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MORSE INDEX OF APPROXIMATING PERIODIC SOLUTIONS FOR THE BILLIARD PROBLEM. APPLICATION TO EXISTENCE RESULTS

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ABSTRACT. This paper deals with periodic solutions for the billiard problem in a bounded open set of $\mathbb{R}^N$ which are limits of regular solutions of Lagrangian systems with a potential well. We give a precise link between the Morse index of approximate solutions (regarded as critical points of Lagrangian functionals) and the properties of the bounce trajectory to which they converge.

1**. Introduction.** Let Ω denote a connex bounded open subset of $\mathbb{R}^N$, such that $\partial\Omega$ is a C^2 hypersurface. A 1-periodic trajectory for the billiard problem in $\overline{\Omega}$ is a continuous nonconstant map $x: S^1 \longrightarrow \overline{\Omega}$ (where $S^1 = \mathbb{R}/\mathbb{Z}$) such that there exists a finite subset $\mathcal{R} = \{t^1, \ldots, t^p\}$ of S^1 such that ($(.,.)$ denoting the standard inner product in $\mathbb{R}^N$):

(i) x is of class C^2 in $S^1 \backslash \mathcal{R}$ and satisfies $\ddot{x}(t) = 0$ for any $t \in S^1 \backslash \mathcal{R}$

(ii) for any $t^i \in \mathcal{R}$, $x(t^i) \in \partial\Omega$ and x has a right derivative $\dot{x}_+(t^i)$ and a left derivative $\dot{x}_-(t^i)$ at t^i which satisfy:

(a) $\dot{x}_+(t^i) - \Big(\dot{x}_+(t^i), n\big(x(t^i)\big)\Big) n\big(x(t^i)\big) = \dot{x}_-(t^i) - \Big(\dot{x}_-(t^i), n\big(x(t^i)\big)\Big) n\big(x(t^i)\big)$

(b) $\Big(\dot{x}_+(t^i), n\big(x(t^i)\big)\Big) = -\Big(\dot{x}_-(t^i), n\big(x(t^i)\big)\Big) \neq 0$, $n(x)$ denoting the interior unit normal to $\partial\Omega$ at x.

$t^1, \ldots, t^p$ are then called the bounce instants and $x(t^1), \ldots, x(t^p)$ the bounce points (the number of bounce points is p).

Thus a bounce periodic trajectory is a continuous periodic piecewise linear path, with corner points only on $\partial\Omega$, the usual laws of reflection on the boundary being satisfied.

REMARK 1. It may happen that $x(t) \in \partial\Omega$ although t is not a bounce instant: $\dot{x}(t)$ is then tangent to $\partial\Omega$.

REMARK 2. A bounce periodic trajectory has at least two bounce points. For $p \geq 2$, we define $L_p: (\partial\Omega)^p \longrightarrow \mathbb{R}$ by

$$L_p(M_1, \ldots, M_p) = M_1M_2 + M_2M_3 + \cdots + M_{p-1}M_p + M_pM_1$$

(M_iM_j denoting the Euclidean distance between M_i and M_j).

If x is a periodic bounce trajectory with bounce instants $t^1, \ldots, t^p$, $\big(x(t^1), \ldots, x(t^p)\big)$ is a critical point of L_p. Conversely, if $M_1, \ldots, M_p$ are p points in $\partial\Omega$ such that $M_i \neq M_{i+1}$ and $M_1 \neq M_p$, if $(M_1, \ldots, M_p)$ is a critical point of L_p and if the segments

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$[M_1M_2], [M_2M_3], \ldots, [M_pM_1]$ are included in $\overline{\Omega}$, then we can define a periodic bounce trajectory containing $M_1, M_2, \ldots, M_p$ and with bounce points only in $\{M_1, \ldots, M_p\}$.

When Ω is convex, Remark 2 allows us to prove the existence of bounce periodic trajectories by considering critical points of L_p. Moreover, multiplicity results can be obtained. For example, in [1] this variational formulation is applied to convex billiards in $\mathbb{R}^3$. We also refer to [6] for the existence of periodic trajectories of special type. We should recall that the first theorem about multiple periodic trajectories was proved by Birkhoff for convex billiards in $\mathbb{R}^2$ ([4], [8]). When Ω is not convex, there still exist multiple critical points of L_p but they do not necessarily correspond to periodic bounce trajectories because a segment joining two points of $\nabla\Omega$ may not lie in $\overline{\Omega}$. There are examples of non-convex billiards in $\mathbb{R}^2$ for which there is no bounce periodic trajectory with only two bounce points (see [5], [7]). In [3], V. Benci and F. Giannoni used a penalization method to achieve a general existence result. Their result states that any bounded open subset Ω of $\mathbb{R}^N$ of class C^2 contains at least one bounce periodic trajectory with at most $N + 1$ bounce points (They show in fact a more general result, adding a smooth potential V; the trajectory then solves equation $\ddot{x} = -\nabla V(x)$ between two bounce points).

For convex billiards, the index of a bounce trajectory with p bounce points $(M_1, \ldots, M_p)$ is generally defined as the Morse index of $(M_1, \ldots, M_p)$, regarded as a critical point of L_p (see [8] for the interest of this index). If x is a 1-periodic bounce trajectory of a non-convex billiard with p bounce instants $t^1, \ldots, t^p$ (and p bounce points $x(t^1), \ldots, x(t^p)$), using Remark 2, we can still define an index for x, which will be denoted by $\bar{\imath}(x)$. We can in the same way define a nullity for x, $\bar{m}(x)$, which is the nullity of $\big(x(t^1), \ldots, x(t^p)\big)$ as a critical point of L_p.

In [3], approximate bounce trajectories x_n are obtained, which converge to a bounce trajectory x. The approximate bounce trajectories are critical points of functionals J_n which are defined on an open subset of $H^1(S^1; \mathbb{R}^N)$. These critical points have finite Morse index $i_n(x_n)$ and finite nullities $m_n(x_n)$. The first aim of this paper is to give exact links between $\lim_{n\to+\infty} i_n(x_n)$, $\lim_{n\to+\infty} m_n(x_n)$, $\bar{\imath}(x)$ and $\bar{m}(x)$. Since $i_n(x_n)$ can generally be known (or at least estimated) this will lead to a better understanding of the limiting bounce trajectory. Moreover we think that this might help to get multiplicity results for non-convex billiards in certain cases by the penalization method. We shall prove in the last section a result of this type.

Before stating our results we shall give some details on the variational framework which is used in [3] to get the approximate bounce trajectories x_n (see also [2]).

V. Benci and F. Giannoni consider for $\epsilon > 0$ the equation

$$(B_\epsilon): \quad \ddot{x} = -\epsilon \nabla U(x),$$

where U is a function defined and of class C^2 on Ω, which satisfies $U(x) = 1/h^2(x)$ in a neighbourhood of $\partial\Omega$, where $h(x) = d(x, \partial\Omega)$ is the Euclidean distance from x to $\partial\Omega$. They find 1-periodic solutions x_ϵ of (B_ϵ) with energy $E_\epsilon = \frac{1}{2}|\dot{x}_\epsilon|^2 + \epsilon U(x_\epsilon)$ bounded independently of ϵ, and show that there is a sequence (x_{ϵ_n}) (with $\epsilon_n \longrightarrow 0$) which converges in $H^1(S^1, \overline{\Omega})$ to a bounce periodic trajectory; x_ϵ is obtained as a critical point of a functional J_ϵ, which is of class C^2 on the open subset Λ of $H^1(S^1, \mathbb{R}^N)$, where

$$\Lambda = \left\{x \in H^1(S^1, \mathbb{R}^N) \mid x(S^1) \subset \Omega\right\},$$

and is defined by:

$$J_\epsilon(x) = \int_{S^1} \frac{1}{2}|\dot{x}(t)|^2 - \epsilon U\big(x(t)\big)\, dt.$$

We have, for $v, w \in H^1(S^1, \mathbb{R}^N)$,

$$J'_\epsilon(x) \cdot v = \int_{S^1} (\dot{x}, \dot{v}) - \epsilon\big(\nabla U(x), v\big)\, dt = \int_{S^1} \big(-\ddot{x} - \epsilon \nabla U(x), v\big)\, dt,$$
$$J''_\epsilon(x) \cdot v \cdot w = \int_{S^1} (\dot{v}, \dot{w}) - \epsilon \nabla^2 U(x) v \cdot w\, dt.$$

If x_ϵ is a critical point of J_ϵ, we denote by $m_\epsilon(x_\epsilon)$ and $i_\epsilon(x_\epsilon)$ the nullity and the Morse index (which are always finite) of x_ϵ: $m_\epsilon(x_\epsilon) = \dim \operatorname{Ker} J''_\epsilon(x_\epsilon)$; $i_\epsilon(x_\epsilon)$ is the dimension of the linear subspace of $H^1(S^1, \mathbb{R}^N)$ spanned by the eigenvectors of $J''_\epsilon(x_\epsilon)$ associated with the strictly negative eigenvalues.

In this paper, we consider a sequence x_{ϵ_n} (with $\lim_{n\to+\infty} \epsilon_n = 0$) of critical points of J_{ϵ_n}, satisfying $c \le E_{\epsilon_n} \le C$, where c and C are strictly positive constants and $E_{\epsilon_n} = \frac{1}{2}|\dot{x}_{\epsilon_n}|^2 + \epsilon_n U(x_{\epsilon_n})$ is the energy of x_{ϵ_n}, which converges in $\overline{\Lambda} = H^1(S^1, \overline{\Omega})$ to $x \in \overline{\Lambda}$, a 1-periodic bounce trajectory. Note (see [3]) that

$$\lim_{n\to+\infty} E_{\epsilon_n} = E, \quad \text{where } E = |x(t)|^2/2.$$

E is the energy of the bounce trajectory. We shall use the abbreviations $x_n = x_{\epsilon_n}$, $E_n = E_{\epsilon_n}$, $J_n = J_{\epsilon_n}$, $i_n(x_n) = i_{\epsilon_n}(x_{\epsilon_n})$, $m_n(x_n) = m_{\epsilon_n}(x_{\epsilon_n})$. We assume that $\liminf_{n\to+\infty} i_n(x_n) = i < +\infty$. In [3], it is proved that x has at most i bounce points; as the Morse index of the critical points obtained by V. Benci and G. Giannoni is less than $N+1$, this property implies that their limiting bounce trajectory has at most $N+1$ bounce points.

The Morse index of x_n thus gives some important information about the trajectory obtained when taking limit (an upper bound of the number of bounce points). In this paper we shall get further information about this bounce trajectory.

Set $C(x) = \{t \in S^1 \mid x(t) \in \partial\Omega\}$. We shall assume:

(H1) $C(x)$ is finite and all the elements of $C(x)$ are bounce instants.

Thus we have, for all $t \in C(x)$, $\Big(\dot{x}_+(t), n\big(x(t)\big)\Big) > 0$ and the case of a trajectory which is tangent to $\partial\Omega$ at some point is excluded.

We recall that $t^1, \ldots, t^p$ denote the bounce instants and that $\bar{\imath}(x)$ and $\overline{m}(x)$ denote the Morse index and the nullity of $\big(x(t^1), \ldots, x(t^p)\big) \in (\partial\Omega)^p$ regarded as a critical point of the function L_p defined in Remark 2.

