Bull. Aust. Math. Soc. **95** (2017), 149–156
doi:10.1017/S0004972716000708

BEST PROXIMITY POINT THEOREMS FOR CYCLIC QUASI-CONTRACTION MAPS IN UNIFORMLY CONVEX BANACH SPACES

NGUYEN VAN DUNG✉ and VO THI LE HANG

(Received 26 June 2016; accepted 7 July 2016; first published online 13 October 2016)

Abstract

In this paper we first give a negative answer to a question of Amini-Harandi ['Best proximity point theorems for cyclic strongly quasi-contraction mappings', *J. Global Optim.* **56** (2013), 1667–1674] on a best proximity point theorem for cyclic quasi-contraction maps. Then we prove some new results on best proximity point theorems that show that results of Amini-Harandi for cyclic strongly quasi-contractions are true under weaker assumptions.

2010 *Mathematics subject classification*: primary 47H09; secondary 54E05, 54H25.

Keywords and phrases: cyclic quasi-contraction, best proximity point, uniformly convex Banach space.

1. Introduction and preliminaries

In 2003, Kirk *et al.* studied fixed points for maps satisfying cyclical contractive conditions and presented an interesting extension of the Banach contraction principle [8, Theorem 1.1]. The cyclical contractive condition was then studied by many authors and was applied to optimisation and approximation (see [2, 5, 7, 9] and the references given there). For some critical remarks on generalisations of cyclic contractions, the reader may refer to [3].

In 2013, Amini-Harandi [1] introduced a new class of maps, called cyclic strongly quasi-contractions, which contains the cyclic contractions as a subclass. Amini-Harandi gave some convergence and existence results of best proximity point theorems for cyclic strongly quasi-contraction maps in uniformly convex Banach spaces and posed an open question on a best proximity point theorem for cyclic quasi-contraction maps. Dung and Radenović [4] found an error in the proof of [1, Theorem 2.3] and presented a counterexample.

In this paper we first give a negative answer to Amini-Harandi's question on a best proximity point theorem for cyclic quasi-contraction maps in uniformly convex Banach spaces [1, Question 2.8]. Then we prove some new results on best proximity point theorems that show that results for cyclic strongly quasi-contractions in [1] remain true under weaker assumptions. These results give affirmative answers to an open question on best proximity point theorems for cyclic strongly quasi-contraction maps in uniformly convex Banach spaces [4, Question 2.6]. For best proximity points and applications, the reader may refer to [6, 9, 10].

We now recall some definitions and properties which are useful in what follows.

Definition 1.1 [5, Definition 2.3]. Let A and B be nonempty subsets of a metric space (X, d) and let $T : A \cup B \longrightarrow A \cup B$ be a map. Then T is called a *cyclic contraction* if:

(1) $T(A) \subset B$ and $T(B) \subset A$;

(2) $d(Tx, Ty) \le kd(x, y) + (1 - k)d(A, B)$ for all $x \in A$, $y \in B$ and some $k \in [0, 1)$, where $d(A, B) = \inf\{d(x, y) : x \in A, y \in B\}$.

Definition 1.2 [1, Definitions 2.2 and 2.3]. Let A and B be nonempty subsets of a complete metric space (X, d) and let $T : A \cup B \longrightarrow A \cup B$ be such that $T(A) \subset B$ and $T(B) \subset A$. Then:

(1) T is called a *cyclic quasi-contraction* if for all $x \in A$ and $y \in B$ and some $c \in [0, 1)$,

$$d(Tx, Ty) \le c \max\{d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\} + (1 - c)d(A, B); \tag{1.1}$$

(2) T is called a *cyclic strongly quasi-contraction* if it is a cyclic quasi-contraction and for all $x \in A$ and $y \in B$,

$$d(T^2x, T^2y) \le cd(x, y) + (1 - c)d(A, B). \tag{1.2}$$

Amini-Harandi proved the following two best proximity point results on cyclic strongly quasi-contraction maps.

