



Derivatives of symplectic eigenvalues and a Lidskii type theorem

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Abstract. Associated with every $2n \times 2n$ real positive definite matrix A , there exist n positive numbers called the symplectic eigenvalues of A , and a basis of \mathbb{R}^{2n} called the symplectic eigenbasis of A corresponding to these numbers. In this paper, we discuss differentiability and analyticity of the symplectic eigenvalues and corresponding symplectic eigenbasis and compute their derivatives. We then derive an analogue of Lidskii's theorem for symplectic eigenvalues as an application.

1 Introduction

Let J be the $2n \times 2n$ matrix

$$(1.1) \quad J = \begin{bmatrix} O & I_n \\ -I_n & O \end{bmatrix},$$

where I_n is the $n \times n$ identity matrix. A $2n \times 2n$ real matrix M is called a *symplectic matrix* if

$$M^T J M = J.$$

The set of all symplectic matrices forms a group under multiplication and is denoted by $Sp(2n)$. A result on symplectic matrices, generally known as *Williamson's theorem*, says that for every $2n \times 2n$ real positive definite matrix A there exists a symplectic matrix M such that

$$(1.2) \quad M^T A M = \begin{bmatrix} D & O \\ O & D \end{bmatrix},$$

where D is an $n \times n$ positive diagonal matrix with diagonal entries $d_1(A) \leq \dots \leq d_n(A)$, [9, 10]. The positive numbers $d_1(A), \dots, d_n(A)$ are uniquely determined. We call these numbers the *symplectic eigenvalues* of A . These are the complete invariants of A under the action of the symplectic group $Sp(2n)$. Symplectic eigenvalues occur in different areas of mathematics and physics such as symplectic geometry, symplectic topology, and both classical and quantum mechanics. See [9, 11, 19, 26]. Recently, there has been a heightened interest in the study of symplectic eigenvalues by both physicists and mathematicians. A particular reason for this being their growing

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importance and applications in quantum information. See, for instance, [10, 23]. A positive number d is a symplectic eigenvalue of A if and only if $\pm d$ is an eigenvalue of the Hermitian matrix $iA^{1/2}JA^{1/2}$ [9, 23]. In principle, it could be possible to derive the properties of symplectic eigenvalues from the well-known properties of eigenvalues of Hermitian matrices. But due to the complicated form of the Hermitian matrix $iA^{1/2}JA^{1/2}$, it is often not feasible to obtain results for symplectic eigenvalues from the well-developed theory for eigenvalues of Hermitian matrices. So, it is necessary as well as helpful to develop independent techniques and theory for symplectic eigenvalues. Some fundamental inequalities and variational principles on these are given in [7]. In this paper, we study some questions on symplectic eigenvalues analogous to some fundamental questions on eigenvalues of Hermitian matrices that have been studied for long.

Eigenvalue problems for Hermitian matrices have a long and rich history. We can classify these problems to be qualitative and quantitative in nature. An example of qualitative problems is the study of continuity, differentiability, and analyticity of eigenvalues and eigenvectors as functions of Hermitian matrices when the matrices depend smoothly on a parameter. These problems have been extensively studied, (see e.g., [15–17, 21, 24, 25, 28]) and are of much importance in perturbation theory, differential equations, numerical analysis, and physics. See [18, 27, 28]. The quantitative problems include variational principles, eigenvalues of functions of matrices, majorization inequalities, and computation of eigenvalues and eigenvectors. There has been much interest in the study of relationships between the eigenvalues of Hermitian matrices A and B and those of their sum $A + B$. Suppose $\lambda^\uparrow(A) = (\lambda_1^\uparrow(A), \dots, \lambda_n^\uparrow(A))$ denote the tuple of eigenvalues of an $n \times n$ Hermitian matrix A arranged in increasing order. In 1912, H. Weyl discovered several relationships between the eigenvalues of sums of Hermitian matrices. These include the inequalities:

$$(1.3) \quad \lambda_j^\uparrow(A + B) \geq \lambda_j^\uparrow(A) + \lambda_1^\uparrow(B) \quad 1 \leq j \leq n.$$

The maximum principle given by Ky Fan in 1949 implies that for all $1 \leq k \leq n$,

$$(1.4) \quad \sum_{j=1}^k \lambda_j^\uparrow(A + B) \geq \sum_{j=1}^k \lambda_j^\uparrow(A) + \sum_{j=1}^k \lambda_j^\uparrow(B).$$

In 1950, V. B. Lidskii proved the inequalities

$$(1.5) \quad \sum_{j=1}^k \lambda_{i_j}^\uparrow(A + B) \geq \sum_{j=1}^k \lambda_{i_j}^\uparrow(A) + \sum_{j=1}^k \lambda_j^\uparrow(B)$$

for all $k = 1, \dots, n$ and $1 \leq i_1 < i_2 < \dots < i_k \leq n$. Inequalities (1.3) and (1.4) are special cases of (1.5). Lidskii’s inequalities played a fundamental role in the study of eigenvalues of sums of matrices and proved to be an important stimulant for the much celebrated Horn’s conjecture. See, for instance, [4, 12]. These inequalities have attracted much attention, and a number of different proofs for these are now available in literature. See [3, 5, 20]. But all the proofs are generally more difficult than those for the earlier two families of inequalities (1.3) and (1.4).

In this paper, we address both the qualitative as well as quantitative problems on symplectic eigenvalues. We study differentiability of symplectic eigenvalues and

also derive a relationship analogous to Lidskii's theorem for these numbers. Let $\mathbb{P}(n)$ denote the set of all $n \times n$ real positive definite matrices. For a matrix P in $\mathbb{P}(2n)$, we shall always denote by $d_1(P) \leq \dots \leq d_n(P)$, its symplectic eigenvalues arranged in increasing order. We know that each map $P \mapsto d_j(P)$ is continuous. See [7, 14]. But this map need not be differentiable, as is shown by the following example.

Example 1 Let I_4 denote the 4×4 identity matrix. Clearly $d_1(I_4) = d_2(I_4) = 1$. We show that the maps $P \mapsto d_1(P)$ and $P \mapsto d_2(P)$ are not even Gateaux differentiable at I_4 . Let B be the 4×4 matrix

$$(1.6) \quad B = I_2 \otimes \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

For any real number t with $|t| < 1$, $I_4 + tB$ is the matrix

$$I_4 + tB = I_2 \otimes \begin{bmatrix} 1 & 0 \\ 0 & 1+t \end{bmatrix}.$$

The symplectic eigenvalues of $I_4 + tB$, are

$$d_1(I_4 + tB) = \begin{cases} 1+t & -1 < t < 0 \\ 1 & 0 \leq t < 1, \end{cases}$$

and

$$d_2(I_4 + tB) = \begin{cases} 1 & -1 < t < 0 \\ 1+t & 0 \leq t < 1. \end{cases}$$

It is easy to see that

$$\lim_{t \rightarrow 0^+} \frac{d_1(I_4 + tB) - d_1(I_4)}{t} = 0$$

and

$$\lim_{t \rightarrow 0^-} \frac{d_1(I_4 + tB) - d_1(I_4)}{t} = 1.$$

This shows that the map d_1 is not differentiable. Similarly, we can see that the map d_2 is not differentiable at I_4 .

A symplectic eigenvalue d of A has *multiplicity* m if the set $\{i : d_i(A) = d\}$ has exactly m elements, and is *simple* if $m = 1$. We see in Example 1, the symplectic eigenvalue d_1 of I_4 has multiplicity 2 and is not differentiable at I_4 . We show in Theorem 3.3 that if $d_j(A)$ is a simple symplectic eigenvalue of A , then the map $P \mapsto d_j(P)$ and the corresponding symplectic eigenvector pair maps are infinitely differentiable at A . We calculate the first derivatives of these maps in Theorem 3.4.

We also study the differentiability and analyticity of symplectic eigenvalues of positive definite matrices that are dependent on a real parameter. Let $A(t)$ be the analytic curve of 4×4 real positive definite matrices given by $A(t) = I_4 + tB$, $t \in (-1, 1)$, where B is the matrix given by (1.6). Let $d_1(t), d_2(t)$ be the symplectic eigenvalues of $A(t)$ arranged in increasing order. From Example 1, we can see that the

maps d_1 and d_2 are not differentiable at $t = 0$. However, if we rearrange the symplectic eigenvalues as $\tilde{d}_1(t) = 1$ and $\tilde{d}_2(t) = 1 + t$, $-1 < t < 1$, we obtain analytic maps \tilde{d}_1 and \tilde{d}_2 of symplectic eigenvalues. In Theorem 4.6, we see that for any real analytic curve in $\mathbb{P}(2n)$, the symplectic eigenvalues can be suitably ordered so that the corresponding symplectic eigenvalue maps and the symplectic eigenbasis map are analytic. We also show in Theorem 4.7 that the maps $t \mapsto d_j(A(t))$ of symplectic eigenvalues arranged in increasing order are piecewise real analytic. These regularity properties of symplectic eigenvalues can be obtained by using the corresponding properties of eigenvalues of Hermitian matrices and the properties of the matrix square root map $A \mapsto A^{1/2}$. But the computation of derivatives requires an independent theory on symplectic eigenvalues.

We now describe the quantitative problems that we study in this paper. Recently, there has been much interest in finding relationships between the symplectic eigenvalues of sums of positive definite matrices and those of individual matrices. T. Hiroshima in [13] proved the following relationship for symplectic eigenvalues that is analogous to (1.4).

$$\sum_{j=1}^k d_j(A + B) \geq \sum_{j=1}^k d_j(A) + \sum_{j=1}^k d_j(B) \quad 1 \leq k \leq n.$$

In [6], R. Bhatia addressed the inequality analogous to (1.3). He showed that

$$d_j(A + B) \geq d_j(A) + d_1(B)$$

for all $j = 1, \dots, n$ when A and B are of the form

$$A = \begin{bmatrix} D & O \\ O & D \end{bmatrix}, \quad B = \begin{bmatrix} X & O \\ O & X^{-1} \end{bmatrix},$$

where D is the diagonal matrix $\text{diag}(d_1(A), \dots, d_n(A))$ and X is any $n \times n$ positive definite matrix. As an application of our results on analyticity of symplectic eigenvalues, we derive relationships analogous to Lidskii’s inequalities (Theorem 5.5). More precisely, we show that for all $k = 1, \dots, n$ and all $1 \leq i_1 < \dots < i_k \leq n$,

$$\sum_{j=1}^k d_{i_j}(A + B) \geq \sum_{j=1}^k d_{i_j}(A) + \sum_{j=1}^k d_j(B).$$

As for the case of eigenvalues of Hermitian matrices, these greatly generalize the inequalities given in [6] and [13]. In this process, we introduce a notion similar to the notion of projections, that we call “symplectic projections,” and give an equivalent statement for Williamson’s theorem in terms of symplectic projections.

The paper is organized as follows. Some definitions and preliminary results on symplectic eigenvalues are summarized in Section 2. In Section 3, we study the differentiability of symplectic eigenvalues and symplectic eigenvectors maps when the symplectic eigenvalues are simple, and compute their first-order derivatives. In Section 4, we discuss differentiability and analyticity of these maps for curves of positive definite matrices when the symplectic eigenvalues are not necessarily simple. As applications of our results, we derive a symplectic analogue of Lidskii’s theorem and give a perturbation bound in Section 5.

2 Preliminaries

Let \mathbb{R}^n denote the space of all n tuples over the real numbers, and let $\mathbb{M}(n)$ denote the space of all $n \times n$ real matrices. The bilinear form (\cdot, \cdot) on \mathbb{R}^{2n} given by

$$(2.1) \quad (x, y) = \sum_{i=1}^n (x_i y_{n+i} - x_{n+i} y_i)$$

is called the *symplectic inner product* on \mathbb{R}^{2n} . This can also be written as

$$(x, y) = \langle x, Jy \rangle.$$

Here, J is the $2n \times 2n$ matrix given by (1.1), and $\langle \cdot, \cdot \rangle$ denotes the Euclidean inner product on \mathbb{R}^{2n} . It is easy to see that a matrix $M \in Sp(2n)$ if and only if it preserves the symplectic inner product on \mathbb{R}^{2n} , that is,

$$(Mx, My) = \langle Mx, JM y \rangle = \langle x, Jy \rangle = (x, y).$$

A pair of vectors (u, v) is called *normalized* if $\langle u, Jv \rangle = 1$. Two pairs of vectors (u_1, v_1) and (u_2, v_2) are called *symplectically orthogonal* if

$$\langle u_i, Jv_j \rangle = \langle u_i, Ju_j \rangle = \langle v_i, Jv_j \rangle = 0$$

for all $i \neq j$, $i, j = 1, 2$. A subset $\{u_1, \dots, u_m, v_1, \dots, v_m\}$ of \mathbb{R}^{2n} is called a *symplectically orthogonal (orthonormal) set* if the pairs of vectors (u_i, v_i) are mutually symplectically orthogonal (and normalized). If $m = n$, then the symplectically orthonormal set is called a *symplectic basis* of \mathbb{R}^{2n} .