THEOREM 1. *Under hypothesis (H1) there exists $n_0 \in \mathbb{N}$ such that for $n \ge n_0$:*

(i) $1 \le m_n(x_n) \le \overline{m}(x) + 1$

(ii) $i_n(x_n) \ge \bar{\imath}(x) + p$

(iii) $i_n(x_n) + m_n(x_n) \le 1 + \overline{m}(x) + \bar{\imath}(x) + p$.

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COROLLARY 1. *For all* $n \geq n_0$, $\overline{\imath}(x) + p \leq i_n(x_n) \leq \overline{m}(x) + \overline{\imath}(x) + p$.

If $\overline{m}(x) = 0$ then for all $n \geq n_0$, $i_n(x_n) = \overline{\imath}(x) + p$ and $m_n(x_n) = 1$.

Note that, as J_ϵ is invariant by the S^1 action on Λ defined by $\theta \cdot x = x(\theta + .)$, it is always true that $m_n(x_n) \geq 1$ ($\dot{x}_n \in \operatorname{Ker} J_n''$). So $\overline{m}(x) = 0$ implies that for $n \geq n_0$, the equivariant nullity of x_n is 0 (the circle $\{x_n(\theta + .); \theta \in S^1\}$ is then a critical non-degenerate circle of J_n).

REMARK 3. U is assumed to be $\frac{1}{h^2}$ (with $h(x) = d(x, \partial\Omega)$) in a neighbourhood of $\partial\Omega$. However this hypothesis is useful only at a precise point of the proof. Elsewhere we shall just assume that $U(x) = g\big(h(x)\big)$ in a neighbourhood of $\partial\Omega$, g being a smooth function from $\mathbb{R}_+^*$ to $\mathbb{R}$ which satisfies:

$$g'(h) < 0, \quad \lim_{h \to 0^+} g(h) = +\infty \text{ and } \lim_{h \to 0^+} \frac{g'(h)}{g(h)} = \lim_{h \to 0^+} \frac{g''(h)}{g'(h)} = -\infty.$$

As an application of Theorem 1 we shall prove the following (we call grazing trajectory a bounce trajectory which is tangent to $\partial\Omega$ at some point):

THEOREM 2. *Assume that there is no grazing periodic bounce trajectory with at most* $N + 1 + 2q$ *bounce instants, where* $q \in \mathbb{N}$. *Then for each* $k \leq q$ *there exists a periodic bounce trajectory* x_k *such that* $N + 1 + 2k - \overline{m}(x_k) \leq p(x_k) + \overline{\imath}(x_k) \leq N + 1 + 2k$, *where the number of bounce instants of* x *is denoted by* $p(x)$.

COROLLARY 2. *Assume that all the periodic bounce trajectories with at most* $N+1+2q$ *bounce instants are non-grazing and correspond to non-degenerate critical points of* L_p. *Then for each* $k \leq q$ *there exists a periodic bounce trajectory* x_k *such that* $p(x_k) + \overline{\imath}(x_k) = N + 1 + 2k$.

REMARK 4. Of course the bounce trajectories obtained by Theorem 2 may not be geometrically distinct: some of them may be iterates of others.

In Section 2 we give some preliminary lemmas concerning the properties of the approximate trajectories in a neighbourhood of a bounce instant. In Section 3 we prove Theorem 1 thanks to these lemmas. In Section 4 we give the proof of the preliminary lemmas. Section 5 is devoted to the proof of Theorem 2.

2. **Preliminary lemmas.** The proofs of the results which are stated in this section will be given in Section 4.

We assume that (H1) holds and we set $\{t^1, \ldots, t^p\} = \{t \in S^1 \mid x(t) \in \partial\Omega\}$. x_n converges to x in $C^0(S^1, \mathbb{R}^N)$, hence there exists $\delta_1 > 0$ such that for n large enough $h(x_n)$ is of class C^2 in the intervals $(t^i - \delta_1, t^i + \delta_1)$; moreover δ_1 is chosen such that $2\delta_1$ is smaller than the distance between two distinct bounce instants. We can write: for all $\delta \in (0, \delta_1)$, there exists $a_\delta > 0$ such that

$$\forall t \in S^1 \backslash \bigcup_{i=1}^{p} (t^i - \delta, t^i + \delta) \quad \forall n \in \mathbb{N} \quad h\big(x_n(t)\big) \geq a_\delta. \tag{2.1}$$

In order to simplify notations, we now consider one bounce point of x and we suppose that the bounce instant is 0; we set $h_n(t) = h\big(x_n(t)\big)$.

Lemma 2.1 and Lemma 2.2 describe some elementary properties of the approximate bounce trajectory in a neighbourhood of the bounce instant.

LEMMA 2.1. *There exist $e > 0, \delta_2 > 0$ ($\delta_2 \le \delta_1$) and $n_2 \in \mathbb{N}$ such that if $t \in (-\delta_2, \delta_2)$ and $n \ge n_2$ then $\frac{1}{2}\big(\dot h_n(t)\big)^2 + \epsilon_n g\big(h_n(t)\big) \ge e$ and $g''\big(h_n(t)\big) > 0$.*

Let t_n, for all $n \ge n_2$, be such that $\mathrm{Min}\{h_n(t); t \in [-\delta_2, \delta_2]\} = h_n(t_n)$; what follows implies the uniqueness of t_n for n large enough.

LEMMA 2.2.

(a) $\lim_{n\to+\infty} t_n = 0$; $\lim_{n\to+\infty} h_n(t_n) = 0$.

(b) There exist $n_3 \in \mathbb{N}$ ($n_3 \ge n_2$) and $\delta_3 > 0$ ($\delta_3 < \delta_2$) such that, for $n \ge n_3$,

- *(i)* $(t_n - \delta_3, t_n + \delta_3) \subset (-\delta_2, \delta_2)$;
- *(ii)* $\dot h_n < 0$ *on* $[t_n - \delta_3, t_n)$ *and* $\dot h_n > 0$ *on* $(t_n, t_n + \delta_3]$;
- *(iii) for all* $\delta \in (0, \delta_3)$ $\lim_{n\to+\infty} \dot x_n(t_n + \delta) = \dot x_+(0)$ *and* $\lim_{n\to+\infty} \dot x_n(t_n - \delta) = \dot x_-(0)$.

(c) For n large enough and $\delta < \delta_3$, we have

$$\forall t \in [t_n, t_n + \delta] \quad \dot h_n(t) \ge \alpha(\delta, n)(t - t_n) \text{ and } \forall t \in [t_n - \delta, t_n] \quad \dot h_n(t) \le \alpha(\delta, n)(t - t_n)$$

with $\lim_{\delta\to 0, n\to+\infty} \alpha(\delta, n) = +\infty$.

(d) There exist two positive constants β and γ such that (for δ small enough and n large enough)

$$\forall t \in [t_n - \delta, t_n + \delta] \quad \beta|\dot h_n(t)| \le |(\dot x_n(t), n_n)| \le \gamma|\dot h_n(t)|,$$

where $n_n = \nabla h\big(x_n(t_n)\big)$; furthermore $\dot h_n(t)$ and $\big(\dot x_n(t), n_n\big)$ have the same sign on $[t_n - \delta, t_n + \delta]$.

In the sequel, we shall denote by $r(\delta)$ (respectively $r(n), r(\delta, n)$) any function depending on δ (respectively on n; on δ and n) which satisfies $\lim_{\delta\to 0+} r(\delta) = 0$ (respectively $\lim_{n\to+\infty} r(n) = 0$; $\lim_{\substack{n\to+\infty \\ \delta\to 0^+}} r(\delta, n) = 0$); we shall denote by $s(\delta, n)$ any function depending on δ and on n which satisfies: for all fixed $\delta > 0$ $\lim_{n\to+\infty} s(\delta, n) = 0$.

We denote by I_n^δ the interval $(t_n - \delta, t_n + \delta)$. The angle θ is defined by $\theta \in [0, \frac{\pi}{2})$ and $\cos\theta = \Big(\dot x_+(0), n\big(x(0)\big)\Big)$ ($\cos\theta > 0$ because of (H1)).

The next lemmas provide estimates which will prove useful to compute the Morse index of x_n.

LEMMA 2.3. *There is a constant C_1 such that for all $n \in \mathbb{N}$, $\int_{I_n^{\delta_3}} \big|\epsilon_n g'\big(h_n(t)\big)\big|\, dt \le C_1$; moreover for $0 < \delta < \delta_3$, we have $\int_{I_n^\delta} -\epsilon_n g'\big(h_n(t)\big)\, dt = 2\sqrt{2E}\cos\theta + r(\delta) + s(\delta, n)$.*

LEMMA 2.4.

$$\int_{I_n^\delta} \epsilon_n g''\big(h_n(t)\big)|t - t_n|^2\, dt = r(\delta, n).$$

For $0 < \delta < \delta_3$ $\lim_{n\to+\infty} \int_{I_n^\delta} \epsilon_n g''\big(h_n(t)\big)\dot h_n(t)^2\, dt = \lim_{n\to+\infty} \int_{I_n^\delta} \epsilon_n g''\big(h_n(t)\big)\, dt = +\infty$.

LEMMA 2.5. *There exists a constant $C_2 > 0$ and, for any $\delta \in (0, \delta_3)$, there exists $n(\delta) \geq n_3$ such that if $n \geq n(\delta)$ and if $\lambda \in H^1(I_n^\delta; \mathbb{R}^N)$ satisfies*

$$\int_{I_n^\delta} g''\big(h_n(t)\big)\lambda(t)\,dt = \int_{I_n^\delta} g''\big(h_n(t)\big)\dot h_n(t)\lambda(t)\,dt = 0,$$

then

$$\int_{I_n^\delta} |\dot\lambda|^2 - \epsilon_n g''\big(h_n(t)\big)\lambda^2(t)\,dt \geq C_2 \int_{I_n^\delta} |\dot\lambda(t)|^2\,dt.$$

This is the only lemma where we use the hypothesis $g(h) = \frac{1}{h^2}$ for small h.

We now introduce some notations which will be used in the next lemmas. Remember that $n_n = \nabla h\big(x_n(t_n)\big)$ ($\lim_{n\to+\infty} n_n = n\big(x(0)\big)$).

Let F_n be the linear subspace of $\mathbb{R}^N$ defined by $F_n = [n_n]^\perp$ and let F be the linear subspace of $\mathbb{R}^N$ defined by $F = \big[n\big(x(0)\big)\big]^\perp$.

Let C_n be the endomorphism of $\mathbb{R}^N$ defined by $C_n = \nabla^2 h\big(x_n(t_n)\big)$. Note that, since $|\nabla h(x)|^2 = 1$ for all x (in a neighbourhood of $\partial\Omega$), $C_n \cdot n_n = 0$ and $\operatorname{Im} C_n \subset F_n$.

The map n which associates with $x \in \partial\Omega$ its interior unit normal $n(x) \in \mathbb{R}^N$ is differentiable of differential Tn; $\operatorname{Im}\Big(Tn\big(x(0)\big)\Big) \subset F$.

Let C be the endomorphism of $\mathbb{R}^N$ defined by $C \cdot n\big(x(0)\big) = 0$ and $C_{|F} = Tn\big(x(0)\big)$. We have $\lim_{n\to+\infty} C_n = C$.

