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THE THEORY OF COMPOSITIONS (I): THE ORDERED FACTORIZATIONS OF n AND A CONJECTURE OF C. LONG

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1. **Introduction.** Several years ago, C. Long wrote two papers ([3], [4]) that related to $F(n)$ the number of ordered factorizations of n. The second of these papers [4] was devoted entirely to a discussion of conjectured formula for $F(n)$. In this paper, Long's conjecture will be proved as

THEOREM 3 (LONG'S CONJECTURE). *If* $1<n=p_1^{\alpha_1}p_2^{\alpha_2}\dots p_r^{\alpha_r}$ *is the prime factorization of* n, *then* $F(n)$ *is the number obtained if the polynomial*

$$2^{\alpha_1-1}\prod_{i=2}^{r}\{x_1x_2\cdots x_{i-1}+(1+x_1)(1+x_2)\cdots(1+x_{i-1})\}^{\alpha_i}$$

is fully expanded and then each x_i^k *is replaced by* $\binom{\alpha_i}{\alpha_{i+1}+\cdots+\alpha_r-k}$ for $1\le i\le r-1$, $0\le k\le \alpha_{i+1}+\alpha_{i+2}+\cdots+\alpha_r$.

In Section 2 we shall prove analytically the following result which will be the essential key to the proof of Long's conjecture.

THEOREM 1. *If* $1<n=p_1^{\alpha_1}p_2^{\alpha_2}\dots p_r^{\alpha_r}$ *is the prime factorization of* n, *then*

$$F(n)=\frac{1}{2}\sum_{h_1\ge 0\cdots h_r\ge 0}\binom{\alpha_1}{h_1}\binom{\alpha_2}{h_2}\cdots\binom{\alpha_r}{h_r}\times\binom{\alpha_1+h_2+\cdots+h_r}{h_2+\cdots+h_r}\binom{\alpha_2+h_3+\cdots+h_r}{h_3+\cdots+h_r}\cdots\binom{\alpha_{r-1}+h_r}{h_r}\tag{1.1}$$

In Section 3 we shall prove Theorem 1 combinatorially. Section 4 considers a refinement of Theorem 1 to $F(n;\rho)$, the number of ordered factorizations of n with ρ factors. In Section 5, we prove Long's conjecture.

2. **Analytic proof of Theorem 1.** Let $g(\alpha_1,\dots,\alpha_r)$ denote the right hand side of (1.1). Then by the binomial series

$$\begin{aligned}G(t_1,t_2,\dots,t_r)&\sum_{\alpha_1\ge 0\cdots\alpha_r\ge 0}g(\alpha_1,\dots,\alpha_r)t_1^{\alpha_1}\cdots t_r^{\alpha_r}\\&=\frac{1}{2}\sum_{h_1\ge 0\cdots h_r\ge 0}\binom{h_1+\cdots+h_r}{h_1}\binom{h_2+\cdots+h_r}{h_2}\cdots\binom{h_{r-1}+h_r}{h_{r-1}}t_1^{h_1}\cdots t_r^{h_r}\\&\quad\times(1-t_1)^{-h_1-h_2\cdots-h_r-1}(1-t_2)^{-h_2-\cdots-h_r-1}\cdots(1-t_r)^{-h_r-1}.\end{aligned}\tag{2.1}$$

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I claim now that for each j with $1 \leq j \leq r+1$

$$G(t_1, \ldots, t_r) \tag{2.2}$$
$$= \frac{1}{2} \sum_{\substack{h_j \geq 0 \\ \vdots \\ h_r \geq 0}} \binom{h_j + \cdots + h_r}{h_j} \cdots \binom{h_{r-1} + h_r}{h_{r-1}} \binom{h_r}{h_r} t_j^{h_j} \cdots t_r^{h_r} (1-t_j)^{-h_j - \cdots - h_r - 1} \cdot$$
$$\times (1-t_{j+1})^{-h_{j+1} - \cdots - h_r - 1} \cdots (1-t_r)^{-h_r - 1} \left\{ 2 \prod_{h=1}^{j-1} (1-t_h) - 1 \right\}^{-h_j - \cdots - h_r - 1}.$$