The following proposition is an easy consequence of Williamson’s Theorem.

Proposition 2.1 *Let A be a $2n \times 2n$ real positive definite matrix with symplectic eigenvalues d_1, \dots, d_n . There exists a symplectic basis $\{u_1, \dots, u_n, v_1, \dots, v_n\}$ of \mathbb{R}^{2n} such that for each $i = 1, \dots, n$,*

$$(2.2) \quad Au_i = d_i Jv_i, \quad Av_i = -d_i Ju_i.$$

A pair of vectors (u_i, v_i) that satisfies (2.2) is called a *symplectic eigenvector pair* of A corresponding to the symplectic eigenvalue d_i . If the pair, in addition, is normalized, it is called a *normalized symplectic eigenvector pair* of A .

The proofs of the next two results are straightforward and left to the reader.

Lemma 2.2 *Let $A \in \mathbb{P}(2n)$, and let d be a positive number. The following statements are equivalent.*

- (i) d is a symplectic eigenvalue of A and (u, v) is a corresponding symplectic eigenvector pair.
- (ii) $\pm d$ is an eigenvalue of ιJA and $u \mp \iota v$ is a corresponding eigenvector.
- (iii) $\pm d$ is an eigenvalue of $\iota A^{1/2} J A^{1/2}$ and $A^{1/2} u \mp \iota A^{1/2} v$ is a corresponding eigenvector.

Since d_1, \dots, d_n denote the symplectic eigenvalues arranged in increasing order, we usually denote any collection of symplectic eigenvalues by $\tilde{d}_1, \dots, \tilde{d}_n$.

Proposition 2.3 For A in $\mathbb{P}(2n)$, the set $\{(\tilde{u}_j, \tilde{v}_j) : j = 1, \dots, m\}$ is a symplectically orthogonal set of symplectic eigenvector pairs of A corresponding to the symplectic eigenvalues $\tilde{d}_1, \dots, \tilde{d}_m$, respectively, if and only if $\{A^{1/2}\tilde{u}_j - \iota A^{1/2}\tilde{v}_j : j = 1, \dots, m\}$ is an orthogonal set of eigenvectors of $\iota A^{1/2}JA^{1/2}$ corresponding to the eigenvalues $\tilde{d}_1, \dots, \tilde{d}_m$ respectively. Furthermore, for each $j = 1, \dots, m$

$$(2.3) \quad \|A^{1/2}\tilde{u}_j - \iota A^{1/2}\tilde{v}_j\|^2 = 2\tilde{d}_j \langle \tilde{u}_j, J\tilde{v}_j \rangle.$$

Corollary 2.4 Any two symplectic eigenvector pairs corresponding to two distinct symplectic eigenvalues of a real positive definite matrix are symplectically orthogonal.

Proposition 2.5 Let A be a $2n \times 2n$ real positive definite matrix, and let d be a symplectic eigenvalue of A with multiplicity m . Let $r_0 = \min\{|d - \tilde{d}| : \tilde{d} \text{ is a symplectic eigenvalue of } A, \tilde{d} \neq d\}$. Then, for any positive number $r < r_0$, there exists an open neighborhood U of A in $\mathbb{P}(2n)$ such that every P in U has exactly m symplectic eigenvalues (counted with multiplicities) contained in $(d - r, d + r)$.

Proof Let $d_1(A) \leq \dots \leq d_i(A) < d_{i+1}(A) = \dots = d_{i+m}(A) < d_{i+m+1}(A) \leq \dots \leq d_n(A)$ be the n symplectic eigenvalues of A with $d_{i+1}(A) = \dots = d_{i+m}(A) = d$. By our choice of r we see that

$$d_i(A) < d - r < d + r < d_{i+m+1}(A).$$

Since each d_j is continuous, we can find an open neighborhood U of A such that for every $P \in U$,

$$d_{i+1}(P), \dots, d_{i+m}(P) \in (d - r, d + r),$$

$$d_i(P) < d - r \text{ and } d_{i+m+1}(P) > d + r.$$

Thus, for every $P \in U$, there are exactly m symplectic eigenvalues $d_{i+1}(P), \dots, d_{i+m}(P)$ of P that are contained in $(d - r, d + r)$. The cases $d = d_1$ and $d = d_n$ can be proved in a similar way. ■

A subspace W of \mathbb{R}^{2n} is called a *symplectic subspace* of \mathbb{R}^{2n} if for every $x \in W$ there exists a $y \in W$ such that $\langle x, Jy \rangle \neq 0$. (See [9], Section 1.2.1.) If W is a symplectic subspace of \mathbb{R}^{2n} , then its dimension is an even number and there exists a symplectically orthonormal set that spans it. Let d be a symplectic eigenvalue of A , and let S be the set of all symplectic eigenvector pairs of A corresponding to d . Suppose W is the span of the set $\{u, v : (u, v) \in S\}$. It is easy to see that W is a symplectic subspace of \mathbb{R}^{2n} . If d has multiplicity k , then the dimension of W is $2k$.

We end this section with an observation on the extension of Williamson’s theorem and the notion of symplectic eigenvalues to positive semidefinite matrices.

Remark 2.6 Let A be a $2n \times 2n$ real positive semidefinite matrix. Then, there exists a symplectic matrix M such that (1.2) holds for some $n \times n$ nonnegative diagonal matrix D if and only if the kernel of A is a symplectic subspace of \mathbb{R}^{2n} . If $\dim \text{Ker } A = 2m$, then exactly m diagonal entries of A are zero. In this case, we call the nonnegative

diagonal entries of D to be the symplectic eigenvalues of the positive semidefinite matrix A .

Let $\mathbb{P}_s(2n)$ be the set of all $2n \times 2n$ real positive semidefinite matrices A such that $\text{Ker } A$ is a symplectic subspace of \mathbb{R}^{2n} . We can see from the proof of Theorem 7 of [7] that the maps d_j taking A to $d_j(A)$ are continuous on $\mathbb{P}_s(2n)$ for all $j = 1, \dots, n$.

3 Simple symplectic eigenvalues

The following theorem is the key result that will be used to prove the main theorem of this section.

Theorem 3.1 *Let A be a $2n \times 2n$ real positive definite matrix. Suppose d_0 is a simple symplectic eigenvalue of A with corresponding normalized symplectic eigenvector pair (u_0, v_0) . Then, there exists an open subset U of $\mathbb{P}(2n)$ containing A , and C^∞ maps $d : U \rightarrow \mathbb{R}$ and $u, v : U \rightarrow \mathbb{R}^{2n}$ that satisfy the following conditions.*

- (i) *For every $P \in U$, $d(P)$ is a simple symplectic eigenvalue of P with the corresponding normalized symplectic eigenvector pair $(u(P), v(P))$.*
- (ii) *$d(A) = d_0$, $u(A) = u_0$ and $v(A) = v_0$.*
- (iii)

$$(3.1) \quad \langle u_0, Ju(P) \rangle + \langle v_0, Jv(P) \rangle = 0.$$

Proof Since d_0 is a simple symplectic eigenvalue of A with symplectic eigenvector pair (u_0, v_0) , by Lemma 2.2, it is a simple eigenvalue of ιJA with eigenvector $x_0 = u_0 - \iota v_0$. Also $\langle x_0, Jx_0 \rangle = -2\iota \langle u_0, Jv_0 \rangle = -2\iota$. Define the map $\varphi : \mathbb{P}(2n) \times \mathbb{C}^{2n} \times \mathbb{C} \rightarrow \mathbb{C}^{2n} \times \mathbb{C}$ as

$$\varphi(P, x, d) = ((\iota JP - d)x, \langle x_0, Jx \rangle + 2\iota).$$

Clearly, φ is a C^∞ map and $\varphi(A, x_0, d_0) = 0$. Let $D_2\varphi$ denote the partial derivative of φ with respect to (x, d) . Then

$$D_2\varphi(A, x_0, d_0) = \begin{bmatrix} \iota JA - d_0 & -x_0 \\ x_0^* J & 0 \end{bmatrix}.$$

Thus, $\det D_2\varphi(A, x_0, d_0) = -\langle x_0, J(\iota JA - d_0)^{\text{adj}} x_0 \rangle$. For any $m \times m$ matrix X , X^{adj} denotes the adjoint of X . This is the $m \times m$ matrix with the ij th entry $(-1)^{i+j} X(j, i)$, where $X(j, i)$ is the (j, i) minor of X . Since d_0 is a simple eigenvalue of ιJA , 0 is a simple eigenvalue of $\iota JA - d_0$. So, we have $(\iota JA - d_0)^{\text{adj}} x_0 = cx_0$, where c is the product of all nonzero eigenvalues of $\iota JA - d_0$. This gives

$$\langle x_0, J(\iota JA - d_0)^{\text{adj}} x_0 \rangle = c \langle x_0, Jx_0 \rangle = -2\iota c \neq 0.$$

Hence, by the implicit function theorem, there exists an open subset U of $\mathbb{P}(2n)$ containing A , and C^∞ maps $d : U \rightarrow \mathbb{C}$ and $x : U \rightarrow \mathbb{C}^{2n}$ that satisfy $\iota JPx(P) = d(P)x(P)$, $\langle x_0, Jx(P) \rangle = -2\iota$, $x(A) = x_0$ and $d(A) = d_0$. Clearly $x(P) \neq 0$, and hence $d(P)$ is an eigenvalue of ιJP . All eigenvalues of ιJP are real. Hence $d(P)$ is real. Since $d_0 > 0$, we can assume that $d(P) > 0$ for all $P \in U$. By Lemma 2.2, we see that

$d(P)$ is a symplectic eigenvalue of P for every $P \in U$. Also, since $D_2\varphi(P, x(P), d(P))$ is invertible, $(\iota JP - d(P))^{\text{adj}} \neq 0$ and this implies that $d(P)$ has multiplicity 1. Let $x(P) = \tilde{u}(P) - \iota\tilde{v}(P)$ be the Cartesian decomposition of $x(P)$. By Lemma 2.2, we see that $(\tilde{u}(P), \tilde{v}(P))$ is a symplectic eigenvector pair of P corresponding to $d(P)$. Also, the maps $P \mapsto \tilde{u}(P)$ and $P \mapsto \tilde{v}(P)$ are C^∞ on U , and $\tilde{u}(A) = u_0$ and $\tilde{v}(A) = v_0$. We know that $\langle u_0, Jv_0 \rangle = 1$. Hence, we can assume that $\langle \tilde{u}(P), J\tilde{v}(P) \rangle > 0$ for all $P \in U$. This implies that the map $P \mapsto \langle \tilde{u}(P), J\tilde{v}(P) \rangle^{-1/2}$ is C^∞ on U . Define the maps $u, v : U \rightarrow \mathbb{R}^{2n}$ as

$$u(P) = \langle \tilde{u}(P), J\tilde{v}(P) \rangle^{-1/2} \tilde{u}(P)$$

and

$$v(P) = \langle \tilde{u}(P), J\tilde{v}(P) \rangle^{-1/2} \tilde{v}(P).$$

The maps u and v are C^∞ and $(u(P), v(P))$ forms a normalized symplectic eigenvector pair of P corresponding to $d(P)$. This shows the existence of infinitely differentiable maps d, u, v on U that satisfy (i) and (ii). Moreover, since the real part of $\langle x_0, Jx(P) \rangle$ is zero,

$$\langle u_0, Ju(P) \rangle + \langle v_0, Jv(P) \rangle = 0.$$

This proves (iii). ■

Remark 3.2 Since d_0 is a simple symplectic eigenvalue of A if and only if it is a simple eigenvalue of $\iota A^{1/2} J A^{1/2}$, (see Proposition 2.3) and the square root map is infinitely differentiable on real positive definite matrices, we can obtain (i) and (ii) of Theorem 3.1 from the corresponding result on eigenvalues in [24]. But, we give an independent proof as (3.1) is required in the computation of the derivatives of symplectic eigenvector pairs in Theorem 3.4.

