WEAKLY STABLE BANACH SPACES AND THE BANACH-SAKS PROPERTIES

by THEODOSIS ZACHARIADES

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1. Introduction. In [**9**] J. L. Krivine and B. Maurey introduced the class of stable Banach spaces: a separable Banach space is called *stable* if for every pair of bounded sequences $(x_n)_n$, $(y_n)_n$ and for every pair of ultrafilters $\mathfrak{U}$, $\mathfrak{V}$ on the natural numbers we have

$$\lim_{\substack{n\\ \mathfrak{U}}} \lim_{\substack{m\\ \mathfrak{V}}} \|x_n + y_m\| = \lim_{\substack{m\\ \mathfrak{V}}} \lim_{\substack{n\\ \mathfrak{U}}} \|x_n + y_m\|.$$

In the paper above it was proved that every stable Banach space contains some $l^p(1 \leq p < \infty)$ almost isometrically. It was also shown that if X is stable then so is $L^p(X)$ for $1 \leq p < \infty$. It is easy to see that c_0 is not a stable space. S. Guerre and J. T. Lapreste in [**8**] have proved that every stable Banach space X is weakly sequentially complete and has the weak Banach–Saks (w.B.S.) property (namely, every weakly null sequence $(x_n)_n$ has a subsequence $(x_{k_n})_n$ whose Cesaro means $\frac{1}{n}\sum_{m=1}^{n} x_{k_m}$ are norm convergent). If in addition X does not contain an isomorphic copy of l^1 then X has the Banach–Saks (B.S.) property (namely, the statements above holds for every bounded sequence $(x_n)_n$).

S. Argyros, S. Negrepontis and the present author in [**2**] and independently, D. J. H. Garling in an earlier unpublished version of [**7**], have introduced the wider class of weakly stable Banach spaces. A separable Banach space is called *weakly stable* if for every pair of sequences $(x_n)_n$, $(y_n)_n$ contained in a weakly compact subset of X, and for every pair of ultrafilters $\mathfrak{U}$, $\mathfrak{V}$ on the natural numbers we have

$$\lim_{\substack{n\\ \mathfrak{U}}} \lim_{\substack{m\\ \mathfrak{V}}} \|x_n + y_m\| = \lim_{\substack{m\\ \mathfrak{V}}} \lim_{\substack{n\\ \mathfrak{U}}} \|x_n + y_m\|.$$

In [**3**], it was proved that c_0 is a weakly stable Banach space (this was also proved by D. J. H. Garling in the afore-mentioned paper) and that every weakly stable Banach space contains c_0 or some $l^p(1 \leq p < \infty)$ almost isometrically.

A weakly stable Banach space X, every subspace of which contains l^2 almost isometrically but which is not reflexive, was constructed in [**2**]. It follows from [**8**] that this space does not admit an equivalent stable norm.

In this paper we prove, using techniques from [**8**] (also see [**4**]), that every weakly stable Banach space X has the w.B.S. property. If in addition X contains no isomorphic copy of l^1, then X has the Alternate-signs Banach–Saks (A.B.S.) property (namely, every bounded sequence $(x_n)_n$ in X has a subsequence $(x_{k_n})_n$ such that the alternate signs Cesaro means $\frac{1}{n}\sum_{m=1}^{n}(-1)^m x_{n_m}$ are norm convergent). It is known that B.S. implies A.B.S. which in turn implies W.B.S. [**3**]. Since l^1 has W.B.S. but not A.B.S. and c_0 has A.B.S. but not B.S. the results above are the best possible. Finally from the example of [**2**] it follows that a weakly stable Banach space not containing l_1 or c_0 does not necessarily have B.S.

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The main theorem has two interesting corollaries. The first is that $L^2(c_0)$ does not admit an equivalent weakly stable norm. Thus the result in **[9]** on L^p spaces of stable Banach spaces that was mentioned above does not extend to the class of weakly stable Banach spaces. Another corollary is a characterization of $C(K)$-spaces (for compact, metrizable K) which admit an equivalent weakly stable norm: they are precisely the ones which are isomorphic to c_0.