LEMMA 2.6. *For $W \in F_n$,*

$$\int_{I_n^\delta} -\epsilon_n U''\big(x_n(t)\big) W \cdot W\,dt = (C_n W, W)\cos\theta 2\sqrt{2E} + \big(r(\delta, n) + s(\delta, n)\big)|W|^2.$$

Moreover there is a constant C_3 and for all $\delta \in (0, \delta_3)$ there is a sequence $(u_n^\delta) \longrightarrow +\infty$ such that

$$\int_{I_n^\delta} -\epsilon_n U''\big(x_n(t)\big) n_n \cdot n_n\,dt \leq -u_n^\delta + C_3$$

and

$$\left|\int_{I_n^\delta} -\epsilon_n U''\big(x_n(t)\big) W \cdot n_n\,dt\right| \leq |W|\big(C_3 + r(\delta, n)\sqrt{u_n^\delta}\big)$$

for all $W \in F_n$.

LEMMA 2.7. *If $\lambda \in H^1(I_n^\delta; \mathbb{R})$ satisfies*

$$\int_{I_n^\delta} g''\big(h_n(t)\big)\lambda(t)\,dt = 0, \quad \int_{I_n^\delta} g''\big(h_n(t)\big)\dot h_n(t)\lambda(t)\,dt = 0,$$

and if $\mu \in H^1(I_n^\delta; F_n)$ satisfies $\mu(t_n) = 0$, then, setting $l = \lambda n_n + \mu$, the following estimates hold:

(i) $\left|\int_{I_n^\delta} \epsilon_n U''\big(x_n(t)\big)\mu(t) \cdot \mu(t)\,dt\right| = r(\delta, n)|\mu|_1^2$;

(ii) $\left|\int_{I_n^\delta} \epsilon_n U''\big(x_n(t)\big)\mu(t) \cdot \lambda(t) n_n\,dt\right| = \big(r(\delta, n) + s(\delta, n)\big)|\mu|_1|\lambda|_1$;

(iii) $\left|\int_{I_n^\delta} \epsilon_n g'\big(h_n(t)\big)\lambda^2(t)\,dt\right| = r(\delta)|\lambda|_1^2$;

(iv) $\left|\int_{I_n^\delta} \epsilon_n U''\big(x_n(t)\big) W \cdot l(t)\,dt\right| = \big(r(\delta, n) + s(\delta, n)\big)|W|\,|l|_1$ *for $W \in F_n$,*

where $|k|_1^2 = \int_{I_n^\delta} \dot k(t)^2\,dt$.

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3. **Proof of Theorem 1.** In the sequel, whenever k is defined on S^1, $|k|_1$ will denote $\left(\int_{S^1} \dot{k}(t)^2\,dt\right)^{1/2}$.

We shall split $H^1(S^1;\mathbb{R}^N)$ into a sum of 3 subspaces. We first introduce further notations. C_n^i, F_n^i, n_n^i, C^i, F^i, n^i, t_n^i, …, defined in Section 2 correspond now to the i-th bouncing (of bounce instant t^i); for $\delta \in (0,\delta_3)$, set $I_n^{i,\delta} = (t_n^i - \delta, t_n^i + \delta)$. We define the linear map $\varphi_n^\delta \colon \mathbb{R}^p \times F_n^1 \times \cdots \times F_n^p \longrightarrow H^1(S^1;\mathbb{R}^N)$ by $\varphi_n^\delta(\alpha_1,\ldots,\alpha_p,W_1,\ldots,W_p) = w$, where w is the element of $H^1(S^1;\mathbb{R}^N)$ (so w is a continuous function on S^1) which satisfies:

- in $I_n^{i,\delta}$, $w(t) = \alpha_i \dot{x}_n(t) + W_i$;
- in every interval of $S^1 \setminus \bigcup_{i=1}^p I_n^{i,\delta}$, w is linear ($\dot{w}(t)$ is constant).

Let $\psi_n^\delta \colon \mathbb{R}^p \longrightarrow H^1(S^1;\mathbb{R}^N)$, be the linear map defined by $\psi_n^\delta(\gamma_1,\ldots,\gamma_p) = w$ where $w(t) = \gamma_i n_n^i$ if $t \in I_n^{i,\delta}$ and w is linear on every interval of $S^1 \setminus \bigcup_{i=1}^p I_n^{i,\delta}$.

Set $B_n^\delta = \operatorname{Im}(\varphi_n^\delta)$ and $D_n^\delta = \operatorname{Im}\psi_n^\delta$; so $\dim B_n^\delta = Np$ and $\dim D_n^\delta = p$.

We denote by Π_n^i the orthogonal projection onto F_n^i and by G_n^δ the set of $w \in H^1(S^1;\mathbb{R}^N)$ which satisfy for $1 \le i \le p$:

(i) $\Pi_n^i\, w(t_n^i) = 0$;
(ii) $\int_{I_n^{i,\delta}} g''\big(h_n(t)\big)\big(w(t), n_n^i\big)\,dt = 0$;
(iii) $\int_{I_n^{i,\delta}} g''\big(h_n(t)\big)\dot{h}_n(t)\big(w(t), n_n^i\big)\,dt = 0$.

It is clear that G_n^δ is a closed subspace of $H^1(S^1;\mathbb{R}^N)$.

LEMMA 3.1. *$\delta \in (0,\delta_3)$ being fixed, for n large enough* $H^1(S^1;\mathbb{R}^N) = B_n^\delta \oplus D_n^\delta \oplus G_n^\delta$.

PROOF. As one can easily see, it is enough to prove that for n large enough, the two equalities

$$\gamma_i \int_{I_n^{i,\delta}} \epsilon_n g''\big(h_n(t)\big)\,dt + \alpha_i \int_{I_n^{i,\delta}} \epsilon_n g''\big(h_n(t)\big)\big(\dot{x}_n(t), n_n^i\big)\,dt = 0$$

and

$$\gamma_i \int_{I_n^{i,\delta}} \epsilon_n g''\big(h_n(t)\big)\dot{h}_n(t)\,dt + \alpha_i \int_{I_n^{i,\delta}} \epsilon_n g''\big(h_n(t)\big)\big(\dot{x}_n(t), n_n^i\big)\dot{h}_n(t)\,dt = 0$$

imply $\alpha_i = \gamma_i = 0$, *i.e.*, that $a(n,\delta) - b(n,\delta) \neq 0$, where

$$a(n,\delta) = \int_{I_n^{i,\delta}} \epsilon_n g''\big(h_n(t)\big)\,dt \int_{I_n^{i,\delta}} \epsilon_n g''\big(h_n(t)\big)\dot{h}_n(t)\big(\dot{x}_n(t), n_n^i\big)\,dt$$

and

$$b(n,\delta) = \int_{I_n^{i,\delta}} \epsilon_n g''\big(h_n(t)\big)\dot{h}_n(t)\,dt \int_{I_n^{i,\delta}} \epsilon_n g''\big(h_n(t)\big)\big(\dot{x}_n(t), n_n^i\big)\,dt.$$

By Lemma 2.2(d), $a(n,\delta) \ge \beta \int_{I_n^{i,\delta}} \epsilon_n g''\big(h_n(t)\big)\,dt \int_{I_n^{i,\delta}} \epsilon_n g''\big(h_n(t)\big)\dot{h}_n(t)^2\,dt$ and, by Lemma 2.4, $\lim_{n\to+\infty} a(n,\delta) = +\infty$.

By Lemmas 2.2, 2.4, and the Cauchy-Schwarz inequality, $\left|\int_{I_n^{i,\delta}} \epsilon_n g''\big(h_n(t)\big)\big(\dot{x}_n(t), n_n^i\big)\,dt\right| \le \gamma\, a(n,\delta)^{\frac12}$. Moreover,

$$\int_{I_n^{i,\delta}} \epsilon_n g''\big(h_n(t)\big)\dot{h}_n(t)\,dt = \epsilon_n\Big[g'\Big(h_n(t_n^i + \delta)\Big) - g'\Big(h_n(t_n^i - \delta)\Big)\Big].$$

From (2.1) $\lim_{n\to+\infty} \epsilon_n g'\big(h_n(t_n^i \pm \delta)\big) = 0$. So we have ($\delta$ being fixed) $\lim_{n\to+\infty} b(n,\delta)/a(n,\delta)^{\frac12} = 0$. Hence $\lim_{n\to+\infty} a(n,\delta) - b(n,\delta) = +\infty$, which proves Lemma 3.1.

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We now consider the restriction of J_n'' to B_n^δ.

We recall that throughout this paper $r(\delta, n)$ (resp. $r(n)$, $r(\delta)$) denotes any function of δ, n (resp. of n, of δ) which tends to 0 as $n \longrightarrow +\infty$ and $\delta \longrightarrow 0_+$ (resp. $n \longrightarrow +\infty$, $\delta \longrightarrow 0_+$). In addition $s(\delta, n)$ denotes any function of δ and n which tends to 0 when $n \longrightarrow +\infty$, $\delta > 0$ being fixed.

From now we shall often use that

$$\int_{S^1 \setminus \bigcup_{i=1}^p I_n^{i,\delta}} \epsilon_n U''\big(x_n(t)\big) v_1(t) v_2(t)\, dt = s(n,\delta) |v_1|_{L_2} |v_2|_{L_2}, \tag{3.1}$$

which is an obvious consequence of (2.1), since U'' is bounded away from $\partial\Omega$.

For $1 \leq i \leq p-1$, set $X_{i,i+1} = \dot{x}_+(t^i) = \dot{x}_-(t^{i+1})$; $X_{p,1} = \dot{x}_+(t^p) = \dot{x}_-(t^1)$. For $a \in \mathbb{R}^p \times F_n^1 \times \cdots \times F_n^p$ set

$$Q_n^\delta(a) = J_n''(x_n) \cdot \varphi_n^\delta(a) \cdot \varphi_n^\delta(a).$$

LEMMA 3.2.

$$\begin{aligned} Q_n^\delta(\alpha_1, \ldots, \alpha_p, W_1, \ldots, W_p) \;=\; & 2\sqrt{2E} \sum_{i=1}^p (C^i W_i, W_i) \cos\theta^i \\ & + \sum_{i=1}^p \frac{|(\alpha_{i+1} - \alpha_i) X_{i,i+1} + W_{i+1} - W_i|^2}{t^{i+1} - t^i} \\ & + \big(r(\delta, n) + s(\delta, n)\big) \sum_{i=1}^p (\alpha_i^2 + |W_i|^2) \end{aligned}$$

(here, when $i = p$, $i+1$ is identified with 1).