The main theorem of this section is as follows:

Theorem 3.3 *Let $A \in \mathbb{P}(2n)$, and suppose that $d_j(A)$ is a simple symplectic eigenvalue of A . Then, there exists a neighborhood U of A in $\mathbb{P}(2n)$ such that for every $P \in U$, $d_j(P)$ is a simple symplectic eigenvalue of P and the map $P \mapsto d_j(P)$ is infinitely differentiable on U . Furthermore, if (u_0, v_0) is a normalized symplectic eigenvector pair of A corresponding to $d_j(A)$, then there exist infinitely differentiable maps $u_j, v_j : U \rightarrow \mathbb{R}^{2n}$ such that for every P in U $(u_j(P), v_j(P))$ is a normalized symplectic eigenvector pair of P corresponding to $d_j(P)$, $u_j(A) = u_0$ and $v_j(A) = v_0$, and $u_j(P), v_j(P)$ satisfy (3.1).*

Proof If $d_j(A)$ is a simple symplectic eigenvalue of A , then by Theorem 3.1, we can find an open neighbourhood V of A in $\mathbb{P}(2n)$, and C^∞ maps $d : V \rightarrow \mathbb{R}$ and $u, v : V \rightarrow \mathbb{R}^{2n}$ that satisfy (i)–(iii) of Theorem 3.1; that is, $d(P)$ is a simple symplectic eigenvalue of P and $(u(P), v(P))$ is a corresponding normalized symplectic eigenvector pair such that $d(A) = d_j(A)$, $u(A) = u_0$, $v(A) = v_0$, and $u(P), v(P)$ satisfy (3.1). Let r be a positive number with $r < \min\{d_{j+1}(A) - d_j(A), d_j(A) - d_{j-1}(A)\}$. By the continuity of the map $P \mapsto d(P)$ and Proposition 2.5, we can assume that for every P in V , $d(P)$ is the only symplectic eigenvalue of P contained in $(d_j(A) - r, d_j(A) + r)$.

By [7, Theorem 7], we know that the map $P \mapsto d_j(P)$ is continuous. Hence, there exists an open neighborhood W of A such that $d_j(P) \in (d_j(A) - r, d_j(A) + r)$ for every P in W . But this implies that $d(P) = d_j(P)$ for every $P \in V \cap W$. Take $U = V \cap W$. Hence, the map d_j is infinitely differentiable on U with the corresponding normalized symplectic eigenvector maps u, v that satisfy the required conditions. ■

Next, we compute the derivatives of the symplectic eigenvalue map d_j and its corresponding symplectic eigenvector pair map at A when $d_j(A)$ has multiplicity 1. We note here that if (u, v) is a normalized symplectic eigenvector pair of A corresponding to a simple symplectic eigenvalue d , then any normalized symplectic eigenvector pair (x, y) corresponding to d is of the form

$$x = au - bv \text{ and } y = bu + av$$

where a, b are real numbers satisfying $a^2 + b^2 = 1$.

Theorem 3.4 *Let $A \in \mathbb{P}(2n)$ be such that $d_j(A)$ is simple, and let (u_j, v_j) be a normalized symplectic eigenvector pair map through $(u_j(A), v_j(A))$ obtained from Theorem 3.3. Suppose M is any symplectic matrix given by (1.2). Then the derivatives $Dd_j(A)$, $Du_j(A)$, and $Dv_j(A)$ at a $2n \times 2n$ symmetric matrix B are given by*

$$(3.2) \quad Dd_j(A)(B) = \frac{\langle u_j(A), Bu_j(A) \rangle + \langle v_j(A), Bv_j(A) \rangle}{2},$$

$$(3.3) \quad Du_j(A)(B) = M\hat{D}M^T Bu_j(A) + M\bar{D}JM^T Bv_j(A),$$

and

$$(3.4) \quad Dv_j(A)(B) = M\hat{D}M^T Bv_j(A) - M\bar{D}JM^T Bu_j(A),$$

where \hat{D} and \bar{D} are the $2n \times 2n$ diagonal matrices with respective diagonal entries given by

$$(3.5) \quad (\hat{D})_{kk} = \begin{cases} \frac{d_k(A)}{d_j^2(A) - d_k^2(A)} & k \neq j, 1 \leq k \leq n \\ -\frac{1}{4d_j(A)} & k = j, 1 \leq k \leq n \\ (\hat{D})_{ii} & k = n + i, 1 \leq i \leq n, \end{cases}$$

and

$$(3.6) \quad (\bar{D})_{kk} = \begin{cases} \frac{d_j(A)}{d_j^2(A) - d_k^2(A)} & k \neq j, 1 \leq k \leq n \\ \frac{1}{4d_j(A)} & k = j, 1 \leq k \leq n \\ (\bar{D})_{ii} & k = n + i, 1 \leq i \leq n. \end{cases}$$

Proof Since $d_j(A)$ is simple, by Theorem 3.3, we know that the map d_j is infinitely differentiable at A . Since (u_j, v_j) is a normalized symplectic eigenvector pair map obtained from Theorem 3.3, we have

$$(3.7) \quad Pu_j(P) = d_j(P)Jv_j(P),$$

$$(3.8) \quad Pv_j(P) = -d_j(P)Ju_j(P),$$

$$(3.9) \quad \langle u_j(P), Jv_j(P) \rangle = 1,$$

$$(3.10) \quad \langle u_j(A), Ju_j(P) \rangle + \langle v_j(A), Jv_j(P) \rangle = 0.$$

Differentiating (3.7) and (3.8) at A , we see that for every $2n \times 2n$ real symmetric matrix B

$$(3.11) \quad Bu_j(A) + ADu_j(A)(B) = Dd_j(A)(B)Jv_j(A) + d_j(A)JDv_j(A)(B),$$

and

$$(3.12) \quad Bv_j(A) + ADv_j(A)(B) = -Dd_j(A)(B)Ju_j(A) - d_j(A)JDv_j(A)(B).$$

Taking the inner product of (3.11) with $u_j(A)$ and using the fact that $\langle u_j(A), Jv_j(A) \rangle = 1$, we get

$$(3.13) \quad \begin{aligned} \langle u_j(A), Bu_j(A) \rangle + \langle u_j(A), ADu_j(A)(B) \rangle \\ = Dd_j(A)(B) + \langle u_j(A), d_j(A)JDv_j(A)(B) \rangle. \end{aligned}$$

Since

$$\begin{aligned} \langle u_j(A), ADu_j(A)(B) \rangle &= \langle Au_j(A), Du_j(A)(B) \rangle \\ &= d_j(A)\langle Du_j(A)(B), Jv_j(A) \rangle, \end{aligned}$$

we can write (3.13) as

$$(3.14) \quad \begin{aligned} Dd_j(A)(B) &= \langle u_j(A), Bu_j(A) \rangle + d_j(A)\langle Du_j(A)(B), Jv_j(A) \rangle \\ &\quad - d_j(A)\langle u_j(A), JDv_j(A)(B) \rangle. \end{aligned}$$

Similarly, taking the inner product of (3.12) with $v_j(A)$, we get

$$(3.15) \quad \begin{aligned} Dd_j(A)(B) &= \langle v_j(A), Bv_j(A) \rangle - d_j(A)\langle Du_j(A)(B), Jv_j(A) \rangle \\ &\quad + d_j(A)\langle u_j(A), JDv_j(A)(B) \rangle. \end{aligned}$$

Adding (3.14) and (3.15) finally gives (3.2).

We next compute the derivatives $Du_j(A)$ and $Dv_j(A)$.

Let the columns of M be $\tilde{u}_1, \dots, \tilde{u}_n, \tilde{v}_1, \dots, \tilde{v}_n$. Clearly, these vectors form a symplectic eigenbasis of \mathbb{R}^{2n} corresponding to A . We can express $Du_j(A)(B)$ and $Dv_j(A)(B)$ uniquely as

$$Du_j(A)(B) = \sum_{k=1}^n \alpha_k \tilde{u}_k + \sum_{k=1}^n \beta_k \tilde{v}_k$$

and

$$Dv_j(A)(B) = \sum_{k=1}^n \gamma_k \tilde{u}_k + \sum_{k=1}^n \delta_k \tilde{v}_k,$$

where $\alpha_k = \langle Du_j(A)(B), J\tilde{v}_k \rangle$, $\beta_k = -\langle Du_j(A)(B), J\tilde{u}_k \rangle$, $\gamma_k = \langle Dv_j(A)(B), J\tilde{v}_k \rangle$ and $\delta_k = -\langle Dv_j(A)(B), J\tilde{u}_k \rangle$ for all $k = 1, \dots, n$. Since $d_j(A)$ is simple, we can assume

that $\tilde{u}_j = au_j(A) - bv_j(A)$ and $\tilde{v}_j = bu_j(A) + av_j(A)$ for some $a, b \in \mathbb{R}$ with $a^2 + b^2 = 1$. Thus

$$(3.16) \quad \langle \tilde{u}_k, Jv_j(A) \rangle = \langle u_j(A), J\tilde{v}_k \rangle = \delta_{kj}a$$

and

$$(3.17) \quad \langle \tilde{u}_k, Ju_j(A) \rangle = \langle \tilde{v}_k, Jv_j(A) \rangle = \delta_{kj}b$$

for all $k = 1, \dots, n$. Here, $\delta_{jk} = 0$ if $j \neq k$ and $\delta_{jk} = 1$ otherwise. Taking inner product of (3.11) with \tilde{u}_k we get

$$\begin{aligned} & \langle \tilde{u}_k, Bu_j(A) \rangle + \langle \tilde{u}_k, ADu_j(A)(B) \rangle \\ & = Dd_j(A)(B)\langle \tilde{u}_k, Jv_j(A) \rangle + d_j(A)\langle \tilde{u}_k, JDv_j(A)(B) \rangle. \end{aligned}$$

Using (3.16) and the values of α_k and δ_k , this reduces to

$$(3.18) \quad d_k(A)\alpha_k - d_j(A)\delta_k = aDd_j(A)(B)\delta_{kj} - \langle \tilde{u}_k, Bu_j(A) \rangle.$$

Similarly, taking inner products of (3.11) with \tilde{v}_k , and of (3.12) with \tilde{u}_k and \tilde{v}_k , and using (3.16) and (3.17), we obtain the expressions

$$(3.19) \quad d_k(A)\beta_k + d_j(A)\gamma_k = bDd_j(A)(B)\delta_{kj} - \langle \tilde{v}_k, Bu_j(A) \rangle,$$

$$(3.20) \quad d_j(A)\beta_k + d_k(A)\gamma_k = -bDd_j(A)(B)\delta_{kj} - \langle \tilde{u}_k, Bv_j(A) \rangle,$$

$$(3.21) \quad -d_j(A)\alpha_k + d_k(A)\delta_k = aDd_j(A)(B)\delta_{kj} - \langle \tilde{v}_k, Bv_j(A) \rangle.$$

Thus, for each $k = 1, \dots, n$, we have a system of four linear equations in four unknowns $\alpha_k, \beta_k, \gamma_k$, and δ_k . When $k \neq j$, this system is

$$\begin{bmatrix} d_k(A) & 0 & 0 & -d_j(A) \\ 0 & d_k(A) & d_j(A) & 0 \\ 0 & d_j(A) & d_k(A) & 0 \\ -d_j(A) & 0 & 0 & d_k(A) \end{bmatrix} \begin{bmatrix} \alpha_k \\ \beta_k \\ \gamma_k \\ \delta_k \end{bmatrix} = - \begin{bmatrix} \langle \tilde{u}_k, Bu_j(A) \rangle \\ \langle \tilde{v}_k, Bu_j(A) \rangle \\ \langle \tilde{u}_k, Bv_j(A) \rangle \\ \langle \tilde{v}_k, Bv_j(A) \rangle \end{bmatrix}.$$

