Hecke algebra is known to follow from it.

However, some later refinements due to Diamond [10] and Fujiwara [13] give an axiomatization of the Taylor–Wiles construction which allows one to prove that the Hecke algebra is a universal deformation ring without assuming the multiplicity one result. Furthermore, multiplicity one becomes a consequence of this construction.

As Diamond points out in [10], in addition to simplifying the arguments of Wiles and Taylor and Wiles, this approach makes these methods applicable in situations

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where one cannot appeal a priori to a q-expansion principle or to the Gorenstein property of the Hecke algebra, for instance, the case of Hilbert modular forms (treated by Fujiwara) or the case of quaternionic modular forms, arising from the cohomology of Shimura curves. In [10], Diamond gives an application of his method which produces multiplicity results for the ℓ-adic cohomology of Shimura curves arising from quaternion algebras unramified at ℓ. Some multiplicity one results of this kind were previously proved by Ribet in [25] by different methods. In his work, Ribet also obtains negative results, i.e., cases where the cohomology fails to be free over the Hecke algebra.

One of the main problems in dealing with Shimura curves arising from quaternion algebras ramified at ℓ is that the Galois representations associated to the cohomology of such curves are very ramified at ℓ. In general, we cannot appeal to a theory analogous to that of Fontaine and Laffaille which allows one to calculate the dimension of the tangent space of the local deformation functor at ℓ. Therefore, the techniques of Wiles and Taylor and Wiles are not applicable as they are.

However, for the case of representations arising from ℓ-divisible groups over certain tamely ramified extensions of $\mathbf{Q}_\ell$, the work of B. Conrad [4, 3] allows one to do this calculation. The results of B. Conrad have already been used in [5] to prove the modularity of some ℓ-adic Galois representation (whose reduction modulo ℓ is known to be modular) which are not semistable at ℓ but only potentially semistable. A generalization of Conrad's results has been recently obtained by Savitt [28].

In this paper, we combine the method of Diamond and Fujiwara with Conrad's result (as in [5]) to deal with the Hecke algebra acting on some local component of the ℓ-adic cohomology of Shimura curves ramified at ℓ.

More precisely we fix a prime $\ell > 3$. Let Δ' be the product of an odd number of primes, $\Delta = \ell\Delta'$; N be a square-free integer, $(N, \Delta) = 1$. Let B denote the indefinite quaternion algebra over $\mathbf{Q}$ of discriminant Δ, and $R(N)$ be an Eichler order of level N in B.

We assume the existence of a new form $f \in S_2(\Gamma_0(N\Delta'\ell^2))$, associated to an automorphic representation π of $GL_2(\mathbf{A})$ coming, by Jacquet–Langlands correspondence, from a representation π' of $B_{\mathbf{A}}^\times$. The local representation π'_ℓ is then associated to a regular character χ of $R(N)_\ell^\times/1 + u_\ell R(N)_\ell \simeq \mathbf{F}_{\ell^2}^\times$, where u_ℓ is an uniformizer of $B_\ell^\times$. We suppose that the order e of χ is $\leqslant \ell - 1$.

Let K be a finite extension of $\mathbf{Q}_\ell$ containing $\mathbf{Q}_{\ell^2}$ and the eigenvalues of f, $\mathcal{O}$ be its ℓ-adic integer ring, λ a uniformizer of $\mathcal{O}$, k the residue field. For simplicity, we discuss in this introduction the case where the group $B^\times \cap (\mathrm{GL}_2^+(\mathbf{R}) \times \prod_p R(N)_p^\times)$ has not elliptic elements (this depends on the congruence class mod 4 of primes dividing Δ and N, see [38, IV.3.A] for the precise statement); in the general case, by an argument of Diamond and Taylor [11] an auxiliary prime s can be added in the level. Let $\mathbf{X}$ be the adelic Shimura curve associated to the compact open subgroup $\prod_{p\neq\ell} R(N)_p^\times \times (1 + u_\ell R(N)_\ell)$ of $B_{\mathbf{A}}^{\times,\infty}$. The module $H^1(\mathbf{X}, \mathcal{O})$ is equipped with an action of $\mathbf{F}_{\ell^2}^\times$ and with an action of the Hecke algebra generated by the operators T_p with $p \neq \ell$. The two actions commute. Then we can consider the sub-Hecke module $H^1(\mathbf{X}, \mathcal{O})^\chi$ on which $\mathbf{F}_{\ell^2}^\times$ acts by the character χ; we let $\mathbf{T}^\chi$ denote the image of the Hecke algebra in the endomorphims of $H^1(\mathbf{X}, \mathcal{O})^\chi$.

The form f determines a character $\mathbf{T}^\chi \to k$ whose kernel is a maximal ideal $\mathfrak{m}$ of $\mathbf{T}^\chi$. We define $M = H^1(\mathbf{X}, \mathcal{O})^\chi_\mathfrak{m}$, $\mathbf{T} = \mathbf{T}^\chi_\mathfrak{m}$.

Let $\rho\colon \mathrm{Gal}(\bar{\mathbf{Q}}/\mathbf{Q}) \to \mathrm{GL}_2(\mathcal{O})$ be the Galois representation associated to f, $\bar{\rho}$ the semi-simplification of its reduction mod λ. We assume that $\bar{\rho}$ is absolutely irreducible and ramified at primes p dividing N. We impose the further conditions that $p^2 \not\equiv 1 \bmod \ell$ if p is a prime dividing Δ' such that $\bar{\rho}$ is unramified at p and that the centralizer of $\bar{\rho}|_{G_\ell}$ is trivial. Under these hypotheses on $\bar{\rho}$, we construct a Taylor–Wiles system consisting of quaternionic cohomological modules, which allows one to characterize $\mathbf{T}$ as the universal solution of a deformation problem for $\bar{\rho}$ and to assert that $\mathbf{T}$ is complete intersection (Theorem 3.1). This construction provides also the multiplicity one result for the module M. In order to define the right deformation condition at ℓ, we make use of the property of 'being weakly of type χ' for a deformation, introduced in [5]. At primes p dividing Δ' such that $\bar{\rho}$ is unramified at p we had to define a deformation condition which excludes deformations arising from modular forms unramified at p; if $p^2 \not\equiv 1 \bmod \ell$ we found that an appropriate condition is given by

$$\mathrm{trace}(\rho(F))^2 = (p+1)^2 \tag{1}$$

for a lift F of Frob_p in G_p. For a deformation to $\mathcal{O}/\lambda^n$ this condition is equivalent to that of being of the 'desired form' in the sense of Ramakrishna [23, Section 3].

Let Δ_1 be the set of primes p dividing Δ' such that $\bar{\rho}$ is ramified at p. By an abuse of notation, if S is a set of primes, we shall sometimes denote by S also the product of the primes in this set.

In Section 4 we assume the existence of a newform g in $S_2(\Gamma_0(\Delta_1\ell^2))$ supercuspidal of type χ at ℓ and such that $\bar{\rho}_g = \bar{\rho}$. In other words, we are assuming that the representation $\bar{\rho}$ occurs in type χ and minimal level. We choose a pair of disjoint finite sets S_1, S_2 of primes p such that $\ell \nmid p(p^2-1)\Delta_1$. We assume that Δ_1 is not empty. We slightly modify the deformation problem of $\bar{\rho}$ described above by imposing condition 1 for primes in S_2 and allowing ramification at primes in S_1; in this way we define a deformation ring $\mathcal{R}_{S_1,S_2}$ and a local Hecke algebra $\mathbf{T}_{S_1,S_2}$ acting on the forms which are supercuspidal of type χ at ℓ, special at each prime in S_2 and congruent to $g \bmod \ell$. By combining Theorem 3.1 with Theorem 5.4.2 of [5], we prove (Theorem 4.5) that the natural map $\mathcal{R}_{S_1,S_2} \to \mathbf{T}_{S_1,S_2}$ is an isomorphism of complete intersections. Let h be a newform in $S_2(\Gamma_0(\Delta_1 S_2 \ell^2))$ supercuspidal of type χ at ℓ and congruent to $g \bmod \ell$. Let $\theta_{h,S_1,S_2}\colon \mathbf{T}_{S_1,S_2} \to \mathcal{O}$ be the section associated to h and η_{h,S_1,S_2} be the corresponding congruence ideal. We show that

$$\eta_{h,S_1S_2,\emptyset} = \left(\prod_{p\in S_2} y_p(h)\right)\eta_{h,S_1,S_2}$$

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where $(y_p(h))$ is the ideal generated by the highest power λ^n of λ such that ρ_h is unramified $\bmod \lambda^n\mathcal{O}$. This ideal can be interpreted in terms of the group of components of the fiber at p of the Néron model of the abelian variety associated to h, so that the above formula gives a generalization of the main theorem of [26] and [33] in the 'type χ' context.

1. Shimura Curves and Cohomology

In this section, ℓ is a prime > 2.

Let Δ' be a product of an odd number of primes, different from ℓ. We put $\Delta = \ell\Delta'$. Let B be the indefinite quaternion algebra over $\mathbf{Q}$ of discriminant Δ. Let R be a maximal order in B. For a rational place v of $\mathbf{Q}$ we put $B_v = B \otimes_{\mathbf{Q}} \mathbf{Q}_v$; if p is a finite place we put $R_p = R \otimes_{\mathbf{Z}} \mathbf{Z}_p$; $B_{\mathbf{A}}$ denotes the adelization of B, $B_{\mathbf{A}}^{\times,\infty}$ the subgroup of finite ideles. The reduced norm and trace in B will be noted ν and t respectively; $\alpha \to \alpha^c$ is the principal involution in B. For every rational place v of $\mathbf{Q}$ not dividing Δ, we fix an isomorphism $i_v\colon B_v \xrightarrow{\sim} \mathrm{M}_2(\mathbf{Q}_v)$, such that $i_p(R_p) = \mathrm{M}_2(\mathbf{Z}_p)$, if $v = p$ is a finite place.

Let N be an integer prime to Δ. If p is a prime not dividing Δ we define

$$K_p^0(N) = i_p^{-1}\left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbf{Z}_p) \,\middle|\, c \equiv 0 \bmod N\right\},$$
$$K_p^1(N) = i_p^{-1}\left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in K_p^0(N) \,\middle|\, a \equiv 1 \bmod N\right\}.$$

For every p dividing N, let K_p be a subgroup of $B_p^\times$ such that $K_p^1(N) \subseteq K_p \subseteq K_p^0(N)$. Write $U = \prod_{p|N} K_p$. We define

$$V_0(N, U) = \prod_{p \nmid N} R_p^\times \times U, \qquad V_1(N, U) = \prod_{p \nmid N\ell} R_p^\times \times U \times (1 + u_\ell R_\ell),$$

where u_ℓ is a uniformizer of $B_\ell^\times$. For $i = 0, 1$, we define also

$$\Phi_i(N, U) = (\mathrm{GL}_2^+(\mathbf{R}) \times V_i(N, U)) \cap B^\times,$$

where $\mathrm{GL}_2^+(\mathbf{R}) = \{g \in \mathrm{GL}_2(\mathbf{R}) \mid \det g > 0\}$. There is an isomorphism

$$V_0(N, U)/V_1(N, U) \simeq \mathbf{F}_{\ell^2}^\times, \tag{2}$$

By this isomorphism $\Phi_0(N, U)/\Phi_1(N, U)$ is identified with the subgroup $G \subseteq \mathbf{F}_{\ell^2}^\times$ of order $\ell + 1$, namely the kernel of the norm from $\mathbf{F}_{\ell^2}^\times$ to $\mathbf{F}_\ell^\times$.

By strong approximation,

$$B_{\mathbf{A}}^\times = B^\times \mathrm{GL}_2^+(\mathbf{R}) V_0(N, U) = \coprod_{i=1}^{\ell-1} B^\times \mathrm{GL}_2^+(\mathbf{R}) t_i V_1(N, U),$$

where the t_i's are representatives in $R_\ell^\times$ of $R_\ell^\times/\{\alpha \in R_\ell^\times \mid \nu(\alpha) \equiv 1 \bmod \ell\} \simeq \mathbf{F}_\ell^\times$. Let $K_\infty^+ = \mathbf{R}^\times \mathrm{SO}_2(\mathbf{R})$. We define the Shimura curves

$$X_0'(N, U) = B^\times \backslash B_{\mathbf{A}}^\times / K_\infty^+ \times V_0(N, U), \qquad X_1'(N, U) = B^\times \backslash B_{\mathbf{A}}^\times / K_\infty^+ \times V_1(N, U).$$

The curve $X'_1(N, U)$ is not connected, since the reduced norm $\nu: 1 + u_\ell R_\ell \to \mathbf{Z}_\ell^\times$ is not surjective. Let $\mathcal{H}$ be the upper complex half-plane. The group $\mathrm{GL}_2^+(\mathbf{R})$ acts on $\mathcal{H}$ by linear fractional transformations. There are isomorphisms (see for example [Proposition 6.1(i)])

$$X'_0(N, U) \simeq \mathcal{H}/\Phi_0(N, U), \qquad X'_1(N, U) \simeq \coprod_{i=1}^{\ell-1} \mathcal{H}/\Phi_1(N, U).$$

We fix a character $\chi: \mathbf{F}_{\ell^2}^\times \to \bar{\mathbf{Q}}^\times$ satisfying the following conditions:

$$\chi|_{\mathbf{F}_\ell^\times} = 1; \tag{3}$$

$$\chi^2 \neq 1. \tag{4}$$

Condition 4 means that χ does not factor by the norm from $\mathbf{F}_{\ell^2}^\times$ to. $\mathbf{F}_\ell^\times$.