PROOF. Set $w = \varphi_n^\delta(\alpha_1, \ldots, \alpha_p, W_1, \ldots, W_p)$. We have

$$\begin{aligned} Q_n^\delta(\alpha_1, \ldots, \alpha_p, W_1, \ldots, W_p) &= \int_{S^1} |\dot{w}(t)|^2 - \epsilon_n U''\big(x_n(t)\big)\, w(t).w(t)\, dt \\ &= q_1(w) + q_2(w), \end{aligned}$$

where

$$q_1(w) = \sum_{i=1}^p \int_{I_n^{i,\delta}} |\dot{w}(t)|^2 - \epsilon_n U''\big(x_n(t)\big) w(t).w(t)\, dt,$$

$$q_2(w) = \int_{S^1 \setminus \bigcup_{i=1}^p I_n^{i,\delta}} |\dot{w}(t)|^2\, dt + s(\delta, n) \int_{S^1} |w(t)|^2\, dt, \quad \text{from (3.1)}.$$

Since $|\dot{x}_n|_\infty$ is bounded, $\int_{S^1} |w(t)|^2\, dt \leq K(\sum_{i=1}^p |\alpha_i|^2 + |W_i|^2)$, where K is some constant. In addition, from the definition of w,

$$\int_{S^1 \setminus \bigcup_{i=1}^p I_n^{i,\delta}} |\dot{w}(t)|^2\, dt = \sum_{i=1}^p \frac{|w(t_n^{i+1} - \delta) - w(t_n^i + \delta)|^2}{t_n^{i+1} - t_n^i - 2\delta}. \tag{3.2}$$

Now, from Lemma 2.2(iii), $\lim_{n\to+\infty} \dot{x}_n(t_n^{i+1} - \delta) = \dot{x}_-(t^{i+1}) = X_{i,i+1}$ and $\lim_{n\to+\infty} \dot{x}_n(t_n^i + \delta) = \dot{x}_+(t^i) = X_{i,i+1}$; thus we have

$$w(t_n^{i+1} - \delta) - w(t_n^i + \delta) = (\alpha_{i+1} - \alpha_i) X_{i,i+1} + (W_{i+1} - W_i) + s(\delta, n)(|\alpha_{i+1}| + |\alpha_i|). \tag{3.3}$$

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Moreover, since $\lim_{n\to+\infty} t_n^i = t^i$,

$$\frac{1}{t_n^{i+1} - t_n^i - 2\delta} = \frac{1}{t^{i+1} - t^i} + r(\delta, n). \tag{3.4}$$

Combining (3.2), (3.3) and (3.4), we get

$$\begin{aligned} q_2(w) \;=\; & \sum_{i=1}^{p} \frac{|(\alpha_{i+1} - \alpha_i)X_{i,i+1} + W_{i+1} - W_i|^2}{t^{i+1} - t^i} \\ & + \big(r(\delta, n) + s(\delta, n)\big) \sum_{i=1}^{p} (|\alpha_i|^2 + |W_i|^2). \end{aligned}$$

For q_1 we have

$$\begin{aligned} q_1(w) \;=\; & \int_{I_n^{i,\delta}} |\alpha_i \ddot{x}_n(t)|^2 - \epsilon_n U''\big(x_n(t)\big)\big(\alpha_i \dot{x}_n(t) + W_i\big).\big(\alpha_i \dot{x}_n(t) + W_i\big)\, dt \\ =\; & \int_{I_n^{i,\delta}} -\epsilon_n U''\big(x_n(t)\big) W_i.W_i\, dt + \alpha_i^2 \int_{I_n^{i,\delta}} |\ddot{x}_n(t)|^2 - \epsilon_n U''\big(x_n(t)\big)\dot{x}_n(t).\dot{x}_n(t)\, dt \\ & - 2\epsilon_n \alpha_i \int_{I_n^{i,\delta}} U''\big(x_n(t)\big)\dot{x}_n(t).W_i\, dt. \end{aligned}$$

We denote by A, B and C the three terms of this sum. By Lemma 2.6

$$\begin{aligned} A \;=\; & (C_n^i W_i, W_i) \cos\theta^i 2\sqrt{2E} + \big(r(\delta, n) + s(\delta, n)\big)|W_i|^2 \\ =\; & (C^i W_i, W_i) \cos\theta^i 2\sqrt{2E} + \big(r(\delta, n) + s(\delta, n)\big)|W_i|^2 \end{aligned}$$

because $\lim_{n\to+\infty} C_n^i = C^i$.

$\dddot{x}_n(t) = -\epsilon_n U''\big(x_n(t)\big)\dot{x}_n(t)$, hence

$$\begin{aligned} B \;=\; & \alpha_i^2 \int_{t_n^i - \delta}^{t_n^i + \delta} |\ddot{x}_n(t)|^2 + \big(\dddot{x}_n(t), \dot{x}_n(t)\big)\, dt \\ =\; & \alpha_i^2 \Big[\big(\ddot{x}_n(t_n^i + \delta), \dot{x}_n(t_n^i + \delta)\big) - \big(\ddot{x}_n(t_n^i - \delta), \dot{x}_n(t_n^i - \delta)\big)\Big], \end{aligned}$$

and

$$C = 2\alpha_i \int_{t_n^i - \delta}^{t_n^i + \delta} \big(\dddot{x}_n(t), W_i\big)\, dt = 2\alpha_i \big(\ddot{x}_n(t_n^i + \delta) - \ddot{x}_n(t_n^i - \delta), W_i\big).$$

$|\dot{x}_n|_\infty$ is bounded and from (2.1) $\ddot{x}_n(t_n^i \pm \delta) = -\epsilon_n \nabla U\big(x_n(t_n^i \pm \delta)\big) = s(\delta, n)$. Hence $B = s(\delta, n)\alpha_i^2$ and $C = s(\delta, n)|\alpha_i|\,|W_i|$. We get

$$q_1(w) = 2\sqrt{2E} \sum_{i=1}^{p} (C^i W_i, W_i) \cos\theta^i + \big(r(\delta, n) + s(\delta, n)\big) \sum_{i=1}^{p} |\alpha_i|^2 + |W_i|^2,$$

and Lemma 3.2 is proved.

A consequence of Lemma 3.2 is

LEMMA 3.3. *There exist a positive constant C_4 and, for all $n \geq n_3$, two linear subspaces of $\mathbb{R}^p \times F_n^1 \times \cdots \times F_n^p$, $\mathcal{A}_n^1$ and $\mathcal{A}_n^2$, which satisfy: for $\delta > 0$ small enough, there is $n'(\delta)\big(n'(\delta) \geq n_3\big)$ such that if $n \geq n'(\delta)$ then $\forall y \in \mathcal{A}_n^1 Q_n^\delta(y) \geq C_4 |y|^2$, $\forall y \in \mathcal{A}_n^2 Q_n^\delta(y) \leq -C_4 |y|^2$ and* $\dim \mathcal{A}_n^1 = Np - \big(\bar{\imath}(x) + \overline{m}(x) + 1\big)$, $\dim \mathcal{A}_n^2 = \bar{\imath}(x)$.

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PROOF. Let $\overline{\Pi}_n^i\colon F^i \longrightarrow F_n^i$ be the restriction to F^i of the orthogonal projection onto F_n^i. Since $\lim_{n\to+\infty} n_n^i = n^i$,

$$|\overline{\Pi}_n^i V - V| = r(n)|V| \tag{3.5}$$

and for n large enough $\overline{\Pi}_n^i$ is an isomorphism.

Let $\overline{Q_n^\delta}$ be the quadratic form defined on $\mathbb{R}^p \times F^1 \times \cdots \times F^p$ by $\overline{Q_n^\delta} = Q_n^\delta \circ P_n$ with $P_n(\alpha_1, \ldots, \alpha_p, V_1, \ldots, V_p) = \big(\alpha_1, \ldots, \alpha_p, \overline{\Pi}_n^1(V_1), \ldots, \overline{\Pi}_n^p(V_p)\big)$. From (3.5) and Lemma 3.2

$$\overline{Q_n^\delta}(y) = \overline{q}(y) + \big(r(\delta, n) + s(\delta, n)\big)|y|^2, \tag{3.6}$$

where

$$\overline{q}(\alpha_1, \ldots, \alpha_p, V_1, \ldots, V_p) = 2\sqrt{2E}\sum_{i=1}^{p}(C^i V_i, V_i)\cos\theta^i + \sum_{i=1}^{p}\frac{|(\alpha_{i+1} - \alpha_i)X_{i,i+1} + V_{i+1} - V_i|^2}{t^{i+1} - t^i}.$$

Let ϕ be the isomorphism of $\mathbb{R}^p \times F^1 \times \cdots \times F^p$ defined by:

$$\phi(\alpha_1, \ldots, \alpha_p, V_1, \ldots, V_p) = (\beta_1, \ldots, \beta_p, V_1, \ldots, V_p),$$

with $\beta_i = \alpha_i + (V_i, \frac{X_{i-1,i}}{2E}) = \alpha_i + (V_i, \frac{X_{i,i+1}}{2E})$ (index 0 and index p are identified). Note that, for all $i \in \{1, \ldots, p\}$, $|X_{i,i+1}|^2 = 2E$, and the laws of reflection at the i-th bouncing imply $(V_i, X_{i-1,i}) = (V_i, X_{i,i+1})$ for $V_i \in F^i$.

Set $\overline{q}' = \overline{q} \circ \phi^{-1}$; let $P_{i,i+1}$ be the orthogonal projection onto the hyperplane $[X_{i,i+1}]^\perp$.

We have

$$\overline{q}'(\beta_1, \ldots, \beta_p, V_1, \ldots, V_p) = \overline{q}_0(V_1, \ldots, V_p) + \sum_{i=1}^{p}\frac{(\beta_{i+1} - \beta_i)^2}{t^{i+1} - t^i}, \tag{3.7}$$

where $\overline{q_0}$ is the quadratic form defined on $F^1 \times \cdots \times F^p$ by

$$\overline{q}_0(V_1, \ldots, V_p) = 2\sqrt{2E}\sum_{i=1}^{p}(C^i V_i, V_i)\cos\theta^i + \sum_{i=1}^{p}\frac{|P_{i,i+1}(V_{i+1} - V_i)|^2}{t^{i+1} - t^i}.$$

Remember that $\big(x(t^1), \ldots, x(t^p)\big)$ is a critical point of L_p; F^i is the tangent space to $\partial\Omega$ at $x(t^i)$.

A quick calculation shows that $d_2L_p\big(x(t^1), \ldots, x(t^p)\big) = \frac{1}{\sqrt{2E}}\overline{q_0}$. Hence the index and the nullity of the quadratic form $\overline{q_0}$ are respectively $\overline{\imath}(x)$ and $\overline{m}(x)$. We derive from (3.7) that $\overline{q}'$ has index $\overline{\imath}(x)$ and nullity $\overline{m}(x) + 1$ ($\dim\operatorname{Ker}\overline{q}' = \dim\operatorname{Ker}\overline{q_0} + 1$ because $\operatorname{Ker}\overline{q}' = \{(\beta_1, \ldots, \beta_p, V_1, \ldots, V_p) \mid (V_1, \ldots, V_p) \in \operatorname{Ker}\overline{q_0} \text{ and } \beta_1 = \cdots = \beta_p\}$).

Thus $\overline{q} = \overline{q}' \circ \phi$ has index $\overline{\imath}(x)$ and nullity $\overline{m}(x)+1$. Hence there exist a constant $K_1 > 0$ and two linear subspaces of $\mathbb{R}^p \times F^1 \times \cdots \times F^p$, A^1 and A^2, of respective dimensions $Np - \big(\overline{\imath}(x) + \overline{m}(x) + 1\big)$ and $\overline{\imath}(x)$, such that

$$\forall y \in A^1, \overline{q}(y) \geq K_1|y|^2 \quad \text{and} \quad \forall y \in A^2, \overline{q}(y) \leq -K_1|y|^2.$$

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Set $A_n^1 = P_n(A^1)$, $A_n^2 = P_n(A^2)$ and $C_4 = \frac{K_1}{2}$. Lemma 3.3 is now an immediate consequence of (3.5) and (3.6).

Set $B_n^{1,\delta} = \varphi_n^\delta(A_n^1)$, $B_n^{2,\delta} = \varphi_n^\delta(A_n^2)$. In the two next lemmas we shall prove that $J_n''(x_n)$ is negative definite on $B_n^{2,\delta} \oplus D_n^\delta$ and positive definite on $B_n^{1,\delta} \oplus G_n^\delta$.