Here, $d_j(A) \neq d_k(A)$, therefore, the coefficient matrix above is invertible and left multiplying by the inverse we get

$$\begin{bmatrix} \alpha_k \\ \beta_k \\ \gamma_k \\ \delta_k \end{bmatrix} = (d_j^2(A) - d_k^2(A))^{-1} \begin{bmatrix} d_k(A) & 0 & 0 & d_j(A) \\ 0 & d_k(A) & -d_j(A) & 0 \\ 0 & -d_j(A) & d_k(A) & 0 \\ d_j(A) & 0 & 0 & d_k(A) \end{bmatrix} \begin{bmatrix} \langle \tilde{u}_k, Bu_j(A) \rangle \\ \langle \tilde{v}_k, Bu_j(A) \rangle \\ \langle \tilde{u}_k, Bv_j(A) \rangle \\ \langle \tilde{v}_k, Bv_j(A) \rangle \end{bmatrix}.$$

The solution is thus given by the following equations

$$(3.22) \quad \alpha_k = \frac{1}{d_j^2(A) - d_k^2(A)} (d_k(A)\langle \tilde{u}_k, Bu_j(A) \rangle + d_j(A)\langle \tilde{v}_k, Bv_j(A) \rangle),$$

$$(3.23) \quad \beta_k = \frac{1}{d_j^2(A) - d_k^2(A)} (d_k(A)\langle \tilde{v}_k, Bu_j(A) \rangle - d_j(A)\langle \tilde{u}_k, Bv_j(A) \rangle),$$

$$(3.24) \quad \gamma_k = \frac{1}{d_j^2(A) - d_k^2(A)} (d_k(A) \langle \tilde{u}_k, Bv_j(A) \rangle - d_j(A) \langle \tilde{v}_k, Bu_j(A) \rangle),$$

$$(3.25) \quad \delta_k = \frac{1}{d_j^2(A) - d_k^2(A)} (d_k(A) \langle \tilde{v}_k, Bv_j(A) \rangle + d_j(A) \langle \tilde{u}_k, Bu_j(A) \rangle).$$

Now, for $k = j$ we have the following system

$$\begin{bmatrix} d_j(A) & 0 & 0 & -d_j(A) \\ 0 & d_j(A) & d_j(A) & 0 \\ 0 & d_j(A) & d_j(A) & 0 \\ -d_j(A) & 0 & 0 & d_j(A) \end{bmatrix} \begin{bmatrix} \alpha_j \\ \beta_j \\ \gamma_j \\ \delta_j \end{bmatrix} = - \begin{bmatrix} \langle \tilde{u}_j, Bu_j(A) \rangle - aDd_j(A)(B) \\ \langle \tilde{v}_j, Bu_j(A) \rangle - bDd_j(A)(B) \\ \langle \tilde{u}_j, Bv_j(A) \rangle + bDd_j(A)(B) \\ \langle \tilde{v}_j, Bv_j(A) \rangle - aDd_j(A)(B) \end{bmatrix}.$$

Using the expression for $Dd_j(A)(B)$, the fact that B is symmetric and the relationship between $(\tilde{u}_j, \tilde{v}_j)$ and $(u_j(A), v_j(A))$ one can see that the solution to the above system exists and is given by

$$(3.26) \quad \alpha_j - \delta_j = \frac{1}{2d_j(A)} (\langle \tilde{v}_j(A), Bv_j(A) \rangle - \langle \tilde{u}_j(A), Bu_j(A) \rangle),$$

$$(3.27) \quad \beta_j + \gamma_j = \frac{-1}{2d_j(A)} (\langle \tilde{v}_j(A), Bu_j(A) \rangle) + \frac{-1}{2d_j(A)} (\langle \tilde{u}_j(A), Bv_j(A) \rangle).$$

Differentiating (3.9) and (3.10), respectively, gives

$$\langle Du_j(A)(B), Jv_j(A) \rangle + \langle u_j(A), JDv_j(A)(B) \rangle = 0$$

and

$$\langle u_j(A), JDu_j(A)(B) \rangle + \langle v_j(A), JDv_j(A)(B) \rangle = 0.$$

These in turn imply $\alpha_j + \delta_j = 0$ and $\beta_j - \gamma_j = 0$. Thus,

$$(3.28) \quad \alpha_j = -\delta_j = \frac{1}{4d_j(A)} (\langle \tilde{v}_j(A), Bv_j(A) \rangle - \langle \tilde{u}_j(A), Bu_j(A) \rangle)$$

and

$$(3.29) \quad \beta_j = \gamma_j = \frac{-1}{4d_j(A)} (\langle \tilde{v}_j(A), Bu_j(A) \rangle) + \frac{-1}{4d_j(A)} (\langle \tilde{u}_j(A), Bv_j(A) \rangle).$$

Simplifying the above expressions we get for $k \neq j$,

$$\begin{aligned} \alpha_k &= \frac{1}{d_j^2(A) - d_k^2(A)} (d_k^2(A) \langle J\tilde{v}_k, A^{-1}Bu_j(A) \rangle + d_j(A) \langle J\tilde{v}_k, JBv_j(A) \rangle), \\ \beta_k &= -\frac{1}{d_j^2(A) - d_k^2(A)} (d_k^2(A) \langle J\tilde{u}_k, A^{-1}Bu_j(A) \rangle + d_j(A) \langle J\tilde{u}_k, JBv_j(A) \rangle), \\ \alpha_j &= -\frac{1}{4} \langle J\tilde{v}_j, A^{-1}Bu_j(A) \rangle + \frac{1}{4d_j(A)} \langle J\tilde{v}_j, JBv_j(A) \rangle, \\ \beta_j &= \frac{1}{4} \langle J\tilde{u}_j, A^{-1}Bu_j(A) \rangle - \frac{1}{4d_j(A)} \langle J\tilde{u}_j, JBv_j(A) \rangle. \end{aligned}$$

Let x be the $2n$ real vector with components $\alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_n$. Then we see that x can be written as

$$x = \hat{D}\tilde{D}M^{-1}A^{-1}Bu_j(A) + \bar{D}M^{-1}JBv_j(A),$$

where \tilde{D} is the $2n \times 2n$ diagonal matrix with diagonal entries the symplectic eigenvalues of $A, d_1(A), \dots, d_n(A), d_1(A), \dots, d_n(A)$, and \hat{D} and \bar{D} are the diagonal matrices given by (3.5) and (3.6), respectively. Therefore,

$$\begin{aligned} Du_j(A)(B) &= M\hat{D}\tilde{D}M^{-1}A^{-1}Bu_j(A) + M\bar{D}M^{-1}JBv_j(A) \\ &= M\hat{D}M^T Bu_j(A) + M\bar{D}JM^T Bv_j(A). \end{aligned}$$

The last equality follows from the fact that $M^TAM = \tilde{D}$ and $M^TJM = J$. This proves (3.3). Similar computations give (3.4). ■

Remark 3.5 Let $A \in \mathbb{P}(2n)$, and let d, u, v be maps on a neighborhood U of A such that $d(P)$ is a symplectic eigenvalue of P and $(u(P), v(P))$ is a normalized symplectic eigenvector pair for all $P \in U$. If d, u, v are differentiable at A , then by following the same steps as those used to prove (3.2), we can compute the derivative of d at A , even if $d(A)$ is not simple, as

$$(3.30) \quad Dd(A)(B) = \frac{1}{2}(\langle u(A), Bu(A) \rangle + \langle v(A), Bv(A) \rangle).$$

Given a map $t \mapsto A(t)$ from an open interval I to $\mathbb{P}(2n)$, we denote the symplectic eigenvalue $d_j(A(t))$ by $d_j(t)$, $1 \leq j \leq n$.

Corollary 3.6 Let $t \mapsto A(t)$ be a map from an open interval I to $\mathbb{P}(2n)$ that is infinitely differentiable at $t_0 \in I$. Suppose that $d_j(t_0)$ is simple. Then there exists an open interval I_0 containing t_0 such that the map d_j is infinitely differentiable on I_0 . If (u_0, v_0) is a corresponding normalized symplectic eigenvector pair of $A(t_0)$, then we can find an infinitely differentiable normalized symplectic eigenvector pair map (u_j, v_j) on I_0 corresponding to $d_j(t)$ such that $(u_j(t_0), v_j(t_0)) = (u_0, v_0)$, and $((u_j(t), v_j(t)))$ satisfies

$$\langle u_0, Ju_j(t) \rangle + \langle v_0, Jv_j(t) \rangle = 0$$

for all $t \in I_0$. Furthermore, for any symplectic matrix M given by the Williamson theorem for $A(t_0)$,

$$(3.31) \quad d'_j(t) = \frac{\langle u_j(t), A'(t)u_j(t) \rangle + \langle u_j(t), A'(t)u_j(t) \rangle}{2} \text{ for all } t \in J,$$

$$(3.32) \quad u'_j(t_0) = M\hat{D}M^T A'(t_0)u_0 + M\bar{D}JM^T A'(t_0)v_0,$$

and

$$(3.33) \quad v'_j(t_0) = M\hat{D}M^T A'(t_0)v_0 - M\bar{D}JM^T A'(t_0)u_0,$$

where \hat{D} and \bar{D} are the diagonal matrices associated with $A(t_0)$ given by (3.5) and (3.6), respectively.

Theorem 3.7 *Following the notations of Corollary 3.6, the second derivative of d_j at t_0 is given by*

$$\begin{aligned}
 d_j''(t_0) &= \frac{1}{2} (\langle u_0, A''(t_0)u_0 \rangle + \langle v_0, A''(t_0)v_0 \rangle) \\
 &\quad + 2\langle A'(t_0)u_0, M\bar{D}JM^T A'(t_0)v_0 \rangle \\
 (3.34) \quad &\quad + \langle A'(t_0)u_0, M\hat{D}M^T A'(t_0)u_0 \rangle + \langle A'(t_0)v_0, M\hat{D}M^T A'(t_0)v_0 \rangle,
 \end{aligned}$$

where \hat{D} and \bar{D} are the diagonal matrices associated with $A(t_0)$ given by (3.5) and (3.6), respectively.

Proof By (3.31), we have

$$(3.35) \quad d_j'(t) = \frac{\langle u_j(t), A'(t)u_j(t) \rangle + \langle v_j(t), A'(t)v_j(t) \rangle}{2}$$

for every t in I_0 . Differentiating (3.35) at $t = t_0$ and using the fact that $A'(t_0)$ is real symmetric, we get

$$\begin{aligned}
 d_j''(t_0) &= \frac{1}{2} (\langle u_0, A''(t_0)u_0 \rangle + \langle v_0, A''(t_0)v_0 \rangle) \\
 (3.36) \quad &\quad + \langle u_j'(t_0), A'(t_0)u_0 \rangle + \langle v_j'(t_0), A'(t_0)v_0 \rangle.
 \end{aligned}$$

Using the expression (3.32) for the derivative $u_j'(t_0)$, we get

$$\begin{aligned}
 \langle u_j'(t_0), A'(t_0)u_0 \rangle &= \langle M\hat{D}M^T A'(t_0)u_0, A'(t_0)u_0 \rangle \\
 (3.37) \quad &\quad + \langle M\bar{D}JM^T A'(t_0)v_0, A'(t_0)u_0 \rangle.
 \end{aligned}$$

Similarly using (3.33), we have

$$\begin{aligned}
 \langle v_j'(t_0), A'(t_0)v_0 \rangle &= \langle M\hat{D}M^T A'(t_0)v_0, A'(t_0)v_0 \rangle \\
 &\quad - \langle M\bar{D}JM^T A'(t_0)u_0, A'(t_0)v_0 \rangle.
 \end{aligned}$$

Since $\bar{D}J = J\bar{D}$, we have

$$\begin{aligned}
 \langle v_j'(t_0), A'(t_0)v_0 \rangle &= \langle M\hat{D}M^T A'(t_0)v_0, A'(t_0)v_0 \rangle \\
 (3.38) \quad &\quad + \langle M\bar{D}JM^T A'(t_0)v_0, A'(t_0)u_0 \rangle.
 \end{aligned}$$

Using (3.37) and (3.38) in (3.36), we obtain (3.34). ■

4 Symplectic eigenvalues of curves of positive definite matrices

In this section, we study the differentiability and analyticity of symplectic eigenvalues of positive definite matrices dependent on a real parameter irrespective of their multiplicities. The matrix square root is an infinitely differentiable map, and the symplectic eigenvalues of A are the positive eigenvalues of the Hermitian matrix $\iota A^{1/2}JA^{1/2}$. So, we obtain the results on the differentiability of symplectic eigenvalues by using the corresponding results on eigenvalues of Hermitian matrices. We similarly derive the results on analyticity of symplectic eigenvalues. For details on the differentiability

and analyticity of eigenvalues and eigenvectors of curves of Hermitian matrices, see [16, 24].

