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ABSTRACT

We study compatible toric Sasaki metrics with constant scalar curvature on co-oriented compact toric contact manifolds of Reeb type of dimension at least five. These metrics come in rays of transversal homothety due to the possible rescaling of the Reeb vector fields. We prove that there exist Reeb vector fields for which the transversal Futaki invariant (restricted to the Lie algebra of the torus) vanishes. Using an existence result of E. Legendre [*Toric geometry of convex quadrilaterals*, J. Symplectic Geom. **9** (2011), 343–385], we show that a co-oriented compact toric contact 5-manifold whose moment cone has four facets admits a finite number of rays of transversal homothetic compatible toric Sasaki metrics with constant scalar curvature. We point out a family of well-known toric contact structures on $S^2 \times S^3$ admitting two non-isometric and non-transversally homothetic compatible toric Sasaki metrics with constant scalar curvature.

1. Introduction

In this paper we study the existence and uniqueness of compatible Sasaki metrics of constant scalar curvature (cscS for short) on a compact co-oriented contact manifold (N, \mathbf{D}) , where the uniqueness should be understood up to a contactomorphism and transversal homothety (rescaling of the Reeb vector field). Sasaki–Einstein metrics, which occur when the first Chern class $c_1(\mathbf{D})$ of the contact distribution \mathbf{D} vanishes, have been intensively studied in recent years by many authors, see [BG08]. On the other hand, the theory of cscS metrics can be viewed as an odd-dimensional analogue of the more classical subject of constant scalar curvature Kähler metrics, which has been actively studied since the pioneering works of Calabi [Cal85]. We will focus in this paper on the special case when the contact structure is toric of Reeb type in the sense of [BG00] and the compatible metric is invariant under the torus action. In this setting, the problem of existence of cscS metrics is very closely related to the theory of constant scalar curvature Kähler metrics on toric varieties, recently developed by Donaldson in [Don02].

Banyaga and Molino, Boyer and Galicki, and Lerman [BM92, BM96, BG00, Ler02, Ler03] classified toric contact manifolds $(N^{2n+1}, \mathbf{D}, \hat{T}^{n+1})$ (in what follows, we suppose $n > 1$). The action of \hat{T} pull-backs to a Hamiltonian action on the symplectization $(M^{2n+2}, \hat{\omega})$ of (N, \mathbf{D}) , commuting with the Liouville vector field τ , see [Ler03]. In particular, the contact moment map $\hat{\mu}: M \rightarrow (\mathbb{R}^{n+1})^*$ refers to the unique moment map on the toric symplectic cone which is homogeneous of degree two with respect to the Liouville vector field τ (i.e. $\mathcal{L}_\tau \hat{\mu} = 2\hat{\mu}$) and $\mathcal{C} = \text{Im } \hat{\mu} \cup \{0\}$ is the moment cone.

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In order to study toric Sasaki metrics, it is not restrictive to consider toric contact manifolds of *Reeb type*, so that there exists a vector $b \in \mathbb{R}^{n+1} = \text{Lie } T^{n+1}$ inducing a Reeb vector field $X_b \in \Gamma(TM)$, see [BG08]. Equivalently, b lies in \mathcal{C}_+^* , the interior of the dual cone of \mathcal{C} (the set of strictly positive linear maps on $\text{Im } \hat{\mu} = \mathcal{C} \setminus \{0\}$). In particular, \mathcal{C} is a *strictly convex* polyhedral cone, that is, \mathcal{C}_+^* is not empty. From [BG00, Ler03], we know that toric contact manifolds of Reeb type of dimension at least five are in correspondence with strictly convex polyhedral cones $\mathcal{C} \subset \mathbb{R}^{n+1}$ which are *good* with respect to a lattice Λ . This means that every set of primitive vectors normal to a face of \mathcal{C} can be completed to a basis of Λ .

Given a strictly convex polyhedral cone \mathcal{C} , which is good with respect to a lattice Λ , one can associate to any $b \in \mathcal{C}_+^*$ the *characteristic labeled polytope*¹ (Δ_b, u_b) , where

$$\Delta_b = \mathcal{C} \cap \{y \mid \langle b, y \rangle = \frac{1}{2}\}$$

is a compact simple polytope and $u_b = \{u_{b_1}, \dots, u_{b_d}\}$ is the set of equivalence classes in $\mathbb{R}^{n+1}/\mathbb{R}b$ of the primitive vectors of Λ which are inward normal to the facets of \mathcal{C} . Here, $\mathbb{R}^{n+1}/\mathbb{R}b$ is identified with the dual vector space of the annihilator of b in $(\mathbb{R}^{n+1})^*$, which, in turn, is identified with the hyperplane $\{y \mid \langle b, y \rangle = \frac{1}{2}\}$.

Remark 1. Referring to (Δ_b, u_b) as a labeled polytope is slightly abusive: when there is a lattice $\Lambda' \subset \mathbb{R}^{n+1}/\mathbb{R}b$ containing the normals u_{b_i} , there exist uniquely determined positive integers m_i such that $(1/m_i)u_{b_i}$ are primitive elements of Λ' . Then $(\Delta_b, m_1, \dots, m_d)$ is a *rational labeled polytope* in the sense of Lerman–Tolman [LT97] and it describes a compact toric symplectic orbifold. This case appears when the Reeb vector field X_b is quasi-regular [BG08].

Recall that on a toric symplectic orbifold a compatible Kähler metric corresponds to a *symplectic potential*, ϕ , that is, a strictly convex smooth function defined on the interior of the moment polytope Δ , which satisfies certain boundary conditions depending on the labeling u [Abr01, ACGT04, Don05, Gui94]. We denote the set of these symplectic potentials by $\mathcal{S}(\Delta, u)$. Similarly, Martelli *et al.* [MSY06] parameterized the set of compatible toric Sasaki metrics in terms of homogeneous smooth functions of degree one on $\mathring{\mathcal{C}}$, the interior of \mathcal{C} , and subject to boundary and convexity conditions. In particular, a Kähler cone metric \hat{g} on $(M, \hat{\omega})$ corresponds to a potential $\hat{\phi}$ on $\mathring{\mathcal{C}}$. According to the Abreu formula [Abr98, Abr10], the scalar curvature $s_{\hat{g}}$ is then the pull-back by $\hat{\mu}$ of

$$S(\hat{\phi}) = - \sum_{i,j=0}^n \frac{\partial^2 \hat{H}_{ij}}{\partial y_i \partial y_j}, \tag{1}$$

where \hat{H}_{ij} is the inverse Hessian of $\hat{\phi}$. Hence, a cscS metric corresponds to a potential $\hat{\phi}$ with $b \in \mathcal{C}_+^*$ such that $S(\hat{\phi})|_{\Delta_b}$ is constant. This correspondence can be equivalently expressed in terms of symplectic potentials on characteristic labeled polytopes. Indeed, any potential $\phi \in \mathcal{S}(\Delta_b, u_b)$ on a characteristic polytope (Δ_b, u_b) canonically determines (and is determined by) a *Boothy–Wang potential* $\hat{\phi}$ corresponding to a Kähler cone metric \hat{g} on M , see [Abr10] and § 2.2 below. The scalar curvature of \hat{g} is the restriction to $N \subset M$ of the pull-back of

$$S(\hat{\phi}) = 4S(\phi) - 4n(n + 1),$$

where $S(\phi) = -\sum_{i,j=1}^n \partial^2 H_{ij} / \partial y_i \partial y_j$ with H_{ij} the inverse Hessian of ϕ , see [Abr10]. Furthermore, the scalar curvature of the Sasaki metric is $s_g = 4S(\phi) - 2n$, see [BG08] and § 2.3.

¹ In [MSY06], Δ is the *characteristic polytope* of a toric Sasaki metric with Reeb vector field X_b .

A primary obstruction to the existence of cscS metrics is given by the *Futaki–Sasaki* or *transversal Futaki invariant* of the Reeb vector field introduced by Boyer *et al.* in [BGS08a]. In the toric case, for any Reeb vector field X_b , one can restrict this invariant to the Lie algebra of the torus and obtain a vector $\mathcal{F}_b \in (\mathbb{R}^{n+1}/\mathbb{R}b)^*$ such that $\mathcal{F}_b = 0$ should a compatible cscS toric metric exist. Thus, we can recast the problem of existence and uniqueness of cscS toric metrics.

Problem 1. Given a strictly convex good cone \mathcal{C} , does there exist $b \in \mathcal{C}_+^*$ such that

$$\mathcal{F}_b = 0, \tag{2}$$

$$\exists \phi \in \mathcal{S}(\Delta_b, u_b), \text{ such that } S(\phi) \text{ is constant?} \tag{3}$$

If it exists, is such a b unique up to rescaling?

Rescaling $b \mapsto \lambda^{-1}b$, equivalently the contact form $\eta_b \mapsto \lambda\eta_b$, leads to a ray of toric Sasaki structures

$$g' = \lambda g + (\lambda^2 - \lambda)\eta_b \otimes \eta_b.$$

Such a deformation, called *transversal homothety*, changes the scalar curvature as

$$s_{g'} = \lambda^{-1}(s_g + 2n) - 2n,$$

see [BG08]. In particular, cscS metrics occur in rays. However, once the Reeb vector $b \in \mathcal{C}_+^*$ is fixed, the uniqueness of cscS metrics follows from uniqueness of solutions of the extremal Kähler equation in $\mathcal{S}(\Delta_b, u_b)$, see [Gun99] and Lemma 2.13 below.

In view of Problem 1, the Donaldson–Tian–Yau conjecture [Don02, Tia90, Yau93] has a straightforward interpretation in the toric Sasaki case using the notion of polystability of labeled polytopes given by Donaldson in [Don02].²

CONJECTURE 1.1. A compact co-oriented toric contact manifold of Reeb type admits a compatible toric cscS metric if and only if there exists $b \in \mathcal{C}_+^*$ such that $\mathcal{F}_b = 0$ and (Δ_b, u_b) is polystable.

Donaldson proved his conjecture [Don02, Don05, Don08b, Don09] for compact convex labeled polytopes in \mathbb{R}^2 . This immediately implies that Conjecture 1.1 holds true for compact five-dimensional toric contact manifolds of Reeb type.

The question of existence of toric Sasaki–Einstein metrics, which makes sense on co-oriented compact toric contact manifolds with Calabi–Yau cone (that is, $c_1(\mathbf{D}) = 0$), is now solved. First, Martelli *et al.* [MSY06] proved that the volume functional, defined on the space of compatible Sasaki metrics, only depends on the Reeb vector field and, up to a multiplicative constant, is

$$W(b) = \int_{\Delta_b} d\varpi.$$

Furthermore, they showed that the Hilbert functional is a linear combination of W and Z , where Z is defined for any $\phi \in \mathcal{S}(\Delta_b, u_b)$ as

$$Z(b) = \int_{\Delta_b} S(\phi) d\varpi$$

and only depends on the Reeb vector field. They also proved that $Z(b)$ coincides with $W(b)$ up to a multiplicative constant, when restricted to a suitable space of normalized Reeb vector fields,

² Donaldson uses a measure on the boundary instead of labels; the two notions are equivalent.

see Remark 2. The unique critical point of (the restriction of) W is then the only normalized Reeb vector field with vanishing transversal Futaki invariant.³

Futaki *et al.* [FOW09] showed, on the other hand, that for such a Reeb vector field Problem 1 has always a solution (corresponding to a Sasaki–Einstein metric).

Remark 2. Unlike cscS metrics, there are no rays of transversal homothetic Sasaki–Einstein metrics. Indeed, since the scalar curvature s_g of a Sasaki–Einstein metric satisfies $s_g = 2n(2n + 1)$, see [BG08], being Sasaki–Einstein prevents the rescaling of the Reeb vector field. In particular, there is an obvious normalization of Reeb vector fields in the search for Sasaki–Einstein metrics. However, Sasaki–Einstein metrics are cscS metrics and thus come in rays of such.

In this paper, we extend the Martelli–Sparks–Yau arguments to toric contact manifolds of Reeb type by showing that, after a suitable normalization of the Reeb vector fields, the critical points of the functional

$$F(b) = \frac{Z(b)^{n+1}}{W(b)^n}$$

coincide with normalized Reeb vectors with vanishing transversal Futaki invariant.