We fix embeddings of $\bar{\mathbf{Q}}$ in $\bar{\mathbf{Q}}_\ell$ and in $\mathbf{C}$ so that we can regard the values of χ in these fields.

Assume now that the group $\Phi_0(N, U)$ has no elliptic elements. Let $\mathbf{Q}_{\ell^2}$ denote the unramified quadratic extension of $\mathbf{Q}_\ell$, $\mathbf{Z}_{\ell^2}$ its ℓ-adic integer ring.

Let K be a finite extension of $\mathbf{Q}_{\ell^2}$. Let $\mathcal{O}$ be the ring of integers of K and λ be a uniformizer of $\mathcal{O}$.

Consider the projection $\pi: X'_1(N, U) \to X'_0(N, U)$. The group $\mathbf{F}_{\ell^2}^\times$ naturally acts on $H^*(X'_1(N, U), \mathcal{O})$ via its action on $X'_1(N, U)$. The cohomology group $H^1(X'_1(N, U), \mathcal{O})$ is also equipped with the action of Hecke operators T_p, for $p \neq \ell$ and diamond operators $\langle n \rangle$ for $n \in (\mathbf{Z}/N\mathbf{Z})^\times$ (for details, see [16, Section 6 and Section 7] and [37, Section 1.12]); if $p|\Delta'$, then the T_p operator is the operator on cohomology associated to the double coset $V_1(N, U)u_p V_1(N, U)$, where u_p is a uniformizer of $B_p^\times$. The Hecke action commutes with the action of $\mathbf{F}_{\ell^2}^\times$, since we do not have a T_ℓ operator. The two actions are $\mathcal{O}$-linear. Since $\mathcal{O}$ contains the $\ell^2 - 1$th roots of unity, and $|\mathbf{F}_{\ell^2}^\times|$ is invertible in $\mathcal{O}$, the action of $\mathbf{F}_{\ell^2}^\times$ decomposes according to the characters of $\mathbf{F}_{\ell^2}^\times$. We denote by $H^1(X'_1(N, U), \mathcal{O})^\chi$ the sub-Hecke module of $H^1(X'_1(N, U), \mathcal{O})$ on which $\mathbf{F}_{\ell^2}^\times$ acts by the character χ. It follows easily from the Hochschild–Serre spectral sequence that

$$H^*(X'_1(N, U), \mathcal{O})^\chi \simeq H^*(X'_0(N, U), \mathcal{O}(\chi)),$$

where $\mathcal{O}(\chi)$ is the sheaf $B^\times \backslash B_{\mathbf{A}}^\times \times \mathcal{O}/K_\infty^+ \times V_0(N, U)$, $B^\times$ acts on $B_{\mathbf{A}}^\times \times \mathcal{O}$ on the left by $\alpha \cdot (g, m) = (\alpha g, m)$ and $K_\infty^+ \times V_0(N, U)$ acts on the right by $(g, m) \cdot v = (gv, \chi(v_\ell)m)$. By translating to the cohomology of groups (see [17, Appendix]), we obtain

PROPOSITION 1.1. *$H^1(X'_1(N, U), \mathcal{O})^\chi \simeq H^1(\Phi_0(N, U), \mathcal{O}(\tilde{\chi}))$, where $\tilde{\chi}$ is the restriction of χ to G and $\mathcal{O}(\tilde{\chi})$ is $\mathcal{O}$ with the action of $\Phi_0(N, U)$ given by $a \mapsto \tilde{\chi}^{-1}(\gamma)a$.*

We give a description of the Hecke action on the group $H^1(\Phi_0(N, U), \mathcal{O}(\tilde{\chi}))$. Let $\alpha \in B_{\mathbf{A}}^{\times,\infty}$ be such that the coset $V_0(N, U)\alpha V_0(N, U)$ defines a Hecke operator. By strong approximation, we can write $\alpha = g_{\mathbf{Q}} g_\infty k$, with $g_{\mathbf{Q}} \in B^\times$, $g_\infty \in \mathrm{GL}_2^+(\mathbf{R})$,

$k \in V_0(N, U)$. Decompose $\Phi_0(N, U)g_{\mathbf{Q}}\Phi_0(N, U) = \coprod_i \Phi_0(N, U)h_i$, with $h_i \in B^\times$. Let $\xi\colon \Phi_0(N, U) \to \mathcal{O}(\tilde{\chi})$ be a cocycle; for $\gamma \in \Phi_0(N, U)$ write $h_i\gamma = \gamma_i h_{j(i)}$; and define

$$\xi|_{[\Phi_0(N,U)g_{\mathbf{Q}}\Phi_0(N,U)]}(\gamma) = \sum_i \chi(h_i)\xi(\gamma_i).$$

Then it is easy to see that $\xi|_{[\Phi_0(N,U)g_{\mathbf{Q}}\Phi_0(N,U)]}$ is a cocycle and that the action of $\Phi_0(N, U)g_{\mathbf{Q}}\Phi_0(N, U)$ on $H^1(\Phi_0(N, U), \mathcal{O}(\tilde{\chi}))$ corresponds to the action of $V_1(N, U)\alpha V_1(N, U)$ on $H^1(X_1'(N, U), \mathcal{O})^\chi$.

Let V be a compact open subgroup of $B_{\mathbf{A}}^{\times,\infty}$. We shall denote by $S_2(V)$ the space of weight 2 automorphic forms on $B_{\mathbf{A}}^\times$ which are right invariant for V (see, for example, [16, Section 2]). If $\psi\colon V \to \mathbf{C}^\times$ is a character with finite order, we shall denote by $S_2(V, \psi)$ the subspace of $S_2(\ker(\psi))$ consisting of the forms φ such that $\varphi(gk) = \psi(k)\varphi(g)$ for any $k \in V, g \in B_{\mathbf{A}}^\times$.

We now describe the structure of the module $H^1(X_1'(N, U), K)^\chi$ over the Hecke algebra. Let $\mathbf{T}_0^\chi(N, U)$ be the $\mathcal{O}$-algebra generated by the Hecke operators T_p, $p \neq \ell$ and the diamond operators, acting on $H^1(X_1'(N, U), \mathcal{O})^\chi$.

PROPOSITION 1.2. *$H^1(X_1'(N, U), K)^\chi$ is free of rank 2 over $\mathbf{T}_0^\chi(N, U) \otimes K$.*

Proof. Let $\mathbf{T}_0^\chi(N, U)_{\mathbf{C}}$ denote the algebra generated over $\mathbf{C}$ by the operators T_p, for $p \neq \ell$ and the diamond operators, acting on $H^1(X_1'(N, U), \mathbf{C})^\chi$. It suffices to show that $H^1(X_1'(N, U), \mathbf{C})^\chi$ is free of rank 2 over $\mathbf{T}_0^\chi(N, U)_{\mathbf{C}}$. We consider the space $S_2(V_1(N, U))$ of weight 2 automorphic forms on $B_{\mathbf{A}}^\times$ which are right invariant for $V_1(N, U)$. By the Matsushima–Shimura isomorphism ([20, §4], see also [16, §6])

$$H^1(X_1'(N, U), \mathbf{C}) \simeq S_2(V_1(N, U)) \oplus \overline{S_2(V_1(N, U))}$$

as Hecke and $\mathbf{F}_{\ell^2}^\times$-modules. By this isomorphism

$$H^1(X_0'(N, U), \mathbf{C}(\chi)) \xrightarrow{\sim} S_2(V_0(N, U), \chi) \oplus \overline{S_2(V_0(N, U), \bar{\chi})},$$

where $S_2(V_0(N, U), \chi)$ is the subspace of $S_2(V_1(N, U))$ consisting of forms φ such that $\varphi(gk) = \chi(k)\varphi(g)$ for all $g \in B_{\mathbf{A}}^\times$ and $k \in V_0(N, U)$. The space $S_2(V_1(N, U))$ decomposes as a direct sum of $V_1(N, U)$-invariants of admissible irreducible representations of $B_{\mathbf{A}}^\times$: $S_2(V_1(N, U)) = \bigoplus_\alpha W_\alpha'$. In an analogous way, there is a decomposition $S_2(\Gamma_0(\Delta'\ell^2) \cap \Gamma_1(N)) = \bigoplus_\beta W_\beta$, where the W_β's are subspaces of irreducible representations of $\mathrm{GL}_2(\mathbf{A})$, invariant by a suitable subgroup. The Jacquet–Langlands correspondence [18] associates injectively a W_α to each W_α'. Observe that

(a) if $p \nmid \Delta$, then the local components $W'_{\alpha,p}$ and $W_{\alpha,p}$ are isomorphic;
(b) if $p|\Delta'$ then $W_{\alpha,p}'$ and $W_{\alpha,p}$ are both one-dimensional, with the same eigenvalue of T_p;
(c) Let $(W_{\alpha,\ell}')^\chi$ be the subspace of $W_{\alpha,\ell}'$ on which $V_0(N, U)$ acts as the character χ; if $(W_{\alpha,\ell}')^\chi \neq 0$ then it is one-dimensional (the corresponding representation of $B_\ell^\times$ has dimension 2; its restriction to $R_\ell^\times$ has the form $\chi \oplus \chi^\sigma$, where σ is

the nontrivial element in $\mathrm{Gal}(\bar{\mathbf{Q}}_{\ell^2}/\mathbf{Q}_\ell)$, see [14, Section 5]). On the other hand, $W_{\alpha,\ell}$ is one-dimensional, with $T_\ell = 0$.

By the above analysis we see that there is an homomorphism

$$\mathrm{JL}: S_2(V_1(N, U)) \longrightarrow S_2(\Gamma_0(\Delta'\ell^2) \cap \Gamma_1(N))$$

satisfying $\mathrm{JL} \circ T_p = T_p \circ \mathrm{JL}$, for every $p \neq \ell$ and $\mathrm{JL} \circ \langle n \rangle = \langle n \rangle \circ \mathrm{JL}$ for every $n \in (\mathbf{Z}/N\mathbf{Z})^\times$ (for details see [37, Section 1]). By c) the restriction of JL to the space $S_2(V_0(N, U), \chi)$ is injective. Let $V_\chi = \mathrm{JL}(S_2(V_0(N, U), \chi))$; then there is an isomorphism between $\mathbf{T}_0^\chi(N, U)_\mathbf{C}$ and the Hecke algebra $\mathbf{T}(V_\chi)$ generated over $\mathbf{C}$ by the Hecke operators T_p with $p \neq \ell$ and $\langle n \rangle$ acting on V_χ. The latter is also equal to the Hecke algebra generated by all the Hecke operators, because the forms occurring in V_χ are supercuspidal at ℓ and so $T_\ell = 0$ on V_χ. In the same way we can deal with $\overline{S_2(V_0(N, U), \bar{\chi})}$. The space V_χ is a direct sum of W_α's; therefore it is a direct summand of $S_2(\Gamma_0(\Delta'\ell^2) \cap \Gamma_1(N))$ as a Hecke module. We can assume that B contains an element γ such that $i_\infty(\gamma) = \begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix}$ for some $a \in \mathbf{R}$. Let γ_∞ be the idèle having 1 in the finite part and γ at the infinite place. Then there is an isomorphism of Hecke modules $S_2(V_0(N, U), \chi) \xrightarrow{\sim} \overline{S_2(V_0(N, U), \bar{\chi})}$, defined by $\varphi \mapsto \bar{\psi}$, where $\psi(g) = \overline{\varphi(g\gamma_\infty)}$; therefore $\overline{V_{\bar{\chi}}} \simeq V_\chi$. Since we know that $H^1(X_1(N\Delta'\ell^2), \mathbf{C})$ is free of rank 2 over the Hecke algebra (see [31, Chap. III]), the result follows. □

2. The Deformation Problem

If K is a field, let $\bar{K}$ denote an algebraic closure of K; we put $G_K = \mathrm{Gal}(\bar{K}/K)$. For a local field K, K^{unr} denotes the maximal unramified extension of K in $\bar{K}$; we put $I_K = \mathrm{Gal}(\bar{K}/K^{\mathrm{unr}})$, the inertia subgroup of G_K. For a prime p we put $G_p = G_{\mathbf{Q}_p}$, $I_p = I_{\mathbf{Q}_p}$; we denote $W_{\mathbf{Q}_p}$, $WD_{\mathbf{Q}_p}$ the Weil group and the Weil–Deligne group over $\mathbf{Q}_p$ respectively, cf. [34]. If ρ is a representation of $G_\mathbf{Q}$ we write ρ_p for the restriction of ρ to a decomposition group at p.