Let $t \mapsto H(t)$ be a map from an open interval I to the space $\mathbb{H}(k)$ of all $k \times k$ Hermitian matrices that is differentiable at $t_0 \in I$. Then, all the eigenvalues of $H(t)$ can be chosen to be differentiable at t_0 . It means that there exist k functions $\lambda_1, \dots, \lambda_k$ in a neighborhood of t_0 that are differentiable at t_0 and $\lambda_1(t), \dots, \lambda_k(t)$ are the k eigenvalues of $H(t)$ counted with multiplicity. Further if the map $t \mapsto H(t)$ is C^1 on I , then we can choose the eigenvalues to be C^1 on the whole of I . See [16, pp. 113–115]. Now, since the square root map $A \mapsto A^{1/2}$ is C^∞ on $\mathbb{P}(2n)$, we can directly obtain the following symplectic analogue. We recall that $\tilde{d}_1, \dots, \tilde{d}_n$ denote any collection of symplectic eigenvalues that are not necessarily ordered in either increasing or decreasing order.

Theorem 4.1 *Let $t \mapsto A(t)$ be a map from an open interval I to $\mathbb{P}(2n)$ that is differentiable at $t_0 \in I$. Then all the symplectic eigenvalues of $A(t)$ can be chosen to be differentiable at t_0 , that is, we can find n functions $\tilde{d}_1, \dots, \tilde{d}_n$ in a neighborhood of t_0 that are differentiable at t_0 such that $\tilde{d}_1(t), \dots, \tilde{d}_n(t)$ are the symplectic eigenvalues of $A(t)$. If, in addition, the map $t \mapsto A(t)$ is C^1 on I , then $\tilde{d}_1, \dots, \tilde{d}_n$ can be chosen to be C^1 on I .*

The continuity of symplectic eigenvector pairs cannot be guaranteed even if the map $t \mapsto A(t)$ is C^∞ on I . This we show by the following example.

Example 2 For each $t \in (-1, 1)$ define the 4×4 positive definite matrix $A(t)$ as

$$A(t) = I_2 \otimes P(t)$$

where

$$P(t) = \begin{bmatrix} 1 - e^{-1/t^2} \cos(2/t) & -e^{-1/t^2} \sin(2/t) \\ -e^{-1/t^2} \sin(2/t) & 1 + e^{-1/t^2} \cos(2/t) \end{bmatrix}$$

for $t \neq 0$ and $P(0) = I_2$. Clearly $t \mapsto A(t)$ is a smooth map. For $t \neq 0$, $d_1(t) = 1 - e^{-1/t^2}$ and $d_2(t) = 1 + e^{-1/t^2}$, and $d_1(0) = d_2(0) = 1$. Let $u_1(t) = e_1 \otimes [\cos(1/t) \ \sin(1/t)]^T$, $v_1(t) = e_2 \otimes [\cos(1/t) \ \sin(1/t)]^T$ and $u_2(t) = e_1 \otimes [\sin(1/t) \ -\cos(1/t)]^T$, $v_2(t) = e_2 \otimes [\sin(1/t) \ -\cos(1/t)]^T$, where e_1, e_2 are the standard unit vectors in \mathbb{R}^2 .

One can see that $(u_1(t), v_1(t))$ (resp. $(u_2(t), v_2(t))$) is a normalized symplectic eigenvector pair corresponding to $d_1(t)$ (resp. $d_2(t)$). Suppose that there exist functions $\tilde{u}, \tilde{v} : (-1, 1) \rightarrow \mathbb{R}^4$, continuous at 0 such that $(\tilde{u}(t), \tilde{v}(t))$ forms a normalized symplectic eigenvector pair of $A(t)$. For each $t \neq 0$ the pair $(\tilde{u}(t), \tilde{v}(t))$ either corresponds to $d_1(t)$, or to $d_2(t)$. Therefore, we can get a sequence $(t_j)_{j \in \mathbb{N}}$ of nonzero terms in $(-1, 1)$ converging to 0 such that for all $j \in \mathbb{N}$ $(\tilde{u}(t_j), \tilde{v}(t_j))$ corresponds either to $d_1(t_j)$ or to $d_2(t_j)$. Consider the case when $(\tilde{u}(t_j), \tilde{v}(t_j))$ corresponds to $d_1(t_j)$ for all j . For each j , $d_1(t_j)$ is a simple symplectic eigenvalue of $A(t_j)$. This implies that the normalized symplectic eigenvector pair $(\tilde{u}(t_j), \tilde{v}(t_j))$ is of the form $\tilde{u}(t_j) = a_j u_1(t_j) - b_j v_1(t_j)$, $\tilde{v}(t_j) = b_j u_1(t_j) + a_j v_1(t_j)$ where $a_j, b_j \in \mathbb{R}$ and

$a_j^2 + b_j^2 = 1$. The continuity of \tilde{u} and \tilde{v} at $t = 0$ implies that the limits $\lim_{j \rightarrow \infty} a_j \sin(1/t_j)$ and $\lim_{j \rightarrow \infty} b_j \sin(1/t_j)$ exist, which in turn imply that $\lim_{j \rightarrow \infty} \sin^2(1/t_j)$ exists. This is a contradiction. We get a similar contradiction in the other case. Therefore, we conclude that there does not exist any continuous selection of normalized symplectic eigenvector pairs.

However, the symplectic eigenvalues and the symplectic eigenvector pairs can be chosen smoothly under an additional condition as shown in the following theorem. The proof follows from the smoothness of the map $A \mapsto A^{1/2}$ on $\mathbb{P}(2n)$, and [1, Theorem 7.6]. We say that two functions f and g continuous at t_0 meet with infinite order if for every $p \in \mathbb{N}$ there exists a function h_p continuous at t_0 such that $f(t) - g(t) = t^p h_p(t)$. See (3.5) in [1].

Theorem 4.2 *Let $t \mapsto A(t)$ be a smooth map from an open interval I to $\mathbb{P}(2n)$ such that for all $1 \leq i \neq j \leq n$ either $d_i(t) = d_j(t)$ for all $t \in I$ or $d_i(t)$ and $d_j(t)$ do not meet with infinite order at any point in I . Then all the symplectic eigenvalues and corresponding symplectic eigenbasis can be chosen smoothly in t on I .*

We now turn to the case when $A(t)$ is a real analytic curve.

Theorem 4.3 *Let $t \mapsto A(t)$ be a map from an open interval I to $\mathbb{P}(2n)$ that is real analytic at $t_0 \in I$.*

- (i) *If d is a symplectic eigenvalue of $A(t_0)$ with multiplicity m , then for some $\varepsilon > 0$, there exist m symplectic eigenvalue maps $\tilde{d}_1, \dots, \tilde{d}_m : (t_0 - \varepsilon, t_0 + \varepsilon) \rightarrow \mathbb{R}$, and m corresponding symplectically orthonormal symplectic eigenvector pair maps $(\tilde{u}_1, \tilde{v}_1), \dots, (\tilde{u}_m, \tilde{v}_m) : (t_0 - \varepsilon, t_0 + \varepsilon) \rightarrow \mathbb{R}^{2n} \times \mathbb{R}^{2n}$ that are real analytic at t_0 with each $\tilde{d}_j(t_0) = d$.*
- (ii) *There exists an $\varepsilon > 0$ such that all the n symplectic eigenvalues of $A(t)$ and a corresponding symplectic eigenbasis can be chosen on $(t_0 - \varepsilon, t_0 + \varepsilon)$ to be real analytic at t_0 .*

Similar to the case of differentiability, we use the results on analyticity of eigenvalues of Hermitian matrices. For this, we need the following proposition. Since we could not find an explicit proof of this in the literature we include its proof in the appendix for the convenience of the reader.

Proposition 4.4 *Let $t \mapsto A(t)$ be a map from an open interval I to $\mathbb{P}(m)$ that is real analytic at $t_0 \in I$. Then the composite map $t \mapsto (A(t))^{1/2}$ is also real analytic at t_0 .*

We use the following result for eigenvalues and eigenvectors for Hermitian matrices to prove Theorem 4.3. See Kato ([16], Ch. II, Sec. 6) and Rellich ([24], Ch. I, Sec. 1, Theorem 1).

Proposition 4.5 *Let $t \mapsto H(t)$ be a map from an open interval I to $\mathbb{H}(k)$ that is real analytic at t_0 . If λ is an eigenvalue of $H(t_0)$ with multiplicity m , then there exists an*

$\varepsilon > 0$ so that we can find m eigenvalue functions $\lambda_1, \dots, \lambda_m : (t_0 - \varepsilon, t_0 + \varepsilon) \rightarrow \mathbb{R}$ and m corresponding orthonormal eigenvector functions $x_1, \dots, x_m : (t_0 - \varepsilon, t_0 + \varepsilon) \rightarrow \mathbb{C}^k$ that are real analytic at t_0 . Also, $\lambda_i(t_0) = \lambda$ for all $i = 1, \dots, m$.

Proof of Theorem 4.3 Let $H(t)$ be the Hermitian matrix ${}^tA^{1/2}(t)JA^{1/2}(t)$. Since $t \mapsto A(t)$ is real analytic at t_0 , by Proposition 4.4, the map $t \mapsto H(t)$ is also real analytic at t_0 . By Proposition 2.3, the multiplicity of the eigenvalue d of $H(t_0)$ is m . Hence by Proposition 4.5, there exists an $\varepsilon > 0$, and m functions $\tilde{d}_1, \dots, \tilde{d}_m : (t_0 - \varepsilon, t_0 + \varepsilon) \rightarrow \mathbb{R}$ and m functions $x_1, \dots, x_m : (t_0 - \varepsilon, t_0 + \varepsilon) \rightarrow \mathbb{C}^{2n}$ that are real analytic at t_0 such that $\tilde{d}_1(t), \dots, \tilde{d}_m(t)$ are m eigenvalues of $H(t)$ and $\{x_1(t), x_2(t), \dots, x_m(t)\}$ is a corresponding orthonormal set of eigenvectors. Also $\tilde{d}_j(t_0) = d$ for all $j = 1, \dots, m$. Since $H(t)$ is invertible for every t and $d > 0$, each $\tilde{d}_j(t) > 0$. Hence $\tilde{d}_j(t)$ is a symplectic eigenvalue of $A(t)$ for every $t \in (t_0 - \varepsilon, t_0 + \varepsilon)$ and $j = 1, \dots, m$. Let $x_j(t) = \bar{u}_j(t) - i\bar{v}_j(t)$ be the Cartesian decomposition of $x_j(t)$. For every $t \in (t_0 - \varepsilon, t_0 + \varepsilon)$ let $\tilde{u}_j(t) = \sqrt{2\tilde{d}_j(t)}A^{-1/2}(t)\bar{u}_j(t)$ and $\tilde{v}_j(t) = \sqrt{2\tilde{d}_j(t)}A^{-1/2}(t)\bar{v}_j(t)$. Since $\tilde{d}_j(t)$ and $A^{-1/2}(t)$ are real analytic at t_0 , $\tilde{u}_j(t)$ and $\tilde{v}_j(t)$ are real analytic at t_0 . Finally by Proposition 2.3, $\{(\tilde{u}_j(t), \tilde{v}_j(t)) : j = 1, \dots, m\}$ is a symplectically orthonormal set of symplectic eigenvector pairs of $A(t)$ corresponding to $\tilde{d}_1(t), \dots, \tilde{d}_m(t)$. This proves (i).