LEMMA 3.4. *For $\delta \in (0, \delta_3)$ small enough, there exists $n''(\delta)$ such that, for $n \geq n''(\delta)$, the restriction of $J_n''(x_n)$ to $B_n^{2,\delta} \oplus D_n^\delta$ is negative definite.*

PROOF. Let $a \in B_n^{2,\delta}$, $a = \varphi_n^\delta(y)$ with $y = (\alpha_1, \ldots, \alpha_p, W_1, \ldots, W_p) \in \mathbb{R}^p \times F_n^1 \times \cdots \times F_n^p$. Let $d \in D_n^\delta$, $d = \phi_n^\delta(\gamma)$ with $\gamma = (\gamma_1, \ldots, \gamma_p) \in \mathbb{R}^p$.

$$J_n''(x_n)(a+d)\cdot(a+d) = Q_n^\delta(y) + J_n''(x_n)d\cdot d + 2J_n''(x_n)a\cdot d.$$

By Lemma 3.3, $Q_n^\delta(y) \leq -C_4\,|y|^2$ (for $\delta \in (0, \delta_3)$ small enough and $n \geq n'(\delta)$).

$$J_n''(x_n)d\cdot d = \sum_{i=1}^p \gamma_i^2 \int_{I_n^{i,\delta}} -\epsilon_n U''\big(x_n(t)\big)n_n^i.n_n^i\,dt + R,$$

where

$$R = \sum_{i=1}^p \frac{|\gamma_{i+1}n_n^{i+1} - \gamma_i n_n^i|^2}{t_n^{i+1} - t_n^i - 2\delta} - \int_{S^1\setminus\bigcup_{i=1}^p I_n^{i,\delta}} \epsilon_n U''\big(x_n(t)\big)d\cdot d\,dt.$$

From (3.1) $|R|$ can be bounded by $C_\delta|\gamma|^2$ (C_δ depending only on δ). Hence, by Lemma 2.6, $J_n''(x_n)d\cdot d \leq |\gamma|^2(C_\delta' - u_n^\delta)$. Set $q(a,d) = J_n''(x_n)\,a.d = q_1(a,d) + q_2(a,d)$, with

$$q_1(a,d) = \sum_{i=1}^p \int_{I_n^{i,\delta}} -\epsilon_n U''\big(x_n(t)\big)a\cdot d\,dt \quad (\dot d(t) = 0 \text{ in } I_n^{i,\delta}),$$

$$q_2(a,d) = \int_{S^1\setminus I_n^{i,\delta}} (\dot a, \dot d) - \epsilon_n U''\big(x_n(t)\big)a\cdot d\,dt.$$

Using (3.1), it is easy to see that $|q_2(a,d)| \leq C_\delta''|\gamma|\,|y|$.

$$q_1(a,d) = \sum_{i=1}^p \gamma_i \int_{I_n^{i,\delta}} -\epsilon_n U''\big(x_n(t)\big)\big(W_i + \alpha_i \dot x_n(t)\big).n_n^i\,dt.$$

By Lemma 2.6,

$$\left|\int_{I_n^{i,\delta}} -\epsilon_n U''\big(x_n(t)\big)\,W_i.n_n^i\,dt\right| \leq |W_i|\big(C_3 + r(\delta,n)\sqrt{u_n^\delta}\big),$$

$$\begin{aligned}\int_{I_n^{i,\delta}} -\epsilon_n U''\big(x_n(t)\big).\dot x_n(t)\,n_n^i\,dt &= \int_{I_n^{i,\delta}} (\dddot x, n_n^i)\,dt \\ &= \Big[\big(\ddot x, n_n^i\big)\Big]_{t_n^i-\delta}^{t_n^i+\delta} \\ &= -\epsilon_n\Big(\nabla U\big(x_n(t_n^i+\delta)\big) - \nabla U\big(x_n(t_n^i-\delta)\big), n_n^i\Big).\end{aligned}$$

Hence from (2.1) $\int_{I_n^{i,\delta}} -\epsilon_n U''\big(x_n(t)\big)\dot x_n(t).n_n^i\,dt = s(n,\delta)$. We get

$$|q_1(a,d)| \leq \sum_{i=1}^p |W_i|\,|\gamma_i|\big[C_3 + r(\delta,n)\sqrt{u_n^\delta}\big] + s(\delta,n)\sum_{i=1}^p |\alpha_i|\,|\gamma_i|$$

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and

$$|q(a,d)| \le \left[C_\delta''' + r(\delta,n)\sqrt{u_n^\delta}\right]|\gamma|\,|y|.$$

Finally, we get

$$J_n''(x_n)\,(a+d).(a+d) \le -C_4|y|^2 + (C_\delta' - u_n^\delta)|\gamma|^2 + 2\left[C_\delta''' + r(\delta,n)\sqrt{u_n^\delta}\right]|\gamma|\,|y|.$$

We can choose $\delta_4 \in (0,\delta_3)$ such that $\forall\delta \in (0,\delta_4)$, $\limsup_{n\to+\infty} r(\delta,n) \le \sqrt{C_4}/2$. Then for fixed $\delta \in (0,\delta_4)$, since $\lim_{n\to+\infty} u_n^\delta = +\infty$, there exist $n''(\delta)$ such that if $n \ge n''(\delta)$ then $[C_\delta''' + r(\delta,n)\sqrt{u_n^\delta}]^2 + C_4(C_\delta' - u_n^\delta) < 0$. So if $n \ge n''(\delta)$ then the restriction of $J_n''(x_n)$ to $B_n^{2,\delta} \oplus D_n^\delta$ is negative definite.

LEMMA 3.5. *For $\delta \in (0,\delta_3)$ small enough, there exists $n'''(\delta)$ such that for $n \ge n'''(\delta)$ the restriction of $J_n''(x_n)$ to $B_n^{1,\delta} \oplus G_n^\delta$ is positive definite.*

PROOF. Let $b \in B_n^{1,\delta}$, $b = \varphi_n^\delta(y)$ with $y = (\alpha_1,\dots,\alpha_p, W_1,\dots,W_p) \in \mathbb{R}^p \times F_n^1 \times \cdots \times F_n^p$; let $l \in G_n^\delta$. Let $\lambda_i \in H^1(I_n^{i,\delta};\mathbb{R})$, $\mu_i \in H^1(I_n^{i,\delta};F_n^i)$ be such that for $t \in I_n^{i,\delta}$, $l(t) = \lambda_i(t)n_n^i + \mu_i(t)$; since $l \in G_n^\delta$ we have

$$\mu_i(t_n^i) = 0 \tag{3.8}$$

$$\int_{I_n^{i,\delta}} g''\big(h_n(t)\big)\lambda_i(t)\,dt = \int_{I_n^{i,\delta}} g''\big(h_n(t)\big)\dot h_n(t)\lambda_i(t)\,dt = 0. \tag{3.9}$$

Note that since $g''\big(h_n(t)\big) > 0$ in $I_n^{i,\delta}$, λ_i vanishes at some point of $I_n^{i,\delta}$, and, since μ_i vanishes at t_n^i, there exists a constant $K_2 \in \mathbb{R}$ such that $|l|_\infty = \sup_{S^1}|l(t)| \le K_2\big(\int_{S^1}|\dot l(t)|^2\,dt\big)^{\frac12}$. We have

$$J_n''(x_n).(b+l).(b+l) = Q_n^\delta(y) + J_n''(x_n).l.l + 2J_n''(x_n).b.l.$$

By Lemma 3.3, $Q_n^\delta(y) \ge C_4|y|^2$.

$$\begin{aligned} J_n''(x_n).l.l = {} & \sum_{i=1}^p \Big(\int_{I_n^{i,\delta}} \big[\dot\lambda_i(t)^2 + |\dot\mu_i(t)|^2\big]\,dt + a^i + b^i + c^i\Big) \\ & + \int_{S^1\setminus\bigcup_{i=1}^p I_n^{i,\delta}} |\dot l(t)|^2 - \epsilon_n U''\big(x_n(t)\big)l(t).l(t)\,dt, \end{aligned}$$

where

$$\begin{aligned} a^i &= \int_{I_n^{i,\delta}} -\epsilon_n\lambda_i(t)^2 U''\big(x_n(t)\big)n_n^i.n_n^i\,dt, \\ b^i &= \int_{I_n^{i,\delta}} -2\epsilon_n\lambda_i(t)U''\big(x_n(t)\big)n_n^i.\mu_i(t)\,dt, \\ c^i &= \int_{I_n^{i,\delta}} -\epsilon_n U''\big(x_n(t)\big)\mu_i(t).\mu_i(t)\,dt. \end{aligned}$$

By Lemma 2.7, $b^i = \big(r(\delta,n) + s(\delta,n)\big)|l|_1^2$ and $c^i = r(\delta,n)|l|_1^2$. Moreover,

$$\begin{aligned} a^i = {} & \int_{I_n^{i,\delta}} -\epsilon_n g''\big(h_n(t)\big)\lambda_i(t)^2\Big(\nabla h\big(x_n(t)\big), n_n^i\Big)^2\,dt \\ & + \int_{I_n^{i,\delta}} -\epsilon_n g'\big(h_n(t)\big)\lambda_i(t)^2\Big(\nabla^2 h\big(x_n(t)\big)\,n_n^i.n_n^i\Big)\,dt. \end{aligned}$$

As $\nabla^2 h$ is bounded in Ω (for $\partial\Omega$ is of class C^2), by Lemma 2.7, the last term can be written as $r(\delta)|l|_1^2$. Since furthermore $\left|\nabla h\big(x_n(t)\big).n_n^i\right| \leq 1$, we get (using (3.1))

$$J_n''(x_n).l.l \geq \sum_{i=1}^{p}\Big(\int_{I_n^{i,\delta}} |\dot\lambda_i(t)|^2 - \epsilon_n g''\big(h_n(t)\big)\lambda_i(t)^2\,dt\Big) + \int_{I_i^{n,\delta}} |\dot\mu_i(t)|^2\,dt + \int_{S^1\setminus\bigcup_{i=1}^p I_n^{i,\delta}} |\dot l(t)|^2 + \big(r(\delta,n)+s(\delta,n)\big)|l|_1^2. \tag{3.10}$$

By Lemma 2.5 $\int_{I_n^{i,\delta}} \dot\lambda_i(t)^2 - \epsilon_n g''\big(h_n(t)\big)\lambda_i(t)^2\,dt \geq C_2\int_{I_n^{i,\delta}} \dot\lambda_i(t)^2\,dt$ for $n \geq n(\delta)$, hence

$$J_n''(x_n).l.l \geq \big(\min(C_2,1)+r(\delta,n)+s(\delta,n)\big)\,|l|_1^2. \tag{3.11}$$

We must estimate $J_n''(x_n).b.l$. On every interval $(t_n^i+\delta, t_n^{i+1}-\delta)$, $\dot b(t) = b^{i,i+1}$, where $b^{i,i+1} = \frac{b(t_n^{i+1}-\delta)-b(t_n^i+\delta)}{t_n^{i+1}-t_n^i-2\delta}$. Since $b = \varphi_n^\delta(y)$, we get $b^{i,i+1} \leq K_3|y|$, where K_3 is a constant. We have

$$\int_{S^1\setminus\bigcup_{i=1}^p I_n^{i,\delta}} \big(\dot b(t),\dot l(t)\big)\,dt = \sum_{i=1}^{p} b^{i,i+1}\big[l(t_n^{i+1}-\delta) - l(t_n^i+\delta)\big].$$

Since λ_i and μ_i vanish somewhere in $I_n^{i,\delta}$, $\sup_{I_n^{i,\delta}}|l(t)| \leq \sqrt{2\delta}\big(\int_{I_n^{i,\delta}} |\dot l(t)|^2\,dt\big)^{\frac12}$. Hence

$$\int_{S^1\setminus\bigcup_{i=1}^p I_n^{i,\delta}} \big(\dot b(t),\dot l(t)\big)\,dt = r(\delta)|y|\,|l|_1.$$