Theorem 1.3 [1, Theorem 2.5]. *Let A and B be nonempty closed subsets of a uniformly convex Banach space X such that A is convex and let $T : A \cup B \longrightarrow A \cup B$ be a cyclic strongly quasi-contraction map. For $x_0 \in A$, define $x_{n+1} = Tx_n$ for each $n \ge 0$. Then, for each $\varepsilon > 0$, there exists n_0 such that for all $m > n \ge n_0$, $\|x_{2m} - x_{2n+1}\| < d(A, B) + \varepsilon$.*

Theorem 1.4 [1, Theorem 2.6]. *Let A and B be nonempty closed convex subsets of a uniformly convex Banach space X and let $T : A \cup B \longrightarrow A \cup B$ be a cyclic strongly quasi-contraction map. For $x_0 \in A$, define $x_{n+1} = Tx_n$ for each $n \ge 0$. Then there exists a unique $x \in A$ such that $\lim_{n\to\infty} x_{2n} = x$, $T^2x = x$ and $\|x - Tx\| = d(A, B)$.*

Amini-Harandi asked whether such results remain true for cyclic quasi-contraction maps.

Question 1.5 [1, Question 2.8]. Does the conclusion of Theorem 1.4 remain true for cyclic quasi-contraction maps?

https://doi.org/10.1017/S0004972716000708 Published online by Cambridge University Press

As noted, Dung and Radenović [4] recently gave a counterexample to some of the statements in [1] which were used in the proofs of the Theorems 1.3 and 1.4.

Example 1.6 [4, Example 2.1]. Let $X = \mathbb{R}^2$ with the Euclidean norm and set

$$M = (0,0), N = (2,0), P = (2,1), Q = (0,1), I = (1, \tfrac{1}{2}).$$

Let $A = [M, P]$, $B = \{N, Q\}$ and define $T : A \cup B \longrightarrow A \cup B$ by

$$Tx = N \quad \text{for } x \in [M, N] \quad \text{and} \quad Tx = Q \quad \text{for } x \in (I, P]$$

and $TN = P$, $TQ = M$. Then:

(1) X is a uniformly convex Banach space;
(2) A and B are nonempty subsets of X, A is convex and $TA \subset B, TB \subset A$;
(3) T is a cyclic quasi-contraction;
(4) there exist $x_0 \in A \cup B$ and $x_{n+1} = Tx_n$ for all $n \geq 0$ such that

$$\lim_{n\to\infty} d(x_n, x_{n+1}) \neq d(A, B), \quad \lim_{n\to\infty} \|x_{2n+2} - x_{2n}\| \neq 0, \quad \lim_{n\to\infty} \|x_{2n+3} - x_{2n+1}\| \neq 0;$$

(5) T and T^2 are fixed point free.

This is not a counterexample to Theorems 1.3 and 1.4 because B is not convex. It therefore suggests a further question.

Question 1.7 [4, Question 2.6]. Prove or disprove Theorems 1.3 and 1.4.

2. Main results

First we show that there exist nonempty closed convex subsets A and B of a uniformly convex Banach space X, a cyclic quasi-contraction map $T : A \cup B \longrightarrow A \cup B$ and some $x_0 \in A$ such that the sequence $\{x_{2n}\}$ defined by $x_{n+1} = Tx_n$ for each $n \geq 0$ is not convergent. So the answer to Question 1.5 is negative.

Example 2.1. Let $X = \mathbb{R}^2$ with the Euclidean norm, $a = (0,0)$, $b = (2,0)$, $c = (2,2)$, $e = (0,2)$, $f = (1,1)$, $A = [a, c]$, $B = [b, e]$ and

$$Ta = b, \quad Tb = c, \quad Tc = e, \quad Te = a, \quad Tx = f \quad \text{for } x \in (a, c) \quad \text{or} \quad x \in (b, e).$$

Then:

(1) X is a uniformly convex Banach space and A and B are nonempty closed convex sets in X;
(2) $T : A \cup B \longrightarrow A \cup B$ is a cyclic quasi-contraction map;
(3) for $x_0 = a \in A$, the sequence $\{x_{2n}\}$ is not convergent, where $x_{n+1} = Tx_n$ for each $n \geq 0$.

Proof.