Let $\bar{d}_1 < \dots < \bar{d}_k$ be distinct symplectic eigenvalues of $A(t_0)$ with multiplicities m_1, \dots, m_k , respectively. By statement (i) of the theorem, we can find an $\varepsilon > 0$ and n symplectic eigenvalue functions $\tilde{d}_{1,1}(t), \dots, \tilde{d}_{1,m_1}(t), \dots, \tilde{d}_{k,1}(t), \dots, \tilde{d}_{k,m_k}(t)$ of $A(t)$ on $(t_0 - \varepsilon, t_0 + \varepsilon)$ that are real analytic at t_0 . Also for each $j = 1, \dots, k$, we can choose corresponding symplectically orthonormal symplectic eigenvector pairs $(\tilde{u}_{j,i}(t), \tilde{v}_{j,i}(t))$, $1 \leq i \leq m_j$, that are real analytic at t_0 . Using Proposition 2.5, we can assume that $\varepsilon > 0$ is small enough so that for all $t \in (t_0 - \varepsilon, t_0 + \varepsilon)$ $\tilde{d}_{r,i}(t) \neq \tilde{d}_{s,j}(t)$ for all $1 \leq i \leq m_r$ and $1 \leq j \leq m_s$, $r \neq s$. Thus, by Corollary 2.4, the symplectic eigenvector pairs $(\tilde{u}_{j,i}(t), \tilde{v}_{j,i}(t))$, $1 \leq i \leq m_j$, $1 \leq j \leq k$, form the required symplectic eigenbasis. ■

By arguing in a similar way as in the proof of Theorem 4.3(i) and using the analogous result for eigenvalues and eigenvectors of Hermitian matrices (see [16], Ch. II, Sec. 6), we can obtain the following theorem.

Theorem 4.6 Let $t \mapsto A(t)$ be a real analytic map from an open interval I to $\mathbb{P}(2n)$. Then we can choose n symplectic eigenvalue functions and corresponding symplectic eigenbasis map such that they are real analytic on I .

We have seen that the ordered tuple $d_1 \leq d_2 \leq \dots \leq d_n$ of symplectic eigenvalues need not be differentiable when the multiplicities are greater than one. But, they can be proved to be piecewise real analytic if the map $t \mapsto A(t)$ is real analytic on I .

Theorem 4.7 Let $t \mapsto A(t)$ be a real analytic map from an open interval I to $\mathbb{P}(2n)$, and let $[a, b]$ be any compact interval contained in I . Then for each $j = 1, \dots, n$, the map $t \mapsto d_j(t) = d_j(A(t))$ is piecewise real analytic on $[a, b]$. Further for each $t \in$

$[a, b]$, we can find a symplectic eigenbasis $\{u_1(t), \dots, u_n(t), v_1(t), \dots, v_n(t)\}$ of $A(t)$ corresponding to $d_1(t), \dots, d_n(t)$ such that the maps $u_1, \dots, u_n, v_1, \dots, v_n$ are also piecewise real analytic on $[a, b]$.

Proof By Theorem 4.6, we can find n symplectic eigenvalues $\tilde{d}_1(t), \dots, \tilde{d}_n(t)$ of $A(t)$ and a corresponding symplectic eigenbasis $\{\tilde{u}_1(t), \dots, \tilde{u}_n(t), \tilde{v}_1(t), \dots, \tilde{v}_n(t)\}$ such that each of the maps \tilde{d}_j, \tilde{u}_j and \tilde{v}_j are real analytic on I .

Define \mathcal{J} to be the set of all ordered pairs $(i, j), 1 \leq i \neq j \leq n$, such that $\tilde{d}_i(t) \neq \tilde{d}_j(t)$ for at least one t in $[a, b]$. Let E be the set of all points t in $[a, b]$ such that $\tilde{d}_i(t) = \tilde{d}_j(t)$ for some $(i, j) \in \mathcal{J}$. By using the real analyticity of the maps $\tilde{d}_1, \dots, \tilde{d}_n$ and the definition of the set \mathcal{J} , we can see that E is finite. Then for every $i = 1, \dots, n$, the multiplicity of $\tilde{d}_i(t)$ is the same for all t in $[a, b] \setminus E$. Hence, $\tilde{d}_1, \dots, \tilde{d}_n$ can be reordered so that $\tilde{d}_i(t) = d_i(t)$ for all $t \in [a, b] \setminus E$. The theorem thus follows by suitably reordering the symplectic eigenvalues $\tilde{d}_1, \dots, \tilde{d}_n$ and correspondingly reordering the symplectic eigenvalue pairs $\tilde{u}_1, \dots, \tilde{u}_n, \tilde{v}_1, \dots, \tilde{v}_n$. ■

5 Symplectic analogue of Lidskii’s theorem and other applications

The main object of this section is to derive a Lidskii type result, that is, a majorization inequality between the symplectic eigenvalues of the sum of two positive definite matrices and those of the two matrices. We start this section by introducing a notion of symplectic projections that is of independent interest and is useful in the proof of our main theorem of this section.

Let $S = \{x_1, \dots, x_k, y_1, \dots, y_k\}$ be a symplectically orthonormal subset of \mathbb{R}^{2n} . Define the map P_S on \mathbb{R}^{2n} as

$$(5.1) \quad P_S(x) = \sum_{i=1}^k (\langle x, Jy_i \rangle Jy_i + \langle x, Jx_i \rangle Jx_i).$$

Suppose M is the $2n \times 2k$ matrix

$$(5.2) \quad M = [Jx_1, \dots, Jx_k, Jy_1, \dots, Jy_k].$$

It is easy to see that $P_S = MM^T$, and so, it is a positive semidefinite matrix. In fact, it can be seen that the kernel of P_S is the symplectic complement of S , and hence $P_S \in \mathbb{P}_S(2n)$ with symplectic eigenvalues 1 and 0 with multiplicities k and $n - k$, respectively. We call P_S to be the *symplectic projection* associated with the set S . If $k = n$, that is, S is a symplectic basis of \mathbb{R}^{2n} , then P_S is a positive definite symplectic matrix with all its symplectic eigenvalues 1.

The symplectic projections associated with two symplectically orthonormal sets that have the same span need not be equal. This can be seen by the following example.

Example 3 Let $S = \{(1, 0)^T, (0, 1)^T\}$ and $T = \{(1, 0)^T, (1, 1)^T\}$. The sets S and T are symplectically orthonormal and span \mathbb{R}^2 . The symplectic projection P_S is the 2×2

identity matrix, whereas the symplectic projection P_T is the matrix $\begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}$.

In the following proposition, we give a necessary and sufficient condition for the equality of two symplectic projections.

Proposition 5.1 *Let $S = \{u_1, \dots, u_k, v_1, \dots, v_k\}$ and $T = \{x_1, \dots, x_m, y_1, \dots, y_m\}$ be two symplectically orthonormal subsets of \mathbb{R}^{2n} , and let P and Q be the symplectic projections associated with them. Let M and N be the $2n \times 2k$ and $2n \times 2m$ matrices given by (5.2) corresponding to the sets S and T , respectively. Then $P = Q$ if and only if $k = m$ and $M = NU$ for some $2k \times 2k$ orthosymplectic (symplectic as well as orthogonal) matrix U .*

Proof If $k = m$ and $M = NU$, the equality $P = Q$ easily follows from the orthogonality of U , and the fact that $P = MM^T$ and $Q = NN^T$.

Conversely, let $P = Q$. Clearly the subspaces spanned by S and T are the same, and hence $k = m$. By (5.1)

$$Px_j = \sum_{i=1}^k (\alpha_{ij}Ju_i + \beta_{ij}Jv_i)$$

for all $j = 1, \dots, k$. Here, $\alpha_{ij} = \langle x_j, Ju_i \rangle$ and $\beta_{ij} = \langle x_j, Jv_i \rangle$, $1 \leq i, j \leq k$. Since $P = Q$, $Px_i = Jy_i$. This gives

$$(5.3) \quad y_j = \sum_{i=1}^k (\alpha_{ij}u_i + \beta_{ij}v_i).$$

Also, since x_j belongs to the span of the symplectically orthonormal vectors $u_1, \dots, u_k, v_1, \dots, v_k$,

$$(5.4) \quad x_j = \sum_{i=1}^k (\langle x_j, Jv_i \rangle u_i - \langle x_j, Ju_i \rangle v_i)$$

$$(5.5) \quad = \sum_{i=1}^k (\beta_{ij}u_i - \alpha_{ij}v_i).$$

Let X and Y be the $k \times k$ matrices $X = [\alpha_{ij}]$ and $Y = [\beta_{ij}]$, and U be the $2k \times 2k$ matrix

$$U = \begin{bmatrix} Y & X \\ -X & Y \end{bmatrix}.$$

Using the fact that $x_1, \dots, x_k, y_1, \dots, y_k$ are symplectically orthonormal, we can see that the columns of U are orthonormal as well as symplectically orthonormal vectors in \mathbb{R}^{2k} . Finally, from (5.3) and (5.4), we obtain $N = MU$. ■

We now give an equivalent statement for Williamson’s theorem in terms of symplectic projections.

Proposition 5.2 *For every B in $\mathbb{P}(2n)$ there exist distinct positive numbers μ_1, \dots, μ_m and symplectic projections P_1, \dots, P_m that satisfy the following conditions.*

- (i) $P_j P_k = 0$ for all $j \neq k, j, k = 1, \dots, m$.

$$(ii) \sum_{k=1}^m P_k J P_k = J.$$

$$(iii) B = \sum_{k=1}^m \mu_k P_k.$$

The numbers μ_1, \dots, μ_m and the symplectic projections P_1, \dots, P_m are uniquely determined by the above conditions. Furthermore, for every $1 \leq j \leq m$, μ_j is a symplectic eigenvalue of B and P_j is the symplectic projection associated with a symplectically orthonormal set of symplectic eigenvector pairs of B corresponding to μ_j .

Proof Let μ_1, \dots, μ_m be the distinct symplectic eigenvalues of B with multiplicities k_1, \dots, k_m , respectively. For every $j = 1, \dots, m$ let $S_j = \{u_{j,1}, \dots, u_{j,k_j}, v_{j,1}, \dots, v_{j,k_j}\}$ be a symplectically orthonormal set of symplectic eigenvector pairs of B corresponding to μ_j . Let P_j be the symplectic projection associated with S_j . By the definition of symplectic projections and Williamson’s theorem, we can see that μ_1, \dots, μ_m and P_1, \dots, P_m satisfy (i)–(iii).

Now, let η_1, \dots, η_l be l distinct positive numbers and Q_1, \dots, Q_l be symplectic projections that also satisfy (i)–(iii). For every $j = 1, \dots, l$, let $T_j = \{x_{j,1}, \dots, x_{j,r_j}, y_{j,1}, \dots, y_{j,r_j}\}$ be a symplectically orthonormal set corresponding to Q_j . By using (i) and (iii), we can see that each η_j is a symplectic eigenvalue of B , and $(x_{j,i}, y_{j,i}), 1 \leq i \leq r_j$, are the symplectically orthonormal symplectic eigenvector pairs corresponding to η_j . Condition (ii) implies that $\{\eta_1, \dots, \eta_l\}$ forms the set of all distinct symplectic eigenvalues of B . By the uniqueness of symplectic eigenvalues, we have $l = m$ and $\{\mu_1, \dots, \mu_m\} = \{\eta_1, \dots, \eta_l\}$. We can assume that $\mu_j = \eta_j$ for all $j = 1, \dots, m$. By (iii), we see that r_j is equal to the multiplicity of μ_j . Since symplectic eigenvector pairs corresponding to different eigenvalues are symplectically orthogonal, S_j is symplectically orthogonal to T_k for all $j \neq k$. Consequently, $P_j x = 0$ for all $x \in T_k$ and for all $k \neq j$. Thus, for every $(x_{j,i}, y_{j,i})$ in T_j , we have

$$\mu_j Q_j x_{j,i} = \mu_j J y_{j,i} = B x_{j,i} = \mu_j P_j x_{j,i}.$$

and since $\mu_j \neq 0, P_j x_{j,i} = Q_j x_{j,i}$. Similarly $P_j y_{j,i} = Q_j y_{j,i}$. Since $\cup T_j$ forms a basis for \mathbb{R}^{2n} , we get $P_j = Q_j$ for all $j = 1, \dots, m$. ■

By using Proposition 5.1 and the uniqueness of symplectic projections in Proposition 5.2, we get the following:

Corollary 5.3 Let $A \in \mathbb{P}(2n)$, and let d be its symplectic eigenvalue with multiplicity m . Let $S = \{u_1, \dots, u_m, v_1, \dots, v_m\}$ be a symplectically orthonormal set of symplectic eigenvector pairs of A corresponding to d . Then the set $T = \{x_1, \dots, x_m, y_1, \dots, y_m\}$ is also a symplectically orthonormal set of symplectic eigenvector pairs corresponding to d if and only if there exists a $2m \times 2m$ orthosymplectic matrix U such that

$$N = MU,$$

where M and N are $2n \times 2m$ matrices with columns $u_1, \dots, u_m, v_1, \dots, v_m$ and $x_1, \dots, x_m, y_1, \dots, y_m$, respectively.