THEOREM 1.2. *A co-oriented toric contact manifold of Reeb type admits at least one ray of Reeb vector fields with vanishing (restricted) transversal Futaki invariant.*

Unlike the Sasaki–Einstein problem, Reeb vector fields with vanishing transversal Futaki invariant do not necessarily lead to cscS metrics, see e.g. [Don02]. However, as we proved in [Leg11], any labeled quadrilateral (Δ, u) with vanishing Futaki invariant admits a symplectic potential $\phi \in \mathcal{S}(\Delta, u)$ for which $S(\phi)$ is constant. Thus, we obtain the following theorem.

THEOREM 1.3. *A co-oriented toric contact five-dimensional manifold of Reeb type whose moment cone has four facets admits at least one and at most seven distinct rays of transversal homothetic compatible Sasaki metrics of constant scalar curvature. Moreover, for each pair of co-prime numbers, (p, q) , such that $p > 5q$, there exist two non-isometric, non-transversally homothetic Sasaki metrics of constant scalar curvature compatible with the same toric contact structure on the Wang–Ziller five-dimensional manifold $M_{p,q}^{1,1}$.*

More precisely, following [Leg11, Corollary 1.6], these metrics are explicitly given in terms of two polynomials of degree at most three. The toric contact structure on $M_{p,q}^{1,1}$ is the one described in [BGS08b]. The first part of Theorem 1.3 partially answers a question of Boyer [Boy11].

The paper is organized as follows. Section 2 contains basic notions of toric Sasakian geometry, emphasizing the boundary conditions required for the potential to induce a smooth Kähler cone metric. We also give the results we need about uniqueness of cscS metrics. In §3 we give a way to check whether or not a labeled polytope is characteristic of a good cone; we then study the properties of the functional F and prove Theorem 1.2. In §4 we specialize our study to the case of cones over quadrilaterals in \mathbb{R}^3 and prove Theorem 1.3.

2. Sasaki and transversal Kähler toric metrics: a quick review

A *labeled polytope*, (Δ, u) , is a simple compact polytope Δ in an n -dimensional vector space \mathfrak{t}^* which has d codimension one faces, called *facets* and denoted by F_1, \dots, F_d , together with a

³ Martelli *et al.* extended their results to the non-toric case in [MSY08].

set $u = \{u_1, \dots, u_d\}$ of vectors in \mathfrak{t} which are inward (with respect to Δ) and such that u_i is normal to F_i . Recall that a polytope is *simple* if each vertex is the intersection of n distinct facets. In what follows, we consider Δ itself as a face. (Δ, u) is *rational* with respect to a lattice Λ if $u \subset \Lambda$.

DEFINITION 2.1. Two labeled polytopes are *equivalent* if one underlying polytope can be mapped on the other via an invertible affine map, $A : \mathfrak{t}^* \rightarrow \mathfrak{s}^*$, such that the normals correspond via the differential's adjoint $(dA)^* : \mathfrak{s} \rightarrow \mathfrak{t}$.

2.1 Toric contact and symplectic geometry

A symplectic cone is a triple (M, ω, τ) , where (M, ω) is a symplectic manifold and $\tau \in \Gamma(TM)$ is a vector field generating a proper action of $\mathbb{R}_{>0}$ on M such that $\mathcal{L}_\tau \omega = 2\omega$.

DEFINITION 2.2. A *toric symplectic manifold* is a symplectic manifold (M, ω) together with an effective Hamiltonian action of a torus T , $\rho : T \hookrightarrow \text{Symp}(M, \omega)$, such that the dimension of T is half the dimension of M and there is a proper T -equivariant smooth map $\mu : M \rightarrow \mathfrak{t}^* = (\text{Lie } T)^*$ satisfying $d\mu(a) = -\iota_{d\rho(a)}\omega$. The map μ is unique up to an additive constant and is called the *moment map*. We recall that at $p \in M$, $d\rho(a) = X_a(p) = (d/dt|_{t=0})(\exp ta) \cdot p$.

Recall that, to a co-oriented compact connected contact manifold, (N^{2n+1}, \mathbf{D}) , there corresponds a symplectic cone over a compact manifold, $(\mathbf{D}_+^o, \hat{\omega}, \tau)$. \mathbf{D}_+^o is a connected component of the complement of the 0-section in \mathbf{D}^o , the annihilator in T^*N of the contact distribution \mathbf{D} . $\hat{\omega} = d\lambda$ is the restriction of the differential of the canonical Liouville form λ of T^*N and τ is the Liouville vector field $\tau_{(p,\alpha)} = (d/ds|_{s=0})e^{2s}\alpha_p$, so that $\mathcal{L}_\tau \hat{\omega} = 2\hat{\omega}$. See [Ler03] for a detailed description.

DEFINITION 2.3. A (compact) *toric contact manifold* $(N^{2n+1}, \mathbf{D}, \hat{T}^{n+1})$ is a co-oriented compact connected contact manifold (N^{2n+1}, \mathbf{D}) endowed with an effective action of a (maximal) torus $\hat{T} \hookrightarrow \text{Diff}(N)$ preserving the contact distribution \mathbf{D} and its co-orientation. Equivalently, the symplectic cone $(\mathbf{D}_+^o, \hat{\omega}, \tau)$ is toric with respect to the action of \hat{T} and the Liouville vector field τ commutes with \hat{T} . We denote by

$$\hat{\mu} : \mathbf{D}_+^o \rightarrow \hat{\mathfrak{t}}^* = (\text{Lie } \hat{T})^*$$

the *contact moment map*, that is, the unique moment map of $(\mathbf{D}_+^o, \hat{\omega}, \hat{T})$ which is homogeneous of degree two with respect to τ , see [Ler03].

DEFINITION 2.4. A polyhedral cone is *good* with respect to a lattice Λ if any facet F_i admits a normal vector, \hat{u}_i , primitive in Λ and, for any face $F_I = \bigcap_{i \in I} F_i$, $\text{span}_{\mathbb{Z}}\{\hat{u}_i \mid i \in I\} = \Lambda \cap \text{span}_{\mathbb{R}}\{\hat{u}_i \mid i \in I\}$ and the \hat{u}_i ($i \in I$) are linearly independent when $F_I \neq \{0\}$.

Lerman [Ler02, Ler03] showed that the image, $\text{Im } \hat{\mu}$, of the contact moment map of a compact toric contact manifold (N, \mathbf{D}, \hat{T}) does not contain 0 and $\mathcal{C} = \text{Im } \hat{\mu} \cup \{0\}$ is a convex, polyhedral cone which is good with respect to the lattice of circle subgroups, $\Lambda \subset \hat{\mathfrak{t}}$. \mathcal{C} is called *the moment cone*.

As mentioned in the introduction, a toric contact manifold of dimension at least five is of *Reeb type* if the moment cone \mathcal{C} is strictly convex, that is,

$$\mathcal{C}_+^* = \{b \in \hat{\mathfrak{t}} \mid \forall x \in \mathcal{C} \setminus \{0\}, \langle b, x \rangle > 0\} \neq \{0\}.$$

Indeed, see [BG08], for any $b \in \mathcal{C}_+^*$, there is a contact form, η_b , (i.e. $\ker \eta_b = \mathbf{D}$) for which X_b is a Reeb vector field, meaning that $\eta_b(X_b) \equiv 1$ and $\mathcal{L}_{X_b} \eta_b \equiv 0$.

Any strictly convex good polyhedral cone is the moment cone of a toric contact manifold of Reeb type, unique up to contactomorphisms, see [Ler03].

DEFINITION 2.5. Let (\mathcal{C}, Λ) be a strictly convex rational polyhedral good cone with d facets. Denote by $\hat{u}_1, \dots, \hat{u}_d$ the set of primitive vectors in Λ normal to the facets of \mathcal{C} . For $b \in \mathcal{C}_+^*$, we define the labeled polytope

$$(\Delta_b, u_b) = (\mathcal{C} \cap \mathfrak{t}_b^*, [\hat{u}_1]_b, \dots, [\hat{u}_d]_b) \tag{4}$$

where \mathfrak{t}_b^* the hyperplane $\mathfrak{t}_b^* = \{x \in \hat{\mathfrak{t}} \mid \langle b, x \rangle = \frac{1}{2}\}$. Up to translation, \mathfrak{t}_b is the annihilator of b in $\hat{\mathfrak{t}}^*$ and, thus, its dual space is identified with $\hat{\mathfrak{t}}/\mathbb{R}b$. The polytope Δ_b is an n -dimensional and simple polytope. We say that (Δ_b, u_b) is the *characteristic labeled polytope* of (\mathcal{C}, Λ) at b .

As Boyer and Galicki showed in [BG00], the space of leaves \mathcal{Z}_b of the Reeb vector field X_b is an orbifold if and only if the orbits of X_b are closed, that is, if and only if $\mathbb{R}b \cap \Lambda \neq \{0\}$. In that case, X_b is said to be *quasi-regular* and b generates a circle subgroup of \hat{T} , $T_b = \mathbb{R}b/(\mathbb{R}b \cap \Lambda)$, which acts on the cone $(\mathbf{D}_+^o, \hat{\omega})$ via the inclusion $\iota_b : T_b \hookrightarrow \hat{T}$, with moment map $\mu_b = \iota_b^* \circ \hat{\mu} : \mathbf{D}_+^o \rightarrow \mathbb{R}$. The space of leaves is identified with the symplectic reduction $\hat{\mu}_b^{-1}(1/2)/T_b$. Via the Delzant–Lerman–Tolman [Del88, LT97] correspondence, $(\mathcal{Z}_b, d\eta_b, T/T_b)$ is the toric symplectic orbifold associated to (Δ_b, u_b) .

Remark 3. Any compact toric symplectic orbifold admits a moment map whose image is a convex polytope Δ in the dual of the Lie algebra of \mathfrak{t}^* . The weights of the action determine a set of normals $u \subset \mathfrak{t}$ so that (Δ, u) is rational with respect to the lattice $\Lambda = \ker \exp(\mathfrak{t} \rightarrow T)$. The Delzant–Lerman–Tolman correspondence states that a compact toric symplectic orbifold is determined by its associated rational labeled polytope, up to a T -invariant symplectomorphism of orbifolds.

Conversely, any rational labeled polytope can be obtained from a toric symplectic orbifold, via *Delzant’s construction*. Two Hamiltonian actions $(\mu, T, \rho), (\mu', T', \rho')$ on a symplectic orbifold (M, ω) are equivalent if and only if their labeled polytopes are equivalent in the sense of Definition 2.1.

A *toric Sasaki* manifold $(N, \mathbf{D}, g, \hat{T})$ is a $(2n + 1)$ -dimensional Sasaki manifold whose underlying contact structure is toric with respect to an $(n + 1)$ -dimensional torus \hat{T} and whose metric is \hat{T} -invariant. Recall that it corresponds to the Kähler toric cone $(M, \hat{\omega}, \hat{g}, \tau, \hat{T}, \hat{\mu})$, where $M = \mathbf{D}_+^o$ and \hat{g} is the cone metric of g , that is, \hat{g} is homogeneous of degree two with respect to the Liouville vector field τ and coincides with g on $N \subset M$, seen as the level set $\hat{g}(\tau, \tau) = 1$. $(\hat{\omega}, \hat{g}, \hat{J})$ is a toric Kähler structure such that τ is real holomorphic. Notice that $\hat{J}\tau$ is induced by an element $b \in \mathcal{C}_+^* \subset \hat{\mathfrak{t}}$, so that $X_b = \hat{J}\tau$ restricts to a Reeb vector field on N . The *transversal Kähler geometry* of g refers to the metric \check{g} induced on \mathbf{D} by $g = \eta_b \otimes \eta_b + \check{g}$.

2.2 Kähler metric in action-angle coordinates

The material of this section is taken from [Abr01, Abr10, BGL08, CDG03, Gui94]. Let $(M^{2n}, \omega, J, g, T, \mu)$ be a Kähler toric orbifold, that is, g is a T -invariant Kähler metric and J is a complex structure such that $g(J \cdot, \cdot) = \omega(\cdot, \cdot)$. We are interested in the cases where $\mathcal{P} = \text{Im } \mu$ is a strictly convex cone or a polytope. We denote by $\mathring{\mathcal{P}}$ the interior of \mathcal{P} . Recall from [Del88, Ler03, LT97] that $\mathring{M} = \mu^{-1}(\mathring{\mathcal{P}})$ is the subset of M where the torus acts freely. The

Kähler metric provides a horizontal distribution for the principal T -bundle $\mu : \overset{\circ}{M} \rightarrow \overset{\circ}{\mathcal{P}}$ which is spanned by the vector fields JX_u , $u \in \mathfrak{t} = \text{Lie } T$. This gives an identification between the tangent space at any point of $\overset{\circ}{M}$ and $\mathfrak{t} \oplus \mathfrak{t}^*$. Usually, one chooses a basis (e_1, \dots, e_n) of \mathfrak{t} to identify $\overset{\circ}{M} \simeq \overset{\circ}{\mathcal{P}} \times T$ using the flows of the induced commuting vector fields $X_{e_1}, \dots, X_{e_n}, JX_{e_1}, \dots, JX_{e_n}$. The action-angle coordinates on $\overset{\circ}{M}$ are local coordinates $(\mu_1, \dots, \mu_d, t_1, \dots, t_d)$ on $\overset{\circ}{M}$ such that $\mu_i = \langle \mu, e_i \rangle$ and $X_{e_i} = \partial/\partial t_i$. The differentials dt_i are real-valued closed 1-forms globally defined on $\overset{\circ}{M}$ as duals of the X_{e_i} (i.e. $dt_i(X_{e_i}) = \delta_{ij}$ and $dt_i(JX_{e_j}) = 0$).

