

ON WEAKLY POSITIVE MATRICES

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A matrix is said to be positive definite if it is hermitian and if all of its characteristic values are positive. It is well known,¹ and easy to prove, that the necessary and sufficient condition for a matrix P to be positive definite is that its hermitian quadratic form

$$(1) \quad (v, Pv) > 0$$

with any vector $v \neq 0$ be positive. (This will imply, in the present article, that it is real.) It is easy to see from (1) that if P_1 and P_2 are positive definite, the same holds of $a_1P_1 + a_2P_2$ if a_1 and a_2 are positive numbers. Similarly, if P is positive definite, so is $X^\dagger PX$ for any non-singular X , where X^\dagger is the hermitian adjoint of X . Indeed, the hermitian quadratic form of $X^\dagger PX$ with the vector v ,

$$(1a) \quad (v, X^\dagger PXv) = (Xv, PXv),$$

is equal to the hermitian quadratic form of P with the vector Xv and is, therefore, positive.

We shall call a matrix W weakly positive if all of its characteristic values are positive and the characteristic vectors form a complete set. An equivalent definition is that W have a diagonal form with (real) positive diagonal elements

$$(2) \quad W = XDX^{-1},$$

where X is a non-singular matrix and D a diagonal matrix with positive diagonal elements.

Clearly, a matrix is positive definite if it is weakly positive and hermitian. In this case, the X of (2) can be assumed to be unitary. Conversely, a matrix which is positive definite is also weakly positive. It is also clear that a weakly positive matrix is necessarily non-singular and that its reciprocal is also weakly positive. Similarly, the reciprocal of a positive definite matrix is also positive definite.

Let us note, finally, that any XPX^{-1} is weakly positive if X is non-singular and P positive definite because a positive definite P can be written as UDU^{-1} with a unitary U so that XPX^{-1} becomes $(XU)D(XU)^{-1}$.

THEOREM 1. *A matrix is weakly positive if and only if it can be written as the product of two positive definite matrices*

$$(3) \quad W = P_1P_2.$$

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¹See, for instance, P. R. Halmos, *Finite dimensional vector spaces* (Princeton University Press, 1942), § 64).

To prove this, one notes that the W of (2) is equal to P_1P_2 , where

$$(4) \quad P_1 = XX^\dagger, \quad P_2 = (X^\dagger)^{-1}DX^{-1}$$

or

$$(5) \quad P_1 = XDX^\dagger, \quad P_2 = (XX^\dagger)^{-1},$$

where the dagger denotes the hermitian adjoint. All P_1, P_2 in (4) and (5) are positive definite because they have the form (1a) with $P = 1$ or $P = D$ and various non-singular X .

The converse, that P_1P_2 is weakly positive if P_1 and P_2 are positive definite can be demonstrated in two steps. First, if λ is a characteristic value of P_1P_2 ,

$$(6) \quad P_1P_2v = \lambda v,$$

one can form the scalar product of (6) with P_2v :

$$(6a) \quad (P_2v, P_1P_2v) = \lambda(P_2v, v) = \lambda(v, P_2v).$$

The left side is the hermitian quadratic form of P_1 with the non-vanishing vector P_2v , and hence positive. Since P_2 is positive definite, (v, P_2v) is also positive and λ becomes the ratio of two positive numbers.

If P_1P_2 had a deficiency of characteristic vectors, there would be a characteristic value λ for which

$$(7) \quad P_1P_2v = \lambda v \quad \text{and} \quad P_1P_2w = \lambda w + v$$

would hold² with a suitable w . The scalar product of the first equation with P_2w is

$$(7a) \quad (P_2w, P_1P_2v) = \lambda(P_2w, v) \quad \text{or} \quad (P_2w, P_1P_2v) = \lambda(w, P_2v).$$

Similarly, the scalar product of the second equation (7) with P_2v is

$$(7b) \quad (P_1P_2w, P_2v) = \lambda(w, P_2v) + (v, P_2v).$$

Since P_1 is hermitian, the left sides of (7b) and of (7a) are equal. It then follows that $(v, P_2v) = 0$ —a contradiction since P_2 is positive definite. Hence, (7) cannot hold and the characteristic vectors of P_1P_2 form a complete set.