We can also verify that if $d_1(B), \dots, d_n(B)$ are the symplectic eigenvalues of B and $\{u_1, \dots, u_n, v_1, \dots, v_n\}$ is a corresponding symplectic eigenbasis, then

$$B = \sum_{j=1}^n d_j(B)P_j,$$

where P_j is the symplectic projection corresponding to $\{u_j, v_j\}$.

For a real vector $x = (x_1, \dots, x_n)$, we denote by x^\uparrow the vector $(x_1^\uparrow, \dots, x_n^\uparrow)$ obtained by rearranging the components of x in increasing order, i.e.,

$$x_1^\uparrow \leq \dots \leq x_n^\uparrow.$$

We say x is *supermajorized* by y , in symbols $x <^w y$, if for $1 \leq k \leq n$

$$(5.6) \quad \sum_{j=1}^k x_j^\uparrow \geq \sum_{j=1}^k y_j^\uparrow.$$

We say that x is *majorized* by y (or y *majorizes* x) if the two sides in the above inequalities are equal when $k = n$.

An $n \times n$ matrix $B = [b_{ij}]$ is called *doubly superstochastic* if there exists an $n \times n$ doubly stochastic matrix $A = [a_{ij}]$ such that $b_{ij} \geq a_{ij}$ for all $i, j = 1, \dots, n$. See [2]. It can be seen that the set of all doubly superstochastic matrices is a closed and convex subset of $\mathbb{M}(n)$. In order to prove Theorem 5.5, we will use the following fundamental result in the theory of majorization, (see [2], Corollary 3.4).

Lemma 5.4 *The following two conditions are equivalent:*

- (i) *An $n \times n$ matrix A is doubly superstochastic.*
- (ii) *$Ax <^w x$ for every positive n -vector x .*

For a positive definite matrix A , we denote by $d^\uparrow(A)$ the n -tuple of symplectic eigenvalues arranged in increasing order, that is,

$$d^\uparrow(A) = (d_1(A), \dots, d_n(A)).$$

Theorem 5.5 *Let A, B be two $2n \times 2n$ positive definite matrices. Then*

$$(5.7) \quad d^\uparrow(A + B) - d^\uparrow(A) <^w d^\uparrow(B).$$

Proof Define the map $\varphi : [0, 1] \rightarrow \mathbb{P}(2n)$ as

$$\varphi(t) = A + tB.$$

Clearly, φ is real analytic with $\varphi'(t) = B$. Let $1 \leq j \leq n$, and let $d_j(t) = d_j(\varphi(t))$. By Theorem 4.7, d_j is piecewise real analytic. Also by the same theorem, we can find a piecewise real analytic symplectic eigenbasis $\beta(t) = \{u_1(t), \dots, u_n(t), v_1(t), \dots, v_n(t)\}$ of $\varphi(t)$ corresponding to $d_1(t), \dots, d_n(t)$. For any t in $[0, 1]$ at which d_j, u_j , and v_j are real analytic, we have

$$(5.8) \quad d'_j(t) = \frac{1}{2} (\langle u_j(t), Bu_j(t) \rangle + \langle v_j(t), Bv_j(t) \rangle).$$

Let $\mu_1 \leq \dots \leq \mu_n$ be the symplectic eigenvalues of B and $\beta = \{x_1, \dots, x_n, y_1, \dots, y_n\}$ be a corresponding symplectic eigenbasis. Let P_j be the symplectic projection corresponding to (x_j, y_j) . Then $B = \sum_{j=1}^n \mu_j P_j$. Thus, by using this expression for B and using (5.1) for P_k in (5.8), we get

$$\begin{aligned}
 d'_j(t) &= \sum_{k=1}^n \frac{\mu_k}{2} (\langle u_j(t), P_k u_j(t) \rangle + \langle v_j(t), P_k v_j(t) \rangle) \\
 &= \sum_{k=1}^n \frac{\mu_k}{2} (\langle u_j(t), Jy_k \rangle^2 + \langle u_j(t), Jx_k \rangle^2 \\
 (5.9) \quad &+ \langle v_j(t), Jy_k \rangle^2 + \langle v_j(t), Jx_k \rangle^2).
 \end{aligned}$$

Since $\beta(t)$ and β are symplectic bases of $\mathbb{P}(2n)$, the matrix $M(t)$ with rs th entry

$$m_{rs}(t) = \begin{cases} \langle u_j(t), Jx_k \rangle & r = j, s = k, 1 \leq j, k \leq n \\ \langle u_j(t), Jy_k \rangle & r = j, s = n + k, 1 \leq j, k \leq n \\ \langle v_j(t), Jx_k \rangle & r = n + j, s = k, 1 \leq j, k \leq n \\ \langle v_j(t), Jy_k \rangle & r = n + j, s = n + k, 1 \leq j, k \leq n \end{cases}$$

is a symplectic matrix. Let $\tilde{M}(t)$ be the $n \times n$ matrix with jk th entry

$$\frac{m_{jk}^2(t) + m_{j(n+k)}^2(t) + m_{(n+j)k}^2(t) + m_{(n+j)(n+k)}^2(t)}{2}.$$

Then by (5.9), we see that $d'_j(t)$ is the j th component of the vector $\tilde{M}(t)d^\dagger(B)$, that is,

$$(5.10) \quad d'(t) = \tilde{M}(t)d^\dagger(B).$$

where $d'(t) = (d'_1(t), \dots, d'_n(t))^T$. Since d_j, u_j, v_j are piecewise real analytic on $[0, 1]$, the maps d_j and \tilde{M} are integrable on $[0, 1]$. Denote by \bar{M} , the $n \times n$ matrix

$$\bar{M} = \int_0^1 \tilde{M}(t) dt.$$

By [7, Theorem 6], each $\tilde{M}(t)$ is doubly superstochastic. Since the set of doubly superstochastic matrices is closed and convex, \bar{M} is also doubly superstochastic. Integrating (5.10), we get

$$d^\dagger(A + B) - d^\dagger(A) = \bar{M}d^\dagger(B).$$

We finally obtain (5.7) by Lemma 5.4. ■

Corollary 5.6 *Let $A, B \in \mathbb{P}(2n)$ and let $k \in \{1, \dots, n\}$. For all $1 \leq i_1 < \dots < i_k \leq n$, we have*

$$(5.11) \quad \sum_{j=1}^k d_{i_j}(A + B) \geq \sum_{j=1}^k d_{i_j}(A) + \sum_{j=1}^k d_j(B).$$

In particular,

$$(5.12) \quad d_j(A + B) \geq d_j(A) + d_1(B),$$

and

$$d_j(A + I) \geq d_j(A) + 1.$$

Here, I denotes the $2n \times 2n$ identity matrix.

When $\{i_1, \dots, i_k\}$ is the set $\{1, \dots, k\}$ in (5.11), we obtain the inequalities first proved by Hiroshima. See [7, 13]. The inequalities (5.12) were proved recently by R. Bhatia in [6] in the case when A and B are of some specific form.

We also point out that the supermajorization in (5.7) cannot be replaced by majorization. Let $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ and $B = I_2$, the 2×2 identity matrix. The only symplectic eigenvalues of A, B and $A + B$ are

$$d_1(A) = \sqrt{3}, \quad d_1(B) = 1 \quad \text{and} \quad d_1(A + B) = 2\sqrt{2}.$$

Clearly, $d_1(A + B) > d_1(A) + d_1(B)$.

Following is a simple application of Theorem 5.5.

Corollary 5.7 For all $k = 1, \dots, n$ and $1 \leq i_1 < \dots < i_k \leq n$, the map $A \mapsto \sum_{j=1}^k d_{i_j}(A)$ on $\mathbb{P}(2n)$ has neither a local minimizer nor a local maximizer in $\mathbb{P}(2n)$. In particular, for every $j = 1, \dots, n$, the map $A \mapsto d_j(A)$ has neither a local minimizer nor a local maximizer in $\mathbb{P}(2n)$.

Proof Let I denote the $2n \times 2n$ identity matrix. Let $A \in \mathbb{P}(2n)$ and $\varepsilon > 0$ be such that $A \pm \varepsilon I \in \mathbb{P}(2n)$. Then replacing B by εI in (5.11) we get

$$\sum_{j=1}^k d_{i_j}(A + \varepsilon I) \geq \sum_{j=1}^k d_{i_j}(A) + k\varepsilon.$$

Similarly, replacing A by $A - \varepsilon I$ and B by εI , we get

$$\sum_{j=1}^k d_{i_j}(A) \geq \sum_{j=1}^k d_{i_j}(A - \varepsilon I) + k\varepsilon.$$

Consequently, we get

$$\sum_{j=1}^k d_{i_j}(A + \varepsilon I) > \sum_{j=1}^k d_{i_j}(A) > \sum_{j=1}^k d_{i_j}(A - \varepsilon I). \quad \blacksquare$$

A $2n \times 2n$ real positive definite matrix A is a covariance matrix corresponding to a Gaussian state (or a Gaussian covariance matrix) if and only if it satisfies

$$A + \frac{i}{2}J \geq 0.$$

This is equivalent to saying that all the symplectic eigenvalues $d_j(A) \geq 1/2$. The von Neumann entropy of a Gaussian state with covariance matrix A is given by

$$(5.13) \quad S(A) = \sum_{i=1}^n \left[\left(d_i + \frac{1}{2} \right) \log \left(d_i + \frac{1}{2} \right) - \left(d_i - \frac{1}{2} \right) \log \left(d_i - \frac{1}{2} \right) \right].$$

Theorem 5.8 *Let $t \mapsto A(t)$ be a real analytic map from an open interval I to the set of Gaussian covariance matrices. Then the entropy map $S(t) = S(A(t))$ is monotonically increasing (decreasing) on I if $A'(t)$ is positive (negative) semidefinite for all t in I .*

Proof Since $t \mapsto A(t)$ is real analytic on I , by Theorem 4.6, we can choose the symplectic eigenvalues $\tilde{d}_1(t), \dots, \tilde{d}_n(t)$, and a corresponding symplectic eigenbasis $\{\tilde{u}_1(t), \dots, \tilde{u}_n(t), \tilde{v}_1(t), \dots, \tilde{v}_n(t)\}$ of $A(t)$ to be real analytic on I . By Remark 3.5, we have

$$\tilde{d}'_j(t) = \frac{1}{2} (\langle \tilde{u}_j(t), A'(t)\tilde{u}_j(t) \rangle + \langle \tilde{v}_j(t), A'(t)\tilde{v}_j(t) \rangle).$$

If $A'(t)$ is positive semidefinite, then each $\tilde{d}'_j(t) \geq 0$. Since the maps \tilde{d}_j are continuous and S is a continuous map of \tilde{d}_j , $t \rightarrow S(t)$ is continuous on I . The matrices $A(t)$ are Gaussian covariance matrices for all t . Hence, $\tilde{d}_j(t) \geq 1/2$ for all $1 \leq j \leq n$ and for all $t \in I$. Let F be the set $\{i : \tilde{d}_i(t) = 1/2 \text{ for all } t \in I\}$. If $F = \{1, \dots, n\}$, then $S(t) = 0$ for all $t \in I$. So, let $F \neq \{1, \dots, n\}$. Let $I_0 \subseteq I$ be any open bounded interval. Clearly, it suffices to show that $S(t)$ is monotonically increasing on I_0 . Consider the set $E = \{t \in I_0 : \tilde{d}_j(t) = 1/2, 1 \leq j \leq n, j \notin F\}$. By the analyticity of \tilde{d}_j , we know that E is finite. For all $t \in I_0 \setminus E$, we have

$$S'(t) = \sum_{\substack{1 \leq j \leq n \\ j \notin F}} \log \left(\frac{2\tilde{d}_j(t) + 1}{2\tilde{d}_j(t) - 1} \right) \tilde{d}'_j(t).$$

Hence, $S'(t) \geq 0$ if $A'(t) \geq 0$ for all $t \in I_0 \setminus E$. The above fact together with the continuity of $S(t)$ proves the theorem. ■

For a matrix A we denote by $\kappa(A)$ the condition number of A , that is $\kappa(A) = \|A\| \|A^{-1}\|$. In our final result, we give a perturbation bound for symplectic eigenvalues. Different perturbation bounds have been given in [7] and [14] using very different techniques than ours.