In action-angle coordinates, the symplectic form becomes $\omega = \sum_{i=1}^n d\mu_i \wedge dt_i$. It is well known [Gui94] that any toric Kähler metric is written as

$$g = \sum_{s,r} G_{rs} d\mu_r \otimes d\mu_s + H_{rs} dt_r \otimes dt_s, \tag{5}$$

where the matrix-valued functions (G_{rs}) and (H_{rs}) are smooth on $\overset{\circ}{\mathcal{P}}$, symmetric, positive definite, and inverse to each other. Following [ACGT04], we define the $S^2\mathfrak{t}^*$ -valued function $\mathbf{H} : \overset{\circ}{\mathcal{P}} \rightarrow \mathfrak{t}^* \otimes \mathfrak{t}^*$ by $\mathbf{H}_{\mu(p)}(u, v) = g_p(X_u, X_v)$, and put $H_{rs} = \mathbf{H}(e_r, e_s)$; $\mathbf{G} : \overset{\circ}{\mathcal{P}} \rightarrow \mathfrak{t} \otimes \mathfrak{t}$ is defined similarly.

When M is compact, necessary and sufficient conditions for a $S^2\mathfrak{t}^*$ -valued function \mathbf{H} to be induced by a globally defined toric Kähler metric on M were established in [Abr01, ACGT04, Don05]. We are going to adapt the point of view of [ACGT04] to Kähler cones. Let (Δ, u) be a labeled polytope. For a non-empty face $F = F_I = \bigcap_{i \in I} F_i$ of Δ , denote $\mathfrak{t}_F = \text{span}_{\mathbb{R}}\{u_i \mid i \in I\}$. Its annihilator in \mathfrak{t}^* , denoted \mathfrak{t}_F° , is naturally identified with $(\mathfrak{t}/\mathfrak{t}_F)^*$. We use the identification $T_\mu^*\mathfrak{t}^* \simeq \mathfrak{t}$.

DEFINITION 2.6. The set of symplectic potentials $\mathcal{S}(\Delta, u)$ is the space of smooth strictly convex functions on $\overset{\circ}{\Delta}$ for which $\mathbf{H} = (\text{Hess } \phi)^{-1}$ is the restriction to $\overset{\circ}{\Delta}$ of a smooth $S^2\mathfrak{t}^*$ -valued function on Δ , still denoted by \mathbf{H} , which verifies the boundary condition: for every y in the interior of the facet $F_i \subset \Delta$,

$$\mathbf{H}_y(u_i, \cdot) = 0 \quad \text{and} \quad d\mathbf{H}_y(u_i, u_i) = 2u_i, \tag{6}$$

and the positivity condition: the restriction of \mathbf{H} to the interior of any face $F \subset \Delta$ is a positive-definite $S^2(\mathfrak{t}/\mathfrak{t}_F)^*$ -valued function.

PROPOSITION 2.7 [ACGT04, Proposition 1]. *Let \mathbf{H} be a positive-definite $S^2\mathfrak{t}^*$ -valued function on $\overset{\circ}{\Delta}$. \mathbf{H} comes from a Kähler metric on M if and only if there exists a symplectic potential $\phi \in \mathcal{S}(\Delta, u)$ such that $\mathbf{H} = (\text{Hess } \phi)^{-1}$.*

This result follows from two lemmas: [ACGT04, Lemma 2], which holds regardless of the completeness of the metric and thus can be used as such here (Lemma 2.8 below), and [ACGT04, Lemma 3], which we adapt to toric Kähler cones (Lemma 2.9 below). Then we prove Proposition 2.11, which is the adaptation of [ACGT04, Proposition 1].

Remark 4. For this adaptation, it is more natural to work with $S^2\mathfrak{t}$ -valued functions $\mathbf{G} = (G_{ij})$ without requiring that they are the Hessian of a potential $\phi \in C^\infty(\overset{\circ}{\mathcal{P}})$. The metric g defined via (5) is then an *almost Kähler* metric.

LEMMA 2.8 [ACGT04, Lemma 2]. *Let (M, ω) be a toric symplectic $2n$ -manifold or orbifold with moment map $\mu : M \rightarrow \mathcal{P} \subset \mathfrak{t}^*$ and suppose that $(g_0, J_0), (g, J)$ are compatible almost Kähler metrics on $\overset{\circ}{M} = \mu^{-1}(\overset{\circ}{\mathcal{P}})$ of the form (5), given by \mathbf{G}_0, \mathbf{G} , and the same angular coordinates, and such that (g_0, J_0) extends to an almost Kähler metric on M . Then (g, J) extends to an almost Kähler metric on M provided that $\mathbf{G}\mathbf{G}_0$ and $\mathbf{G}_0\mathbf{H}\mathbf{G}_0 - \mathbf{G}_0$ are smooth on \mathcal{P} .*

LEMMA 2.9. Let $(M, \hat{\omega}, \tau)$ be a toric symplectic cone over a compact co-oriented contact manifold of Reeb type with two cone Kähler metrics inducing the same $S^2\hat{\mathfrak{t}}$ -valued function $\hat{\mathbf{G}}$ on the interior of the moment cone. Then there exists an equivariant symplectomorphism of $(M, \hat{\omega})$ commuting with τ and sending one metric to the other.

Proof. Let \hat{g} and \hat{g}' be two such metrics; they share the same level set

$$\{p \in M \mid \hat{g}_p(\tau, \tau) = 1\} = \{p \in M \mid \hat{g}'_p(\tau, \tau) = 1\} \simeq N.$$

Indeed, the part $(\hat{\mu}_1, \dots, \hat{\mu}_n)$ of the action-angle coordinates does not depend on the metric, the Liouville vector field is $\tau = \sum_i 2\hat{\mu}_i(\partial/\partial\hat{\mu}_i)$, and thus

$$\hat{g}(\tau, \tau) = 4 \sum_{i,j=0}^n \hat{\mu}_i \hat{\mu}_j \hat{G}_{ij} = \hat{g}'(\tau, \tau).$$

The equivariant symplectomorphism, say ψ , of $(\mathring{M}, \hat{\omega})$ sending one set of action-angle coordinates to the other commutes with τ and sends one metric to the other. Moreover, it restricts to \mathring{N} as an equivariant isometry between g and g' which can then be uniquely extended to a smooth equivariant isometry on N by a standard argument. Finally, since g and g' determine the respective cone metrics, this isometry pull-backs to a global isometry on M which coincides with ψ on \mathring{M} . □

The fact that $(\hat{\omega}, \hat{g}, \hat{J})$ is a cone Kähler structure with respect to τ reads, in terms of $\hat{\mathbf{H}}$, as follows.

LEMMA 2.10 [MSY06]. For any $0 \leq i, j \leq n$, \hat{H}_{ij} is homogeneous of degree one with respect to the dilatation in \mathfrak{t}^* . Moreover, if $b \in \hat{\mathfrak{t}}$ induces the Reeb vector field $X_b = J\tau$, then for all $\hat{\mu} \in \mathcal{C}$, $\hat{\mathbf{H}}_{\hat{\mu}}(b, \cdot) = 2\hat{\mu}$.

PROPOSITION 2.11. Let $(M, \hat{\omega}, \hat{T}, \tau)$ be a toric symplectic cone associated to a strictly convex good polyhedral cone (\mathcal{C}, Λ) having inward primitive normals $\hat{u}_1, \dots, \hat{u}_d$. Let $\hat{\mathbf{H}}$ be a positive-definite $S^2\hat{\mathfrak{t}}$ -valued function on $\mathring{\mathcal{C}}$. $\hat{\mathbf{H}}$ comes from a \hat{T} -invariant almost Kähler cone metric \hat{g} on M if and only if:

- $\hat{\mathbf{H}}$ is the restriction to $\mathring{\mathcal{C}}$ of a smooth $S^2\hat{\mathfrak{t}}^*$ -valued function on \mathcal{C} ;
- for every y in the interior of the facet $\hat{F}_i \subset \mathcal{C}$,

$$\hat{\mathbf{H}}_y(\hat{u}_i, \cdot) = 0 \quad \text{and} \quad d\hat{\mathbf{H}}_y(\hat{u}_i, \hat{u}_i) = 2\hat{u}_i; \tag{7}$$

- the restriction of $\hat{\mathbf{H}}$ to the interior of any face $\hat{F} \subset \mathcal{C}$ is a positive-definite $S^2(\hat{\mathfrak{t}}/\hat{\mathfrak{t}}_F)^*$ -valued function;
- $\hat{\mathbf{H}}$ is homogeneous of degree one with respect to the dilatation in \mathfrak{t}^* .

Proof. The necessary part follows from [ACGT04, Proposition 1] and Lemma 2.10 above. For the converse, take a basis of \mathfrak{t} consisting of the Reeb vector b (already determined by $\hat{\mathbf{H}}$ on \mathring{M}) and the normals associated to an edge of \mathcal{C} . Following the proof of [ACGT04, Proposition 1], we consider the first few terms of the Taylor series of the entries of $\hat{\mathbf{H}}$ in this basis. Then we conclude the proof with Lemmas 2.8 and 2.9. □

For $b \in \mathcal{C}_+^*$, the Boothy–Wang symplectic potential $\hat{\phi}$ of $\phi \in \mathcal{S}(\Delta_b, u_b)$ is the function

$$\hat{\phi}(\hat{\mu}) = 2\langle \hat{\mu}, b \rangle \cdot \phi\left(\frac{\hat{\mu}}{2\langle \hat{\mu}, b \rangle}\right) + \frac{\langle \hat{\mu}, b \rangle}{2} \log \langle \hat{\mu}, b \rangle$$

defined on the interior of the cone \mathcal{C} , see e.g. [Abr10]. The following proposition adapts the relation between potentials via reduction given in [CDG03] to the case of cone metrics and non-necessarily quasi-regular Reeb vector fields. We give a proof below, since this statement does not appear in this form in the literature and is central in our study.

PROPOSITION 2.12. *Let (\mathcal{C}, Λ) be a strictly convex good cone with $b \in \mathcal{C}_+^*$. Denote by (Δ_b, u_b) the characteristic labeled polytope of (\mathcal{C}, Λ) at b . $\hat{\mathbf{H}}$ is a positive-definite $S^2\hat{\mathfrak{t}}^*$ -valued function on \mathcal{C} coming from a \hat{T} -invariant almost Kähler cone metric \hat{g} on M with Reeb vector field X_b if and only if*

$$\hat{\mathbf{H}}_{\hat{\mu}} = 2\langle \hat{\mu}, b \rangle \mathbf{H}_{\mu}^b + 2 \frac{\hat{\mu} \otimes \hat{\mu}}{\langle \hat{\mu}, b \rangle}, \tag{8}$$

where $\mu = \hat{\mu}/2\langle \hat{\mu}, b \rangle \in \Delta_b$ and \mathbf{H}^b is a positive-definite $S^2(\hat{\mathfrak{t}}/\mathbb{R}b)^*$ -valued function on Δ_b satisfying the conditions of Proposition 2.7 with respect to (Δ_b, u_b) .

Moreover, in that case, $\hat{\mathbf{H}}$ is the Hessian's inverse of a symplectic potential $\hat{\phi}$ if and only if $\hat{\phi}$ is the Boothy–Wang symplectic potential of $\phi \in \mathcal{S}(\Delta_b, u_b)$ whose Hessian's inverse is \mathbf{H}^b . Finally, $\check{g}(X_a, X_c) = \mathbf{H}^b(a, c)$ on N .