In the rest of this paper, ℓ is a fixed prime > 3. We fix a character $\chi: \mathbf{F}_{\ell^2}^\times \to \bar{\mathbf{Q}}^\times$, trivial over $\mathbf{F}_\ell^\times$ and such that

$$2 < \mathrm{ord}(\chi) \leqslant \ell - 1. \tag{5}$$

By composing with the reduction $\bmod \ell$ we can view χ as a character of $\mathbf{Z}_{\ell^2}^\times \to \bar{\mathbf{Q}}^\times$ and extend it to $\mathbf{Q}_{\ell^2}^\times$ by putting $\chi(\ell) = -1$; the above conditions imply that χ is trivial over $\mathbf{Z}_\ell^\times$ and that it does not factor through the norm from $\mathbf{Q}_{\ell^2}^\times$ to $\mathbf{Q}_\ell^\times$. By [14, Section 3] we can associate to χ a supercuspidal representation $\pi_\ell(\chi)$ of $\mathrm{GL}_2(\mathbf{Q}_\ell)$ having conductor ℓ^2 and trivial central character. Let $WD(\pi_\ell(\chi))$ be the two-dimensional representation of the Weil–Deligne group at ℓ associated to $\pi_\ell(\chi)$ by local Langlands correspondence. Here we normalize $WD(\pi_\ell(\chi))$ by following the conventions in [2], but twisted by the character $|\ |_\ell^{-1}$. Then we have, by [2, Section 11.3], $WD(\pi_\ell(\chi)) = \mathrm{Ind}_{W_{\mathbf{Q}_{\ell^2}}}^{W_{\mathbf{Q}_\ell}}(\chi) \otimes |\ |_\ell^{-1/2}$.

Let $M \neq 1$ be a square-free integer not divisible by ℓ; and $f = \sum_{n=1}^{\infty} a_n(f)q^n$ be a normalized newform in $S_2(\Gamma_0(M\ell^2))$. Let $\pi_f = \bigotimes_v \pi_{f,v}$ be the automorphic representation of $\mathrm{GL}_2(\mathbf{A})$ associated to f; then $\pi_{f,p}$ is special, for all $p|M$. We assume that f is supercuspidal of type χ at ℓ, that is $\pi_{f,\ell} = \pi_\ell(\chi)$, (see [37, Section 3.16] for some conditions on M assuring that such a form f exists). Let $\rho_f: G_{\mathbf{Q}} \longrightarrow \mathrm{GL}_2(\bar{\mathbf{Q}}_\ell)$ be the Galois representation associated to f and $\bar{\rho}: G_{\mathbf{Q}} \longrightarrow \mathrm{GL}_2(\bar{\mathbf{F}}_\ell)$ be its reduction modulo ℓ.

We fix a factorization $M = N\Delta'$, where Δ' is a product of an odd number of primes. We impose the following conditions on $\bar{\rho}$:

$$\bar{\rho} \text{ is absolutely irreducible;} \tag{6}$$

$$\text{if } p|N, \text{ then } \bar{\rho}(I_p) \neq 1; \tag{7}$$

$$\text{if } p|\Delta' \text{ and } p^2 \equiv 1 \bmod \ell, \text{ then } \bar{\rho}(I_p) \neq 1; \tag{8}$$

$$\mathrm{End}_{\bar{\mathbf{F}}_\ell[G_\ell]}(\bar{\rho}_\ell) = \bar{\mathbf{F}}_\ell. \tag{9}$$

Let $K = K(f)$ be a finite extension of $\mathbf{Q}_\ell$ containing $\mathbf{Q}_{\ell^2}$ and the eigenvalues for f of all Hecke operators. Let $\mathcal{O}$ be the ring of integers of K, λ be a uniformizer of $\mathcal{O}$, $k = \mathcal{O}/(\lambda)$ be the residue field.

Let $\mathcal{B}$ denote the set of normalized newforms h in $S_2(\Gamma_0(M\ell^2))$ which are supercuspidal of type χ at ℓ and whose associated representation ρ_h is a deformation of $\bar{\rho}$. For $h \in \mathcal{B}$, let $h = \sum_{n=1}^{\infty} a_n(h)q^n$ be the q-expansion of h and let $\mathcal{O}_h$ be the $\mathcal{O}$-algebra generated in $\bar{\mathbf{Q}}_\ell$ by the Fourier coefficients of h. Let $\mathbf{T}$ denote the sub-$\mathcal{O}$-algebra of $\prod_{h\in\mathcal{B}} \mathcal{O}_h$ generated by the elements $\tilde{T}_p = (a_p(h))_{h\in\mathcal{B}}$ for $p \nmid M\ell$.

Our next goal is to state a deformation condition of $\bar{\rho}$ which is a good candidate for having $\mathbf{T}$ as universal deformation ring.

2.1. LOCAL DEFORMATIONS AT ℓ: THE TYPE τ

We use the terminology and the results in [5].

We can regard χ as a character of I_ℓ by local classfield theory:

$$I_\ell = \mathrm{Gal}(\overline{\mathbf{Q}}_\ell/\mathbf{Q}_\ell^{unr}) \to \mathrm{Gal}(\mathbf{Q}_{\ell^2}^{ab}/\mathbf{Q}_\ell^{unr}) \xrightarrow{\sim} \mathbf{Z}_{\ell^2}^\times \to \mathbf{F}_{\ell^2}^\times \xrightarrow{\chi} \bar{\mathbf{Q}}_\ell^\times.$$

Consider the type $\tau = \chi \oplus \chi^\sigma: I_\ell \to \mathrm{GL}_2(\bar{\mathbf{Q}}_\ell)$. The representation $\rho_{f,\ell}$ is of type τ, since by [27] or [5, Appendix B], $WD(\rho_{f,\ell}) \simeq \mathrm{Ind}_{W_{\mathbf{Q}_{\ell^2}}}^{W_{\mathbf{Q}_\ell}}(\chi) \otimes |\ |_\ell^{-1/2}$, so that $WD(\rho_{f,\ell})|_{I_\ell} \simeq \chi \oplus \chi^\sigma$; moreover $\rho_{f,\ell}$ is Barsotti–Tate over any finite extension L of $\mathbf{Q}_\ell$ such that $\chi|_{I_L}$ is trivial.

Let e be the order of χ. The kernel H of the above map is an open normal subgroup of I_ℓ, therefore it fixes a finite extension F' of $\mathbf{Q}_\ell^{unr}$. Then we have $F' = F \cdot \mathbf{Q}_\ell^{unr}$ for a finite extension F of $\mathbf{Q}_\ell$ of ramification index e. Since $I_F = \mathrm{Gal}(\overline{\mathbf{Q}}_\ell/F') = H$, χ is trivial over the inertia of F.

Then there is an ℓ-divisible group Γ over $\mathcal{O}_F$ with an action of $\mathcal{O}$ such that $\rho_{f,\ell}|_{G_F}$ is isomorphic to the representation defined by the action of $\mathcal{O}[G_F]$ on the Tate module

Λ of Γ. By [24, Cor. 3.3.6] the group scheme $G = \Gamma[\lambda] \simeq \Gamma[\ell]/\lambda\Gamma[\ell]$ is finite flat over $\mathcal{O}_F$. Since $\bar{\rho}_\ell|_{G_F}$ is isomorphic to $G(\overline{\mathbf{Q}}_\ell)$ as a $k[G_F]$-module, it is finite and flat over $\mathcal{O}_F$.

We show that G is connected. The canonical connected-étale sequence for Γ gives rise to an exact sequence $0 \to \Lambda^0 \to \Lambda \to \Lambda^{\text{ét}} \to 0$ of free $\mathcal{O}$-modules with an action of G_F, and I_F acts trivially on $\Lambda^{\text{ét}}$. Since $\rho_{f,\ell}$ is ramified, $\Lambda^0 \neq 0$. If $\operatorname{rank}_{\mathcal{O}}(\Lambda^0) = 1$, then by the exactness of the functor WD and the fact that $WD(\rho_{f,\ell}|_{G_F}) = WD(\rho_{f,\ell})|_{W_F}$, the representation $WD(\rho_{f,\ell})|_{W_F} = (\operatorname{Ind}_{W_{\mathbf{Q}_{\ell^2}}}^{W_{\mathbf{Q}_\ell}}(\chi) \otimes |\ |_\ell^{-1/2})|_{W_F}$ would have a one-dimensional subrepresentation of the form $\eta|\ |_\ell^{-1}$ where η is an unramified character of W_F with values in $\mathcal{O}^\times$, a contradiction. Then $\Lambda = \Lambda^0$ and thus Γ is connected, so that G is connected. A similar argument applied to the dual ℓ-divisible group Γ^D shows that the Cartier dual of G is also connected.

Suppose first that $\bar{\rho}_\ell$ is irreducible. In this case, by [30, Section 2], $\bar{\rho}_\ell|_{I_\ell} \simeq \omega_2^m \oplus \omega_2^{\ell m}$ where ω_2 is a fundamental character of level 2, $m \equiv 1 \bmod \ell - 1$ and $m \not\equiv 0 \bmod \ell + 1$. Replacing m by ℓm if necessary we can write $em \equiv a + \ell b \bmod \ell^2 - 1$, where $0 \leqslant b \leqslant a \leqslant \ell - 1$. By Raynaud's classification [24, Théorème 3.4.3] the $\mathcal{O}_F$-flatness condition gives the constraint $a, b \leqslant e$. Since $e|\ell + 1$ we have $a \equiv b \bmod e$ and thus either $a = b$ or $b = 0, a = e$. However, if $a = b$, then $e \equiv 2a \bmod \ell - 1$ and since $a \leqslant e < \ell - 1$, $e = 2a$, which implies $(\ell - 1)/2 \equiv 0 \bmod \ell - 1$, a contradiction. Therefore $em \equiv e \bmod \ell^2 - 1$.

Suppose that $\bar{\rho}_\ell$ is reducible. Then $\bar{\rho}_\ell \sim \begin{pmatrix} \eta\omega^m & * \\ 0 & \eta^{-1}\omega^n \end{pmatrix}$ where η is an unramified character of G_ℓ, ω is the cyclotomic character $\bmod \ell$ and $m + n \equiv 1 \bmod \ell - 1$. By $\mathcal{O}_F$-flatness and the connectedness of G and its Cartier dual, Raynaud's classification gives $ne \equiv i + 1 \bmod \ell - 1$ with $0 \leqslant i \leqslant e - 2$. If $\ell \not\equiv 1 \bmod 4$ or $e \neq (\ell + 1)/2$ or $n \neq (\ell - 1)/2 \bmod \ell - 1$ or $\eta(\mathrm{Frob}_\ell) \neq \pm 1$, then there is exactly one such representation (up to isomorphism).

Suppose finally

$$\ell \equiv 1 \bmod 4, \qquad e = m = \frac{\ell + 1}{2}, \qquad n = \frac{\ell - 1}{2}, \qquad \eta(\mathrm{Frob}_\ell) = \pm 1.$$

Then $\bar{\rho}_\ell \otimes \eta\omega^{-n} \sim \begin{pmatrix} \omega & * \\ 0 & 1 \end{pmatrix}$ becomes flat over the ring of integers of a tamely ramified extension of F, so that $*$ must be peu ramifié by [30, Section 2.8]. Then we see that in any case $\bar{\rho}_\ell$ is included in the classification of [4, Theorem 0.1]. Let $\mathcal{R}_{\mathcal{O},\ell}^\chi$ be the universal deformation ring for $\bar{\rho}_\ell$ with respect to the property of being weakly of type τ; by [5, Corollary 2.2.2] there is a surjective homomorphism of local $\mathcal{O}$-algebras

$$\mathcal{O}[[X]] \to \mathcal{R}_{\mathcal{O},\ell}^\chi. \tag{10}$$

2.2. LOCAL DEFORMATIONS AT PRIMES DIVIDING M

Let g be a weight two eigenform with trivial character such that $\bar{\rho}_g \sim \bar{\rho}$. By the results of Deligne, Langlands and Carayol [2], the local component $\pi_{g,p}$ is special of conductor p if and only if $\rho_g|_{I_p} \sim \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}$ with $*$ ramified. Hence if $\bar{\rho}(I_p) \neq 1$

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we get a suitable deformation condition at p by requiring the restriction to I_p to be unipotent (the condition of minimal ramification at p, [22]).

On the other hand, if $\bar{\rho}(I_p) = 1$ we have to rule out those deformations of $\bar{\rho}$ arising from modular forms which are not special at p.