Note that, while the B.S. properties are isomorphic invariants, the stability properties are only isometric invariants. It is an open problem, to find conditions under which a separable Banach space has an equivalent stable or weakly stable norm.

2. Preliminaries. Let X be a separable Banach space. A *type on* X is a function $\tau: X \to \mathbb{R}^+$, for which there exists a sequence $(x_n)_n$ in X such that $\tau(x) = \lim_n \|x_n + x\|$ for every $x \in X$. Then we say that $(x_n)_n$ *generates* τ. The set of all types on X is denoted by $\mathcal{T}(X)$ and is provided with the topology of pointwise convergence. We set $\mathcal{T}^1(X) = \{\tau \in \mathcal{T}(X): \tau(0) \le 1\}$. It is obvious that $\mathcal{T}^1(X)$ is a compact space and $\mathcal{T}(X)$ is locally compact and σ-compact. We denote by $\mathcal{T}_w(X)$ (respectively $\mathcal{T}_{wn}(X)$) the set of types on X which are generated by a weakly convergent (respectively weakly null) sequence of X. An element of $\mathcal{T}_w(X)$ (respectively $\mathcal{T}_{wn}(X)$) is called a *weak* (respectively *weakly null*) type on X. A type $\tau \in \mathcal{T}(X)$ is *symmetric* if $\tau(x) = \tau(-x)$ for every $x \in X$. We denote by $\mathcal{T}^{\mathcal{S}}(X)$, $\mathcal{T}^{\mathcal{S}}_w(X)$, $\mathcal{T}^{\mathcal{S}}_{wn}(X)$ the set of all symmetric, weak and symmetric, weakly null and symmetric types on X respectively. For $\tau \in \mathcal{T}(X)$ and $\lambda \in \mathbb{R}$ we set $\lambda\tau = 0$ if $\lambda = 0$, and $(\lambda\tau)(x) = |\lambda|\tau\left(\frac{x}{\lambda}\right)$ if $\lambda \neq 0$ for $x \in X$.

Let X be a weakly stable space. If $\sigma, \tau \in \mathcal{T}_w(X)$ we set $(\tau * \sigma)(x) = \lim_n \lim_m \|x_n + y_m + x\|$ for every $x \in X$, where $(x_n)_n$, $(y_n)_n$ are two sequences which generate the types τ, σ, respectively. In **[2]** it is shows that $\tau * \sigma \in \mathcal{T}_w(X)$ and if $\tau,\ \sigma \in \mathcal{T}_{wn}(X)$ then $\tau * \sigma \in \mathcal{T}_{wn}(X)$. A subset $\mathcal{C}$ of $\mathcal{T}^{\mathcal{S}}_{wn}(X)$ is called a *conic class* if $\mathcal{C} \neq \emptyset$, $\mathcal{C} \neq \{0\}$, $\mathcal{C}$ is a closed subset of $\mathcal{T}^{\mathcal{S}}_{wn}(X)$, and $\sigma * \tau \in \mathcal{C}$, $\lambda\tau \in \mathcal{C}$ for $\sigma,\ \tau \in \mathcal{C}$ and $\lambda \in \mathbb{R}$, $\lambda \ge 0$. If $\mathcal{C}$ is a conic class in $\mathcal{T}^{\mathcal{S}}_{wn}(X)$, $\sigma \in \mathcal{C}$ and $\alpha,\ \beta > 0$ then σ is called an $(\alpha, \beta, \mathcal{C})$*-approximating type* if for every $\varepsilon > 0$ and every neighborhood V of σ there is $\tau \in \mathcal{C} \cap V$ such that

$$|(\tau * \alpha\tau)(x) - (\beta\tau)(x)| \le \varepsilon \quad \text{for} \quad x \in X.$$

We denote by $\Gamma_{\alpha,\beta,\mathcal{C}}$ the set of $(\alpha, \beta, \mathcal{C})$-approximating types.