(1) It is clear.

(2) By definition of T, we have $TA \subset B$ and $TB \subset A$. Moreover, $d(A, B) = 0$. We will check that T satisfies (1.1) by exhausting the following cases.

https://doi.org/10.1017/S0004972716000708 Published online by Cambridge University Press

Case 1. $x = a, y = b$. Then $d(Tx, Ty) = d(b, c) = 2$ and $d(x, Ty) = d(a, c) = 2\sqrt{2}$. So $d(Tx, Ty) \le \frac{1}{\sqrt{2}} d(x, Ty)$.

Case 2. $x = a, y \in (b, e)$. Then $d(Tx, Ty) = d(b, f) = \sqrt{2}$ and $d(x, Tx) = d(a, b) = 2$. So $d(Tx, Ty) \le \frac{1}{\sqrt{2}} d(x, Tx)$.

Case 3. $x = a, y = e$. Then $d(Tx, Ty) = d(b, a) = 2$ and $d(y, Tx) = d(e, b) = 2\sqrt{2}$. So $d(Tx, Ty) \le \frac{1}{\sqrt{2}} d(y, Tx)$.

Case 4. $x \in (a, c), y = b$. Then $d(Tx, Ty) = d(f, c) = \sqrt{2}$ and $d(y, Ty) = d(b, c) = 2$. So $d(Tx, Ty) \le \frac{1}{\sqrt{2}} d(y, Ty)$.

Case 5. $x \in (a, c), y \in (b, e)$. Then $d(Tx, Ty) = d(f, f) = 0$.

Case 6. $x \in (a, c), y = e$. Then $d(Tx, Ty) = d(f, a) = \sqrt{2}$ and $d(y, Ty) = d(e, a) = 2$. So $d(Tx, Ty) \le \frac{1}{\sqrt{2}} d(y, Ty)$.

Case 7. $x = c, y = b$. Then $d(Tx, Ty) = d(e, c) = 2$ and $d(y, Tx) = d(b, e) = 2\sqrt{2}$. So $d(Tx, Ty) \le \frac{1}{\sqrt{2}} d(y, Tx)$.

Case 8. $x = c, y \in (b, e)$. Then $d(Tx, Ty) = d(e, f) = \sqrt{2}$ and $d(x, Tx) = d(c, e) = 2$. So $d(Tx, Ty) \le \frac{1}{\sqrt{2}} d(x, Tx)$.

Case 9. $x = c, y = e$. Then $d(Tx, Ty) = d(e, a) = 2$ and $d(x, Ty) = d(c, a) = 2\sqrt{2}$. So $d(Tx, Ty) \le \frac{1}{\sqrt{2}} d(x, Ty)$.

By the above nine cases, $d(Tx, Ty) \le \frac{1}{\sqrt{2}} \max\{d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\}$. So (1.1) holds for any $c \in [\frac{1}{\sqrt{2}}, 1)$.

(3) For $x_0 = a \in A$, we have $x_1 = Ta = b$, $x_2 = Tb = c$, $x_3 = Tx = e$, $x_4 = Te = a, \ldots, x_{4n} = a$, $x_{4n+1} = b$, $x_{4n+2} = c$, $x_{4n+3} = e, \ldots$. This proves that the sequence $\{x_{2n}\}$ is not convergent. □

Next we prove that the conclusions of Theorems 1.3 and 1.4 remain true for maps satisfying only the contraction condition (1.2) in uniformly convex Banach spaces. This answers Question 1.7 and proves Theorems 1.3 and 1.4 for this wider class of maps.

The following two lemmas state particular properties of uniformly convex Banach spaces that will be used later.

Lemma 2.2 [5, Lemma 3.7]. *Let A be a nonempty closed convex subset and B a nonempty closed subset of a uniformly convex Banach space. Let $\{x_n\}$, $\{z_n\}$ be two sequences in A and $\{y_n\}$ a sequence in B such that:*

(1) $\lim_{n\to\infty} \|z_n - y_n\| = d(A, B)$;

(2) *for each $\varepsilon > 0$, there exists n_0 such that for all $m > n \ge n_0$,*

$$\|x_m - y_n\| \le d(A, B) + \varepsilon.$$

Then, for each $\varepsilon > 0$, there exists n_1 such that for all $m > n \ge n_1$, $\|x_m - z_n\| \le \varepsilon$.