Proof. The necessary part of the first affirmation follows from [Leg11]. For the converse, take a basis (e_0, e_1, \dots, e_n) of $\hat{\mathfrak{t}}$ such that $e_0 = b$, the Reeb vector. Consider the corresponding coordinates (y_0, y_1, \dots, y_n) on $\hat{\mathfrak{t}}^*$ and write $\hat{\mathbf{H}}$ and its inverse $\hat{\mathbf{G}}$ as matrices using (8):

$$\hat{\mathbf{H}} = 2 \left(\begin{array}{c|ccc} y_0 & & & y_1 \cdots y_n \\ \hline & y_1 & & \\ & \vdots & & \\ & & y_0 \mathbf{H}^b + \frac{y_i y_j}{y_0} & \\ \hline & & & y_n \end{array} \right), \quad \hat{\mathbf{G}} = \frac{1}{2y_0^3} \left(\begin{array}{c|ccc} y_0^2 + y_i y_j G_{ij} & -y_0 y_i G_{i1} & \cdots & -y_0 y_i G_{in} \\ \hline -y_0 y_j G_{1j} & & & \\ \vdots & & & \\ -y_0 y_j G_{nj} & & & y_0^2 \mathbf{G}^b \end{array} \right),$$

where \mathbf{G}^b stands for the inverse of \mathbf{H}^b . It is then elementary to check that $\hat{\mathbf{H}}$ satisfies the conditions of Proposition 2.11 as soon as \mathbf{H}^b satisfies the conditions of Proposition 2.7 and that $\hat{\mathbf{G}}$ is the Hessian of $\hat{\phi}$ if \mathbf{G}^b is the Hessian of ϕ . \square

Remark 5. If $\mathbb{R}b \cap \Lambda \neq \{0\}$, then the transversal metric \check{g} induced by \hat{g} on the orbifold \mathcal{Z}_b is associated to $2\langle \hat{\mu}, b \rangle \mathbf{H}^b$, see [Leg11].

2.3 Toric cscS metrics and uniqueness

A Sasaki structure (N, \mathbf{D}, g) implies three Riemannian structures: the Riemannian metric g on the contact manifold N , the transversal metric \check{g} on the contact bundle, and the Riemannian metric \hat{g} on the symplectization $M = \mathbf{D}_+^o$. The respective scalar curvatures are related to each other by

$$s_{\hat{g}|_N} = s_{\check{g}} - 4n(n + 1) = s_g - 2n(2n + 1),$$

see [BG08]. On the other hand, the scalar curvature of a toric Kähler cone metric \hat{g} with symplectic potential $\hat{\phi}$ is given by the pull-back of $S(\hat{\phi})$ defined by the Abreu formula (1) and, if $\hat{\phi}$ is the Boothy–Wang potential of $\phi \in \mathcal{S}(\Delta_b, u_b)$, then

$$S(\hat{\phi})(\hat{\mu}) = \frac{1}{\langle \hat{\mu}, b \rangle} \left(2S(\phi) \left(\frac{\hat{\mu}}{\langle \hat{\mu}, b \rangle} \right) - 2n(n + 1) \right),$$

see [Abr98, Abr10]. We now give a straightforward corollary of the uniqueness of the solutions to Abreu’s equation on labeled polytopes, due to Guan [Gua99] and the formulas above.

LEMMA 2.13. *Up to an equivariant contactomorphism, for each $b \in \mathcal{C}_+^*$ there exists at most one toric cscS metric with Reeb vector field X_b .*

Proof. If $\hat{\phi}$ and $\hat{\phi}'$ are potentials on the cone \mathcal{C} associated to Kähler cone metrics having the same Reeb vector field X_b , then $\hat{\phi}$ and $\hat{\phi}'$ are the Boothby–Wang potentials of functions ϕ and $\phi' \in \mathcal{S}(\Delta_b, u_b)$. $\hat{\phi}$ (respectively $\hat{\phi}'$) defines a toric cscS metric if and only if $S(\phi)$ (respectively $S(\phi')$) is constant. Now, thanks to Guan’s uniqueness result [Gua99] (recast in terms of symplectic potentials on labeled polytopes in [Don09]), if $S(\phi)$ and $S(\phi')$ are both constant then $\phi - \phi'$ is affine-linear. In that case, $\hat{\phi} - \hat{\phi}'$ is affine-linear and thus $\hat{\phi}$ and $\hat{\phi}'$ define the same Kähler cone metric. \square

PROPOSITION 2.14. *Let (N, \mathbf{D}, \hat{T}) be a co-oriented compact toric contact manifold of Reeb type with contact moment map $\hat{\mu} : \mathbf{D}_+^o \rightarrow \hat{\mathfrak{t}}^*$ and moment cone \mathcal{C} . Let g_a and g_b be compatible \hat{T} -invariant Sasaki metrics on N with respective vector fields X_a and X_b . We suppose that N is not a sphere, that its dimension is at least five, and that (N, \mathbf{D}, g_a) and (N, \mathbf{D}, g_b) are not 3-Sasaki. If $\varphi : N \rightarrow N$ is a diffeomorphism such that $\varphi^*g_a = g_b$, then φ is a contactomorphism and there exist $\psi \in \text{Isom}(N, g_b)$ and $A \in \text{Gl}(\hat{\mathfrak{t}})$, preserving the lattice $\Lambda = \ker(\exp : \hat{\mathfrak{t}} \rightarrow \hat{T})$, so that*

$$\hat{\mu} \circ (\varphi \circ \psi)^* = A^* \circ \hat{\mu}.$$

In particular, A^ is an automorphism of \mathcal{C} and $Ab = a$.*

*Conversely, any linear automorphism of \mathcal{C} whose adjoint preserves the lattice gives rise to a T -equivariant contactomorphism ψ such that if g is a compatible toric Sasaki metric on (N, \mathbf{D}, \hat{T}) then so is ψ^*g .*

Proof. Since 3-Sasaki manifolds, spheres, and 3-manifolds are the only manifolds carrying Riemannian metrics which are Sasakian with respect to more than one contact structure, see [BG08], under our assumptions, $\text{Isom}(N, g_b) \subset \text{Con}(N, \mathbf{D})$. Thus, \hat{T} and $\varphi^{-1} \circ \hat{T} \circ \varphi$ are tori in $\text{Con}(N, \mathbf{D})$. Since a Hamiltonian action on the symplectization $(\mathbf{D}_+^o, \hat{\omega})$ induces isotropic distributions, \hat{T} and $\varphi^{-1} \circ \hat{T} \circ \varphi$ are maximal tori in $\text{Isom}(N, g_b)$, which is a compact Lie group since N is compact. In particular, they are conjugate, that is, there exists $\psi \in \text{Isom}(N, g_b)$ such that

$$\hat{T} = (\varphi \circ \psi)^{-1} \circ \hat{T} \circ \varphi \circ \psi.$$

The differential A at $1 \in \hat{T}$ of the automorphism $\tau \mapsto (\varphi \circ \psi)^{-1} \circ \tau \circ \varphi \circ \psi$ is linear and preserves the lattice. Moreover, $\hat{\mu} \circ (\varphi \circ \psi)^*$ and $A^* \circ \hat{\mu}$ are moment maps for the same Hamiltonian action of \hat{T} on $(\mathbf{D}_+^o, \hat{\omega})$ and they are both homogeneous of degree two with respect to the Liouville vector field. Hence, $\hat{\mu} \circ (\varphi \circ \psi)^* = A^* \circ \hat{\mu}$.

Note that $(\varphi \circ \psi)_*X_b = X_a$, since $(\varphi \circ \psi)^*g_a = g_b$ and $\varphi \circ \psi \in \text{Con}(N, \mathbf{D})$. The converse follows from the Delzant–Lerman construction. \square

In the proof of Proposition 2.14, the hypothesis that (N, \mathbf{D}, g_b) is not 3-Sasaki, a sphere, or a 3-manifold is used to deduce that $\varphi^{-1} \circ \hat{T} \circ \varphi$ is included in $\text{Con}(N, \mathbf{D})$. Thus, we can remove this hypothesis by assuming that φ is a \hat{T} -equivariant contactomorphism. Combined with Lemma 2.13, we get the following proposition.

PROPOSITION 2.15. Let (N, \mathbf{D}, \hat{T}) be a compact toric contact manifold of Reeb type of dimension at least five. Two toric cscS metrics g_a and g_b , with respective Reeb vector fields X_a and X_b , coincide up to a combination of \hat{T} -equivariant contactomorphism and transversal homothety if and only if there exists $\lambda > 0$ such that $(\lambda\Delta_b, u_b)$ and (Δ_a, u_a) are equivalent in the sense of Definition 2.1.

2.4 Transversal Futaki invariant and extremal affine function

Let (Δ, u) be a labeled polytope, $\Delta \subset \mathfrak{t}^*$, and $u \subset \mathfrak{t}$. Choosing a basis (e_1, \dots, e_n) of \mathfrak{t} gives a basis $\mu_0 = 1, \mu_1 = \langle e_1, \cdot \rangle, \dots, \mu_n = \langle e_n, \cdot \rangle$ of affine-linear functions.

DEFINITION 2.16. Let the vector $\zeta = (\zeta_0, \dots, \zeta_n) \in \mathbb{R}^{n+1}$ be the unique solution of the linear system

$$\sum_{j=0}^n W_{ij}(\Delta)\zeta_j = Z_i(\Delta, u), \quad i = 0, \dots, n \tag{9}$$

with

$$W_{ij}(\Delta) = \int_{\Delta} \mu_i \mu_j d\varpi \quad \text{and} \quad Z_i(\Delta, u) = 2 \int_{\partial\Delta} \mu_i d\sigma,$$

where the volume form $d\varpi = d\mu_1 \wedge \dots \wedge d\mu_n$ and the measure $d\sigma$ on $\partial\Delta$ are related by the equality $u_j \wedge d\sigma = -d\varpi$ on the facet F_j . We call $\zeta_{(\Delta, u)} = \sum_{i=0}^n \zeta_i \mu_i$ the *extremal affine function*.

Remark 6. One can also define $\zeta_{(\Delta, u)}$ as the unique affine function such that $\mathcal{L}_{\Delta, u}(f) = \int_{\partial\Delta} f d\sigma - \frac{1}{2} \int_{\Delta} f \zeta_{(\Delta, u)} d\varpi = 0$ for any smooth function f , see [Don02].

The extremal affine function $\zeta_{(\Delta, u)}$ is the $L^2(\Delta, d\varpi)$ -projection of ‘the scalar curvature’ $S(\phi)$ to the space of affine-linear functions, for any symplectic potential $\phi \in \mathcal{S}(\Delta, u)$. Indeed, integrating (1) and using (6), we get

$$Z_i(\Delta, u) = \int_{\Delta} S(\phi)\mu_i d\varpi = 2 \int_{\partial\Delta} \mu_i d\sigma. \tag{10}$$

In view of this, if there exists a symplectic potential $\phi \in \mathcal{S}(\Delta, u)$ such that $S(\phi)$ is an affine-linear function, then $S(\phi) = \zeta_{(\Delta, u)}$. In that case, the corresponding toric Sasaki metric is *extremal* in the sense of [BGS08a]. Thus, we get the following corollary.

COROLLARY 2.17. On a compact toric contact manifold with good moment cone (\mathcal{C}, Λ) , an extremal compatible toric Sasaki metric with Reeb vector field X_b has constant scalar curvature if and only if $\zeta_{(\Delta_b, u_b)}$ is constant, where (Δ_b, u_b) is the characteristic labeled polytope of (\mathcal{C}, Λ) at b .

For any $\phi \in \mathcal{S}(\Delta, u)$, we define the linear functional $\mathcal{F}_{(\Delta, u)} : \mathfrak{t} \rightarrow \mathbb{R}$ as

$$\mathcal{F}_{(\Delta, u)}(a) = \int_{\Delta} f_a S(\phi) d\varpi,$$

where $f_a(\mu) = \langle a, \mu \rangle \int_{\Delta} d\varpi - \int_{\Delta} \langle a, \mu \rangle d\varpi$ has mean value 0. Via (10), $\mathcal{F}_{(\Delta, u)}$ does not depend on the choice of ϕ . Indeed, setting $\bar{Z} = (1/\int_{\Delta} d\varpi)Z_0(\Delta, u)$,

$$\mathcal{F}_{(\Delta, u)}(a) = \int_{\Delta} f_a \zeta_{(\Delta, u)} d\varpi = \int_{\Delta} f_a (\zeta_{(\Delta, u)} - \bar{Z}) d\varpi.$$

In particular, $\mathcal{F}_{(\Delta, u)} = 0$ if and only if $\zeta_{(\Delta, u)}$ is constant.