The *spreading model* of $\sigma \in \mathcal{T}^{\mathcal{S}}_w(X)$ is a Banach space $Y \supset X$, where Y is spanned by $X \cup \{e_n : n = 1, 2, \ldots,\}$ and such that

$$\|x + \lambda_1 e_1 + \ldots + \lambda_n e_n\| = (\lambda_1\sigma * \ldots * \lambda_n\sigma)(x)$$

for $n = 1, 2, \ldots, \lambda_1, \ldots, \lambda_n \in \mathbb{R}$, $x \in X$. The spreading model of σ is unique up to isometry and the sequence $(e_n)_n$ is called the *fundamental sequence of* σ. A type $\sigma \in \mathcal{T}^{\mathcal{S}}_w(X)$ is called an l^p*-type*, for some $1 \le p < \infty$, (respectively a c_0*-type*) if $\alpha\sigma * \beta\sigma = (\alpha^p + \beta^p)^{1/p}\sigma$ (respectively $\alpha\sigma * \beta\sigma = \max(\alpha, \beta)\sigma$) for every $\alpha, \beta \ge 0$. It is clear that if a type $\sigma \in \mathcal{T}^{\mathcal{S}}_w(X)$ is an l^p-type or a c_0-type, then the fundamental sequence of σ is equivalent with the usual basis of l^p or c_0.

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3. The main result. In what follows, X will denote a fixed weakly stable space. For $\sigma \in \mathcal{T}^{\mathcal{S}}_{wn}(X)$, we set

$$D_\sigma = \{\tau \in \mathcal{T}(X) : \text{there exist } \lambda_1, \ldots, \lambda_n \in \mathbb{R} \quad \text{with} \quad \tau = \lambda_1\sigma * \ldots * \lambda_n\sigma\}$$

and $\mathcal{C}_\sigma = \bar{D}_\sigma$. It is clear that $D_\sigma \subset \mathcal{T}^{\mathcal{S}}_{wn}(X)$ and $\mathcal{C}_\sigma$ is a closed subset of $\mathcal{T}^{\mathcal{S}}(X)$.

LEMMA 3.1. *Let σ be a non-trivial, weakly null and symmetric type on X with fundamental sequence equivalent with the usual basis of l^1. Then we have the following*:

(i) $\mathcal{C}_\sigma \subset \mathcal{T}^{\mathcal{S}}_{wn}(X)$.

(ii) *The function* $* : \mathcal{C}_\sigma \times \mathcal{C}_\sigma \to \mathcal{T}^{\mathcal{S}}_{wn}(X)$ *is separately continuous.*

(iii) $\mathcal{C}_\sigma$ *is a conic class.*

(iv) *Every non-trivial type τ that belongs to $\mathcal{C}_\sigma$ has fundamental sequence equivalent with the usual basis of l^1.*

Proof. (i) Let $\tau \in \mathcal{C}_\sigma$. Without loss of generality we suppose that $\tau(0) = 1$. Then there is a sequence $\tau_n = \lambda_1^{(n)}\sigma * \ldots * \lambda_{k_n}^{(n)}\sigma (n = 1, 2, \ldots)$ with $\tau_n(0) = 1$, such that $\lim_n \tau_n = \tau$. Since the fundamental sequence of σ is equivalent with the usual basis of l^1, there is a $K > 0$ such that

$$\frac{1}{K} \leq \sum_{i=1}^{k_n} |\lambda_i^{(n)}| \leq K \tag{1}$$

for $n = 1, 2, \ldots$. If (x_m) is a weakly null sequence which generates the type σ, as in the proof of Proposition 9, Ch. VII of **[4]**, for every $n = 1, 2, \ldots$, there exists a block $(y_m^{(n)})_m$ of (x_m), with $y_m^{(n)} = \sum_{i=1}^{k_n} \lambda_i^{(n)} x_{l_{mk_n+i}}$, which generates the type τ_n, and there are sequences $n_1 < \ldots < n_k < \ldots$ and $m_1 < \ldots < m_k < \ldots$ such that the sequence $(y_{m_k}^{(n_k)})_k$ is a block of (x_m) which generates the type τ. From (1) we have $\lim_k y_{m_k}^{(n_k)} = 0$ weakly. Thus $\tau \in \mathcal{T}^{\mathcal{S}}_{wn}(X)$.