We denote by $\mathcal{C}_{\mathcal{O}}$ the category of local complete Noetherian $\mathcal{O}$-algebras with residue field k. Let $\epsilon\colon G_p \to \mathbf{Z}_\ell^\times$ be the cyclotomic character and $\omega\colon G_p \to \mathbf{F}_\ell^\times$ be its reduction mod ℓ. The following lemma gives a characterization of the deformations of $\bar{\rho}_p$ in the unramified case, if $p^2 \not\equiv 1 \bmod \ell$:

LEMMA 2.1. *Let p be a prime such that $\ell \nmid p(p^2-1)$. Let $\bar{\rho}\colon G_p \to \mathrm{GL}_2(k)$ be an unramified representation. Assume that $\bar{\rho}(\mathrm{Frob}_p) = \pm\left(\begin{smallmatrix} p & 0 \\ 0 & 1\end{smallmatrix}\right)$. Then every deformation ρ of $\bar{\rho}$ over an $\mathcal{O}$-algebra $A \in \mathcal{C}_{\mathcal{O}}$ is strictly equivalent to an upper triangular representation ρ such that $\rho(I_p) \subseteq \left(\begin{smallmatrix} 1 & * \\ 0 & 1\end{smallmatrix}\right)$.*

Proof. Let $\mathfrak{m}_A$ be the maximal ideal of A. Since $\rho(I_p) \subseteq 1 + \mathfrak{m}_A$, the wild inertia group acts trivially. Let F be a lift of Frob_p in G_p, σ be a topological generator of I_p^{tame}. Since $p \not\equiv 1 \bmod \ell$ we see that ρ is strictly equivalent to a representation (which we denote by ρ again) such that $\rho(F)$ is diagonal: $\rho(F) = \left(\begin{smallmatrix} a & 0 \\ 0 & b\end{smallmatrix}\right)$ with $a \equiv \pm p$, $b \equiv \pm 1 \bmod \mathfrak{m}_A$. We prove that $\rho(\sigma)$ has the form $\left(\begin{smallmatrix} 1 & \delta \\ 0 & 1\end{smallmatrix}\right)$ for some $\delta \in \mathfrak{m}_A$. By induction on n write $\rho(\sigma) = \left(\begin{smallmatrix} 1 & \delta_n \\ 0 & 1\end{smallmatrix}\right) + N_n$, with $N_n \equiv 0 \bmod \mathfrak{m}_A^n$, $N_n = \left(\begin{smallmatrix} x_n & y_n \\ z_n & w_n\end{smallmatrix}\right)$. The relation $F\sigma F^{-1} \equiv \sigma^p \bmod I_p^{\mathrm{wild}}$ implies

$$\rho(F\sigma F^{-1}) = \left(\begin{pmatrix} 1 & \delta_n \\ 0 & 1 \end{pmatrix} + N_n\right)^p \equiv \begin{pmatrix} 1 & p\delta_n \\ 0 & 1 \end{pmatrix} + pN_n \bmod \mathfrak{m}_A^{n+1}$$

because $\left(\begin{smallmatrix} 1 & \delta_n \\ 0 & 1\end{smallmatrix}\right)$ and N_n commute $\bmod \mathfrak{m}_A^{n+1}$. The above equality, under the hypothesis $p^2 \not\equiv 1 \bmod \ell$, gives $x_n, w_n, z_n \in \mathfrak{m}_A^{n+1}$. □

By the previous lemma, every class of strict equivalence of deformations ρ of $\bar{\rho}_p$ over A with determinant ϵ is determined by a pair of elements (γ, δ) in $\mathfrak{m}_A$, given by

$$a = \pm p + \gamma, \qquad b = p/a, \qquad \rho(F) = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}, \qquad \rho(\sigma) = \begin{pmatrix} 1 & \delta \\ 0 & 1 \end{pmatrix},$$

satisfying

$$\begin{pmatrix} a & 0 \\ 0 & p/a \end{pmatrix}\begin{pmatrix} 1 & \delta \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a^{-1} & 0 \\ 0 & a/p \end{pmatrix} = \begin{pmatrix} 1 & p\delta \\ 0 & 1 \end{pmatrix},$$

that is $\gamma\delta = 0$. Moreover, two deformations ρ_1, ρ_2 corresponding to the pairs (γ_1, δ_1) and (γ_2, δ_2) respectively are strictly equivalent if and only if $\gamma_1 = \gamma_2$ and $\delta_2 = (1+m)\delta_1$, for some $m \in \mathfrak{m}_A$. Then we see that $\mathcal{R}'_p = \mathcal{O}[[X, Y]]/(XY)$ is the versal deformation ring of $\bar{\rho}$. If we assume $\bar{\rho}_p$ has been suitable diagonalized, then the versal deformation ρ^v over $\mathcal{R}'_p$ is such that

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$$\rho^v(F) = \begin{pmatrix} \pm p + X & 0 \\ 0 & p/(\pm p + X) \end{pmatrix}, \qquad \rho^v(\sigma) = \begin{pmatrix} 1 & Y \\ 0 & 1 \end{pmatrix}.$$

DEFINITION 2.2. Let p be a prime such that $\ell \nmid p(p^2-1)$ and $\bar{\rho}$ is unramified at p. We say that a deformation ρ of $\bar{\rho}|_{G_p}$ over a $\mathcal{O}$-algebra $A \in \mathcal{C}_{\mathcal{O}}$ satisfies the *sp-condition* if every homomorphism $\varphi: \mathcal{R}'_p \to A$ associated to ρ has $\varphi(X) = 0$.

It is immediate to see that the sp-condition is equivalent to the condition that

$$\mathrm{trace}(\rho(F))^2 = (p+1)^2$$

for a lift F of Frob_p in G_p.

Remark 2.3. There is some connection here with Ramakrishna's work [23, Section 3]. Though the application is quite different, a key role is played there by lifts of $\bar{\rho}|_{G_p}$ to quotients of $W(k)$ satisfying the condition of being of the 'desired form', which is equivalent to the sp-condition.

Remark 2.4. Suppose that $p|\Delta'$ and $\bar{\rho}$ is unramified at p; then by condition 8, $p^2 \not\equiv 1 \bmod \ell$. Let g be a modular form (weight 2, trivial Nebentypus) such that $\bar{\rho}_{g,p} \sim \bar{\rho}_p$. If g is special at p then $\rho_{g,p} \sim \psi \otimes \left(\begin{smallmatrix} \epsilon & * \\ 0 & 1 \end{smallmatrix}\right)$ with an unramified quadratic character ψ. Therefore $\rho_{g,p}$ satisfies the sp-condition. On the other hand, if g is not special at p then by Lemma 2.1 it must be principal unramified at p. Then the representation $\rho_{g,p}$ cannot satisfy the sp-condition: otherwise $a_p(g)^2 = (p+1)^2$, in contradiction with the Ramanujan–Petersson conjecture, proved by Deligne.

In the hypotheses of the above remark, we consider the deformations of $\bar{\rho}_p$ satisfying the sp-condition. This space includes the restrictions to G_p of representations coming from forms in $S_2(\Gamma_0(N\Delta'\ell^2))$ which are special at p, but it does not contain those coming from principal forms in $S_2(\Gamma_0(N\Delta'\ell^2))$. The corresponding versal ring is

$$\mathcal{O}[[X, Y]]/(X, XY) = \mathcal{O}[[Y]]. \tag{11}$$

2.3. THE GLOBAL DEFORMATION CONDITION

We let Δ_1 be the product of primes $p \mid \Delta'$ such that $\bar{\rho}(I_p) \neq 1$, and Δ_2 be the product of primes $p \mid \Delta'$ such that $\bar{\rho}(I_p) = 1$.

DEFINITION 2.5. Let Q be a square-free integer, prime to $M\ell$. We consider the functor $\mathcal{F}_Q$ from $\mathcal{C}_{\mathcal{O}}$ to the category of sets which associate to an object A in $\mathcal{C}_{\mathcal{O}}$ the set of strict equivalence classes of continuous homomorphisms ρ: $G_{\mathbf{Q}} \to \mathrm{GL}_2(A)$ lifting $\bar{\rho}$ and satisfying the following conditions:

(a_Q) ρ is unramified outside $MQ\ell$;
(b) if $p \mid \Delta_1 N$ then $\rho \mid_{I_p}$ is unipotent;

(c) if $p \mid \Delta_2$ then ρ_p satisfies the sp-condition;
(d) ρ_ℓ is weakly of type τ;
(e) $\det(\rho)$ is the cyclotomic character $\epsilon\colon G_{\mathbf{Q}} \to \mathbf{Z}_\ell^\times$.

PROPOSITION 2.6. *The functor $\mathcal{F}_Q$ is representable.*

Proof. By the hypothesis of absolute irreducibility of $\bar\rho$, there is in $\mathcal{C}_\mathcal{O}$ the universal deformation ring $\tilde{\mathcal{R}}_Q$ of $\bar\rho$ with condition (a_Q) [22, Section 20, Prop. 2 and Section 21]. Then we can use Proposition 6.1 in [8] for checking the representability of the deformation subfunctor $\mathcal{F}_Q$. Let $\mathcal{F}'_Q$ be the functor corresponding to conditions (a_Q), (b), (e). We know that it is representable (see, for example, [22, Section 29]). On the other hand one easily checks that the subset of deformations having properties (c) and (d) in Definition 2.5 satisfies the representability criterion in [8, Proposition 6.1]: then there is a closed ideal $\mathfrak{a}_Q$ of $\tilde{\mathcal{R}}_Q$ such that the ring $\mathcal{R}_Q = \tilde{\mathcal{R}}_Q/\mathfrak{a}_Q$ represents the functor $\mathcal{F}_Q$ in $\mathcal{C}_\mathcal{O}$. □

Let $\mathcal{R}_Q$ be the universal ring associated to the functor $\mathcal{F}_Q$. We put $\mathcal{F} = \mathcal{F}_\emptyset$, $\mathcal{R} = \mathcal{R}_\emptyset$.

3. Construction of a Taylor–Wiles System

We set $\Delta = \ell\Delta'$; let B be the indefinite quaternion algebra over $\mathbf{Q}$ of discriminant Δ. Let R be a maximal order in B.

It is convenient to choose an auxiliary prime $s \nmid M\ell$, $s > 3$ such that no lift of $\bar\rho$ can be ramified at s; such a prime exists by [11, Lemma 2]. With the notation of Section 1, we put $U = \prod_{p\mid N} K_p^0(N) \times K_s^1(s^2)$, $\Phi_0 = \Phi_0(Ns, U)$; it is easy to verify that the group Φ_0 has not elliptic elements.

There exists an eigenform $\tilde f$ in $S_2(\Gamma_0(Ms^2\ell^2))$ such that $\rho_f = \rho_{\tilde f}$ and $T_s \tilde f = 0$. By the Jacquet–Langlands correspondence, the form $\tilde f$ determines a character $\mathbf{T}_0^\chi(Ns^2, U) \longrightarrow k$ sending the operator t in the class mod λ of the eigenvalue of t for $\tilde f$. The kernel of this character is a maximal ideal $\mathfrak{m}$ in $\mathbf{T}_0^\chi(Ns^2, U)$. We define $M = H^1(X'_1(Ns^2, U), \mathcal{O})^\chi_\mathfrak{m}$. By combining Proposition 4.7 of [6.7] with the Jacquet–Langlands correspondence we see that there is a natural isomorphism $\mathbf{T} \simeq \mathbf{T}_0^\chi(Ns^2, U)_\mathfrak{m}$. Therefore by Proposition 1.2

$$M \otimes_\mathcal{O} K \text{ is free of rank 2 over } \mathbf{T} \otimes_\mathcal{O} K. \tag{12}$$

Since $\mathcal{R}$ is topologically generated by traces, the map $\mathcal{R} \to \prod_{h\in\mathcal{B}} \mathcal{O}_h$ has image $\mathbf{T}$. Thus there is a surjective homomorphism of $\mathcal{O}$-algebras $\Phi\colon \mathcal{R} \longrightarrow \mathbf{T}$.

Our goal is to prove the following

THEOREM 3.1. (a) *$\mathcal{R}$ is complete intersection of dimension* 1*;* (b) Φ: $\mathcal{R} \to \mathbf{T}$ *is an isomorphism;* (c) *M is a free* $\mathbf{T}$*-module of* rank 2.

In order to prove Theorem 3.1, we shall apply the Taylor–Wiles criterion in the version of Diamond [10] and Fujiwara [13].

We shall prove the existence of a family $\mathcal{Q}$ of finite sets Q of prime numbers, not dividing $M\ell$, and of an $\mathcal{R}_Q$-module M_Q for each $Q \in \mathcal{Q}$ such that the system $(\mathcal{R}_Q, M_Q)_{Q\in\mathcal{Q}}$ satisfies the following conditions:

(TWS1) For every $Q \in \mathcal{Q}$ and every $q \in Q$, $q \equiv 1 \bmod \ell$; for such a q, let Δ_q be the ℓ-Sylow of $(\mathbf{Z}/q\mathbf{Z})^\times$ and define $\Delta_Q = \prod_{q\in Q} \Delta_q$. Let I_Q be the augmentation ideal of $\mathcal{O}[\Delta_Q]$. Then $\mathcal{R}_Q$ is a local complete $\mathcal{O}[\Delta_Q]$-algebra and $\mathcal{R}_Q/I_Q\mathcal{R}_Q \simeq \mathcal{R}$;
(TWS2) M_Q is $\mathcal{O}[\Delta_Q]$-free of finite rank α independent of Q;
(TWS3) for every positive integer m there exists $Q_m \in \mathcal{Q}$ such that $q \equiv 1 \bmod \ell^m$ for any prime q in Q_m;
(TWS4) $r = |Q|$ does not depend on $Q \in \mathcal{Q}$;
(TWS5) for any $Q \in \mathcal{Q}$, $\mathcal{R}_Q$ is generated by at most r elements as local complete $\mathcal{O}$-algebra;
(TWS6) M_Q/I_QM_Q is isomorphic to M as $\mathcal{R}$ modules, for every $Q \in \mathcal{Q}$.