Also from (3.1)

$$\int_{S^1\setminus\bigcup_{i=1}^p I_n^{i,\delta}} \epsilon_n U''\big(x_n(t)\big)b(t).l(t)\,dt = s(\delta,n)|b|_\infty|l|_\infty = s(\delta,n)|y|\,|l|_{H^1}.$$

There remains to estimate $d^i = \int_{I_n^{i,\delta}} \big(\dot b(t),\dot l(t)\big) - \epsilon_n U''\big(x_n(t)\big)b(t).l(t)\,dt$: $b(t) = \alpha_i\dot x_n(t) + W_i$ in $I_n^{i,\delta}$; $\dddot x_n(t) = -\epsilon_n U''\big(x_n(t)\big)\dot x_n(t)$, hence

$$d^i = \int_{I_n^{i,\delta}} \alpha_i\Big[\big(\ddot x_n(t),\dot l(t)\big) + \big(\dddot x_n(t), l(t)\big)\Big]\,dt + \int_{I_n^\delta} -\epsilon_n U''\big(x_n(t)\big)W_i.l(t)\,dt.$$

From Lemma 2.7, the latter term in this sum can be written $\big(r(\delta,n)+s(\delta,n)\big)|y|\,|l|_1$. The former is equal to $\alpha_i\Big[\big(\ddot x_n(t), l(t)\big)\Big]_{t_n^i-\delta}^{t_n^i+\delta}$. Since $\ddot x_n(t_n^i\pm\delta) = -\epsilon_n\nabla U\big(x_n(t_n^i\pm\delta)\big) = s(\delta,n)$ (from (2.1)), it can be written as $\alpha_i|l|_\infty s(\delta,n)$, or $|y|\,|l|_1 s(\delta,n)$; hence $d^i = \big(r(\delta,n) + s(\delta,n)\big)|y|\,|l|_1$. Finally $J_n''(x_n).b.l = \big(r(\delta,n)+s(\delta,n)\big)|y|\,|l|_1$ and we get

$$J_n''(x_n).(b+l).(b+l) \geq C_4\,|y|^2 + \big(\min(1,C_2)+r(\delta,n)+s(\delta,n)\big)|l|_1^2 + \big(r(\delta,n)+s(\delta,n)\big)|y|\,|l|_1,$$

which concludes the proof of Lemma 3.5.

PROOF OF THE THEOREM. Firstly, $\dot{x}_n \in \operatorname{Ker} J''_n(x_n)$ hence $m_{\varepsilon_n}(x_n) \geq 1$. Let $\delta \in (0, \delta_3)$ be small enough so that Lemmas 3.3 and 3.4 can be applied. Let $n_0 \geq \max\big(n'(\delta), n''(\delta), n'''(\delta)\big)$ be large enough so that Lemma 3.1 can be applied. By Lemma 3.4, for $n \geq n_0$, $i_{\varepsilon_n}(x_n) \geq \dim B_n^{2,\delta} + \dim D_n^\delta$, hence

$$i_{\varepsilon_n}(x_n) \geq \overline{\imath}(x) + p. \tag{3.12}$$

By Lemma 3.5, for $n \geq n_0$, $i_{\varepsilon_n}(x_n) + m_{\varepsilon_n}(x_n) \leq \operatorname{codim}(B_n^{1,\delta} \oplus G_n^\delta)$. Now

$$\begin{aligned} \operatorname{codim}(B_n^{1,\delta} \oplus G_n^\delta) &= \dim D_n^\delta + \dim B_n^\delta - \dim B_n^{1,\delta} \qquad \text{(by Lemma 3.1)} \\ &= p + Np - \Big(Np - \big[\overline{\imath}(x) + \overline{m}(x) + 1\big]\Big). \end{aligned}$$

Thus

$$i_{\varepsilon_n}(x_n) + m_{\varepsilon_n}(x_n) \leq \overline{\imath}(x) + p + \overline{m}(x) + 1 \tag{3.13}$$

From (3.12) and (3.13) we derive $m_{\varepsilon_n}(x_n) \leq \overline{m}(x) + 1$, which completes the proof of the theorem.

4. Proof of the preliminary lemmas.

PROOF OF LEMMA 2.1.

- For n large enough, $h_n(t) = h\big(x_n(t)\big)$ is of class C^2 on $(-\delta_1, \delta_1)$, and

$$|\dot{h}_n(t)| \leq |\dot{x}_n(t)| \leq \sqrt{2E_n} \leq K_4,$$

where E_n is the energy $\frac{1}{2}|\dot{x}_n(t)|^2 + \epsilon_n U\big(x_n(t)\big)$ and K_4 is a constant. Hence, for $t \in (-\delta_1, \delta_1)$,

$$h_n(t) \leq h_n(0) + K_4|t| \tag{4.1}$$

$\lim_{h \to 0} \frac{g''(h)}{g'(h)} = -\infty$ and $g'(h) < 0$ on $\mathbb{R}_+^\star$ hence there exists $\eta > 0$ such that if $0 < h < \eta$ then $g''(h) > 0$.

From (4.1), since $\lim_{n \to +\infty} h_n(0) = 0$, it is clear that there exist $\overline{\delta_2} \in (0, \delta_1)$ and $\overline{n_2} \in \mathbb{N}$ such that if $t \in (-\overline{\delta}_2, \overline{\delta}_2)$ and $n \geq \overline{n_2}$ then $h_n(t) < \eta$, and thus $g''\big(h_n(t)\big) > 0$.

- For $t \in (-\delta_1, \delta_1)$, set $z_n(t) = \dot{x}_n(t) - \Big(\dot{x}_n(t), \nabla h\big(x_n(t)\big)\Big)\nabla h\big(x_n(t)\big)$. We have $\dot{h}_n(t) = \Big(\dot{x}_n(t), \nabla h\big(x_n(t)\big)\Big)$ and $|\nabla h(x)| = 1$ hence $|\dot{x}_n(t)|^2 = |\dot{h}_n(t)|^2 + |z_n(t)|^2$; z_n is of class C^1 in $(-\delta_1, \delta_1)$, and

$$\begin{aligned} \dot{z}_n(t) &= \ddot{x}_n(t) - \Big(\ddot{x}_n(t), \nabla h\big(x_n(t)\big)\Big)\nabla h\big(x_n(t)\big) \\ &\quad - \Big(\dot{x}_n(t), \nabla^2 h\big(x_n(t)\big)\dot{x}_n(t)\Big)\nabla h\big(x_n(t)\big) \\ &\quad - \Big(\dot{x}_n(t), \nabla h\big(x_n(t)\big)\Big)\nabla^2 h\big(x_n(t)\big)\dot{x}_n(t). \end{aligned} \tag{4.2}$$

$\ddot{x}_n(t) = -\epsilon_n \nabla U\big(x_n(t)\big) = -\epsilon_n g'\big(h_n(t)\big)\nabla h\big(x_n(t)\big)$ hence
$\ddot{x}_n(t) - \Big(\ddot{x}_n(t), \nabla h\big(x_n(t)\big)\Big)\nabla h\big(x_n(t)\big) = 0.$

Now, since $\partial\Omega$ is of class C^2, $|\nabla^2 h|$ is bounded in Ω; since $|\dot{x}_n(t)| \leq K_4$, (4.2) implies the existence of a constant K_5 such that for all $t \in (-\delta_1, \delta_1)$ $|\dot{z}_n(t)| \leq K_5$; so z_n converges uniformly in the interval $(-\delta_1, \delta_1)$ to z, defined by

$$z(t) = \dot{x}(t) - \Big(\dot{x}(t), \nabla h\big(x(t)\big)\Big)\nabla h\big(x(t)\big)$$

(∇h can be continuously extended to $\overline{\Omega}$: for $x \in \partial\Omega$ $\nabla h(x) = n(x)$, interior unit normal to $\partial\Omega$ at x).

We have assumed that $\Big(\dot{x}_+(0), n\big(x(0)\big)\Big) > 0$ (there is a "genuine bouncing" at the instant 0 with bounce point $x(0)$). Hence $|z(0)| < |\dot{x} \pm (0)|$ and $E - \frac{1}{2}|z(0)|^2 > 0$, $E = \frac{1}{2}|\dot{x} \pm (0)|^2$ being the energy of x. As z is continuous in $(-\delta_1, \delta_1)$, there exist $\delta_2 > 0$ (with $\delta_2 < \overline{\delta}_2$) and a positive constant e_0 such that for $t \in (-\delta_2, \delta_2)$, $E - \frac{1}{2}|z(t)|^2 > e_0$; set $e = \frac{e_0}{2}$; since $\lim_{n\to\infty} E_n = E$ and z_n converges uniformly to z in $(-\delta_2, \delta_2)$, there exists $n_2 \in \mathbb{N}$ ($n_2 \geq \overline{n}_2$) such that if $n \geq n_2$ then $E_n - \frac{1}{2}|z_n(t)|^2 > e$ for all $t \in (-\delta_2, \delta_2)$. Now, $E_n - \frac{1}{2}|z_n(t)|^2 = \frac{1}{2}|\dot{h}_n(t)|^2 + \epsilon_n g\big(h_n(t)\big)$. Thus Lemma 2.1 is proved.

PROOF OF LEMMA 2.2. (a) $\lim_{n\to\infty} h_n(0) = h\big(x(0)\big) = 0$ hence $\lim_{n\to\infty} h_n(t_n) = 0$. From (2.1), this implies that $\lim_{n\to+\infty} t_n = 0$.

(b) (i) is a trivial consequence of $\lim_{n\to+\infty} t_n = 0$. For n large enough, $t_n \in (-\delta_2, \delta_2)$ and $\dot{h}_n(t_n) = 0$; h_n is a function of class C^2 in $(-\delta_2, \delta_2)$, and, since $\dot{h}_n(t) = \Big(\dot{x}_n(t), \nabla h\big(x_n(t)\big)\Big)$, $\ddot{h}_n(t) = -\epsilon_n g'(h_n(t)) + \Big(\dot{x}_n(t), \nabla^2 h\big(x_n(t)\big)\dot{x}_n(t)\Big)$.

Hence there exists a constant K_6 such that

$$\forall t \in (-\delta_2, \delta_2) \quad \Big|\ddot{h}_n(t) + \epsilon_n g'\big(h_n(t)\big)\Big| \leq K_6. \tag{4.3}$$

On the other hand

$$\frac{1}{2}\dot{h}_n(t)^2 + \epsilon_n g\big(h_n(t)\big) \geq e. \tag{4.4}$$

From (4.3) and (4.4), if $\dot{h}_n(t) = 0$ (with $t \in (-\delta_2, \delta_2)$) then

$$g\big(h_n(t)\big) \geq \frac{e}{\epsilon_n} \tag{4.5}$$

and

$$\ddot{h}_n(t) \geq -\frac{g'\big(h_n(t)\big)}{g\big(h_n(t)\big)}e - K_6. \tag{4.6}$$

Since $\lim_{h\to 0} -\frac{g'(h)}{g(h)} = +\infty$, there exists h_0 such that if $\dot{h}_n(t) = 0$ and $h_n(t) < h_0$ then $\ddot{h}_n(t) > 0$. Now, since (4.5) holds, for n large enough, $t \in (-\delta_2, \delta_2)$ and $\dot{h}_n(t) = 0$ imply $h_n(t) < h_0$. Hence there exists $n_3 \in \mathbb{N}$ ($n_3 \geq n_2$) such that if $n \geq n_3$, $t \in (-\delta_2, \delta_2)$ and $\dot{h}_n(t) = 0$ then $\ddot{h}_n(t) > 0$. As a consequence $\dot{h}_n$ can vanish only at t_n in the interval $(-\delta_2, \delta_2)$; since h_n has a minimum at t_n, $\dot{h}_n < 0$ on $(\delta_2, t_n]$, and $\dot{h}_n > 0$ on $[t_n, \delta_2)$. This proves (ii) (δ_3 is chosen such that $(t_n - \delta_3, t_n + \delta_3) \subset (-\delta_2, \delta_2)$ for any $n \geq n_3$).