Given any compatible \hat{T} -invariant CR structure on a toric contact manifold with Reeb vector field X_b , $\mathcal{F}_b = \mathcal{F}_{(\Delta_b, u_b)}$ is the restriction of the transversal Futaki invariant of [BGS08a] to the space of real transversally holomorphic vector fields which are induced by the toric action. However, our definition of $\mathcal{F}_{(\Delta_b, u_b)}$ is independent of the choice of a compatible CR structure, relating this invariant to the symplectic version of the Futaki invariant introduced in [Lej10].

3. The Reeb family of a labeled polytope

A *labeled cone* (\mathcal{C}, L) consists of a polyhedral cone, \mathcal{C} , with d facets in some vector space V and $L = \{L_1, \dots, L_d\} \subset V^*$, so that $\mathcal{C} = \{y \mid \langle y, L_i \rangle \geq 0, i = 1, \dots, d\}$. Moreover, a labeled polytope (Δ, u) is *characteristic* of (\mathcal{C}, L) at $b \in V^*$ if $\Delta = \mathcal{C} \cap \{y \mid \langle y, b \rangle = 1\}$ and $u_i = [L_i] \in V^*/\mathbb{R}b$. The interesting case is when L_1, \dots, L_d span a lattice $\Lambda \subset V^*$ for which \mathcal{C} is good, since in that case (Δ, u) determines the transversal geometry associated to the Reeb vector X_b on the toric contact manifold associated to (\mathcal{C}, Λ) .

Convention 1. From now on, we set $\Delta = \mathcal{C} \cap \{y \mid \langle y, b \rangle = 1\}$ instead of $\Delta = \mathcal{C} \cap \{y \mid \langle y, b \rangle = 1/2\}$. This convention facilitates the calculations of the next sections and geometrically corresponds to normalizing the Liouville vector field so that the symplectic form of the symplectic cone is homogeneous of degree one instead of two.

DEFINITION 3.1. A *Reeb family* is a set of equivalence classes of labeled polytopes characteristic of a given labeled cone (\mathcal{C}, L) .

3.1 The cone associated to a polytope

Let (Δ, u) be a labeled polytope with *defining functions*

$$L_i = \langle \cdot, u_i \rangle + \lambda_i \in \text{Aff}(\mathfrak{t}^*, \mathbb{R}).$$

That is, $\Delta = \{x \in \mathfrak{t}^* \mid L_i(x) \geq 0, i = 1, \dots, d\}$, $F_i = \{x \in \mathfrak{t}^* \mid L_i(x) = 0\}$, and $u_i = dL_i$ via the identification $T_\mu^* \mathfrak{t}^* \simeq \mathfrak{t}$.

The defining functions L_1, \dots, L_d determine a cone

$$\mathcal{C}(\Delta) = \{y \in \text{Aff}(\mathfrak{t}^*, \mathbb{R})^* \mid \langle y, L_i \rangle \geq 0, i = 1, \dots, d\}$$

and its dual $\mathcal{C}^*(\Delta) = \{L \in \text{Aff}(\mathfrak{t}^*, \mathbb{R}) \mid \langle y, L \rangle \geq 0 \forall y \in \mathcal{C}(\Delta)\}$. By translating the polytope $\Delta + \mu$, we translate the defining functions $L_i^\mu = L_i - \langle \mu, u_i \rangle$, producing a linear equivalence between the cones $\mathcal{C}(\Delta)$ and $\mathcal{C}(\Delta + \mu)$. More generally, we have the following lemma.

LEMMA 3.2. If (Δ, u) and (Δ', u') are equivalent by an invertible affine map A , then so are $(\mathcal{C}(\Delta'), L')$ on $(\mathcal{C}(\Delta), L)$ via the adjoint map of the pull-back of A .

PROPOSITION 3.3. A labeled polytope (Δ, u) is characteristic of a good cone if and only if the defining functions L_1, \dots, L_d of (Δ, u) span a lattice in $\text{Aff}(\mathfrak{t}^*, \mathbb{R})$ with respect to which the cone $\mathcal{C}(\Delta)$ is good.

This proposition follows from the next lemma, where $\mathbf{1} \in \text{Aff}(\mathfrak{t}^*, \mathbb{R})$ denotes the constant function equal to 1.

LEMMA 3.4. Let (Δ, u) be a labeled polytope with defining functions L_1, \dots, L_d . $\mathcal{C}(\Delta)$ is a non-empty, strictly convex, polyhedral cone. Moreover, (Δ, u) is characteristic of

$(\mathcal{C}(\Delta), \{L_1, \dots, L_d\})$ at $\mathbf{1} \in \text{Aff}(\mathfrak{t}^*, \mathbb{R})$ and, up to linear equivalence, this is the unique labeled cone for which (Δ, u) is characteristic.

Proof. Consider the evaluation map $e : \mathfrak{t}^* \hookrightarrow \text{Aff}(\mathfrak{t}^*, \mathbb{R})^*$, that is, for $\mu \in \mathfrak{t}^*$, the map $e_\mu : L \rightarrow L(\mu)$ is linear. We have $\text{Im } e = \{y \mid \langle y, \mathbf{1} \rangle = 1\}$ and then $e(\Delta) = \text{Im } e \cap \mathcal{C}(\Delta)$. For any $y \in \text{Aff}(\mathfrak{t}^*, \mathbb{R})^*$, there exist a unique $\mu \in \mathfrak{t}^*$ and a unique $r = \langle y, \mathbf{1} \rangle \in \mathbb{R}$ such that $y = e_\mu - e_0 + re_0$, where e is the evaluation map as above. This gives an identification $\text{Aff}(\mathfrak{t}^*, \mathbb{R})^* \simeq \mathfrak{t}^* \times \mathbb{R}$, leading to

$$\mathcal{C}(\Delta) \setminus \{0\} = \{e_{r\mu} - e_0 + re_0 \mid \mu \in \Delta, r > 0\} \simeq \Delta \times \mathbb{R}_{>0}. \tag{11}$$

By using again the identification $T_\mu^* \mathfrak{t}^* \simeq \mathfrak{t}$, the differential maps $\text{Aff}(\mathfrak{t}^*, \mathbb{R})$ to \mathfrak{t} . The differential corresponds also to the quotient map of $\text{Aff}(\mathfrak{t}^*, \mathbb{R})$ by the linear subspace of constant functions. Thus, $dL_i = [L_i] = u_i \in \text{Aff}(\mathfrak{t}^*, \mathbb{R})/\mathbb{R}\mathbf{1}$. The uniqueness part of Lemma 3.4 is straightforward. \square

Remark 7. For a given labeled (simple) polytope (Δ, u) with vertices ν_1, \dots, ν_N and facets F_1, \dots, F_d , one can define N lattices of $\text{Aff}(\mathfrak{t}^*, \mathbb{R})$ as $\Lambda_i = \langle L_l \mid \nu_i \in F_l \rangle_{\mathbb{Z}}$. They are free groups of rank n and one can prove that, assuming that $\Lambda = \langle L_1, \dots, L_d \rangle_{\mathbb{Z}}$ is a lattice in $\text{Aff}(\mathfrak{t}^*, \mathbb{R})$, the cone $(\mathcal{C}(\Delta), \Lambda)$ is good if and only if all the groups $\Lambda/\Lambda_1, \dots, \Lambda/\Lambda_N$ are free.

3.2 The Reeb family of a labeled polytope

Let (Δ, u) be a labeled polytope with defining functions L_1, \dots, L_d . Denote the interior of $\mathcal{C}^*(\Delta)$ by $\mathcal{C}_+^*(\Delta)$. It is the set of affine maps $b \in \text{Aff}(\mathfrak{t}^*, \mathbb{R})$ which are strictly positive on $\mathcal{C}(\Delta)$. $\mathcal{C}_+^*(\Delta)$ is non-empty and open ($\mathcal{C}(\Delta)$ is a closed subset of $\text{Aff}(\mathfrak{t}^*, \mathbb{R})^*$) and, by using (11), is given by

$$\mathcal{C}_+^*(\Delta) = \{b \in \text{Aff}(\mathfrak{t}^*, \mathbb{R}) \mid \langle b, \mu \rangle > 0 \ \forall \mu \in \Delta\}.$$

Using Lemma 3.4, we get that: *if (Δ, u) and (Δ', u') are in the same Reeb family then there exists a unique vector $b \in \mathcal{C}_+^*(\Delta)$ such that (Δ', u') is characteristic of $(\mathcal{C}(\Delta), \{L_1, \dots, L_d\})$ at b .* In particular, the cone $\mathcal{C}_+^*(\Delta)$ provides an effective parametrization of the Reeb family of (Δ, u) . The affine hyperplane

$$\mathfrak{t}_b^* = \{y \in \text{Aff}(\mathfrak{t}^*, \mathbb{R})^* \mid \langle y, b \rangle = 1\}$$

is identified with the annihilator of b in $\text{Aff}(\mathfrak{t}^*, \mathbb{R})^*$ via a translation and then its dual vector space is the quotient $\mathfrak{t}_b = \text{Aff}(\mathfrak{t}^*, \mathbb{R})/\mathbb{R}b$.

PROPOSITION 3.5. *The Reeb family of (Δ, u) is parameterized by*

$$\{(\Delta_b, u_b) = (\Psi_b(\Delta), [L_1]_b, \dots, [L_d]_b) \mid b \in \mathcal{C}_+^*(\Delta)\},$$

where $[L_l]_b$ is the equivalence class of L_l in \mathfrak{t}_b and the map $\Psi_b : \Delta \rightarrow \mathfrak{t}_b^*$ is defined as $\Psi_b(\mu) = e_\mu/b(\mu)$.

Proof. Using the decomposition (11), we see that

$$\mathfrak{t}_b^* \cap \mathcal{C}(\Delta) = \left\{ e_{\mu/b(\mu)} - e_0 + \frac{e_0}{b(\mu)} = \frac{e_\mu}{b(\mu)} \mid \mu \in \Delta \right\}. \tag{12}$$

The map $\Psi_b(\mu) = e_\mu/b(\mu)$ is well defined and injective on any set where b is positive.

A basis, (v_1, \dots, v_n) , of \mathfrak{t} provides coordinates on \mathfrak{t}^* via $\mu_i = \langle \mu, v_i \rangle$ and so we write $\mu = (\mu_1, \dots, \mu_n)$, as well as a basis of $\text{Aff}(\mathfrak{t}^*, \mathbb{R})$, that is,

$$(\mathbf{1}, \langle \cdot, v_1 \rangle, \dots, \langle \cdot, v_n \rangle), \tag{13}$$

which, in turn, gives coordinates on $\text{Aff}(\mathfrak{t}^*, \mathbb{R})^*$ as $y = (y_0, y_1, \dots, y_n)$, where $y_0 = \langle y, \mathbf{1} \rangle$ and $y_i = \langle y, v_i \rangle$.

In this system of coordinates, we have $e_\mu = (1, \mu_1, \dots, \mu_n)$ and thus

$$\Psi_b(\mu) = \frac{1}{b(\mu)}(1, \mu_1, \dots, \mu_n).$$

Denoting $b = b(0)\mathbf{1} + \sum_{i=1}^n b_i v_i$, the differential of Ψ_b at μ is

$$d_\mu \Psi_b = \frac{1}{b(\mu)^2} \left(-\frac{\partial}{\partial y_0} \otimes db + \sum_{i,j=1}^n \left(b(\mu) \frac{\partial}{\partial y_i} \otimes d\mu_i - b_j \mu_i \frac{\partial}{\partial y_i} \otimes d\mu_j \right) \right), \tag{14}$$

where db is identified with an element of \mathfrak{t} and thus with a linear map in $\text{Aff}(\mathfrak{t}^*, \mathbb{R})$. For a given vector $X = \sum_{i=1}^n X_i(\partial/\partial \mu_i) \in T_\mu \Delta$, we compute that $d_\mu \Psi_b(X) = 0$ if and only if $db(X) = 0$ and $b(\mu)X = 0$, that is, Ψ_b is an immersion. \square

3.3 The Futaki invariant of a Reeb family

The purpose of this paragraph is to find a functional over the Reeb family of (Δ, u) , whose critical points are the Reeb vector $b \in \mathcal{C}_+(\Delta)$ such that the extremal affine function $\zeta_{(\Delta_b, u_b)}$ is constant (i.e. the corresponding restricted Futaki invariant \mathcal{F}_b vanishes, see § 2.4).