Theorem 5.9 *Let $A, B \in \mathbb{P}(2n)$. Then*

$$(5.14) \quad \max_{1 \leq j \leq n} |d_j(A) - d_j(B)| \leq K(A, B) \|A - B\|,$$

where $K(A, B) = \int_0^1 \kappa(A + t(B - A)) dt$.

Proof Define $\varphi : [0, 1] \rightarrow \mathbb{P}(2n)$ as

$$\varphi(t) = A + t(B - A).$$

As in the proof of Theorem 5.5, we see that $d_j(t) = d_j(\varphi(t))$ is piecewise real analytic on $[0, 1]$, and we can choose a corresponding piecewise real analytic symplectic eigenbasis $\beta(t) = \{u_1(t), \dots, u_n(t), v_1(t), \dots, v_n(t)\}$. Then for t where d_j, u_j, v_j are real analytic, we have

$$d'_j(t) = \frac{1}{2}(\langle u_j(t), (B - A)u_j(t) \rangle + \langle v_j(t), (B - A)v_j(t) \rangle).$$

Integrating the above equation, we get

$$\begin{aligned} & |d_j(B) - d_j(A)| \\ &= \left| \int_0^1 d'_j(t) dt \right| \\ &\leq \frac{1}{2} \int_0^1 |\langle u_j(t), (B - A)u_j(t) \rangle + \langle v_j(t), (B - A)v_j(t) \rangle| dt \\ (5.15) \quad &\leq \frac{1}{2} \int_0^1 (\|u_j(t)\|^2 + \|v_j(t)\|^2) dt \|A - B\|. \end{aligned}$$

Since $(u_j(t), v_j(t))$ is a normalized symplectic eigenvector pair of $\varphi(t)$ corresponding to $d_j(t)$,

$$\begin{aligned} \|u_j(t)\|^2 + \|v_j(t)\|^2 &\leq \|\varphi(t)^{-1}\| (\|\varphi(t)^{1/2}u_j(t)\|^2 + \|\varphi(t)^{1/2}v_j(t)\|^2) \\ &= \|\varphi(t)^{-1}\| 2d_j(t) \leq 2\kappa(\varphi(t)). \end{aligned}$$

Thus (5.15) gives (5.14). ■

Appendix: Proof of Proposition 4.4

Lemma 5.10 *Let \mathcal{X} and \mathcal{Y} be Banach spaces, and let $T : \mathcal{X}^k \rightarrow \mathcal{Y}$ be a bounded k -linear map. Suppose $\sum_{n=0}^\infty a_{jn}$ is an absolutely convergent series in \mathcal{X} with sum a_j for all $j = 1, \dots, k$. For each n , let $c_n = \sum_{j_1+\dots+j_k=n} T(a_{1j_1}, \dots, a_{kj_k})$. Then the series $\sum_{n=0}^\infty c_n$ is absolutely convergent in \mathcal{Y} and has sum $T(a_1, \dots, a_k)$.*

Proof The absolute convergence of the series $\sum_{n=0}^\infty c_n$ follows from Merten’s theorem for Cauchy products of series of real numbers. We shall prove that its sum is $T(a_1, \dots, a_k)$ by induction on k . When $k = 1$, the statement directly follows from the boundedness and linearity of T . Assume that the result holds for k . Let $\sum_{n=0}^\infty a_{jn}$ ($1 \leq j \leq k$) and $\sum_{n=0}^\infty b_n$ be absolutely convergent series in \mathcal{X} such that $a_j = \sum_{n=0}^\infty a_{jn}$ and $b = \sum_{n=0}^\infty b_n$.

For each m , define the map \tilde{T}_m from $\mathcal{X} \rightarrow \mathcal{Y}$ as

$$\tilde{T}_m(x) = \sum_{j_1+\dots+j_k=m} T(a_{1j_1}, \dots, a_{kj_k}, x).$$

It is easy to see that \tilde{T}_m is linear and bounded with $\|\tilde{T}_m\| \leq \|T\| \sum_{j_1+\dots+j_k=m} \|a_{1j_1}\| \cdots \|a_{kj_k}\|$. Since each $\sum_{n=0}^{\infty} \|a_{jn}\|$ is convergent, by Merten's theorem for Cauchy products of series of real numbers, we see that $\sum_{m=0}^{\infty} \|\tilde{T}_m\|$ converges. Let $K = \sum_{m=0}^{\infty} \|\tilde{T}_m\|$. For each $j \geq 0$, let

$$x_j = \tilde{T}_j(b),$$

and

$$c_j = \sum_{l=0}^j \tilde{T}_{j-l}(b_l).$$

Clearly $c_j = \sum_{j_1+\dots+j_k+l=j} T(a_{1j_1}, \dots, a_{kj_k}, b_l)$. We need to show that $\sum_{j=0}^{\infty} c_j$ is convergent to $T(a_1, \dots, a_k, b)$. Let (X_n) , (C_n) , and (B_n) be the sequences of partial sums of the series $\sum_{j=0}^{\infty} x_j$, $\sum_{j=0}^{\infty} c_j$ and $\sum_{j=0}^{\infty} b_j$, respectively. By induction hypothesis, $\sum_{j=0}^{\infty} x_j$ is absolutely convergent and its sum equals $T(a_1, \dots, a_j, b)$. Take $d_n = b - B_n$ and $E_n = \sum_{j=0}^n \tilde{T}_j(d_{n-j})$. We have

$$\begin{aligned} C_n &= \sum_{j=0}^n \sum_{l=0}^j \tilde{T}_l(b_{j-l}) \\ &= \sum_{l=0}^n \sum_{j=l}^n \tilde{T}_l(b_{j-l}) \\ &= \sum_{l=0}^n \tilde{T}_l\left(\sum_{j=0}^{n-l} b_j\right) = \sum_{l=0}^n \tilde{T}_l(B_{n-l}) \\ &= \sum_{l=0}^n \tilde{T}_l(b) - \sum_{l=0}^n \tilde{T}_l(d_{n-l}) \\ &= X_n - E_n. \end{aligned}$$

It suffices to show that $E_n \rightarrow 0$ as $n \rightarrow \infty$. Since $d_n \rightarrow 0$, we can find a positive number M such that $\|d_n\| \leq M$ for all $n \geq 0$. Given an $\varepsilon > 0$, choose N in \mathbb{N} such that for all $n \geq N$

$$\|d_n\| < \frac{\varepsilon}{2(K+1)}$$

and

$$\sum_{j=n+1}^{\infty} \|\tilde{T}_j\| < \frac{\varepsilon}{2M}.$$

Then for all $n > 2N$, we can write

$$\begin{aligned} \|E_n\| &\leq \sum_{j=0}^N \|\tilde{T}_j\| \|d_{n-j}\| + \sum_{j=N+1}^n \|\tilde{T}_j\| \|d_{n-j}\| \\ &< \frac{\varepsilon}{2(K+1)} \sum_{j=0}^N \|\tilde{T}_j\| + M \sum_{j=N+1}^n \|\tilde{T}_j\| \\ &< \frac{\varepsilon}{2(K+1)} K + M \frac{\varepsilon}{2M} \leq \varepsilon. \end{aligned}$$

This proves $\lim_{n \rightarrow \infty} C_n = \lim_{n \rightarrow \infty} X_n = T(a_1, \dots, a_k, b)$. ■

Proof of Proposition 4.4 Without loss of generality, we can assume that the interval $I = (-1, 1)$ and $t_0 = 0$. Since $t \mapsto A(t)$ is real analytic at $t = 0$, there exists an $r > 0$ such that $A(t)$ can be expressed as $A(t) = A(0) + \sum_{j=1}^{\infty} C_j t^j$ for all $|t| < r$.

Here, $\sum_{j=1}^{\infty} C_j t^j$ is absolutely convergent for $|t| < r$. Let $f(A) = A^{1/2}$ be the square root map. Since each k th order derivative $D^k f(A(0))$ is k -linear and bounded, by Lemma 5.10 we have

$$\begin{aligned} D^k f(A(0))(A(t) - A(0), \dots, A(t) - A(0)) \\ = \sum_{n=0}^{\infty} \sum_{j_1 + \dots + j_k = n} t^n D^k f(A(0))(C_{j_1}, \dots, C_{j_k}). \end{aligned}$$

Let $B_{k,n}$ denote the matrix $\sum_{j_1 + \dots + j_k = n} D^k f(A(0))(C_{j_1}, \dots, C_{j_k})$. For $n < k$, $B_{k,n}$ be the zero matrix. We have the following Taylor expansion of f at $A(0)$ in a neighborhood $U \subseteq \mathbb{P}(m)$. See [22].

$$f(A) = f(A(0)) + \sum_{k=1}^{\infty} \frac{1}{k!} D^k f(A(0))(A - A(0), \dots, A - A(0)).$$

Let λ_0 be the minimum eigenvalue of $A(0)$. Since $\lambda_0 > 0$, the square root function f is real analytic at λ_0 , that is, there exists an $r_0 > 0$ such that the series $\sum_{k=1}^{\infty} \frac{1}{k!} f^{(k)}(\lambda_0)(t - \lambda_0)^k$ is absolutely and locally uniformly convergent in $(\lambda_0 - r_0, \lambda_0 + r_0)$. Choose $\delta, 0 < \delta < r$ such that $\sum_{j=1}^{\infty} \|C_j\| \delta^j < r_0$ and $A(t) \in U$ for all $t \in (-\delta, \delta)$. Thus, for all $|t| < \delta$,

$$(5.16) \quad f(A(t)) = f(A(0)) + \sum_{k=1}^{\infty} \frac{1}{k!} \sum_{n=k}^{\infty} B_{k,n} t^n.$$

We show that the iterated sum $\sum_{k=1}^{\infty} \frac{1}{k!} \sum_{n=k}^{\infty} \|B_{k,n}\| |t|^n < \infty$. Let C be the sum $\sum_{j=1}^{\infty} \|C_j\| \delta^j$.

For $|t| < \delta$, we have

$$\begin{aligned} \sum_{n=k}^{\infty} \|B_{k,n}\| |t|^n &\leq \sum_{n=k}^{\infty} \|B_{k,n}\| \delta^n \\ &\leq \sum_{n=k}^{\infty} \sum_{j_1+\dots+j_k=n} \|D^k f(A(0))(C_{j_1}, \dots, C_{j_k})\| \delta^n \\ &\leq \|D^k f(A(0))\| \sum_{n=k}^{\infty} \sum_{j_1+\dots+j_k=n} (\|C_{j_1}\| \delta^{j_1}) \dots (\|C_{j_k}\| \delta^{j_k}) \\ &= \|D^k f(A(0))\| C^k. \end{aligned}$$

The last equality follows from the convergence of Cauchy product of the series $\sum_{j=1}^{\infty} \|C_j\| \delta^j$. By [8]

$$\|D^k f(A(0))\| = \|f^{(k)}(A(0))\| = |f^{(k)}(\lambda_0)|.$$

For $|t| < \delta$, we have $C < r_0$ and hence

$$\sum_{k=1}^{\infty} \frac{1}{k!} \sum_{n=k}^{\infty} \|B_{k,n}\| |t|^n \leq \sum_{k=1}^{\infty} \frac{1}{k!} |f^{(k)}(\lambda_0)| C^k < \infty.$$

This implies that the iterated sum on the right hand side of (5.16) is equal to the sum $\sum_{n=1}^{\infty} \sum_{k=1}^n \frac{1}{k!} B_{k,n} t^n$. This shows that $\sqrt{A(t)}$ can be expressed as the power series

$$\sqrt{A(t)} = \sqrt{A(0)} + \sum_{n=1}^{\infty} \left(\sum_{k=1}^n \frac{1}{k!} B_{k,n} \right) t^n \text{ for all } |t| < \delta. \quad \blacksquare$$

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