Then Theorem 3.1 will follow from the isomorphism criterion in [10, Theorem 2.1] and [13, Theorem 1.2].

3.1. THE ACTION OF Δ_Q ON $\mathcal{R}_Q$

Let Q be a finite set of prime numbers not dividing $N\Delta$ and such that

(A) $q \equiv 1 \bmod \ell, \forall q \in Q$;
(B) if $q \in Q$, $\bar\rho(\mathrm{Frob}_q)$ has distinct eigenvalues α_q, β_q contained in k.

Let $\tilde\alpha_q$ and $\tilde\beta_q$ be the two roots in $\mathcal{O}$ of the polynomial $X^2 - a_q(f)X + q$ reducing to α_q, β_q, respectively. Let Δ_q, Δ_Q, I_Q as in condition (TWS1) above. The ring $\mathcal{R}_Q$ defined in Section 2.3 is naturally equipped with a structure of $\mathcal{O}[\Delta_Q]$-module. In fact for every deformation ρ of $\bar\rho$ with determinant ϵ and for every $q \in Q$,

$$\rho|_{G_q} \sim \begin{pmatrix} \xi_q & 0 \\ 0 & \epsilon\xi_q^{-1} \end{pmatrix} \tag{13}$$

for some character ξ_q such that $\bar\xi_q(\mathrm{Frob}_q) = \alpha_q$, [36, Appendix, Lemma 7]. Let χ_q: $G_{\mathbf{Q}} \to \Delta_q$ be the composite of the cyclotomic character modulo q: $G_{\mathbf{Q}} \to (\mathbf{Z}/q\mathbf{Z})^\times$ and the projection on the ℓ-part $(\mathbf{Z}/q\mathbf{Z})^\times \to \Delta_q$; we put $\chi_Q = \prod_{q|Q} \chi_q$. The map $I_q \to \mathcal{R}_Q^\times$, $\sigma \mapsto \xi_q(\sigma)$, factors through χ_q: $\xi_q\,|_{I_q} = \phi_q \circ \chi_q\,|_{I_q}$, where ϕ_q is a character $\Delta_q \to 1 + \mathcal{M}_{\mathcal{R}_Q}$ ([7], Corollary 3). Consider the character $\tilde\phi = \prod_{q|Q} \phi_q^2$: $\Delta_Q \to \mathcal{R}_Q^\times$. Its $\mathcal{O}$-linearization gives the structural map $\mathcal{O}[\Delta_Q] \to \mathcal{R}_Q$.

PROPOSITION 3.2. *There is a canonical isomorphism $\mathcal{R}_Q/I_Q\mathcal{R}_Q \simeq \mathcal{R}$.*

Proof. The deformation associated to the quotient $\mathcal{R}_Q \to \mathcal{R}_Q/I_Q\mathcal{R}_Q$ is unramified at every $q \in Q$; properties (b)-(e) in Definition 2.5 are stable by quotients;

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therefore there exists a map $\mathcal{R} \to \mathcal{R}_Q/I_Q\mathcal{R}_Q$ which is the inverse of the evident map $\mathcal{R}_Q/I_Q\mathcal{R}_Q \to \mathcal{R}$. □

3.2. THE HECKE ALGEBRAS $\mathbf{T}_Q$

Let $\mathcal{B}_Q$ denote the set of new forms h of level dividing $MQ\ell^2$ special at primes p dividing M, supercuspidal of type χ at ℓ, such that $\bar{\rho}_h \sim \bar{\rho}$ and whose nebentypus ψ_h factors through the map $(\mathbf{Z}/MQ\ell^2\mathbf{Z})^\times \to \Delta_Q$. Let $\mathbf{T}_Q$ denote the sub-$\mathcal{O}$-algebra of $\prod_{h\in\mathcal{B}_Q}\mathcal{O}_h$ generated by the elements $\tilde{\tilde{T}}_p = (a_p(h))_{h\in\mathcal{B}_Q}$ for $p \nmid MQ\ell$ and $(\psi_h(n))_{h\in\mathcal{B}_Q}$ for $n \in \Delta_Q$. Then $\mathbf{T}_Q$ is naturally an $\mathcal{O}[\Delta_Q]$-algebra.

PROPOSITION 3.3. *There is a surjective homomorphism of* $\mathcal{O}[\Delta_Q]$-*algebras* $\Phi_Q\colon \mathcal{R}_Q \to \mathbf{T}_Q$.

Proof. For each h in $\mathcal{B}_Q$, the Galois representation ρ_h is a deformation of $\bar{\rho}$, unramified outside $MQ\ell$ and such that $\det(\rho_h(\mathrm{Frob}_p)) = \psi_h(p)p$, if $p \nmid MQ\ell$. By Čebotarev, $\det(\rho_h) = (\psi_h \circ \chi_Q)\epsilon$. Define $\rho'_h = (\psi_h \circ \chi_Q^{-1/2}) \otimes \rho_h$ (since Δ_Q is an ℓ-group, the square root makes sense). Then ρ'_h is a deformation of $\bar{\rho}$ unramified outside $MQ\ell$, with determinant ϵ. By the results of Deligne, Langlands and Carayol, if $p|M$, then

$$\rho_h|_{G_p} \sim \begin{pmatrix} \alpha^{-1}(\psi_h \circ \chi_Q)\epsilon & * \\ 0 & \alpha \end{pmatrix},$$

where α is an unramified character, $\alpha^2 = \psi_h \circ \chi_Q$, $*$ ramified. Therefore ρ'_h satisfies conditions b) and c) in definition 2.5. Since $\chi_Q|_{G_\ell}$ is unramified, the type of $\rho'_h|_{G_\ell}$ is equal to the type of $\rho_h|_{G_\ell}$; hence condition d) is fulfilled by ρ'_h. By the universality of $\mathcal{R}_Q$ there exists an homomorphism of $\mathcal{O}$-algebras $\Phi_Q\colon \mathcal{R}_Q \to \prod_{h\in\mathcal{B}_Q}\mathcal{O}_h$; since $\mathcal{R}_Q$ is generated by traces, the image of this homomorphism is in $\mathbf{T}_Q$. Again by Deligne–Langlands–Carayol, if $q|Q$ then

$$\rho_h|_{G_q} \sim \begin{pmatrix} \alpha^{-1}(\psi_h \circ \chi_q)\epsilon & 0 \\ 0 & \alpha \end{pmatrix},$$

where α is unramified. Therefore Φ_Q brings $\phi_q|_{I_q}$ to $\chi_q^{1/2}|_{I_q}$ and so it is $\mathcal{O}[\Delta_Q]$-linear and surjective. □

3.3. DEFINITION OF THE MODULES M_Q

If $q \in Q$, we put

$$K'_q = \left\{ \alpha \in R_q^\times | i_q(\alpha) \in \begin{pmatrix} H_q & * \\ q\mathbf{Z}_q & * \end{pmatrix} \right\},$$

where H_q is the subgroup of $(\mathbf{Z}/q\mathbf{Z})^\times$ consisting of elements of order prime to ℓ. We define

$$U_Q = \prod_{p|N} K_p^0(N) \times K_s^1(s^2) \times \prod_{q|Q} K'_q; \qquad \Phi'_Q = \Phi_0(NQs^2, U_Q);$$
$$V_Q = \prod_{p|NQ} K_p^0(NQ) \times K_s^1(s^2); \qquad \Phi_Q = \Phi_0(NQs^2, V_Q).$$

Then $\Phi_Q/\Phi'_Q \simeq \Delta_Q$ acts on $H^1(\Phi'_Q, \mathcal{O}(\tilde{\chi}))$. Consider the Hecke algebras $\mathbf{T}_0^{\chi}(NQs^2, U_Q)$ and $\mathbf{T}_0^{\chi}(NQs^2, V_Q)$ defined in Section 1. There is a natural surjection σ_Q: $\mathbf{T}_0^{\chi}(NQs^2, U_Q) \to \mathbf{T}_0^{\chi}(NQs^2, V_Q)$. Since the diamond operator $\langle n \rangle$ depends only on the image of n in Δ_Q, $\mathbf{T}_0^{\chi}(NQs^2, U_Q)$ is naturally an $\mathcal{O}[\Delta_Q]$-algebra.

As in [6, Sect. 4.2] we see that there exists a unique eigenform $\tilde{f}_Q \in S_2(\Gamma_0(MQs^2\ell^2))$ such that $\rho_{\tilde{f}_Q} = \rho_f$, $a_s(\tilde{f}_Q) = 0$, $a_q(\tilde{f}_Q) = \tilde{\beta}_q$ for $q|Q$.

By the Jacquet–Langlands correspondence, the form $\tilde{f}_Q$ determines a character θ_Q: $\mathbf{T}_0^{\chi}(NQs^2, V_Q) \to k$, sending T_p to $a_p(\tilde{f}_Q) \bmod \lambda$ and the diamond operators to 1. We define

$$\tilde{\mathfrak{m}}_Q = \ker\theta_Q, \qquad \mathfrak{m}_Q = \sigma_Q^{-1}(\tilde{\mathfrak{m}}_Q), \quad \text{and} \quad M_Q = H^1(\Phi'_Q, \mathcal{O}(\tilde{\chi}))_{\mathfrak{m}_Q}.$$

Then the map σ_Q induces a surjective homomorphism $\mathbf{T}_0^{\chi}(NQs^2, U_Q)_{\mathfrak{m}_Q} \to T_0^{\chi}(NQs^2, V_Q)_{\tilde{\mathfrak{m}}_Q}$ whose kernel contains $I_Q(\mathbf{T}_0^{\chi}(NQs^2, U_Q))_{\mathfrak{m}_Q}$. By combining the Jacquet–Langlands correspondence with the discussion in Section 4.2 of [6] we obtain:

PROPOSITION 3.4. *There is an isomorphism of $\mathcal{O}[\Delta_Q]$-algebras*

$$\mathbf{T}_Q \simeq (\mathbf{T}_0^{\chi}(NQs^2, U_Q))_{\mathfrak{m}_Q}$$

sending $\tilde{T}_p$ to T_p for each prime p not dividing $MQs\ell$.

PROPOSITION 3.5. (*a*) *M_Q is free over $\mathcal{O}[\Delta_Q]$;* (*b*) *$M_Q/I_QM_Q = H^1(\Phi_Q, \mathcal{O}(\tilde{\chi}))_{\tilde{\mathfrak{m}}_Q}$;* (*c*) *$rk_{\mathcal{O}[\Delta_Q]}M_Q$ does not depend on Q;* (*d*) *There is an isomorphism of $\mathcal{R}$-modules $M_Q/I_QM_Q \simeq M$.*

Proof. (a) We shall prove that $H^1(\Phi'_Q, \mathcal{O}(\tilde{\chi}))$ is free over $\mathcal{O}[\Delta_Q]$. Remark that $H^i(\Phi_Q, \mathcal{O}(\tilde{\chi})) = H^i(\Phi'_Q, \mathcal{O}(\tilde{\chi})) = 0$ if $i \neq 1$: in fact $H^0 = 0$ since $\tilde{\chi}$ is nontrivial. By [31, Props. 8.1 and 8.2], if $G = \Phi_Q$ or Φ'_Q, $H^2(G, \mathcal{O}(\tilde{\chi})) = \mathcal{O}/I$ where I is the $\mathcal{O}$-ideal generated by $\tilde{\chi}(\gamma) - 1$ for all $\gamma \in G$. Since $\tilde{\chi}$ is not trivial and $\mathrm{Im}(\tilde{\chi})$ consists of $\ell^2 - 1th$ roots of unity, $I = \mathcal{O}$, so that $H^2(G, \mathcal{O}(\tilde{\chi})) = 0$. Moreover, if $i > 2$, $H^i = 0$, because Φ_Q and Φ'_Q have cohomological dimension 2, [29, Prop. 18.a]). Since $H^0(\Phi'_Q, K/\mathcal{O}(\tilde{\chi})) = 0$, $H^1(\Phi'_Q, \mathcal{O}(\tilde{\chi}))$ is free over $\mathcal{O}$. Then it suffices to prove that $H^i(\Delta_Q, H^1(\Phi'_Q, \mathcal{O}(\tilde{\chi})) = 0$ if $i > 0$ (see for example [1, VI.8.10]). Recall the Hochschild–Serre spectral sequence: $E_2^{p,q} = H^p(\Delta_Q, H^q(\Phi'_Q, \mathcal{O}(\tilde{\chi}))) \Rightarrow H^n(\Phi_Q, \mathcal{O}(\tilde{\chi}))$; by the previous considerations $E_2^{p,q} = 0$ if $q \neq 1$. Therefore $E_\infty^{p,q} = E_2^{p,q}$. Since $H^n(\Phi_Q, \mathcal{O}(\tilde{\chi})) = 0$ if $n > 1$, we obtain $E_2^{p,q} = 0$ if $p > 0$.