LEMMA 2.3 [5, Lemma 3.8]. *Let A be a nonempty closed convex subset and B a nonempty closed subset of a uniformly convex Banach space. Let $\{x_n\}$, $\{z_n\}$ be two sequences in A and $\{y_n\}$ a sequence in B such that*

$$\lim_{n\to\infty} \|x_n - y_n\| = \lim_{n\to\infty} \|z_n - y_n\| = d(A, B).$$

Then $\lim_{n\to\infty} \|x_n - z_n\| = 0$.

THEOREM 2.4. *Let A and B be nonempty closed subsets of a uniformly convex Banach space X such that A is convex and let $T : A \cup B \longrightarrow A \cup B$ be a map satisfying (1.2). For $x_0 \in A$, define $x_{n+1} = Tx_n$ for each $n \geq 0$. Then:*

(1) *for each $\varepsilon > 0$, there exists n_0 such that for all $m > n \geq n_0$,*

$$\|x_{2m} - x_{2n+1}\| < d(A, B) + \varepsilon;$$

(2) *there exists a unique $x \in A$ such that $\lim_{n\to\infty} x_{2n} = x$, $T^2x = x$.*

PROOF. (1) By (1.2),

$$\begin{aligned} d(x_{2n}, x_{2n+1}) &= d(T^{2n}x_0, T^{2n+1}x_0) = d(T^2T^{2n-2}x_0, T^2T^{2n-2}Tx_0) \\ &\leq cd(T^{2n-2}x_0, T^{2n-2}Tx_0) + (1-c)d(A, B) \\ &\leq c[cd(T^{2n-4}x_0, T^{2n-4}Tx_0) + (1-c)d(A, B)] + (1-c)d(A, B) \\ &= c^2d(x_{2n-4}, x_{2n-3}) + (1-c^2)d(A, B). \end{aligned} \tag{2.1}$$

By induction, $d(x_{2n}, x_{2n+1}) \leq c^n d(x_0, x_1) + (1 - c^n)d(A, B)$. Letting $n \to \infty$ yields

$$\lim_{n\to\infty} d(x_{2n}, x_{2n+1}) = d(A, B). \tag{2.2}$$

Similarly,

$$\begin{aligned} d(x_{2n+2}, x_{2n+1}) &= d(T^{2n+2}x_0, T^{2n+1}x_0) = d(T^2T^{2n}x_0, T^2T^{2n-2}Tx_0) \\ &\leq cd(T^{2n}x_0, T^{2n-2}Tx_0) + (1-c)d(A, B) \\ &\leq c[cd(T^{2n-2}x_0, T^{2n-4}Tx_0) + (1-c)d(A, B)] + (1-c)d(A, B) \\ &= c^2d(x_{2n-2}, x_{2n-3}) + (1-c^2)d(A, B). \end{aligned}$$

By induction, $d(x_{2n+2}, x_{2n+1}) \leq c^n d(x_2, x_1) + (1 - c^n)d(A, B)$ and letting $n \to \infty$ yields

$$\lim_{n\to\infty} \|x_{2n+2} - x_{2n+1}\| = \lim_{n\to\infty} d(x_{2n+2}, x_{2n+1}) = d(A, B). \tag{2.3}$$

From (2.2), (2.3) and Lemma 2.3,

$$\lim_{n\to\infty} \|x_{2n+2} - x_{2n}\| = 0. \tag{2.4}$$

Repeating the argument with Tx_0 playing the role of x_0, we also find that

$$\lim_{n\to\infty} \|x_{2n+3} - x_{2n+1}\| = 0. \tag{2.5}$$

https://doi.org/10.1017/S0004972716000708 Published online by Cambridge University Press