Notation 1. Fix a basis of \mathfrak{t} giving coordinates (y_0, \dots, y_n) on $\text{Aff}(\mathfrak{t}_b^*, \mathbb{R})^*$ as above and set $d\varpi = dy_0 \wedge dy_1 \wedge \dots \wedge dy_n$. For $b \in \mathcal{C}_+(\Delta)$, put $\zeta(b) = \zeta_{(\Delta_b, u_b)}$, where (Δ_b, u_b) is the labeled polytope in the Reeb family of (Δ, u) given by b via the parametrization of Proposition 3.5. Denote by $d\varpi_b$ the volume form on $\mathfrak{t}_b^* = \{y \mid \langle y, b \rangle = 1\} \subset \text{Aff}(\mathfrak{t}^*, \mathbb{R})^*$, satisfying $b \wedge d\varpi_b = d\varpi$, and $d\sigma_b$ the measure on $\partial\Delta_b$ determined by the equality $L_l \wedge d\sigma_b = -d\varpi_b$ on the facet $F_{b,l} = \Delta_b \cap L_l^\circ$. In the system of coordinates induced from (y_0, \dots, y_n) on \mathfrak{t}_b^* , the affine extremal function is written $\zeta(b) = \zeta_0(b) + \sum_{i=1}^n \zeta_i(b)y_i \in \text{Aff}(\mathfrak{t}_b^*, \mathbb{R})$, where $(\zeta_0(b), \zeta_1(b), \dots, \zeta_n(b)) \in \mathbb{R}^{n+1}$ is the solution of a linear system (9) involving the functions

$$W_{ij}(b) = W_{ij}(\Delta_b) \quad \text{and} \quad Z_i(b) = Z_i(\Delta_b, u_b)$$

computed using $d\varpi_b$.

Remark 8. In the case where L_1, \dots, L_d span a lattice Λ for which $\mathcal{C}(\Delta)$ is good, there is a contact manifold (N, D) associated to $(\mathcal{C}(\Delta), \Lambda)$. Then, as in [MSY06], one can compute that up to a positive multiplicative constant depending only on the dimension of N , $W_{00}(b)$ is the volume of N with respect to the volume form $\eta_b \wedge (d\eta_b)^n$, where η_b is the contact form of X_b .

LEMMA 3.6. For $i, j = 1, \dots, n$,

$$W_{00}(b) = \int_{\Delta} \frac{1}{b(\mu)^{n+1}} d\varpi, \quad W_{ij}(b) = \int_{\Delta} \frac{\mu_i \mu_j}{b(\mu)^{n+3}} d\varpi,$$

$$W_{i0}(b) = W_{0i}(b) = \int_{\Delta} \frac{\mu_i}{b(\mu)^{n+2}} d\varpi$$

and

$$Z_0(b) = 2 \int_{\partial\Delta} \frac{1}{b(\mu)^n} d\sigma, \quad Z_i(b) = 2 \int_{\partial\Delta} \frac{\mu_i}{b(\mu)^{n+1}} d\sigma.$$

Proof. We use the coordinate systems on \mathfrak{t}^* and $\text{Aff}(\mathfrak{t}^*, \mathbb{R})$ introduced in the proof of Proposition 3.5.

Choose $p = (p_1, \dots, p_n) \in \Delta$ and denote

$$d\varpi_b = \frac{1}{b(p)} \left(dy_1 \wedge \dots \wedge dy_n + \sum_{i=1}^n (-1)^{i+1} p_i dy_0 \wedge \dots \wedge \widehat{dy_i} \wedge \dots \wedge dy_n \right).$$

Thus, $b \wedge d\varpi_b = dy_0 \wedge dy_1 \wedge \dots \wedge dy_n$, since $b = b(0) dy_0 + \sum_{i=1}^n b_i dy_i$ when viewed as an element of $T_y^* \text{Aff}(\mathfrak{t}^*, \mathbb{R})^*$. By setting $\alpha = \sum_{k=1}^n \alpha_k$ with $\alpha_k = (-1)^{k+1} p_k dy_1 \wedge \dots \wedge \widehat{dy_k} \wedge \dots \wedge dy_n$, we get

$$d\varpi_b = \frac{1}{b(p)} (dy_1 \wedge \dots \wedge dy_n + dy_0 \wedge \alpha).$$

On the other hand, put $A_\mu = \sum_{i,j=1}^n (b(\mu)(\partial/\partial y_i) \otimes d\mu_i - b_i \mu_j (\partial/\partial y_i) \otimes d\mu_j)$. In view of the expression of $d_\mu \Psi_b$ (14), $A_\mu = b(\mu)^2 d_\mu \Psi_b + (\partial/\partial y_0) \otimes db$. Moreover, A_μ is a morphism between $T_\mu \Delta$ and the kernel of dy_0 in the tangent space of $\text{Aff}(\mathfrak{t}_b^*, \mathbb{R})^*$ and one can prove that⁴

$$\det A_\mu = b(0)b(\mu)^{n-1} \quad \text{and} \quad db \wedge A_\mu^* \alpha_k = -b_k p_k b(\mu)^{n-1} d\mu_1 \wedge \dots \wedge d\mu_n.$$

Hence, since α is an $(n - 2)$ -form on $\text{Aff}(\mathfrak{t}^*, \mathbb{R})$ such that $\alpha(\partial/\partial y_0) = 0$, we have

$$(\Psi_b^* d\varpi_b)_\mu = \frac{1}{b(p)b(\mu)^{2n}} (A_\mu^* dy_1 \wedge \dots \wedge dy_n - db \wedge A_\mu^* \alpha) = \frac{1}{b(\mu)^{n+1}} d\varpi,$$

where $db = \sum_{i=1}^n b_i d\mu_i$, via the identification $T_\mu^* \mathfrak{t}^* \simeq \mathfrak{t}$. The first part of Lemma 3.6 then follows easily and it remains to prove the statement concerning the functions Z_0 and Z_i . Note that $\Psi_b^* L_l = (1/b(\mu))u_l$ and then

$$u_l \wedge (\Psi_b^* d\sigma_b)_\mu = b(\mu) \Psi_b^* (L_l \wedge d\sigma_b)_\mu = -b(\nu) (\Psi_b^* d\varpi)_\mu = -\frac{1}{b(\mu)^n} d\varpi.$$

This shows that $(\Psi_b^* d\sigma_b)_\mu = (1/b(\mu)^n) d\sigma$ for $\mu \in \partial\Delta$, which concludes the proof. □

Convention 2. For now on, we suppose $b(0) = 1$. There is no loss of generality since, in view of the defining equations (9), for $r > 0$ we have

$$\zeta(rb) = r\zeta_0(b) + r^2 \sum_{i=1}^n \zeta_i(b)\mu_i.$$

Note that $\Omega = \{b \in \mathcal{C}_+^*(\Delta) \mid b(0) = 1\}$ is relatively compact in $\text{Aff}(\mathfrak{t}^*, \mathbb{R})$.

PROPOSITION 3.7. *The critical points of the functional $F : \Omega \rightarrow \mathbb{R}$, defined as*

$$F(b) = \frac{Z_0(b)^{n+1}}{W_{00}(b)^n},$$

are the affine-linear functions b for which $\zeta(b)$ is constant.

Proof. Notice that $\zeta(b)$ is constant if and only if $\zeta(b) = \zeta_0(b)$, which happens if and only if $\zeta_0(b)$ is a solution of the linear system

$$W_{i0}(b)\zeta_0(b) = Z_i(b), \quad i = 0, \dots, n. \tag{15}$$

In that case, $\zeta_0(b) = Z_0(b)/W_{00}(b)$ with b a solution of

$$W_{i0}(b)Z_0(b) - W_{00}(b)Z_i(b) = 0, \quad i = 1, \dots, n. \tag{16}$$

⁴ For example, a proof can use induction on n when viewing A_μ as an $n \times n$ matrix depending on two vectors $\mu \in \mathbb{R}^n$ and $b \in \mathbb{R}^{n+1}$.

By noticing that for $i = 1, \dots, n$,

$$\frac{\partial}{\partial b_i} W_{00}(b) = -(n + 1)W_{i0}(b) \quad \text{and} \quad \frac{\partial}{\partial b_i} Z_0(b) = -nZ_i(b),$$

we compute the differential of F at $b \in \Omega$:

$$d_b F = \frac{-n(n + 1)Z_0(b)^n}{W_{00}(b)^{n+1}} \sum_{i=1}^n (W_{i0}(b)Z_0(b) - W_{00}(b)Z_i(b)) db_i,$$

which concludes the proof. □

LEMMA 3.8. *Let $b \in \Omega$. Then*

$$W_{00}(b) = n \sum_l \int_{F_l} \frac{L_l(0)}{b^n} d\sigma.$$

Proof. For $b \in \Omega$, put $X = \sum_i (\mu_i/b^n)(\partial/\partial \mu_i)$, so we have $\text{div } X = n/b^{n+1}$. Thus,

$$\frac{1}{n} W_{00}(b) = \int_{\partial \Delta} \iota_X d\varpi = -\sum_l \int_{F_l} \frac{\langle u_l, \mu \rangle}{b^n} d\sigma = \sum_l \int_{F_l} \frac{L_l(0)}{b^n} d\sigma$$

since $u_l \wedge \sum_i \iota_{\partial/\partial \mu_i} d\varpi = \langle u_l, \mu \rangle d\varpi$. □

Consider the map $x = (x_1, \dots, x_d) : \Omega \rightarrow \mathbb{R}^d$ whose components are the strictly convex and strictly positive functions

$$x_l = x_l(b) = \int_{F_l} \frac{1}{b^n} d\sigma.$$

With this notation and by putting $\lambda_l = L_l(0)$, the functionals W_{00} , Z_0 , and F are

$$W_{00}(b) = n \sum_{i=1}^d \lambda_i x_i, \quad Z_0(b) = 2 \sum_{i=1}^d x_i, \quad \text{and} \quad F(b) = \frac{(2 \sum x_l)^{n+1}}{(\sum n \lambda_l x_l)^n}. \tag{17}$$

For each $l = 1, \dots, d$, F_l is an $(n - 1)$ -dimensional polytope whose volume is x_l , up to a positive multiplicative constant determined by u_l . This suggests applying Lemma 3.8 recursively. Note that $\int_E (1/b^2) d\sigma(E) = 1/b(p_E)b(q_E)$, where $d\sigma(E)$ is the Lebesgue measure on the edge E and p_E and q_E denote the vertices of Δ lying in E . Therefore, for each edge there are suitable constants $\alpha(E)$, $\beta(E)$ so that

$$W_{00}(b) = \sum_{E \in \text{edges}(\Delta)} \frac{\alpha(E)}{b(p_E)b(q_E)} \quad \text{and} \quad Z_0(b) = \sum_{E \in \text{edges}(\Delta)} \frac{\beta(E)}{b(p_E)b(q_E)}. \tag{18}$$

Moreover, F is a rational function of the values of b on the vertices. More precisely,

$$F(b) = \left(\prod_{\nu \in \text{vert}(\Delta)} \frac{1}{b(\nu)} \right) \cdot \frac{(\sum \beta(E) \prod_{\nu \notin E} b(\nu))^{n+1}}{(\sum \alpha(E) \prod_{\nu \notin E} b(\nu))^n}, \tag{19}$$

where the sums are taken over edges of Δ .

Proof of Theorem 1.2. Formulas (17) imply that $\lambda = \max\{\lambda_l \mid l = 1, \dots, d\} > 0$ and

$$F(b) \geq \frac{(2 \sum x_l)^{n+1}}{(n\lambda \sum x_l)^n} = \frac{2^{n+1}}{(\lambda n)^n} \sum x_l = \frac{2^n}{(\lambda n)^n} Z_0(b).$$

Moreover, $\partial\Omega$ is the set of affine-linear functions vanishing on the boundary of Δ (but not on the interior). In particular, they vanish on some vertices of Δ . Since $Z_0(b)$ only depends on the

value of b on the vertices, see (18), $Z_0(b)$ and thus $F(b)$ converge to infinity when b converges to a point of $\partial\Omega$.