(b) We have $M_Q/I_QM_Q = H_0(\Delta_Q, H^1(\Phi'_Q, \mathcal{O}(\tilde{\chi})))_{\mathfrak{m}_Q}$. We have proved in (a) that the Δ_Q-module $N = H^1(\Phi'_Q, \mathcal{O}(\tilde{\chi}))$ is cohomologically trivial; from the exact sequence

$$0 \to \hat{H}^{-1}(\Delta_Q, N) \to H_0(\Delta_Q, N) \to H^0(\Delta_Q, N) \to \hat{H}^0(\Delta_Q, N) \to 0$$

we deduce

$$H_0(\Delta_Q, N) \simeq H^0(\Delta_Q, N) = H^0(\Delta_Q, H^1(\Phi'_Q, \mathcal{O}(\tilde{\chi}))).$$

Again by the Hochschild–Serre spectral sequence the latter is isomorphic to $H^1(\Phi_Q, \mathcal{O}(\tilde{\chi}))$. The trace map in the sequence is compatible with the Hecke operators, as is the map $H^1(\Phi_Q, \mathcal{O}(\tilde{\chi})) \to H^1(\Phi'_Q, \mathcal{O}(\tilde{\chi}))$, so that, after localization, we get the result.

(c) It is sufficient to show that the rank over $\mathcal{O}$ of the module $M_Q/I_Q M_Q = H^1(\Phi_Q, \mathcal{O}(\tilde{\chi}))_{\tilde{\mathfrak{m}}_Q}$ does not depend on Q. Let $\mathcal{B}'_Q$ be the set of forms in $\mathcal{B}_Q$ with trivial character. By Proposition 1.2

$$\mathrm{rank}_{\mathcal{O}} H^1(\Phi_Q, \mathcal{O}(\tilde{\chi}))_{\tilde{\mathfrak{m}}_Q} = 2\dim_K(\mathbf{T}_0^{\chi}(NQs^2, V_Q)_{\tilde{\mathfrak{m}}_Q} \otimes_{\mathcal{O}} K) = 2|\mathcal{B}'_Q|.$$

By 13 every form in $\mathcal{B}'_Q$ is principal at each q dividing Q; therefore it is unramified at these places, by [19]. So these forms are Q-old; therefore $\mathcal{B}'_Q = \mathcal{B}$ and $\mathrm{rank}_{\mathcal{O}}(M_Q/I_Q M_Q) = \mathrm{rank}_{\mathcal{O}}(M)$.

(d) We show that if $Q' = Q \cup \{q\}$ there is an isomorphism of $\mathcal{R}$-modules $M_{Q'}/I_{Q'}M_{Q'} \simeq M_Q/I_Q M_Q$. Let $e \in \mathbf{T}_0^{\chi}(NQ's^2, V_{Q'})$ be the projection on the $\tilde{\mathfrak{m}}_{Q'}$-component. We define a map of $\mathcal{R}$-modules:

$$\begin{aligned} H^1(\Phi_Q, \mathcal{O}(\tilde{\chi}))_{\tilde{\mathfrak{m}}_Q} &\longrightarrow H^1(\Phi_{Q'}, \mathcal{O}(\tilde{\chi}))_{\tilde{\mathfrak{m}}_{Q'}} \\ x &\longmapsto e(\mathrm{res}_{\Phi_Q/\Phi_{Q'}} x) \end{aligned}$$

By (c) and Nakayama's lemma, it is an isomorphism if and only if its reduction modulo λ

$$F\colon\ H^1(\Phi_Q, k(\tilde{\chi}))_{\tilde{\mathfrak{m}}_Q} \longrightarrow H^1(\Phi_{Q'}, k(\tilde{\chi}))_{\tilde{\mathfrak{m}}_{Q'}}$$

is injective.

Notice that the restriction map $\mathrm{res}_{\Phi_Q/\Phi_{Q'}}$ is injective on $H^1(\Phi_Q, k(\tilde{\chi}))$, because $\ell \nmid q+1$.

Let η_q be the idèle in $B_{\mathbf{A}}^{\times}$ defined by $\eta_{q,v} = 1$ if $v \neq q$ and $\eta_{q,q} = i_q^{-1}\left(\begin{smallmatrix} q & 0 \\ 0 & 1 \end{smallmatrix}\right)$. By strong approximation, write $\eta_q = \delta_q g_\infty u$ with $\delta_q \in B^{\times}, g_\infty \in \mathrm{GL}_2^+(\mathbf{R}), u \in V_0(NQ's^2, V_{Q'})$. We define a map

$$\begin{aligned} H^1(\Phi_Q, \mathcal{O}(\tilde{\chi})) &\longrightarrow H^1(\Phi_{Q'}, \mathcal{O}(\tilde{\chi})) \\ x &\longmapsto x|\eta_q \end{aligned}$$

as follows: let ξ be a cocycle representing the cohomology class x in $H^1(\Phi_Q, \mathcal{O}(\tilde{\chi}))$; then $x|\eta_q$ is represented by the cocycle $\xi'(\gamma) = \chi(\delta_q) \cdot \xi(\delta_q \gamma \delta_q^{-1})$. It is straightforward to see that $T_p(x\,|\,\eta_q) = (T_p(x)\,|\,\eta_q)$ if $p \nmid MQ'\ell$, that $T_q(x\,|\,\eta_q) = q\,\mathrm{res}_{\Phi_Q/\Phi_{Q'}} x$, and that $T_q(\mathrm{res}_{\Phi_Q/\Phi_{Q'}} x) = \mathrm{res}_{\Phi_Q/\Phi_{Q'}}(T_q(x)) - x\,|\,\eta_q$.

Let $x \in H^1(\Phi_Q, k(\tilde{\chi}))_{\tilde{\mathfrak{m}}_Q}$. Since $T_q - a_q \in \tilde{\mathfrak{m}}_Q$, there is a smallest integer n such that $(T_q - a_q)^n(x) = 0$. By induction on n, we show that $F(x) = 0$ implies $x = 0$. If $n = 1$ then x is an eigenvector for T_q. Then it is easy to see that

$$(T_q - \beta_q)(\beta_q \mathrm{res}_{\Phi_Q/\Phi_{Q'}} x - x|\eta_q) = 0 \quad \text{and} \quad (T_q - \alpha_q)(\alpha_q \mathrm{res}_{\Phi_Q/\Phi_{Q'}} x - x|\eta_q) = 0,$$

so that

$$F(x) = \frac{1}{\beta_q - \alpha_q}(\beta_q\,\mathrm{res}_{\Phi_Q/\Phi_{Q'}} x - x|\eta_q).$$

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Assume that $\beta_q \mathrm{res}_{\Phi_Q/\Phi_{Q'}} x = x|\eta_q$. Decompose the double coset $\Phi_Q\delta_q\Phi_Q = \coprod_{i=1}^{q+1} \Phi_Q\delta_q h_i$ with $h_i \in \Phi_Q$. If $\gamma \in \Phi_{Q'}$, we put $\delta_q h_i \gamma = \gamma_i \delta_q h_{j(i)}$, where $\gamma_i \in \Phi_Q$. Then we have $h_i \gamma h_{j(i)}^{-1} \in \delta_q^{-1}\Phi_Q\delta_q \cap \Phi_Q = \Phi_{Q'}$. Let ξ be a cocycle representing x in $Z^1(\Phi_Q, k(\tilde{\chi}))$. Then

$$(T_q\xi)(\gamma) = \sum_i \chi(h_i\delta_q)\xi(\gamma_i) = \sum_i \chi(h_i\delta_q)\xi(\delta_q h_i \gamma h_{j(i)}^{-1}\delta_q^{-1})$$
$$= \sum_i \chi(h_i)\xi|\eta_q(h_i\gamma h_{j(i)}^{-1}),$$

and the latter is cohomologous to $\beta_q \sum_i \chi(h_i)\xi(h_i\gamma h_j^{-1})$. From the cocycle relation we know that $\chi(h_i)\xi(h_i\gamma h_j^{-1}) = \xi(\gamma) + \chi^{-1}(\gamma)\xi(h_j^{-1}) - \xi(h_i^{-1})$. Since $i \mapsto j(i)$ is a permutation of $\{1, \ldots, q+1\}$ we find

$$(T_q\xi)(\gamma) = \beta_q(q+1)\xi(\gamma) + \beta_q\Big(\sum_i \chi^{-1}(\gamma)\xi(h_i^{-1}) - \xi(h_i^{-1})\Big).$$

The sum on the right side is a coboundary, so that

$$\mathrm{res}_{\Phi_Q/\Phi_{Q'}} T_q x = 2\beta_q \mathrm{res}_{\Phi_Q/\Phi_{Q'}} x,$$

since $q \equiv 1 \bmod \ell$. This shows that $a_q - 2\beta_q$ kills $\mathrm{res}_{\Phi_Q/\Phi_{Q'}} x$. Since α_q and β_q are distinct $\bmod \ell$, $a_q - 2\beta_q$ is a unit, so $\mathrm{res}_{\Phi_Q/\Phi_{Q'}} x = 0$ and thus $x = 0$.

Suppose now the result to be true for n and $(T_q - a_q)^{n+1}x = 0$ with $F(x) = 0$. Then $eT_q(x|\eta_q) = 0$ and so $e(x|\eta_q) = 0$, since $T_q \notin \tilde{\mathfrak{m}}_{Q'}$. Let $y = (T_q - a_q)x$. Then $(T_q - a_q)^n(y) = 0$ and $F(y) = F(T_q(x)) = T_q(F(x)) + e(x|\eta_q) = 0$. By induction hypothesis $y = 0$, so that x is an eigenvector for T_q and the above argument shows that $x = 0$. □

3.4. CALCULATIONS ON SELMER GROUPS

Propositions 3.2 and 3.5 show that if $\mathcal{Q}$ is a family of finite sets Q of primes satisfying conditions (A) and (B), then conditions TWS1, TWS2 and TWS6 hold for the system $(\mathcal{R}_Q, M_Q)_{Q\in\mathcal{Q}}$. The existence of a family $\mathcal{Q}$ realizing simultaneously conditions (TWS3), (TWS4), (TWS5) is proved by the same methods as in [6, Section 6 and Theorem 2.49] or [7, Sections 4, 5]; we confine ourselves to show that in our situation the dimensions of the cohomological subgroups defining the local conditions at $p \mid \Delta_2\ell$ allow one to apply that technique.

We let $ad^0\bar{\rho}$ denote the subrepresentation of the adjoint representation of $\bar{\rho}$ over the space of the trace-0 endomorphisms. Local deformation conditions (a_Q), (b), (c), (d) in Section 2.3 allow one to define for each place v of $\mathbf{Q}$, a subgroup L_v of $H^1(G_v, ad^0\bar{\rho})$, see [22, Section 23]. If p divides Δ_2, L_p is the kernel of the restriction map to $H^1(\langle F\rangle, ad^0\bar{\rho})$, for a lift F of Frob_p in G_p. Then

- $\dim_k L_p = 1$ (formula 11)
- $\dim_k H^0(G_p, ad^0\bar{\rho}) = 1$, because the eigenvalues of $\bar{\rho}(\mathrm{Frob}_p)$ are distinct, by hypothesis 8.

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By Conrad's result (formula 10),

- $\dim_k L_\ell = 1$; and
- $\dim_k H^0(G_\ell, ad^0\bar{\rho}) = 0$, because of hypothesis 9

Theorem 3.1 is then proved.

4. The Quotient between Classical and Quaternionic Congruence Ideals

Let Δ_1 be a set of primes, disjoint from ℓ. By an abuse of notation, we shall sometimes denote by Δ_1 also the product of the primes in this set.

Let g be a newform in $S_2(\Gamma_0(\Delta_1\ell^2))$ supercuspidal of type χ at ℓ. As above, let τ be the type $\tau = (\chi \oplus \chi^\sigma)|_{I_\ell}$. Let $\bar{\rho} = \bar{\rho}_g$: $G_{\mathbf{Q}} \to k, k \subseteq \bar{\mathbf{F}}_{\ell 1}$ be the residue representation associated to g and suppose that $\bar{\rho}$ is ramified at every prime in Δ_1. In other words, we are assuming that the representation $\bar{\rho}$ occurs with type τ and minimal level. This happens for example if the type τ is 'strongly acceptable' for $\bar{\rho}$ in the sense of Conrad, Diamond and Taylor [5, pp. 524–525 and Proposition 5.4.1].

We assume that the character χ satisfies conditions (3) and (4) in Section 1, that $\bar{\rho}$ is absolutely irreducible and that $\bar{\rho}_\ell$ has a trivial centralizer.