Now, suppose contrary to (1), that there exists $\varepsilon > 0$ such that for all k there are $m_k > n_k \geq k$ satisfying $\|x_{2m_k} - x_{2n_k+1}\| \geq d(A, B) + \varepsilon$. We can choose m_k such that it is the least integer greater than n_k to satisfy the above inequality. Consequently, we find that $\|x_{2m_k-2} - x_{2n_k+1}\| < d(A, B) + \varepsilon$ and so

$$\begin{aligned} d(A, B) + \varepsilon &\leq \|x_{2m_k} - x_{2n_k+1}\| \\ &\leq \|x_{2m_k} - x_{2m_k-2}\| + \|x_{2m_k-2} - x_{2n_k+1}\| \\ &\leq \|x_{2m_k} - x_{2m_k-2}\| + d(A, B) + \varepsilon. \end{aligned}$$

Letting $k \to \infty$ and using (2.4),

$$\lim_{n\to\infty} \|x_{2m_k} - x_{2n_k+1}\| = d(A, B) + \varepsilon. \tag{2.6}$$

By (2.1),

$$\begin{aligned} \|x_{2m_k} - x_{2n_k+1}\| &\leq \|x_{2m_k} - x_{2m_k+2}\| + \|x_{2m_k+2} - x_{2n_k+3}\| + \|x_{2m_k+3} - x_{2n_k+1}\| \\ &\leq \|x_{2m_k} - x_{2m_k+2}\| + c^2\|x_{2m_k} - x_{2n_k+1}\| + (1 - c^2)d(A, B) + \|x_{2m_k+3} - x_{2n_k+1}\|. \end{aligned}$$

Letting $k \to \infty$ and using (2.4), (2.5) and (2.6),

$$d(A, B) + \varepsilon \leq c^2[d(A, B) + \varepsilon] + (1 - c^2)d(A, B) = d(A, B) + c^2\varepsilon.$$

This is a contradiction. Therefore, for each $\varepsilon > 0$, there exists n_0 such that for all $m > n \geq n_0$, $\|x_{2m} - x_{2n+1}\| < d(A, B) + \varepsilon$.

(2) By (1), for each $\varepsilon > 0$, there exists n_0 such that for all $m > n \geq n_0$, $\|x_{2m} - x_{2n+1}\| < d(A, B) + \varepsilon$. By (2.3), $\lim_{n\to\infty} \|x_{2n+2} - x_{2n+1}\| = d(A, B)$. By Lemma 2.2, we conclude that for each $\varepsilon > 0$ there exists n_1 such that for all $m > n \geq n_1$, $\|x_{2m} - x_{2n+2}\| \leq \varepsilon$. Thus, $\{x_{2n}\}$ is a Cauchy sequence in X and there exists $x \in X$ such that

$$\lim_{n\to\infty} x_{2n} = x. \tag{2.7}$$

Since $x_{2n} \in A$ for all n and A is closed, $x \in A$. So

$$d(A, B) \leq d(x_{2n+1}, T^2x) \leq d(x_{2n+1}, x_{2n}) + d(x_{2n}, T^2x) \leq d(x_{2n+1}, x_{2n}) + d(x_{2n-2}, x).$$

Letting $n \to \infty$ and using (2.3),

$$\lim_{n\to\infty} d(x_{2n+1}, T^2x) = d(A, B). \tag{2.8}$$

From (2.3), (2.8) and Lemma 2.2, we find that for each $\varepsilon > 0$ there exists n_2 such that for all $n \geq n_2$, $\|x_{2n+2} - T^2x\| \leq \varepsilon$. It implies that

$$\lim_{n\to\infty} x_{2n+2} = T^2x. \tag{2.9}$$

From (2.7) and (2.9), we deduce that $x = T^2x$. □

Corollary 2.5 [1, Theorem 2.6]. *Let A and B be nonempty closed convex subsets of a uniformly convex Banach space X and $T : A \cup B \longrightarrow A \cup B$ a cyclic strongly quasi-contraction map. For $x_0 \in A$, define $x_{n+1} = Tx_n$ for each $n \geq 0$. Then there exists a unique $x \in A$ such that* $\lim_{n\to\infty} x_{2n} = x$, *$T^2x = x$ and $\|x - Tx\| = d(A, B)$.*

https://doi.org/10.1017/S0004972716000708 Published online by Cambridge University Press