We note that (2.2) is true for $j=1$ since in this case it is just (2.1) (the expression inside the curly brackets is equal to 1). Assuming (2.2) true for a fixed j; and noting that

$$\sum_{h_j \geq 0} \binom{h_j + \cdots + h_r}{h_j} t_j^{h_j} (1-t_j)^{-h_j} \left\{ 2 \prod_{h=1}^{j-1} (1-t_h) - 1^{-h_j} \right\}$$
$$= \left(1 - \frac{t_j}{(1-t_j) \left\{ 2 \prod_{h=1}^{j-1} (1-t_h) - 1 \right\}} \right)^{-h_{j+1} - \cdots - h_r - 1}$$

we derive

$$G(t_1, \ldots, t_r) = \frac{1}{2} \sum_{\substack{h_{j+1} \geq 0 \\ \vdots \\ h_r \geq 0}} \binom{h_{j+1} + \cdots + h_r}{h_j} \cdots \binom{h_{r-1} + h_r}{h_{r-1}} \binom{h_r}{h_r} t_{j+1}^{h_{j+1}} \cdots t_r^{h_r}$$
$$\times (1-t_{j+1})^{-h_{j+1} - \cdots - h_r - 1} \cdots (1-t_r)^{-h_r - 1} \left\{ 2 \prod_{h=1}^{j} (1-t_h) - 1 \right\}^{-h_{j+1} - \cdots - h_r - 1}$$

which is just (2.5) with j replaced by $j+1$. Since the above process may be iterated as long as $j \leq r$, we see that the final application when $j=r$ produces

$$G(t_1, \ldots, t_r) = \frac{1}{2} \left(2 \prod_{h=1}^{r} (1-t_h) - 1 \right)^{-1} \tag{2.3}$$
$$= \sum_{\alpha_1 \geq 0 \cdots \alpha_1 \geq 0} F(p_1^{\alpha_1} \cdots p_r^{\alpha_r}) t_1^{\alpha_1} \cdots t_r^{\alpha_r} \qquad [5; \text{p. } 156].$$

3. **Combinatorial proof of Theorem 1.** Let $\mathscr{P}_j$ denote the j-dimensional plane in R^r given by $X_{j+1} = X_{j+2} = \cdots = X_r = 0$. We let $F_r(h_1, \ldots, h_r; \alpha_1, \ldots, \alpha_r)$ denote the number of (monotone) lattice paths starting at the origin and ending somewhere in the parallelepiped $0 \leq x_i \leq \alpha_i$ $(1 \leq i \leq \alpha_i)$ wherein exactly h_j edges are parallel to $\mathscr{P}_j$ but not to $\mathscr{P}_{j-1}$. Such paths have two kinds of edges, (i) the h_1 edges parallel to the x_1-axis, and (ii) the other $h_2 + \cdots + h_r$ edges. Suppose for a path P the type (i) edges terminate on the hyperplanes $x_1 = a_i$ $(1 \leq i \leq h_1)$ and the type (ii) edges terminate on the hyperplanes $x_1 = b_j$ $(1 \leq j \leq h_2 + \cdots + h_r)$. Then P is uniquely determined by its projection on the hyperplane $x_1 = 0$, together with the numbers a_1 and b_j. Since the a_i are chosen from $\{1, 2, \ldots, \alpha_1\}$ without repeats, while the b_j are chosen from $\{0, 1, \ldots, \alpha_1\}$ with repeats, we see that

$$F_r(h_1, \ldots, h_r; \alpha_1, \ldots, \alpha_r) \tag{3.1}$$
$$= \binom{\alpha_1}{h_1} \binom{\alpha_1 + h_2 + \cdots + h_r}{h_2 + \cdots + h_r} \cdot F_{r-1}(h_2, \ldots, h_r; \alpha_2, \ldots, \alpha_r).$$

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Iteration of (3.1) yields

$$F_r(h_1, \ldots, h_r, \alpha_1, \ldots, \alpha_r) \tag{3.2}$$
$$= \binom{\alpha_1}{h_1}\cdots\binom{\alpha_r}{h_r}\binom{\alpha_1+h_2+\cdots+h_r}{h_2+\cdots+h_r}\binom{\alpha_2+h_3+\cdots+h_r}{h_3+\cdots+h_r}\cdots\binom{\alpha_{r-1}+h_r}{h_r}.$$

Thus we see that

$$F(p_1^{\alpha_1}\cdots p_r^{\alpha_r})$$
$$=\frac{1}{2}\sum_{h_1\geq 0\cdots h_r\geq 0}\binom{\alpha_1}{h_1}\cdots\binom{\alpha_r}{h_r}\binom{\alpha_1+h_2+\cdots+h_r}{h_2+\cdots+h_r}\binom{\alpha_2+h_3+\cdots+h_r}{h_3+\cdots+h_r}\cdots\binom{\alpha_{r-1}+h_r}{h_r},$$

by the correspondence of the two lattice paths $\{(a_1, \ldots, a_r), (b_1, \ldots, b_r), \ldots, (z_1, \ldots, z_r)\}$ and $\{(a_1, \ldots, z_r), (b_1, \ldots, b_r), \ldots, (z_1, \ldots, z_r), (\alpha_1, \ldots, \alpha_r)\}$ with the ordered factorization of n:

$$n = (p_1^{a_1}\cdots p_r^{a_r})(p_1^{b_1-a_1}\cdots p_r^{b_r-a_r})\cdots(p_1^{\alpha_1-z_1}\cdots p_r^{\alpha_r-z_r}).$$