Let Δ_2 be a finite set of primes p, not dividing $\Delta_1\ell$ such that $p^2 \not\equiv 1 \bmod \ell$ and $\mathrm{trace}(\bar{\rho}(\mathrm{Frob}_p))^2 \equiv (p+1)^2 \bmod \ell$. We let $\mathcal{B}_{\Delta_2}$ denote the set of new forms h of weight 2, trivial character and level dividing $\Delta_1\Delta_2\ell$ which are special at Δ_1, supercuspidal of type χ at ℓ and such that $\bar{\rho}_h = \bar{\rho}$. We choose an ℓ-adic ring $\mathcal{O}$ with residue field k, sufficiently large, so that every representation ρ_h for $h \in \mathcal{B}_{\Delta_2}$ is defined over $\mathcal{O}$. For every pair of disjoint subsets S_1, S_2 of Δ_2 we denote by $\mathcal{R}_{S_1,S_2}$ the universal solution over $\mathcal{O}$ for the deformation problem of $\bar{\rho}$ consisting of deformations ρ satisfying

(a) ρ is unramified outside $\Delta_1 S_1 S_2 \ell$;
(b) if $p|\Delta_1$ then $\rho|_{I_p}$ is unipotent;
(c) if $p|S_2$ then ρ_p satisfies the sp-condition;
(d) ρ_ℓ is weakly of type τ;
(e) $\det(\rho)$ is the cyclotomic character $\epsilon: G_{\mathbf{Q}} \to \mathbf{Z}_\ell^\times$.

Let $\mathcal{B}_{S_1,S_2}$ be the set of newforms in $\mathcal{B}_{\Delta_2}$ of level dividing $\Delta_1 S_1 S_2 \ell$ which are special at S_2 and let $\mathbf{T}_{S_1,S_2}$ be the sub-$\mathcal{O}$-algebra of $\prod_{h \in \mathcal{B}_{S_1,S_2}} \mathcal{O}$ generated by the elements $\tilde{T}_p = (a_p(h))_{h \in \mathcal{B}_{S_1,S_2}}$ for p not in $\Delta_1 \cup S_1 \cup S_2 \cup \{\ell\}$. Since $\mathcal{R}_{S_1,S_2}$ is generated by traces, we know that there exists a surjective homomorphism of $\mathcal{O}$-algebras $\mathcal{R}_{S_1,S_2} \to \mathbf{T}_{S_1,S_2}$. Moreover, Theorem 5.4.2 of [5] shows that $\mathcal{R}_{S_1,\emptyset} \to \mathbf{T}_{S_1,\emptyset}$ is an isomorphism of complete intersections, for any subset S_1 of Δ_2 (In section 2.1 we verified that the type τ is acceptable for $\bar{\rho}$; in [5] the further hypothesis that τ is strongly acceptable for $\bar{\rho}$ is made in order to prove that $\mathcal{B}_\emptyset \neq \emptyset$, but we can do without it, since we are already assuming the existence of g).

If $\Delta_1 \neq 1$, then each $\mathbf{T}_{\emptyset,S_2}$ acts on a local component of the cohomology of a suitable Shimura curve, obtained by taking an indefinite quaternion algebra of discriminant $S_2\ell$ or $S_2\ell p$ for a prime p in Δ_1. Therefore Theorem 3.1 gives the following

COROLLARY 4.1. *Suppose that* $\Delta_1 \neq 1$ *and that* $\mathcal{B}_{\emptyset,S_2} \neq \emptyset$*; then the map* $\mathcal{R}_{\emptyset,S_2} \to \mathbf{T}_{\emptyset,S_2}$ *is an isomorphism of complete intersections.*

If $p \in S_2$ there is a commutative diagram

$$\begin{array}{ccc} \mathcal{R}_{S_1p,S_2/p} & \longrightarrow & \mathcal{R}_{S_1,S_2} \\ \downarrow & & \downarrow \\ \mathbf{T}_{S_1p,S_2/p} & \longrightarrow & \mathbf{T}_{S_1,S_2} \end{array}$$

where all the arrows are surjections.

For every p dividing Δ_2 the deformation over $\mathcal{R}_{\Delta_2,\emptyset}$ restricted to G_p gives maps $\mathcal{R}'_p = \mathcal{O}[[X, Y]]/(XY) \longrightarrow \mathcal{R}_{\Delta_2,\emptyset}$ as explained in Section 2.2. The image x_p of X and the ideal (y_p) generated by the image y_p of Y in $\mathcal{R}_{\Delta_2,\emptyset}$ do not depend on the choice of the map. By an abuse of notation, we shall call x_p, y_p also the images of x_p, y_p in every quotient of $\mathcal{R}_{\Delta_2,\emptyset}$. If h is a form in $\mathcal{B}_{\Delta_2,\emptyset}$, we denote by $x_p(h), y_p(h) \in \mathcal{O}$ the images of x_p, y_p by the map $\mathcal{R}_{\Delta_2,\emptyset} \to \mathcal{O}$ corresponding to ρ_h.

LEMMA 4.2 *If* $h \in \mathcal{B}_{\Delta_2}$ *and* $p \mid \Delta_2$, *then*

(a) $x_p(h) = 0$ *if and only if h is special at p;*
(b) *if h is unramified at p then* $(x_p(h)) = (a_p(h)^2 - (p+1)^2)$*;*
(c) $y_p(h) = 0$ *if and only if h is unramified at p;*
(d) *if h is special at p, the order at* (λ) *of* $y_p(h)$ *is the greatest positive integer n such that* $\rho_h(I_p) \equiv 1 \bmod \lambda^n$.

Proof. It is an immediate consequence of the discussion in Section 2.2. Statement (b) follows from the fact that

$$a_p(h) = \mathrm{trace}(\rho_h(\mathrm{Frob}_p)) = \pm p + x_p(h) + \frac{p}{\pm p + x_p(h)} \qquad \square$$

LEMMA 4.3. *For every pair of disjoint subsets* S_1, S_2 *of* Δ_2 *and for every* $p \in S_1$

(a) *the map* $\mathcal{R}_{S_1,S_2} \to \mathcal{R}_{S_1/p,S_2p}$ *has* kernel (x_p)*;*
(b) *the map* $\mathcal{R}_{S_1,S_2} \to \mathcal{R}_{S_1/p,S_2}$ *has* kernel (y_p).

Proof. The deformation over $R_{S_1,S_2}/(x_p)$ satisfies the sp-condition at p; thus there is a map $\mathcal{R}_{S_1/p,S_2p} \to \mathcal{R}_{S_1,S_2}/(x_p)$; on the other hand the map $\mathcal{R}_{S_1,S_2} \to \mathcal{R}_{S_1/p,S_2p}$ kills x_p and so it induces a map $\mathcal{R}_{S_1,S_2}/(x_p) \to \mathcal{R}_{S_1/p,S_2p}$. By universality the two maps are inverse each other. An analogous argument holds for assertion b), by replacing x_p by y_p and the sp-condition by the condition of being unramified at p. $\square$

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If h is a form in $\mathcal{B}_{S_1,S_2}$ then there is a character θ_{h,S_1,S_2}: $T_{S_1,S_2} \to \mathcal{O}$ corresponding to h; we denote by π_{h,S_1,S_2}: $R_{S_1,S_2} \to \mathcal{O}$ the composition of θ_{h,S_1,S_2} with the map $R_{S_1,S_2} \to T_{S_1,S_2}$ and by $\mathcal{P}_{h,S_1,S_2}$ the kernel of π_{h,S_1,S_2}.

LEMMA 4.4. *Suppose that p divides S_1 and h belongs to $\mathcal{B}_{S_1/p,S_2p}$. Then*

$$\text{length}_{\mathcal{O}}\left(\frac{\mathcal{P}_{h,S_1,S_2}}{(\mathcal{P}_{h,S_1,S_2})^2}\right) \leqslant \text{length}_{\mathcal{O}}\left(\frac{\mathcal{P}_{h,S_1/p,S_2p}}{(\mathcal{P}_{h,S_1/p,S_2p})^2}\right) + \text{length}_{\mathcal{O}}\left(\frac{\mathcal{O}}{(y_p(h))}\right).$$

Proof. There is a surjective homomorphism

$$\varphi\colon \frac{\mathcal{P}_{h,S_1,S_2}}{(\mathcal{P}_{h,S_1,S_2})^2} \to \frac{\mathcal{P}_{h,S_1/p,S_2p}}{(\mathcal{P}_{h,S_1/p,S_2p})^2}$$

induced by $\mathcal{R}_{S_1,S_2} \to \mathcal{R}_{S_1/p,S_2p}$. By point a) of Lemma 4.3 the kernel of φ is the $\mathcal{O}$-module generated by the image $\tilde{x}_p$ of x_p in $\mathcal{P}_{h,S_1,S_2}/(\mathcal{P}_{h,S_1,S_2})^2$. We choose a map φ_p: $\mathcal{R}'_p \simeq \mathcal{O}[[X, Y]]/(XY) \to \mathcal{R}_{S_1,S_2}$ associated to the restriction to G_p of the universal deformation over $\mathcal{R}_{S_1,S_2}$. Let $\mathcal{P}_p = \varphi_p^{-1}(\mathcal{P}_{h,S_1,S_2})$ be the kernel of $\pi_{h,S_1,S_2} \circ \varphi_p$ and let $\tilde{X}$ be the image of X in $\mathcal{P}_p/\mathcal{P}_p^2$. Then $\mathcal{O}\tilde{X}$ maps surjectively on $\mathcal{O}\tilde{x}_p$ via φ_p. We put $a = \pi_{h,S_1,S_2} \circ \varphi_p(Y) \in \mathcal{O}$, so that $(a) = (y_p(h))$; then $\mathcal{P}_p = (X, Y - a)$ and $\mathcal{P}_p^2 = (X^2, (Y-a)^2, aX)$. The $\mathcal{O}$-module $\mathcal{O}[[X, Y]]/(XY)$ is isomorphic to $X\mathcal{O}[[X]] \oplus \mathcal{O}[[Y]]$; by this isomorphism $\mathcal{P}_p$ is sent on $X\mathcal{O}[[X]] \oplus (Y-a)\mathcal{O}[[Y]]$ and $\mathcal{P}_p^2$ on $XI \oplus (Y-a)^2\mathcal{O}[[Y]]$, where $I = (a, X)$ is the $\mathcal{O}[[X]]$-ideal consisting of series f with $f(0) \in (a)$. Thus, as $\mathcal{O}$-modules, $\mathcal{O}\tilde{X} \simeq \mathcal{O}[[X]]/I \simeq \mathcal{O}/(a)$ and therefore $\text{length}_{\mathcal{O}}(\mathcal{O}\tilde{x}_p) \leqslant \text{length}_{\mathcal{O}}(\mathcal{O}/(y_p(h)))$. □

We now define the congruence ideal of h relatively to $\mathcal{B}_{S_1,S_2}$ as the $\mathcal{O}$-ideal:

$$\eta_{h,S_1,S_2} = \theta_{h,S_1,S_2}(\text{Ann}_{\mathbf{T}_{S_1,S_2}}(\ker \theta_{h,S_1,S_2})).$$

It is known that η_{h,S_1,S_2} controls congruences between h and linear combinations of forms different from h in $\mathcal{B}_{S_1,S_2}$.

THEOREM 4.5. *Suppose $\Delta_1 \neq 1$ and Δ_2 as above. Then*

(a) *$\mathcal{B}_{\emptyset,\Delta_2} \neq \emptyset$; and for every subset $S \subseteq \Delta_2$*
(b) *the map $\mathcal{R}_{S,\Delta_2/S} \to \boldsymbol{T}_{S,\Delta_2/S}$ is an isomorphism of complete intersections;*
(c) *for every $h \in \mathcal{B}_{\emptyset,\Delta_2}$, $\eta_{h,S,\Delta_2/S} = (\prod_{p|S} y_p(h))\eta_{h,\emptyset,\Delta_2}$.*

Proof. By induction on $|\Delta_2|$. If Δ_2 is empty statement (a) is true by the hypothesis $\mathcal{B}_{\emptyset,\emptyset} \neq \emptyset$, statement (b) is the minimal case of Theorem 5.4.2 in [5] and (c) is tautological. Assume now the result to be true for $|\Delta_2| \leqslant n$ and suppose that $|\Delta_2| = n + 1$. Choose a prime p in Δ_2 and define $Q = \Delta_2/p$. Let h be a form in $\mathcal{B}_{\emptyset,Q}$, whose existence is assured by induction hypothesis. Then

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$$\eta_{h,\Delta_2,\emptyset} = (x_p(h))\eta_{h,Q,\emptyset} = \left(x_p(h)\prod_{q|Q} y_q(h)\right)\eta_{h,\emptyset,Q}. \tag{14}$$

Because of the characterization of $x_p(h)$ given in Lemma 4.2, the first equality above follows from [5, Cor. 1.4.3 and Section 5.5]; the second one holds by inductive hypothesis.