The second part of Theorem 1 also follows from the fact that, if P_2 is positive definite, there is a square root q of $P_2 = q^2$ which is also positive definite.³ However,

$$(8) \quad P_1P_2 = P_1q^2 = q^{-1} \cdot qP_1q \cdot q$$

so that, if v' is a characteristic vector of the positive matrix qP_1q ,

$$(8a) \quad qP_1qv' = \lambda v'.$$

$q^{-1}v' = v$ is a characteristic vector of P_1P_2 :

²See any description of the so-called Jordan canonical form, for instance, Halmos, Appendix I, or N. Dunford and J. T. Schwartz, *Linear operators*, Part I (Interscience Publishers, New York, 1958), p. 563, example 17.

³See, for instance, Halmos, § 66, or Dunford and Schwartz, p. 569, Theorem 11.

$$(8b) \quad P_1P_2v = q^{-1} \cdot qP_1q \cdot v' = q^{-1}\lambda v' = \lambda v.$$

Hence, the characteristic values of P_1P_2 and of qP_1q are the same. Further, since every vector can be expressed linearly in terms of the v' , the expression for the vector $q^{-1}x$ in terms of the v' gives the expression for x in terms of the v so that these indeed form a complete set.

THEOREM 2. *If the product of a positive definite and of a weakly positive matrix is hermitian, it is also positive definite. The same holds of the product $P_1P_2P_3$ of three positive definite matrices P_1, P_2, P_3 if it is hermitian.*

Since a weakly positive matrix is the product of two positive definite matrices, it suffices to prove the second part of the theorem. This is a generalization, to three factors, of the well-known theorem⁴ that, if the product of two positive definite matrices is hermitian, it is also positive definite.

In order to prove Theorem 2, consider the matrices

$$(9) \quad H(\lambda) = [(1 - \lambda)P_3 + \lambda P_1]P_2P_3$$

with positive real $\lambda \leq 1$. As a linear combination of two hermitian matrices $P_3P_2P_3$ and $P_1P_2P_3$ with positive coefficients, $H(\lambda)$ is itself hermitian, its characteristic values real. For $\lambda = 0$ they are all positive since P_2 and hence also $P_3P_2P_3$ is positive definite (see (1a)). As λ increases, the characteristic values change continuously and they will remain positive for all $\lambda \leq 1$ unless there is a 0 characteristic value for some intermediate λ . This is not possible, however, because, if v is non-zero, the same holds for P_3v , and hence for P_2P_3v , and finally for $H(\lambda)v$. The last conclusion follows from the positive definite nature of $(1 - \lambda)P_3 + \lambda P_1$ which is (for $0 \leq \lambda \leq 1$) a linear combination of positive definite matrices with positive coefficients.

Remarks. It may be worth noting that $a_1W_1 + a_2W_2$ need not be weakly positive for weakly positive W_1, W_2 and positive a_1, a_2 . Thus, if $a_1 = a_2 = 1$,

$$(10) \quad W_1 = \begin{vmatrix} 10 & 6 \\ 1 & 1 \end{vmatrix}, \quad W_2 = \begin{vmatrix} 10 & 1 \\ 6 & 1 \end{vmatrix},$$

the determinant of $W_1 + W_2$ is negative and hence $W_1 + W_2$ is not weakly positive. It is also easy to find two weakly positive matrices, or four positive definite matrices, the product of which is not positive even though it is hermitian. Two such W , with their product, are

$$\begin{vmatrix} 1 & 1+\epsilon \\ -\epsilon & -\epsilon \end{vmatrix} \cdot \begin{vmatrix} 1 & \epsilon+\epsilon^2 \\ -1 & -\epsilon \end{vmatrix} = \begin{vmatrix} -\epsilon & 0 \\ 0 & -\epsilon^3 \end{vmatrix}$$

with a sufficiently small but positive ϵ .

Theorem 3 will extend Theorems 1 and 2 to bounded linear operators in Hilbert space. For this purpose, the notions of positive definiteness and weak

⁴See, for instance, Halmos, §§ 56, 72.

positivity have to be formulated for such operators. A bounded linear operator will be called positive definite if it is self-adjoint and if it has a positive lower bound so that

$$(11) \quad (v, Pv) > b(v, v), \quad b > 0.$$

Such an operator has a reciprocal which is positive definite and also a square root which is also positive definite.⁵

A bounded linear operator will be called weakly positive if it can be expressed in the form

$$(12) \quad W = XDX^{-1},$$

where D is positive definite and both X and its reciprocal X^{-1} are bounded. Since X can be written⁶ as $X = HU$ with a self-adjoint, in fact positive definite, H , and a unitary U , and since UDU^{-1} is also a positive definite operator, it can be assumed that all three factors in (12) are positive definite.