4. Refinement of Theorem 1.

THEOREM 2. *If* $1<n=p_1^{\alpha_1}\cdots p_r^{\alpha_r}$ *is the prime factorization of* n, *then*

$$F(p_1^{\alpha_1}\cdots p_r^{\alpha_r};\rho)$$
$$=\sum_{h_2\geq 0,\ldots,h_r\geq 0}\binom{\alpha_1-1}{\rho-1-h_2-\cdots-h_r}\binom{\alpha_2}{h_2}\cdots\binom{\alpha_r}{h_r}\binom{\alpha_1+h_2+\cdots+h_r}{h_2+\cdots+h_r}\cdots\binom{\alpha_{r-1}+h_r}{h_r}.$$

Proof. The correspondence described at the end of Section 1 shows that

$$\sum_{\substack{h_1\geq 0\cdots h_r\geq 0\\ h_1+\cdots+h_r=\rho}} F_r(h_1, \ldots, h_r; \alpha_1, \ldots, \alpha_r)$$

counts the number of ordered factorizations of n with either ρ or $\rho+1$ factors. Hence

$$F(p_1^{\alpha_1}\cdots \rho_r^{\alpha_r};\rho)$$
$$=\sum_{j=0}^{\rho-1}(-1)^j\sum_{\substack{h_1\geq 0\cdots h_r\geq 0\\ h_1+\cdots+h_r=\rho-1-j}} F_r(h_1, \ldots, h_r; \alpha_1, \ldots, \alpha_r)$$
$$=\sum_{j=0}^{\rho-1}(-1)^j\sum_{h_2\geq 0\cdots h_r\geq 0}\binom{\alpha_1}{\rho-1-j-h_2-\cdots-h_r}$$
$$\times\binom{\alpha_2}{h_2}\cdots\binom{\alpha_r}{h_r}\binom{\alpha_1+h_2+\cdots+h_r}{h_2+\cdots+h_r}\cdots\binom{\alpha_{r-1}+h_r}{h_1}$$
$$=\sum_{h_2\geq 0,\ldots,h_r\geq 0}\binom{\alpha_1-1}{\rho-1-h_2-\cdots-h_r}\binom{\alpha_2}{h_2}\cdots\binom{\alpha_r}{h_r}\binom{\alpha_1+h_2+\cdots+h_r}{h_2+\cdots+h_r}\cdots\binom{\alpha_{r-1}+h_r}{h_r}$$

by [2; p. 95, eq. (48)].

5. Proof of Long's conjecture.

Here we begin by formalizing the substitutions

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described in Theorem 3. We define r linear operators L_i on the polynomial ring $R[x_1, \dots, x_r]$:

$$L_i : R[x_1, \dots, x_r] \to R[x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_r],$$

where

$$L_i(x_i^k) = \binom{\alpha_i}{\alpha_{i+1}+\cdots+\alpha_r-k}. \tag{5.1}$$

Since $x_i^0, x_i, x_i^2, x_i^3, \dots$ form a basis for $R[x_1, \dots, x_r]$ over $R[x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_r]$ we see that each L_i is well-defined on $R[x_1, \dots, x_r]$.

Next we note that

$$\begin{aligned} L_i(x_i^C(1+x_i)^D) &= L_i\left(\sum_{j=0}^{D}\binom{D}{j}x_i^{C+j}\right) \\ &= \sum_{j=0}^{D}\binom{D}{j}\binom{\alpha_i}{\alpha_{i+1}+\cdots+\alpha_r-C-j} = \binom{\alpha_i+D}{\alpha_{i+1}+\cdots+\alpha_r-C}, \end{aligned} \tag{5.2}$$

where the last line follows from Vandermonde's convolution [6; p. 9].