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Let $\delta' \in (0, \delta_2)$; there exists $a_{\delta'} > 0$ such that for all $t \in (\delta', \delta_2)$ and for all $n \in \mathbb{N}$ $|h_n(t)| \geq a_{\delta'}$. Hence $-\epsilon_n U''\big(x_n(t)\big)$ converges uniformly to 0 in the interval (δ', δ_2). Since $\ddot{x}_n(t) = -\epsilon_n U''\big(x_n(t)\big)$ (and x_n is convergent to x), we can infer that $\dot{x}_n$ converges uniformly to $\dot{x}$ (constant function equal to $\dot{x}_+(0)$) in the interval (δ', δ_2).

Since $\lim_{n\to\infty} t_n = 0$, it is now clear that for all $\delta \in (0, \delta_3)$ $\lim_{n\to\infty} \dot{x}_n(t_n + \delta) = \dot{x}_+(0)$; in the same way we get $\lim_{n\to\infty} \dot{x}_n(t_n - \delta) = \dot{x}_-(0)$; (iii) is proved.

(c) If assertion (c) does not hold, then (possibly considering a subsequence), we can assume that for $n \geq n_3$, there is $s_n \in (0, \delta_3)$, and a constant $K > 0$ such that $\dot{h}_n(t_n + s_n) \leq K s_n$, with $\lim_{n\to+\infty} s_n = 0$. Since $\dot{h}_n(t_n) = 0$, it follows from (4.3) that $K s_n \geq \dot{h}_n(t_n + s_n) \geq \int_0^{s_n} -\epsilon_n g'\big(h_n(t_n + s)\big)\, ds - K_6 s_n$. Hence

$$\int_0^{s_n} -\epsilon_n g'\big(h_n(t_n + s)\big)\, ds \leq (K_6 + K)s_n. \tag{4.7}$$

We derive from Lemmas 2.1 and 2.2(b) (ii) that the function $s \longmapsto -\epsilon_n g'\big(h_n(t_n + s)\big)$ is decreasing on $(0, \delta_3)$, hence (4.7) implies

$$0 \leq -\epsilon_n g'\big(h_n(t_n + s_n)\big) \leq (K_6 + K). \tag{4.8}$$

Now, by Lemma 2.1,

$$\epsilon_n g\big(h_n(t_n + s_n)\big) \geq \left[e - \frac{1}{2}\dot{h}_n(t_n + s_n)^2\right] \geq \left[e - \frac{1}{2}K^2 {s_n}^2\right]. \tag{4.9}$$

Hence $\lim_{n\to\infty} g\big(h_n(t_n+s_n)\big) = +\infty$ and $\lim_{n\to\infty} h_n(t_n+s_n) = 0$. Since $\lim_{h\to 0} \frac{g(h)}{g'(h)} = 0$, we can derive from (4.8) that $\lim_{n\to\infty} \epsilon_n g\big(h_n(t_n + s_n)\big) = 0$, which contradicts (4.9) (because $\lim_{n\to\infty} K^2 {s_n}^2 = 0$). So assertion (c) holds.

(d) We have

$$\Big|\dot{h}_n(t) - \big(\dot{x}_n(t), n_n\big)\Big| = \Big|\Big(\dot{x}_n(t), \nabla h\big(x_n(t)\big) - \nabla h\big(x_n(t_n)\big)\Big)\Big| \leq K_7|t - t_n|, \tag{4.10}$$

where K_7 is an upper bound of $(\sup_\Omega |\nabla^2 h|)|\dot{x}_n|_\infty$. Hence, from (c), there exist $\beta > 0$ and $\gamma > 0$ such that, provided that n is large enough and δ is small enough, for any $t \in [t_n, t_n + \delta)$, $\beta \dot{h}_n(t) \leq \big(\dot{x}_n(t), n_n\big) \leq \gamma \dot{h}_n(t)$. We get similar estimates for $\big(\dot{x}_n(t), n_n\big)$ when $t \in (t_n - \delta, t_n]$.

PROOF OF LEMMA 2.3. Since $g'(h) < 0$, $|\epsilon_n g'\big(h_n(t)\big)| = -\epsilon_n g'\big(h_n(t)\big)$. From (4.3) we get $-\epsilon_n g'\big(h_n(t)\big) \leq \ddot{h}_n(t) + K_6$ and hence $\int_{I_n^{\delta_3}} \big|-\epsilon_n g'\big(h_n(t)\big)\big|\, dt \leq \dot{h}_n(t_n + \delta_3) - \dot{h}_n(t_n - \delta_3) + 2K_6\delta_3$.

We have: $|\dot{h}_n(t)| \leq |\dot{x}_n(t)| \leq \sqrt{2E_n}$. Hence there exists a constant C_1 such that $\int_{I_n^{\delta_3}} \big|\epsilon_n g'\big(h_n(t)\big)\big|\, dt \leq C_1$.

From (4.3)

$$\begin{aligned} \int_{I_n^\delta} -\epsilon_n g'\big(h_n(t)\big)\, dt &= \dot{h}_n(t_n + \delta) - \dot{h}_n(t_n - \delta) + r(\delta) \\ &= \Big(\dot{x}_n(t_n + \delta), \nabla h\big(x_n(t_n + \delta)\big)\Big) - \Big(\dot{x}_n(t_n - \delta), \nabla h\big(x_n(t_n - \delta)\big)\Big) + r(\delta). \end{aligned}$$

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By Lemma 2.2, $\lim_{n\to\infty} \dot{x}_n(t_n \pm \delta) = \dot{x}_\pm(0)$. x_n converges uniformly to x and $\lim_{n\to\infty} t_n = 0$, therefore $\lim_{n\to\infty} \nabla h\big(x_n(t_n \pm \delta)\big) = \nabla h\big(x(\pm\delta)\big)$; hence

$$\int_{I_n^\delta} -\epsilon_n g'\big(h_n(t)\big)\, dt = \Big(\dot{x}_+(0), \nabla h\big(x(\delta)\big)\Big) - \Big(\dot{x}_-(0), \nabla h\big(x(-\delta)\big)\Big) + r(\delta) + s(\delta, n).$$

Now, $\lim_{\delta\to 0} \nabla h\big(x(\pm\delta)\big) = n\big(x(0)\big)$, hence

$$\begin{aligned}\int_{I_n^\delta} -\epsilon_n g'\big(h_n(t)\big)\, dt &= \Big(\dot{x}_+(0) - \dot{x}_-(0), n\big(x(0)\big)\Big) + s(\delta, n) + r(\delta)\\ &= 2\sqrt{2E}\cos\theta + s(\delta, n) + r(\delta),\end{aligned}$$

because $|\dot{x} \pm (0)| = \sqrt{2E}$ and $\cos\theta = \frac{(\dot{x}_+(0), n(x(0)))}{|\dot{x}_+(0)|}$.

PROOF OF LEMMA 2.4. Set $I(\delta, n) = \int_{t_n}^{t_n+\delta} \epsilon_n g''\big(h_n(t)\big)(t - t_n)^2\, dt$ ($\delta \in (0, \delta_3)$ and $n \geq n_3$). By Lemma 2.2(d) (and since $g''\big(h_n(t)\big) > 0$ by Lemma 2.1),

$$0 \leq I(\delta, n) \leq \int_{t_n}^{t_n+\delta} \frac{\epsilon_n}{\alpha(\delta, n)} g''\big(h_n(t)\big)\dot{h}_n(t)(t - t_n)\, dt.$$

Integrating by parts and using the fact that $g' < 0$ we get

$$I(\delta, n) \leq \int_{t_n}^{t_n+\delta} \frac{-\epsilon_n}{\alpha(\delta, n)} g'\big(h_n(t)\big)\, dt.$$

Hence, by Lemma 2.3,

$$0 \leq I(\delta, n) \leq \frac{C_1}{\alpha(\delta, n)},$$

so $I(\delta, n) = r(\delta, n)$ because $\lim_{\delta\to 0, n\to+\infty} \alpha(\delta, n) = +\infty$. In the same way we can prove that $\int_{t_n-\delta}^{t_n} \epsilon_n g''\big(h_n(t)\big)(t - t_n)^2\, dt = r(\delta, n)$.

We now prove the second point of the lemma. For $0 < d \leq \delta$, set

$$K_n(d) = \int_{t_n}^{t_n+d} \epsilon_n g''\big(h_n(t)\big)\dot{h}_n(t)^2\, dt.$$

By Lemma 2.1, $K_n(\delta) \geq K_n(d)$. Since $\lim_{h\to 0^+} \frac{g''(h)}{g'(h)} = -\infty$ and $\lim_{\substack{d\to 0\\ n\to+\infty}} \sup_{[t_n, t_n+d]} h = 0$, we obtain

$$K_n(d) \geq L_n(d) \int_{t_n}^{t_n+d} -\epsilon_n g'\big(h_n(t)\big)\dot{h}_n(t)^2\, dt,$$

with $\lim_{\substack{d\to 0\\ n\to+\infty}} L_n(d) = +\infty$. Hence, from (4.3),

$$K_n(d) \geq L_n(d)\Big[\int_{t_n}^{t_n+d} \ddot{h}_n(t)\dot{h}_n(t)^2\, dt - K_6 d|\dot{h}_n|_\infty^2\Big].$$

We get, for all $d \in (0, \delta)$,

$$\int_{I_n^\delta} \epsilon_n g''\big(h_n(t)\big)\dot{h}_n(t)^2\, dt \geq K_n(d) \geq \overline{L}_n(d)\big[\dot{h}_n(t_n + d)^3 - d\big], \tag{4.11}$$

with $\lim_{\substack{d\to 0\\ n\to+\infty}} \overline{L}_n(d) = +\infty$. Moreover, d being fixed, by Lemma 2.1, for n large enough $\dot{h}_n(t_n + d) \geq \sqrt{e}$. So it is not difficult to check that (4.11) implies $\lim_{n\to\infty} \int_{I_n^\delta} \epsilon_n g''\big(h_n(t)\big)\dot{h}_n(t)^2\, dt = +\infty$. Since $|\dot{h}_n|_{L^\infty(I_n^\delta)}$ is bounded, we conclude that $\lim_{n\to\infty} \int_{I_n^\delta} \epsilon_n g''\big(h_n(t)\big)\, dt = +\infty$, which completes the proof of Lemma 2.4.

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PROOF OF LEMMA 2.5.

REMARK. The fact that $g(h) = \frac{1}{h^2}$ near $\partial\Omega$ is used only in this proof.