Hence, Theorem 1.2 follows from the fact that F is a strictly positive function, defined on a relatively compact open set Ω , and converging to infinity at the boundary. In particular, F must have a critical point somewhere in Ω . □

PROPOSITION 3.9. *Let (Δ, u) be an n -dimensional labeled polytope with N vertices. The set of critical points of F is a real algebraic set given as the common roots of n polynomials in n variables of degree $2N - 3$.*

Proof. By using (18), $d_b F = 0$ if and only if for any linear function $h : \mathfrak{t}^* \rightarrow \mathbb{R}$,

$$\sum_E \frac{(nZ_0\alpha(E) - (n + 1)W_{00}\beta(E))(h(p_E)b(q_E) + h(q_E)b(p_E))}{b(p_E)^2b(q_E)^2} = 0. \tag{20}$$

A linear function is determined by its values on n linearly independent points. Taking a set of n linearly independent vertices of Δ , say p_1, \dots, p_n , (20) may be written as a homogeneous equation in n variables $h(p_1), \dots, h(p_n)$. In particular, if (20) holds for any linear function h , the coefficient P_i of $h(p_i)$ in (20) vanishes for all i . These coefficients P_i are functions of $b(p_1), \dots, b(p_n)$ (since $b(0) = 1$; the affine-linear function b is determined by its value at p_1, \dots, p_n). It is easy to see that, up to a suitable positive multiplicative constant, P_i is a polynomial of degree at most $2N - 3$ in n variables $b(p_1), \dots, b(p_n)$. □

3.3.1 *The Sasaki–Einstein case.* Let (N, \mathbf{D}) be a contact manifold such that $c_1(\mathbf{D})$, the first Chern class of the contact bundle \mathbf{D} , vanishes. In [MSY06], it is shown that the normalized Reeb vector field for which the transversal Futaki invariant is zero corresponds to the critical point of the volume functional and that such a point is unique. In our setting, this implies that, if $c_1(\mathbf{D}) = 0$, the critical point of F is unique and corresponds to the critical point of $W_{00}(b)$ in Ω .

The condition $c_1(\mathbf{D}) = 0$ is a necessary condition for the existence of a Sasaki–Einstein metric and corresponds to the fact that the primitive normals of the moment cone lie in a hyperplane, see [FOW09, MSY06]. Moreover, if X_b is the Reeb vector field of a Sasaki–Einstein metric, then the basic first Chern class $c_1^B = 2(n + 1)[d\eta_b]_B$, which implies that (Δ_b, u_b) is *monotone* in the sense of the following definition.

DEFINITION 3.10. A labeled polytope (Δ, u) is monotone if there exists $\mu \in \mathring{\Delta}$ such that $L_l(\mu) = c$ for all $l = 1, \dots, d$, where c is some positive constant.

In [Don08a], integral polytopes which are monotone (with respect to the normals primitive in the lattice which is the dual of the lattice containing the vertices of the polytope) are called Fano polytopes. This terminology is justified since a smooth toric variety X is Fano in the usual sense (i.e. the anticanonical line bundle $-K_X$ is ample) if and only if the integral Delzant polytope associated to $(X, -K_X)$ is monotone in the sense above. Equivalently, the symplectic manifold associated to this integral Delzant polytope is monotone (i.e. the symplectic class coincides up to a multiplicative constant with the first Chern class). In the orbifold case, one can prove that a rational labeled polytope (Δ, u) is monotone if only if the associated symplectic toric orbifold (M, ω) is monotone. The next lemma is straightforward.

LEMMA 3.11. *The defining functions of a monotone polytope lie in a hyperplane and any labeled polytope in the Reeb family of a monotone labeled polytope is monotone.*

THEOREM 3.12 [MSY06]. *If (Δ, u) is monotone, then F has a unique critical point. Moreover, up to a translation of Δ , there exists a constant $\lambda > 0$ such that $W = \lambda Z$.*

Proof. The functional F depends equivariantly on the representative of the affine equivalence class of a labeled polytope. In particular, the number of critical points of F does not change if we translate the polytope. If (Δ, u) is monotone, there exists $\mu \in \overset{\circ}{\Delta}$ such that $L_l(\mu) = c > 0$; then, by using Lemma 3.8, the function W_{00} associated to $(\Delta - \mu, u)$ is $W_{00}(b) = (n/c)Z_0(b)$. □

4. Examples: the case of quadrilaterals

Up to affine transform, there exists a unique strictly convex cone with four facets in \mathbb{R}^3 . Indeed, $\mathbb{P} \text{Gl}(3, \mathbb{R})$ acts transitively on the set of generic 4-tuples of points of $\mathbb{R}P^2$. In particular, up to linear transform, every Reeb family of quadrilaterals contains the equivalence class of a labeled square. It is then enough to consider the Reeb family of labeled squares in order to study Reeb families of quadrilaterals.

Let Δ_o be the convex hull of $p_1 = (-1, -1)$, $p_2 = (-1, 1)$, $p_3 = (1, 1)$, $p_4 = (1, -1)$ in \mathbb{R}^2 . Δ_o is a square and the vectors normal to its edges are of the form

$$u_1 = \frac{1}{r_1}e_1, \quad u_2 = \frac{-1}{r_2}e_2, \quad u_3 = \frac{-1}{r_3}e_1, \quad u_4 = \frac{1}{r_4}e_2, \tag{21}$$

where $e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \in (\mathbb{R}^2)^*$. The defining functions of (Δ_o, u) are

$$L_l = \langle \cdot, u_l \rangle + \frac{1}{r_l}$$

for $l = 1, \dots, 4$. Let us denote the edges $E_l = L_l^{-1}(0) \cap \Delta_o$. With coordinates (x, y) on \mathbb{R}^2 , we have $u_1 = (1/r_1) dx$, $u_2 = (-1/r_2) dy$, $u_3 = (-1/r_3) dx$, $u_4 = (1/r_4) dy$. The following lemma is straightforward.

LEMMA 4.1. *(Δ_o, u) is monotone if and only if $r_1 + r_3 = r_2 + r_4$.*

Moreover, one can easily check that

$$\Omega = \{b = 1 + b_1e_1 + b_2e_2 \mid |b_1 + b_2| < 1, |b_1 - b_2| < 1\}.$$

By integration, we get

$$W_{00}(b) = \int_{\Delta_o} \frac{1}{b^3} dx dy = \frac{2}{b(p_1)b(p_2)b(p_3)b(p_4)}.$$

In this setting, the measure $d\sigma$ can be made explicit:

$$d\sigma|_{E_1} = -r_1 dy, \quad d\sigma|_{E_2} = -r_2 dx, \quad d\sigma|_{E_3} = r_3 dy, \quad d\sigma|_{E_4} = r_4 dx.$$

Integrating again leads to

$$Z_0(b) = \int_{\partial\Delta_o} \frac{1}{b^2} d\sigma = \frac{2r_1}{b(p_1)b(p_2)} + \frac{2r_2}{b(p_2)b(p_3)} + \frac{2r_3}{b(p_3)b(p_4)} + \frac{2r_4}{b(p_1)b(p_4)}.$$

We then get

$$F(b) = \frac{Z_0(b)^3}{W_{00}(b)^2} = \frac{2(r_1b(p_3)b(p_4) + r_2b(p_1)b(p_4) + r_3b(p_1)b(p_2) + r_4b(p_2)b(p_3))^3}{b(p_1)b(p_2)b(p_3)b(p_4)}.$$

Put $a_1 = b_1 - b_2$ and $a_2 = b_1 + b_2$ so that $b(p_1) = 1 - a_2$, $b(p_2) = 1 - a_1$, $b(p_3) = 1 + a_2$, $b(p_4) = 1 + a_1$, and $\Omega \simeq \{(a_1, a_2) \mid |a_i| < 1\}$. Moreover,

$$F(b) = \frac{2(a_1 a_2 K + a_1(K - 2(r_2 - r_3)) + a_2(K - 2(r_4 - r_3)) + K - 2(r_2 + r_4))^3}{(1 - a_1^2)(1 - a_2^2)}$$

with $K = r_1 + r_3 - r_2 - r_4$. Note that $K = 0$ if (Δ_o, u) is monotone.

One can prove the following lemma using elementary methods.

LEMMA 4.2. *The point $b = 1 + b_1 e_1 + b_2 e_2 \in \Omega$ is a critical point of F if and only if $(a_1, a_2) = (b_1 - b_2, b_1 + b_2) \in \mathbb{R}^2$ is a common root of the polynomials*

$$\begin{aligned} P(a_1, a_2) &= -a_1^2 a_2 K - a_1^2 (K + 2(r_2 - r_3)) + 2a_1 a_2 (K + 2(r_4 - r_3)) \\ &\quad + 2a_1 (K + 2(r_2 + r_4)) + 3(a_2 K + K + 2(r_2 - r_3)), \\ Q(a_1, a_2) &= -a_2^2 a_1 K - a_2^2 (K + 2(r_4 - r_3)) + 2a_1 a_2 (K + 2(r_2 - r_3)) \\ &\quad + 2a_2 (K + 2(r_2 + r_4)) + 3(a_1 K + K + 2(r_4 - r_3)) \end{aligned}$$

lying in the square $\{(a_1, a_2) \mid |a_i| < 1\}$. Consequently, F has at most seven critical points.

LEMMA 4.3. *If $r_1 = r_3$, $r_2 = r_4$, and $r_1 + r_3 > 5(r_2 + r_4)$, then $K > 0$ and P and Q have exactly five distinct common roots: $(0, 0)$, $\pm(a, -a)$ with $0 < a^2 = 1 - 4(r_2 + r_4)/K < 1$, and $\pm(a, a)$ with $a^2 = 5 + 4(r_2 + r_4)/K > 1$.*

In particular, F admits three distinct critical points in Ω .

Proof. If $r_1 = r_3$ and $r_2 = r_4$, then $K + 2(r_4 - r_3) = 0$, $K + 2(r_2 - r_3) = 0$, and

$$\begin{aligned} P(a_1, a_2) &= -a_1^2 a_2 K + 2a_1 (K + 2(r_2 + r_4)) + 3a_2 K, \\ Q(a_1, a_2) &= -a_2^2 a_1 K + 2a_2 (K + 2(r_2 + r_4)) + 3a_1 K. \end{aligned}$$

Write $P(x, a_2) = C(x)a_2 + D(x)$, where $C(x) = -Kx^2 + 3K$ and $D(x) = 2(K + 2(r_2 + r_4))x$. Since C and D do not have any common root, P and Q have at most five common roots of the form

$$\left(a, -\frac{D(a)}{C(a)}\right) \quad \text{with } a \text{ a root of } C^2(x)Q\left(x, -\frac{D(x)}{C(x)}\right).$$

By noticing that

$$\begin{aligned} P(a_1, -a_1) &= -Q(a_1, -a_1) = a_1(a_1^2 K - K + 4(r_2 + r_4)) \\ P(a_1, a_1) &= Q(a_1, a_1) = a_1(-a_1^2 K + 5K + 4(r_2 + r_4)), \end{aligned}$$

we get the desired five distinct common roots.

Since $r_1 + r_3 > 5(r_2 + r_4)$, $0 < 1 - 4(r_2 + r_4)/K < 1$ while $5 + 4(r_2 + r_4)/K > 1$. □

We have to find which examples provided by Lemma 4.3 correspond to Reeb vector fields on a toric contact manifold, that is, which labeled squares of Lemma 4.3 are characteristic of a good cone. Denote by (e_1, e_2, e_3) the standard basis of \mathbb{R}^3 . Consider the polyhedral cone $\mathcal{C}_o = \{x \mid \langle x, \delta_j \rangle \geq 0, j = 1, \dots, 4\} \subset \mathbb{R}^3$, where the rays δ_i are

$$\delta_1 = \mathbb{R}_{>0}(e_3 + e_1), \quad \delta_2 = \mathbb{R}_{>0}(e_3 + e_2), \quad \delta_3 = \mathbb{R}_{>0}(e_3 - e_1), \quad \delta_4 = \mathbb{R}_{>0}(e_3 - e_2).$$

LEMMA 4.4. *Each strictly convex polyhedral cone with four facets in \mathbb{R}^3 is equivalent to \mathcal{C}_o by an affine transform. Moreover, \mathcal{C}_o is a good cone with respect to a lattice $\Lambda \subset \mathbb{R}^3$ if and only if*

there exists $s > 0$ such that the primitive normal vectors inward to \mathcal{C}_o are $\hat{u}_1 = sc_1(e_1 + e_3)$, $\hat{u}_2 = sc_2(e_2 + e_3)$, $\hat{u}_3 = sc_3(e_3 - e_1)$, and $\hat{u}_4 = sc_4(e_3 - e_2)$ for some positive integers c_1, c_2, c_3, c_4 such that

$$\text{lcm}(c_1, c_2) = \text{lcm}(c_1, c_4) = \text{lcm}(c_3, c_2) = \text{lcm}(c_3, c_4). \tag{22}$$

Proof. The first statement follows from the fact that $\mathbb{P} \text{Gl}(3, \mathbb{R})$ acts transitively on the set of generic 4-tuples of points of $\mathbb{R}\mathbb{P}^2$ and that the normal lines of a strictly convex polyhedral cone are generic (any subset of three elements is a basis).