For an element t in $\mathrm{Ann}(\ker\theta_{h,p,Q})$, let $\tilde{t}$ be a lift of t in $\mathbf{T}_{\Delta_2,\emptyset}$. Lemma 4.2c) implies that $\prod_{q|Q} y_q$ annihilates $\ker(\mathbf{T}_{\Delta_2,\emptyset} \to \mathbf{T}_{p,Q})$ so that $\tilde{t}\prod_{q|Q} y_q \in \mathrm{Ann}(\ker\theta_{h,\Delta_2,\emptyset})$, and

$$\eta_{h,p,Q}\left(\prod_{q|Q} y_q(h)\right) \subseteq \eta_{h,\Delta_2,\emptyset}. \tag{15}$$

By 14 and 15 we obtain $\eta_{h,p,Q} \subseteq (x_p(h))\eta_{h,\emptyset,Q}$ and we know that $x_p(h)$ is not invertible in $\mathcal{O}$; thus the map $\mathbf{T}_{p,Q} \to \mathbf{T}_{\emptyset,Q}$ has a non trivial kernel, that is $\mathcal{B}_{\emptyset,\Delta_2}$ is not empty and (a) is proved.

We prove (b) and (c) by induction on $|S|$. Suppose $S = \emptyset$; we know by a) that $\mathcal{B}_{\emptyset,\Delta_2} \neq \emptyset$; then the hypothesis $\Delta_1 \neq 1$ and Corollary 4.1 imply b); c) is tautological. Suppose now the results being true for S and let $r \in \Delta_2 \setminus S$. By inductive hypothesis (on S and Δ_2 respectively) we know that the maps $\mathcal{R}_{S,\Delta_2/S} \longrightarrow T_{S,\Delta_2/S}$ and $\mathcal{R}_{S,\Delta_2/Sr} \longrightarrow \mathbf{T}_{S,\Delta_2/Sr}$ are isomorphisms of complete intersections. Consider the following surjections

$$\begin{aligned}
&\alpha\colon \mathcal{R}_{Sr,\Delta_2/Sr} \longrightarrow \mathbf{T}_{Sr,\Delta_2/Sr}\\
&\beta\colon \mathbf{T}_{Sr,\Delta_2/Sr} \longrightarrow \mathbf{T}_{S,\Delta_2/S} \simeq \mathcal{R}_{S,\Delta_2/S}\\
&\gamma\colon \mathbf{T}_{Sr,\Delta_2/Sr} \longrightarrow \mathbf{T}_{S,\Delta_2/Sr} \simeq \mathcal{R}_{S,\Delta_2/Sr}.
\end{aligned}$$

Since $\mathcal{B}_{Sr,\Delta_2/Sr}$ is the disjoint union of $\mathcal{B}_{S,\Delta_2/S}$ and $\mathcal{B}_{S,\Delta_2/Sr}$, and since $\ker(\gamma\circ\alpha) = (y_r)$ by Lemma 4.3 we have $(y_r) = \ker\gamma = \mathrm{Ann}(\ker\beta)$. Let h be a form in $\mathcal{B}_{S,\Delta_2/S}$. Then $(y_r(h))\eta_{h,S,\Delta_2/S} \subseteq \eta_{h,Sr,\Delta_2/Sr}$ (see the proof of 15 above). We claim that this inclusion is in fact an equality: suppose that $t \in \mathrm{Ann}(\ker\theta_{h,Sr,\Delta_2/Sr})$; then t belongs to $\mathrm{Ann}(\ker\beta) = (y_r)$; write $t = cy_r$, with c in $\mathbf{T}_{Sr,\Delta_2/Sr}$. For every form $g \in \mathcal{B}_{S,\Delta_2/S}$ different from h we have $tg = 0$ and $y_r g \neq 0$, thus $cg = 0$. Then $\beta(c) \in \mathrm{Ann}(\ker\theta_{h,S,\Delta_2/S})$ and so $\theta_{h,Sr,\Delta_2/Sr}(c) \in \eta_{h,S,\Delta_2/S}$. Therefore

$$\eta_{h,Sr,\Delta_2/Sr} = (y_r(h))\eta_{h,S,\Delta_2/S}, \quad \text{for every } h \in \mathcal{B}_{S,\Delta_2/S}. \tag{16}$$

We are now ready to prove b); according with Criterion I of [9] the map α is an isomorphism of complete intersections if and only if

$$\mathrm{length}_{\mathcal{O}}\left(\frac{\mathcal{P}_{h,Sr,\Delta_2/Sr}}{(P_{h,Sr,\Delta_2/Sr})^2}\right) \leqslant \mathrm{length}_{\mathcal{O}}\left(\frac{\mathcal{O}}{\eta_{h,Sr,\Delta_2/Sr}}\right).$$

By applying successively Lemma 4.4, point b) of the inductive hypothesis and equality 16 we obtain

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$$\begin{aligned}\text{length}_{\mathcal{O}}\left(\frac{\mathcal{P}_{h,Sr,\Delta_2/Sr}}{(\mathcal{P}_{h,Sr,\Delta_2/Sr})^2}\right) &\leqslant \text{length}_{\mathcal{O}}\left(\frac{\mathcal{P}_{h,S,\Delta_2/S}}{(\mathcal{P}_{h,S,\Delta_2/S})^2}\right) + \text{length}_{\mathcal{O}}\left(\frac{\mathcal{O}}{(y_r(h))}\right)\\ &\leqslant \text{length}_{\mathcal{O}}\left(\frac{\mathcal{O}}{\eta_{h,S,\Delta_2/S}(y_r(h))}\right)\\ &\leqslant \text{length}_{\mathcal{O}}\left(\frac{\mathcal{O}}{\eta_{h,Sr,\Delta_2/Sr}}\right).\end{aligned}$$

Now we prove c): if $h \in \mathcal{B}_{\emptyset,\Delta_2}$, then the identity 16 combined with the inductive hypothesis gives $\eta_{h,Sr,\Delta_2/Sr} = (y_r(h)\prod_{p|S} y_p(h))\eta_{h,\emptyset,\Delta_2}$. □

Remark 4.6. Statement (a) in Theorem 4.5 determines some 'nonoptimal levels' (in the sense of [12]) for which $\bar{\rho}$ is modular of type χ at ℓ and weight two.

If we combine point (c) of Theorem 4.5 to the results in Section 5.5 of [5] we obtain:

COROLLARY 4.7. *Let* $h \in \mathcal{B}_{S_1,S_2}$. *Then*

$$\eta_{h,\Delta_2,\emptyset} = \prod_{p|\frac{\Delta_2}{S_1S_2}} x_p(h) \prod_{p|S_2} y_p(h)\eta_{h,S_1,S_2}.$$

Remark 4.8. Let h be a weight two eigenform with trivial character, which is p-new for a prime p such that $\ell \not| p(p^2-1)$. Let $\mathcal{O}_0$ be the ring generated over $\mathbf{Z}$ by the Fourier coefficients of h, K_0 its quotient field, and let $\tilde{\mathcal{O}}_0$ be the integral closure of $\mathcal{O}_0$ in K_0. By the work of Shimura [32], there is an Abelian variety A over $\mathbf{Q}$ associated to h, of dimension equal to $[K_0{:}\,\mathbf{Q}]$, such that $\mathcal{O}_0 \subseteq \text{End}\,(A)$.

LEMMA 4.9. *There exists an abelian variety* $\tilde{A}$ *over* $\mathbf{Q}$*, isogenous to* A*, such that* $\tilde{\mathcal{O}}_0 \subseteq \text{End}\,(\tilde{A})$.

Proof. Let $\phi \in \text{End}^0(A) = \text{End}\,(A) \otimes \mathbf{Q}$ be an element in $\tilde{\mathcal{O}}_0$. Then ϕ satisfies a relation of the form $\phi^k = a_{k-1}\phi^{k-1} + \cdots + a_0$ with $a_0, \ldots, a_{k-1} \in \mathcal{O}_0$. Put $J = \{a \in \mathcal{O}_0 | a\phi, \ldots, a\phi^{k-1} \in \mathcal{O}_0\}$; then J is a nonzero ideal of $\mathcal{O}_0$, $J\phi \subseteq J$ and $A[J]$ is finite. We define $A' = A/A[J]$; it is a consequence of Grotendieck's results on quotients of group schemes (see [35, Theorem 3.4 and Section 3.5]) that this quotient is an Abelian variety, defined over $\mathbf{Q}$, with an action of $\mathcal{O}_0$ that the projection $A \to A'$ is an isogeny defined over $\mathbf{Q}$ and $\mathcal{O}_0$-linear. We show that $\phi \in \text{End}\,(A')$. Let x be a non zero element in $\mathcal{O}_0$ such that $x\phi \in \mathcal{O}_0$. Let $P \in A'$ and choose $Q \in A'$ such that $xQ = P$. Define $\phi(P) = (x\phi)(Q)$; it is immediate to see that this definition does not depend on the choice of x and Q, and that $\mathcal{O}_0[\phi] \subseteq \text{End}\,(A')$. By induction on the number of generators of $\tilde{\mathcal{O}}_0$ as an $\mathcal{O}_0$-algebra, we deduce the result. □

The ℓ-adic Tate module $T_\ell(\tilde{A})$ is a free $\mathbf{Z}_\ell \otimes_{\mathbf{Z}} \tilde{\mathcal{O}}_0$-module of rank 2 and $M = T_\ell(\tilde{A}) \otimes_{\mathbf{Z}_\ell \otimes \tilde{\mathcal{O}}_0} \mathcal{O}$ is an $\mathcal{O}$-integral model for ρ_h. Assume that the representation

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$\bar{\rho}_h$ over k is absolutely irreducible; then up to homotheties there is a unique $\mathcal{O}$-lattice stable for $G_{\mathbf{Q}}$ in the space of ρ_h. Therefore the order at (λ) of $y_p(h)$ is the greatest exponent n_0 such that I_p acts trivially over $M \otimes_{\mathcal{O}} \mathcal{O}/(\lambda^{n_0})$.

Since h is special at p, A and $\tilde{A}$ have multiplicative reduction at p: there is an exact sequence of $(\mathbf{Z}_\ell \otimes \tilde{\mathcal{O}}_0)[I_p]$-modules

$$0 \to L_1 \to T_\ell(\tilde{A}) \to L_2 \to 0 \tag{17}$$

where $L_1 = T_\ell(\tilde{A})^{I_p}$ and I_p acts trivially over L_1 and L_2. Let $\Phi_p(\tilde{A})$ be the group of components of the fiber at p of the Néron model of $\tilde{A}$. It is shown in [15, Section 11] that $\Phi_p(\tilde{A}) \otimes_{\mathbf{Z}} \mathbf{Z}_\ell$ is isomorphic to the torsion part of $H^1(I_p, T_\ell(\tilde{A}))$, that is to the cokernel of the coboundary map $\delta\colon L_2 \to \operatorname{Hom}(I_p, L_1)$ associated to sequence 17.

Since $\mathcal{O}$ is flat over $\mathbf{Z}_\ell \otimes \tilde{\mathcal{O}}_0$, we can tensor sequence 17 with $\mathcal{O}$ over $\mathbf{Z}_\ell \otimes \tilde{\mathcal{O}}_0$ and get a sequence of $\mathcal{O}[I_p]$-modules

$$0 \to M_1 \to M \to M_2 \to 0. \tag{18}$$

Then $\Phi_p(\tilde{A}) \otimes_{\tilde{\mathcal{O}}_0} \mathcal{O} \simeq \operatorname{coker}(\delta')$ where $\delta'\colon M_2 \to \operatorname{Hom}(I_p, M_1)$ is the coboundary map associated to sequence 18. On the other hand, it is immediate to see that $\operatorname{coker}(\delta')$ is a cyclic $\mathcal{O}$-module whose annihilator is $(\lambda^{n_0}) = (y_p(h))$. Therefore we obtain the formula

$$\mathcal{O}/(y_p(h)) \simeq \Phi_p(\tilde{A}) \otimes_{\tilde{\mathcal{O}}_0} \mathcal{O}. \tag{19}$$

Now let h be a newform in $S_2(\Gamma_0(M), \mathbf{Q})$ where $M = \Delta N$ is the product of two relatively prime integers Δ and N and Δ is the discriminant of an indefinite quaternion algebra B over $\mathbf{Q}$. Let $X_0^\Delta(N)$ be the Shimura curve associated to B and to an Eichler order of level N in B. Let E be the elliptic curve associated to h and let $\delta(E), \delta^\Delta(E)$ denote the degrees of parametrization of E by $X_0(M)$ and $X_0^\Delta(N)$ respectively; under the hypothesis of the irreducibility of $\bar{\rho}_h$, the main theorem in [26] and [33] implies that

$$\operatorname{ord}_\ell(\delta(E)) = \operatorname{ord}_\ell\left(\delta^\Delta(E) \cdot \prod_{p \mid \Delta} c_p(E)\right), \tag{20}$$

where $c_p(E) = |\Phi_p(E)|$. If $\ell \nmid M$ then the ideal generated by $\delta(E)$ in $\mathbf{Z}_\ell$ is the annihilator of the $\mathbf{Z}_\ell$-module of congruence of h with respect to forms in $S_2(\Gamma_0(M))$, cf. [40, Theorem 3]. Therefore by equality 19 we can regard Corollary 4.7 as an analogue of formula 20 (locally at ℓ) in the 'type χ' context.

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