Now to prove Theorem 3, we are asked to evaluate

$$\begin{aligned} &L_1L_2\cdots L_{r-1}\left\{2^{\alpha_1-1}\prod_{i=2}^{r}(x_1\,x_2\cdots x_{i-1}+(1+x_1)(1+x_2)\cdots(1+x_{i-1}))^{\alpha_i}\right\} \\ &= L_1L_2\cdots L_{r-1}\left\{2^{\alpha_1-1}\sum_{h_2\geq 0\cdots h_r\geq 0}\binom{\alpha_2}{h_2}\cdots\binom{\alpha_r}{h_r}\prod_{j=2}^{r}\{x_{j-1}^{(\alpha_j-h_j)+\cdots+(\alpha_r-h_r)}(1+x_{j-1})^{h_j+\cdots+h_r}\}\right\} \\ &= 2^{\alpha_1-1}\sum_{\substack{h_2\geq 0\\ \vdots \\ h_r\geq 0}}\binom{\alpha_2}{h_2}\cdots\binom{\alpha_r}{h_r}\prod_{j=2}^{r}L_{j-1}\{x_{j-1}^{(\alpha_j-h_j)+\cdots+(\alpha_r-h_r)}(1+x_{j-1})^{h_j+\cdots+h_r}\} \\ &= 2^{\alpha_1-1}\sum_{\substack{h_2\geq 0\\ \vdots \\ h_r\geq 0}}\binom{\alpha_2}{h_2}\cdots\binom{\alpha_r}{h_r}\binom{\alpha_1+h_2+\cdots+h_r}{h_2+\cdots+h_r}\binom{\alpha_2+h_3+\cdots+h_r}{h_3+\cdots+h_r}\cdots\binom{\alpha_{r-1}+h_r}{h_r} \\ &= \frac{1}{2}\sum_{\substack{h_1\geq 0\\ \vdots \\ h_r\geq 0}}\binom{\alpha_1}{h_1}\binom{\alpha_2}{h_2}\cdots\binom{\alpha_r}{h_r}\binom{\alpha_1+h_2+\cdots+h_r}{h_2+\cdots+h_r}\binom{\alpha_2+h_3+\cdots+h_r}{h_3+\cdots+h_r}\cdots\binom{\alpha_{r-1}+h_r}{h_r} \\ &= F(p_1^{\alpha_1}\cdots p_r^{\alpha_r}), \end{aligned}$$

and so Theorem 3 (Long's Conjecture) is established.

6. Conclusion. In [4; p. 335], Long states that, "What may be of considerable importance is that the conjectured method of solution (i.e. Theorem 3) suggests the existence of a transform method of solution which may be applicable to a reasonably large class of partial difference equations."

We point out here that two areas of combinatorics have already been explored by G. C. Rota [7] and G. C. Rota and J. Goldman [1], in which such transform techniques play a substiantial role.

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The first relates to B_n the number of partitions of a set of n elements [7]. Rota [7] considers a linear operator L on $R[u]$ given by

$$L(1) = 1, \qquad L(u(u-1)\cdots(u-k+1)) = 1.$$

He then notes that $L(u^n)=B_n$, and he is thus able to derive in an elegant manner a number of well-known properties of B_n. For example, he derives the exponential generating function for B_n as follows: if $v=e^x-1$, then

$$\sum_{n\geq 0}\frac{B_n x^n}{n!} = \sum_{n\geq 0}\frac{L(u^n)x^n}{n!} = L(e^{ux}) = L((1+v)^u)$$

$$= L\sum_{n\geq 0}\frac{u(u-1)\cdots(u-n+1)}{n!}v^n = \sum_{n\geq 0}\frac{v^n}{n!} = e^v = e^{e^x-1}.$$

The second area concerns the combinatorics of finite vector spaces. Here Rota and Goldman [1] consider G_n the number of subspaces of an n-dimensional vector space over the finite field $GF(q)$. They consider a linear operator L defined by

$$L(1) = 1, \qquad L((x-1)(x-q)\cdots(x-q^{n-1})) = 1.$$

In this case $L(x^n)=G_n$. They then derive a number of results of combinatorial interest involving the Gaussian polynomials.

We close by pointing out that the function $F(p_1^{\alpha_1}\ldots p_r^{\alpha_r})$ has arisen in a number of different contexts in number theory and combinatorics. Besides enumerating the compositions of the multipartite number $(\alpha_1, \alpha_2, \ldots, \alpha_r)$ and the ordered factorizations of n, MacMahon [5] showed that $F(n)$ enumerates the number of perfect partitions of $n-1$.

In [4], C. Long discusses the following problem: Let C be a set of integers. Two subsets A and B of C are said to be complementing subsets of C in case every $c \in C$ is uniquely represented in the sum

$$C = A+B = \{x \mid x = a+b,\ a \in A,\ b \in B\}.$$

Long shows that the number of pairs of complementing subsets of $\{0, 1, \ldots, n-1\}$ is just $F(n)$.

It is hoped that the investigations undertaken in this paper provide some further insights concerning $F(n)$. In a future paper I hope to investigate Simon Newcomb's problem [5; p. 187] utilizing the techniques developed here.

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