In our case, the normal inward vectors are $\hat{u}_1 = c_1(e_1 + e_3)$, $\hat{u}_2 = c_2(e_2 + e_3)$, $\hat{u}_3 = c_3(e_3 - e_1)$, and $\hat{u}_4 = c_4(e_3 - e_2)$. Up to multiplication by a constant $s > 0$, we may assume that $c_i \in \mathbb{Q}$ for all i . Indeed, the fact that (\mathcal{C}_o, Λ) is rational (i.e. it is possible to choose $u_j \in \Lambda$) implies that $c_i/c_j \in \mathbb{Q}$. Moreover, there exists a primitive vector of \mathbb{Z}^4 , say $k = (k_1, k_2, k_3, k_4)$, such that $\sum_j k_j u_j = 0$. This vector is unique up to sign and, putting $C = \text{lcm}(c_1, c_2, c_3, c_4)$, we choose it to be

$$k = (C/c_1, -C/c_2, C/c_3, -C/c_4).$$

The Delzant–Lerman construction [Ler03] implies that the symplectic cone over the toric contact manifold is the quotient of

$$\tilde{X} = \left\{ (z_1, z_2, z_3, z_4) \in \mathbb{C}^4 \setminus \{0\} \mid \sum_{j=1}^4 k_j |z_j|^2 = 0 \right\}$$

by the action of S^1 , defined by $\rho: S^1 \times \tilde{X} \rightarrow \tilde{X}$,

$$\rho(\lambda, z) = ((\lambda^{k_1} z_1, \lambda^{k_3} z_3), (\lambda^{k_2} z_2, \lambda^{k_4} z_4)).$$

Lerman [Ler03] showed that the quotient constructed via this method from a rational polyhedral cone (\mathcal{C}_o, Λ) is smooth if and only if (\mathcal{C}_o, Λ) is a good cone. The stabilizer of a point $z \in \tilde{X}$ is determined by its vanishing components, precisely $\text{Stab}_\rho z = \{\lambda \in S^1 \mid \lambda^{k_j} = 1 \text{ if } z_j \neq 0\}$. One can then verify that the stabilizer group of each point $z \in \tilde{X}$ is trivial if and only if

$$\text{gcd}(k_1, k_2) = \text{gcd}(k_1, k_4) = \text{gcd}(k_3, k_2) = \text{gcd}(k_1, k_4) = 1. \tag{23}$$

Since $k_j = (-1)^{j-1} C/c_j$ and $|ab| = \text{lcm}(a, b)\text{gcd}(a, b)$, the condition (23) is equivalent to $\text{lcm}(c_1, c_2) = \text{lcm}(c_1, c_4) = \text{lcm}(c_3, c_2) = \text{lcm}(c_3, c_4) = C$. \square

Proof of Theorem 1.3. Let $(r_1, r_2, r_3, r_4) \in \mathbb{Z}^4$ with $p = r_1 = r_3$, $q = r_2 = r_4$ being coprime integers such that $p > 5q$. Let u be defined by (21). The labeled cone of (Δ_o, u) is identified with the cone \mathcal{C}_o labeled with the vectors

$$\hat{u}_1 = \frac{1}{p}(e_1 + e_3), \quad \hat{u}_2 = \frac{1}{q}(e_2 + e_3), \quad \hat{u}_3 = \frac{1}{p}(e_3 - e_1), \quad \hat{u}_4 = \frac{1}{q}(e_3 - e_2).$$

These vectors are all contained in a lattice Λ and the rational cone (\mathcal{C}_o, Λ) is good since it satisfies Lemma 4.4. In particular, (\mathcal{C}_o, Λ) is associated to the Wang–Ziller manifold $M_{p,q}^{1,1}$ with the toric contact structure (\mathbf{D}, \hat{T}) described in [BGS08b].

The set of characteristic labeled polytopes of (\mathcal{C}_o, Λ) is the Reeb family of (Δ_o, u) which satisfies the hypothesis of Lemma 4.3. Hence, there exist three Reeb vectors

$$b_o = (0, 0, 1) \quad \text{and} \quad b_\pm = \left(0, \pm \sqrt{1 - \frac{4q}{p-q}}, 1 \right)$$

whose respective characteristic labeled quadrilaterals have constant extremal affine functions. Via [Leg11, Theorem 1.4], there exist symplectic potentials, say $\phi_{b_o} \in \mathcal{S}(\Delta_{b_o}, u_{b_o})$ and $\phi_{\pm b} \in \mathcal{S}(\Delta_{b_{\pm}}, u_{b_{\pm}})$, for which $S(\phi_{b_o})$, $S(\phi_{b_+})$, and $S(\phi_{b_-})$ are constant. In particular, their respective Boothy–Wang symplectic potentials $\hat{\phi}_{b_o}$, $\hat{\phi}_{b_+}$, and $\hat{\phi}_{b_-}$ define toric Kähler cone metrics \hat{g}_{b_o} , \hat{g}_{b_+} , and \hat{g}_{b_-} on the symplectic cone over $(M_{p,q}^{1,1}, \mathbf{D}, \hat{T})$ for which the associated Sasaki metrics g_{b_o} , g_{b_+} , and g_{b_-} on $M_{p,q}^{1,1}$ have constant scalar curvature.

The Wang–Ziller manifold $M_{p,q}^{1,1}$ is diffeomorphic to $S^2 \times S^3$, see [BGS08b], and cannot carry a 3-Sasaki structure due to its dimension. Thus, it satisfies the hypothesis of Proposition 2.14. So, if there exists a diffeomorphism ψ such that $\psi^*g_{b_o}$ is a transversal homothety of g_{b_+} , then there exists a real number $\lambda > 0$ such that $(\lambda\Delta_{b_o}, u_{b_o})$ is equivalent to (Δ_{b_+}, u_{b_+}) in the sense of Definition 2.1. But, this cannot happen since Δ_{b_o} is a square while Δ_{b_+} is a trapezoid. Similarly, g_{b_o} and g_{b_-} are not isometric as Riemannian metrics even up to a transversal homothety.

On the other hand, the linear transform $A = -e_1 \otimes e_1^* - e_2 \otimes e_2^* + e_3 \otimes e_3^*$ preserves the set of normals and exchanges b_+ and b_- . Thus, A^* preserves \mathcal{C}_o and provides a \hat{T} -equivariant contactomorphism sending g_{b_+} to g_{b_-} , thanks to Proposition 2.15. \square

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REFERENCES

Abr98 M. Abreu, *Kähler geometry of toric varieties and extremal metrics*, Internat. J. Math. **9** (1998), 641–651.

Abr01 M. Abreu, *Kähler metrics on toric orbifolds*, J. Differential Geom. **58** (2001), 151–187.

Abr10 M. Abreu, *Kähler–Sasaki geometry of toric symplectic cones in action-angle coordinates*, Port. Math. (N.S.) **67** (2010), 121–153.

ACGT04 V. Apostolov, D. M. J. Calderbank, P. Gauduchon and C. Tønnesen-Friedman, *Hamiltonian 2-forms in Kähler geometry. II. Global classification*, J. Differential Geom. **68** (2004), 277–345.

BM92 A. Banyaga and P. Molino, *Géométrie des formes de contact complètement intégrable de type torique* (Séminaire Gaston Darboux, Montpellier, 1991–1992), 1–25.

BM96 A. Banyaga and P. Molino, *Complete integrability in contact geometry*, Pennsylvania State University Preprint PM 197, 1996.

Boy11 C. P. Boyer, *Extremal Sasakian metrics on S^3 -bundle over S^2* , Math. Res. Lett. **18** (2011), 181–189.

BG00 C. P. Boyer and K. Galicki, *A note on toric contact geometry*, J. Geom. Phys. **35** (2000), 288–298.

BG08 C. P. Boyer and K. Galicki, *Sasakian geometry*, Oxford Mathematical Monographs (Oxford University Press, Oxford, 2008).

BGS08a C. P. Boyer, K. Galicki and S. R. Simanca, *Canonical Sasakian metrics*, Comm. Math. Phys. **279** (2008), 705–733.

BGS08b C. P. Boyer, K. Galicki and S. R. Simanca, *The Sasaki cone and extremal Sasakian metrics*, in *Proceedings of the conference on Riemannian topology*, eds K. Galicki and S. R. Simanca (Birkhäuser, Boston, MA, 2008), 263–290.

BGL08 D. Burns, V. Guillemin and E. Lerman, *Kähler metrics on singular toric varieties*, Pacific J. Math. **238** (2008), 27–40.

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- Cal85 E. Calabi, *Extremal Kähler metrics. II*, in *Differential geometry and complex analysis*, eds I. Chavel and H. M. Farkas (Springer, Berlin, 1985), 95–114.
- CDG03 D. M. J. Calderbank, L. David and P. Gauduchon, *The Guillemin formula and Kähler metrics on toric symplectic manifolds*, *J. Symplectic Geom.* **1** (2003), 767–784.
- Del88 T. Delzant, *Hamiltoniens périodiques et images convexes de l'application moment*, *Bull. Soc. Math. France* **116** (1988), 315–339.
- Don02 S. K. Donaldson, *Scalar curvature and stability of toric varieties*, *J. Differential Geom.* **62** (2002), 289–349.
- Don05 S. K. Donaldson, *Interior estimates for solutions of Abreu's equation*, *Collect. Math.* **56** (2005), 103–142.
- Don08a S. K. Donaldson, *Kähler geometry on toric manifolds, and some other manifolds with large symmetry*, in *Handbook of geometric analysis. No. 1*, Advanced Lectures in Mathematics (ALM), vol. 7 (International Press, Somerville, MA, 2008), 29–75.
- Don08b S. K. Donaldson, *Extremal metrics on toric surfaces: a continuity method*, *J. Differential Geom.* **79** (2008), 389–432.
- Don09 S. K. Donaldson, *Constant scalar curvature metrics on toric surfaces*, *Geom. Funct. Anal.* **19** (2009), 83–136.
- FOW09 A. Futaki, H. Ono and G. Wang, *Transverse Kähler geometry of Sasaki manifolds and toric Sasaki–Einstein manifolds*, *J. Differential Geom.* **83** (2009), 585–636.
- Gun99 D. Guan, *On modified Mabuchi functional and Mabuchi moduli space of Kähler metrics on toric bundles*, *Math. Res. Lett.* **6** (1999), 547–555.
- Gui94 V. Guillemin, *Kähler structures on toric varieties*, *J. Differential Geom.* **40** (1994), 285–309.
- Leg11 E. Legendre, *Toric geometry of convex quadrilaterals*, *J. Symplectic Geom.* **9** (2011), 343–385.
- Lej10 M. Lejmi, *Extremal almost-Kähler metrics*, *Internat. J. Math.* **21** (2010), 1639–1662.
- Ler02 E. Lerman, *A convexity theorem for torus actions on contact manifolds*, *Illinois J. Math.* **46** (2002), 171–184.
- Ler03 E. Lerman, *Toric contact manifolds*, *J. Symplectic Geom.* **1** (2003), 785–828.
- LT97 E. Lerman and S. Tolman, *Hamiltonian torus actions on symplectic orbifolds and toric varieties*, *Trans. Amer. Math. Soc.* **349** (1997), 4201–4230.
- MSY06 D. Martelli, J. Sparks and S.-T. Yau, *The geometric dual of a – maximisation for toric Sasaki–Einstein manifolds*, *Comm. Math. Phys.* **268** (2006), 39–65.
- MSY08 D. Martelli, J. Sparks and S.-T. Yau, *Sasaki–Einstein manifolds and volume minimisation*, *Comm. Math. Phys.* **280** (2008), 611–673.
- Tia90 G. Tian, *On Calabi's conjecture for complex surfaces with positive first Chern class*, *Invent. Math.* **101** (1990), 101–172.
- Yau93 S.-T. Yau, *Open problems in geometry*, in *Differential geometry: partial differential equations on manifolds* (Los Angeles, CA, 1990), Proceedings of Symposia in Pure Mathematics, vol. 54, Part 1 (American Mathematical Society, Providence, RI, 1993), 1–28.

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