(ii) For every $\pi \in \mathcal{T}^{\mathcal{S}}_{w}(X)$ the function $\phi_\pi : \mathcal{C}_\sigma \to \mathbb{R}$, with $\phi_\pi(\tau) = (\pi * \tau)(0)$, is well defined by (i). It is enough to prove that, if $\tau_n \in D_\sigma$, $n = 1, 2, \ldots$, and $\lim_n \tau_n = \tau$, then there exists a subsequence $(\tau_{n_k})_k$ of $(\tau_n)_n$ such that $\lim_k \phi_\pi(\tau_{n_k}) = \phi(\tau)$.

Let $(x_m)_m$ be a weakly null sequence that generates the type σ. Then for every $n \in \mathbb{N}$ there exists a weakly null block $(y_m^{(n)})_m$ of (x_m) which generates the type τ_n. Thus we have $\phi_\pi(\tau_n) = \lim_m \pi(y_m^{(n)})$ for $n = 1, 2, \ldots$, and, without loss of generality, we may suppose that

$$|\phi_\pi(\tau_n) - \pi(y_m^{(n)})| \leq \frac{1}{n} \tag{2}$$

for n, $m = 1, 2, \ldots$. As in (i), there are sequences $n_1 < \ldots < n_k < \ldots$ and $m_1 < \ldots < m_k < \ldots$ such that the sequence $(y_{m_k}^{(n_k)})_k$ is a weakly null block of $(x_m)_m$ which generates the type τ. So we have

$$\phi_\pi(\tau) = \lim_k \pi(y_{m_k}^{(n_k)}). \tag{3}$$

From (2) and (3) it follows that $\lim_k \phi_\pi(\tau_{n_k}) = \phi_\pi(\tau)$.

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(iii) It is obvious that $\mathscr{C}_\sigma \neq \emptyset$, $\mathscr{C}_\sigma \neq \{0\}$, $\mathscr{C}_\sigma$ is a closed subset of $\mathscr{T}^{\mathscr{S}}_{wn}(X)$ and $\lambda\tau \in \mathscr{C}_\sigma$ for $\tau \in \mathscr{C}_\sigma$ and $\lambda \geq 0$. The fact that $\mathscr{C}_\sigma$ is closed under $*$ follows from (ii).

(iv) Let τ be a non-trivial type of $\mathscr{C}_\sigma$, $\tau(0)=1$, and $\tau_n = \lambda_1^n\sigma * \ldots * \lambda_{k_n}^n\sigma$, $\tau_n(0)=1$ for $n = 1, 2, \ldots$, with $\lim_n \tau_n = \tau$. From (1) and (ii) it is easy to see that for every $c_1, \ldots, c_n \in \mathbb{R}$ we have

$$\frac{1}{K^2}\sum_{i=1}^n |c_i| \leq (c_1\tau * \ldots * c_n\tau)(0) \leq K^2 \sum_{i=1}^n |c_i|$$

(see the proof of Proposition 9, Ch. VII of **[4]**). Thus the fundamental sequence of τ is equivalent to the usual basis of l^1.

Proposition 3.2. *Let $\mathscr{C}$ be a conic class of $\mathscr{T}^{\mathscr{S}}_{wn}(X)$ such that*

(i) *$\mathscr{C}$ is a closed subset of $\mathscr{T}^{\mathscr{S}}(X)$, and*

(ii) *the function $*: \mathscr{C} \times \mathscr{C} \to \mathscr{C}$ is separately continuous.*

Then there exists a type $\tau \in \mathscr{C}$, which is an l^p-type, for some $1 < p < \infty$, or a c_0-type.