PROOF. By Theorem 2.4, there exists a unique $x \in A$ such that $\lim_{n\to\infty} x_{2n} = x$, $T^2x = x$. We will prove that $\|x - Tx\| = d(A, B)$. By (1.1),

$$\begin{aligned} d(x_{2n+2}, Tx) &\le d(x_{2n+2}, x_{2n+1}) + d(x_{2n+1}, Tx) \\ &\le d(x_{2n+2}, x_{2n+1}) + c \max\{d(x_{2n}, x), d(x_{2n+1}, x_{2n}), d(x, Tx), d(x_{2n}, Tx), d(x, x_{2n+1})\} \\ &\quad + (1 - c)d(A, B). \end{aligned}$$

Letting $n \to \infty$,

$$\begin{aligned} d(x, Tx) &\le c \max\{0, d(A, B), d(x, Tx), d(x, Tx), 0\} + (1 - c)d(A, B) \\ &\le cd(x, Tx) + (1 - c)d(A, B). \end{aligned} \tag{2.10}$$

If $d(A, B) < d(x, Tx)$, then, from (2.10), $d(x, Tx) < d(x, Tx)$, which is a contradiction. So $d(A, B) = d(x, Tx) = \|x - Tx\|$. □

Acknowledgements

The authors are greatly indebted to the anonymous reviewers for their helpful comments.

References

[1] A. Amini-Harandi, 'Best proximity point theorems for cyclic strongly quasi-contraction mappings', *J. Global Optim.* **56** (2013), 1667–1674.

[2] S. S. Basha, N. Shahzad and R. Jeyaraj, 'Best proximity points: approximation and optimization', *Optim. Lett.* **7**(1) (2013), 145–155.

[3] N. V. Dung and V. T. L. Hang, 'Remarks on cyclic contractions in *b*-metric spaces and applications to integral equations', *Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. RACSAM* (2016), 9 pages doi:10.1007/s13398-016-0291-5.

[4] N. V. Dung and S. Radenović, 'Remarks on theorems for cyclic quasi-contractions in uniformly convex Banach spaces', *Kragujevac J. Math.* **40**(2) (2016), 272–279.

[5] A. A. Eldred and P. Veeramani, 'Existence and convergence of best proximity points', *J. Math. Anal. Appl.* **323** (2006), 1001–1006.

[6] M. Gabeleh, 'Best proximity point theorems via proximal non-self mappings', *J. Optim. Theory Appl.* **164**(2) (2015), 565–576.

[7] E. Karapinar, W. S. Du, P. Kumam, A. Petruşel and S. Romaguera (eds.), *Optimization Problems Via Best Proximity Point Analysis*, Abstract and Applied Analysis (Hindawi, New York, 2014).

[8] W. A. Kirk, P. S. Srinivasan and P. Veeramani, 'Fixed points for mappings satisfying cyclical contractive conditions', *Fixed Point Theory* **4**(1) (2003), 79–89.

[9] C. Mongkolkeha and P. Kumam, 'Best proximity point theorems for generalized cyclic contractions in ordered metric spaces', *J. Optim. Theory Appl.* **155**(1) (2012), 215–226.

[10] B. Samet, 'Some results on best proximity points', *J. Optim. Theory Appl.* **159** (2013), 281–291.

NGUYEN VAN DUNG, Nonlinear Analysis Research Group,
Ton Duc Thang University,
Ho Chi Minh City, Viet Nam

https://doi.org/10.1017/S0004972716000708 Published online by Cambridge University Press

and
Faculty of Mathematics and Statistics,
Ton Duc Thang University, Ho Chi Minh City, Viet Nam
e-mail: nguyenvandung2@tdt.edu.vn

VO THI LE HANG,
Faculty of Mathematics and Information Technology Teacher Education,
Dong Thap University, 783 Pham Huu Lau Street,
Ward 6, Cao Lanh City, Dong Thap Province, Viet Nam
e-mail: vtlhang@dthu.edu.vn