Let E_λ be the self-adjoint projection operators in the spectral decomposition of D ,

$$(13) \quad D = \int \lambda dE_\lambda.$$

W will also have a spectral decomposition

$$(14) \quad W = \int \lambda dF_\lambda,$$

where, however, the projection operators

$$(14a) \quad F_\lambda = XE_\lambda X^{-1}$$

are not, in general, self-adjoint. They have the multiplication properties of the E_λ

$$(14b) \quad F_\lambda F_\mu = F_\mu F_\lambda = F_\lambda \quad \text{for } \lambda \leq \mu.$$

Note that the integrals in (13) and (14) have to be extended only between the positive lower and upper bounds of D , i.e., over a finite positive interval.

THEOREM 3. *Theorems 1 and 2 hold also for bounded linear operators in Hilbert space.*

The first part of the extension of Theorem 1 follows from the fact that the P_1 and P_2 given by (4) or (5) satisfy (3) identically. As was mentioned before, it can be assumed that X is self-adjoint so that

⁵See, for instance, F. Riesz and B. Sz-Nagy, *Leçons d'analyse fonctionnelle* (Akadémiai Kiadó, Budapest, 1953), Section 120, or F. J. Murray, *Introduction to linear transformations in Hilbert space* (Princeton University Press, 1941), Chap. VIII, § 3.

⁶See Riesz and Nagy, Section 110. Since X has a bounded reciprocal, the same holds for $H = (X^\dagger X)^\dagger$ and Hv covers the whole Hilbert space. Hence, according to Riesz and Nagy, U is isometric rather than merely quasi-isometric. Further, if there were a vector W orthogonal to all Uv , then $H^{-1}w$ would be orthogonal to all $HUv = Xv$. This is not possible if X has a reciprocal. Hence U is unitary in the case considered here.

$$(15) \quad W = X^2 \cdot X^{-1}DX^{-1}$$

is a possible decomposition of W into the product of two positive definite operators. Indeed, both X^2 and $X^{-1}DX^{-1}$, as well as the reciprocals of these operators, are bounded. Thus, if b and B are the lower and upper bounds of D ,

$$(15a) \quad b(X^{-1}v, X^{-1}v) < (X^{-1}v, DX^{-1}v) = (v, X^{-1}DX^{-1}v) < B(X^{-1}v, X^{-1}v)$$

and if ξ and Ξ are lower and upper bounds of X ,

$$(15b) \quad \Xi^{-2}(v, v) < (X^{-1}v, X^{-1}v) < \xi^{-2}(v, v)$$

so that

$$(15c) \quad b\Xi^{-2}(v, v) < (v, X^{-1}DX^{-1}v) < \xi^{-2}B(v, v),$$

i.e., $X^{-1}DX^{-1}$ has positive upper and lower bounds. The same can be demonstrated for the other P which occur in (4) or (5).

The second part of the extension of Theorem 1 follows by the argument contained in equations (8). The existence of a positive definite square root of a positive definite operator was mentioned before and bounds for $D = qP_1q$ can be established in the same way as they were established for $X^{-1}DX^{-1}$.

As to the extension of Theorem 2, one can note that a lower bound of $H(\lambda)^2$ is $b^2b_2^2b_3^2$, where b_2 and b_3 are the lower bounds for P_2 and P_3 and b is the lower bound of P_1 or P_3 , whichever is smaller. Hence, b is also a lower bound for $P_\lambda = (1 - \lambda)P_3 + \lambda P_1$ for every λ between 0 and 1. It then follows that b^2 is a lower bound for P_λ^2 , so that

$$\begin{aligned} (v, H(\lambda)^2v) &= (H(\lambda)v, H(\lambda)v) = (P_\lambda P_2 P_3 v, P_\lambda P_2 P_3 v) \\ &= (P_2 P_3 v, P_\lambda^2 P_2 P_3 v) > b^2 (P_2 P_3 v, P_2 P_3 v) \\ &= b^2 (P_3 v, P_2^2 P_3 v) > b^2 b_2^2 (P_3 v, P_3 v) > b^2 b_2^2 b_3^2 (v, v). \end{aligned}$$

It then follows that the spectrum of $H(\lambda)$ does not extend over the interval $(-bb_2b_3, bb_2b_3)$ as long as $0 \leq \lambda \leq 1$. However, the lower bound of $H(\lambda)$ is a continuous function of λ and since it is larger than bb_2b_3 for $\lambda = 0$ and cannot enter the $(-bb_2b_3, bb_2b_3)$ interval, it always stays above this interval. Hence, $H(\lambda)$ and, in particular, $H(1) = P_1P_2P_3$ are positive definite.

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