Proof. By (i), Zorn's Lemma implies that $\mathscr{C}$ contains a minimal conic class. Without loss of generality we may thus suppose that $\mathscr{C}$ is minimal. From (i) we have that $\mathscr{C} \cap \mathscr{T}^1(X)$ is compact and we can prove, as in Lemma IV.4 of **[9]**, that for every $\alpha > 0$ there exists $\beta > 0$ such that $\Gamma_{\alpha,\beta,\mathscr{C}} \neq \emptyset$ and $\Gamma_{\alpha,\beta,\mathscr{C}} \neq \{0\}$. From (ii) we have that $\Gamma_{\alpha,\beta,\mathscr{C}}$ is a conic class (see **[9**, Lemma IV.5**]**) and then, since $\mathscr{C}$ is a minimal conic class, for every $\alpha > 0$ there exists $\beta > 0$ such that $\mathscr{C} = \Gamma_{\alpha,\beta,\mathscr{C}}$. From (i) $\mathscr{C}$ is locally compact and σ-compact; hence Namioka's theorem **[10]**, implies that there exists a dense subset D of $\mathscr{C}$ such that every $(\sigma, \tau) \in D \times \mathscr{C}$ is a point of joint continuity of the convolution $*: \mathscr{C} \times \mathscr{C} \to \mathscr{C}$. From the above we have that for every $\sigma \in D$ and for every $\alpha \geq 0$ there exists $\beta \geq 0$ such that $\sigma * \alpha\sigma = \beta\sigma$. Thus, from Lemma 1.11 of **[2]**, every $\sigma \in D$ is an l^p-type, for some $1 \leq p < \infty$, or a c_0-type, and since $\sigma \in \mathscr{T}^{\mathscr{S}}_{wn}(X)$, σ is not an l^1-type.

Theorem 3.3. *Every weakly stable Banach space has the W.B.S. property.*

Proof. Let X be a weakly stable Banach space and suppose that X does not have the W.B.S. property. Then, from a result of Rosenthal **[11]** (see also Proposition II.1 of **[3]**), there exists a weakly null sequence $(x_n)_n$ in X, which we may suppose generates a type $\pi \in \mathscr{T}_{wn}(X)$, such that the fundamental sequence of π is equivalent with the usual basis of l^1. We set $\sigma = \pi * (-\pi)$ and we have that σ is a non-trivial type of $\mathscr{T}^{\mathscr{S}}_{wn}(X)$ with fundamental sequence equivalent to the usual basis of l^1. Then, from Lemma 3.1 (i), (ii), (iii) and Proposition 3.2 there exists a type $\tau \in \mathscr{C}_\sigma$ which is an l^p-type, for some $1 < p < \infty$, or a c_0-type. Thus we have a contradiction, by Lemma 3.1 (iv).

Corollary 3.4. *Every weakly stable Banach space that does not contain a subspace isomorphic to l^1 has the A.B.S. property.*

Proof. This is a consequence of Theorem 3.3 and Proposition II.3 and Theorem III.1 of **[3]**.

In **[9]** it was proved that if X is a stable Banach space then $L_p(X)$ is a stable Banach space for $1 \leq p < \infty$. From Theorem 3.3 and **[1]** we have that the corresponding result for weakly stable Banach spaces is false.

Corollary 3.5. *The space $L_2(c_0)$ does not admit an equivalent norm for which it is a weakly stable space.*

Proof. It is known [1] that $L_2(c_0)$ does not have the W.B.S. property. The result thus follows from Theorem 3.3.

COROLLARY 3.6. *Let K be a compact metric space. The space $C(K)$ has a renorming in which it is weakly stable Banach space if and only if it is isomorphic to c_0.*

Proof. If $C(K)$ is isomorphic to a weakly stable space, then from Theorem 3.3, $C(K)$ has the W.B.S. property. Thus, from [6], $K^{(\omega)}=\varnothing$, where $K^{(\omega)}$ is the ω-th derived of K. Then [5] implies that $C(K)$ is isomorphic to c_0. The converse is obvious from the fact that c_0 is a weakly stable Banach space [2].

REMARK. Every Banach space which has the B.S. property is reflexive. Thus, from the example of [2], there is a weakly stable space X not containing isomorphic copies of either l^1 or c_0, and not having the B.S. property.

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DEPARTMENT OF MATHEMATICS
UNIVERSITY OF ATHENS
PANEPISTEMIOPOLIS,
15781 ATHENS, GREECE

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