In order to simplify notations, we assume without loss of generality that for any n, $t_n = 0$: thus $I_n^\delta = I^\delta = (-\delta, \delta)$, and $\dot h_n(0) = 0$. We have already seen that for $t \in I^{\delta_3}$ $\ddot h_n(t) = -\epsilon_n g'\big(h_n(t)\big) + r_n(t) = \frac{2\epsilon_n}{h_n(t)^3} + r_n(t)$, where $r_n(t) = \Big(\dot x_n(t), \nabla^2 h\big(x_n(t)\big)\dot x_n(t)\Big)$, $|r_n(t)| \le K_6$.

Set $\frac{\epsilon_n}{h_n(0)^2} = \epsilon_n g\big(h_n(0)\big) = e_n$; as in the proof of Lemma 2.1, we readily verify that $\lim_{n\to+\infty} e_n = e > 0$, where $e = \frac{1}{2}\Big(\dot x_+(0), n\big(x(0)\big)\Big)^2$. Set $h_n(t) = \sqrt{\frac{\epsilon_n}{e_n}} f_n(\frac{e_n t}{\sqrt{\epsilon_n}})$; f_n satisfies:

$$
(4.12)\qquad
\begin{aligned}
&(i)\ \ \ddot f_n(s) = \frac{2}{f_n(s)^3} + \epsilon_n^{\frac{1}{2}} S_n(s) \quad \text{for } s \in \frac{e_n}{\sqrt{\epsilon_n}} I^{\delta_3};\\
&(ii)\ \ \dot f_n(0) = 0;\\
&(iii)\ \ f_n(0) = 1.
\end{aligned}
$$

Here we have set $S_n(s) = r_n(\frac{\sqrt{\epsilon_n}}{e_n}s)e_n^{-\frac{3}{2}}$; $|S_n(s)| \le K_8$, where K_8 is a constant.

As $|\dot h_n|_{L^\infty(I_n^\delta)}$ is bounded, $|\dot f_n|_\infty$ is bounded. Set

$$
H_n^\delta = \Big\{\lambda \in H^1(I_n^\delta; \mathbb{R}^N) \;\Big|\; \int_{I_n^\delta} g''\big(h_n(t)\big)\lambda(t)\,dt = \int_{I_n^\delta} g''\big(h_n(t)\big)\dot h_n(t)\lambda(t)\,dt = 0\Big\}.
$$

For $\lambda \in H_n^\delta$ set

$$
(4.13)\qquad \lambda(t) = l\left(\frac{e_n t}{\sqrt{\epsilon_n}}\right), \quad q_n^\delta(\lambda) = \int_{I_n^\delta} \dot\lambda(t)^2 - \epsilon_n g''\big(h_n(t)\big)\lambda(t)^2\,dt.
$$

We get: $\int_{I_n^\delta} \dot\lambda(t)^2\,dt = \frac{e_n}{\sqrt{\epsilon_n}}\int_{J_n^\delta} \dot l(s)^2\,ds$ and $q_n^\delta(\lambda) = \frac{e_n}{\sqrt{\epsilon_n}}\tilde q_n^\delta(l)$, where

$$
J_n^\delta = \frac{e_n}{\sqrt{\epsilon_n}} I^\delta \quad \text{and} \quad \tilde q_n^\delta(l) = \int_{J_n^\delta} \dot l(s)^2 - \frac{6}{f_n(s)^4} l(s)^2\,ds.
$$

Let $\mathcal{H}_n^\delta = \{l \in H^1(J_n^\delta; \mathbb{R}^N) \mid \int_{J_n^\delta} \frac{l(s)}{f_n(s)^4}\,ds = \int_{J_n^\delta} \frac{\dot f_n(s)}{f_n(s)^4} l(s)\,ds = 0\}$. $\lambda \in H_n^\delta$ iff l defined by (4.13) belongs to $\mathcal{H}_n^\delta$. So we have to prove that there exists a constant C_2 such that if $l \in \mathcal{H}_n^\delta$ then $\tilde q_n^\delta(l) \ge C_2\,|l|_1^2$.

Let f be the function defined on $\mathbb{R}$ by $f(s) = \sqrt{2s^2+1}$. Note that f satisfies $\ddot f(s) = \frac{2}{f(s)^3}$, $\dot f(0) = 0$, $\ddot f(0) = 1$. Let

$$
E = \Big\{l \in H^1_{\mathrm{loc}}(\mathbb{R}; \mathbb{R}) \;\Big|\; \int_{\mathbb{R}} |\dot l(s)|^2\,ds < +\infty\Big\},
$$
$$
\mathcal{H} = \Big\{l \in E \;\Big|\; \int_{\mathbb{R}} \frac{l(s)}{f(s)^4}\,ds = \int_{\mathbb{R}} \frac{\dot f(s)}{f(s)^4} l(s)\,ds = 0\Big\}.
$$

Note that all $l \in E$ satisfies $\int_{\mathbb{R}} \frac{l(s)}{f(s)^4}\,ds < +\infty$, because $|l(s)| \le |l(0)| + \sqrt{|s|}\big(\int_{\mathbb{R}} \dot l(s)^2\,ds\big)^{1/2}$, and that $\dot f \in E$.

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LEMMA 4.1. *There exists a constant* $\overline{C}_2 > 0$ *such that, for all* $l \in H$,

$$\int_{\mathbb{R}} \dot{l}(s)^2 - \frac{6l(s)^2}{f(s)^4}\, ds \geq \overline{C}_2 \int_{\mathbb{R}} \dot{l}(s)^2\, ds.$$

Before proving Lemma 4.1 we shall explain how we can derive Lemma 3.5 from it. Set $C_2 = \frac{\overline{C}_2}{2}$. Let $\delta \in (0, \delta_3)$; we suppose that there does not exist $n(\delta)$ such that $n \geq n(\delta)$ and $l \in H_n^\delta$ imply $\tilde{q}_n^\delta(l) \geq C_2 |l|_1^2$, and we seek a contradiction.

Then (up to a subsequence), we may suppose that there exists, for any $n \in \mathbb{N}$, $l_n \in H_n^\delta$ which satisfies $\tilde{q}_n^\delta(l_n) < C_2 |l_n|_1^2$, and $\int_{J_n^\delta} \frac{l_n(s)^2}{f_n(s)^4}\, ds = 1$. The functions f_n and l_n are defined on $J_n^\delta = (-\frac{\delta e_n}{\sqrt{\epsilon_n}}, \frac{\delta e_n}{\sqrt{\epsilon_n}})$; since $\lim_{n\to+\infty} \frac{\delta e_n}{\sqrt{\epsilon_n}} = +\infty$, $\bigcup_{n\in\mathbb{N}} J_n^\delta = \mathbb{R}$. In addition, by Lemma 2.2, for $t \in I_n^\delta$, $h_n(t) \geq h_n(0)$; hence for $s \in J_n^\delta$, $f_n(s) \geq 1$. It follows by (4.12) that $|\ddot{f}_n|_{L^\infty(J_n^\delta)}$ is bounded independently of n.

So every subsequence of (f_n) has a subsequence which converges in $C^1(I)$ for any bounded interval I of $\mathbb{R}$ (this makes sense because f_n is defined on I when n is large enough). Now, from (4.12) a limit F must satisfy: $\ddot{F}(s) = \frac{2}{F(s)^3}$ for all $s \in \mathbb{R}$, $\dot{F}(0) = 0$ and $F(0) = 1$. Hence $F = f$. We can conclude that f_n converges to f in $C^1_{\text{loc}}(\mathbb{R})$.

Moreover $\tilde{q}_n^\delta(l_n) < C_2 |l_n|_1^2$ hence $(1 - C_2)|l_n|_1^2 \leq \int_{J_n^\delta} 6\frac{l_n(s)^2}{f_n(s)^4}\, ds \leq 6$.

The constant $\overline{C}_2$ defined in Lemma 4.1 clearly is smaller than 1, so $C_2 \leq \frac{1}{2}$; hence $|l_n|_1$ is bounded; moreover, since $\int_{J_n^\delta} \frac{l_n(s)^2}{f_n(s)^4}\, ds$ is bounded, and since f_n converges to f in $C^1_{\text{loc}}(\mathbb{R})$, for any bounded interval I, $\int_I l_n(s)^2\, ds$ is bounded. As a consequence $\|l_n\|_{H^1(I)}$ is bounded for any bounded interval I of $\mathbb{R}$ (l_n is well defined on I for large n). We infer that there exists $l \in H^1_{\text{loc}}(\mathbb{R})$ such that (l_n) (or a subsequence) converges uniformly to l on I for any bounded interval I of $\mathbb{R}$; l_n converges weakly to l in $H^1_{\text{loc}}(\mathbb{R})$ hence, for any bounded interval I,

$$\int_I \dot{l}(s)^2\, ds \leq \liminf_{n\to+\infty} \int_I \dot{l}_n(s)^2\, ds \leq \frac{6}{1 - C_2}.$$

Therefore

$$\int_{\mathbb{R}} \dot{l}(s)^2\, ds \leq \frac{6}{1 - C_2}. \tag{4.14}$$

We next prove that $\lim_{n\to+\infty} \int_{J_n^\delta} \frac{l_n(s)^2}{f_n(s)^4}\, ds = \int_{\mathbb{R}} \frac{l(s)^2}{f(s)^4}\, ds$. For this purpose, further information about f_n is required: we know that f_n (as h_n) is non-decreasing on $[0, \frac{\delta e_n}{\sqrt{\epsilon_n}}]$ and that

$$\ddot{f}_n(s) \geq -\sqrt{\epsilon_n}\, K_8. \tag{4.15}$$

The following hold: $\lim_{n\to+\infty} f_n(2) = f(2) > 2$; $\lim_{n\to+\infty} \dot{f}_n(2) = \dot{f}(2) > 1$. Hence for n large enough $f_n(2) \geq 2$ and $\dot{f}_n(2) \geq 1$, and for any $s \in J_n^\delta$ s.t. $s \geq 2$, by (4.15), $\dot{f}_n(s) \geq \dot{f}_n(2) - \sqrt{\epsilon_n} K_8 s \geq 1 - \sqrt{\epsilon_n} K_8 s$. Let $s_n = \frac{1}{2\sqrt{\epsilon_n} K_8}$; when $s \in J_n^\delta$ and $2 \leq s \leq s_n$, $\dot{f}_n(s) \geq \frac{1}{2}$ and $f_n(s) \geq f_n(2) + \frac{s-2}{2} \geq \frac{s}{2}$ (for n large enough).

Furthermore, since f_n is non-decreasing on J_n^δ, if $\frac{\delta e_n}{\sqrt{\epsilon_n}} > s_n$ then for $s_n \le s \le \frac{\delta e_n}{\sqrt{\epsilon_n}}$ $f_n(s) \ge \frac{s_n}{2}$. Hence, since $\frac{\delta e_n}{\sqrt{\epsilon_n}} \frac{1}{s_n}$ is bounded by a constant independent of δ and n, there is a constant $K_9 > 0$ such that (provided n is large enough), for $s \ge 2$ and $s \in J_n^\delta$,

$$f_n(s) \ge sK_9. \tag{4.16}$$

Moreover, for $0 \le s \le \frac{\delta e_n}{\sqrt{\epsilon_n}}$, $|l_n(s)| \le |l_n(0)| + \int_0^s |\dot{l}_n(u)|\,du \le K_{10} + \sqrt{s}|l_n|_1$. Therefore

$$|l_n(s)| \le K_{10} + \frac{6}{1 - C_2}\sqrt{s}. \tag{4.17}$$

For $M > 0$ set $a(M) = \sup_{n \ge n_3}\left(\int_{J_n^\delta \setminus (-\infty, M]} \frac{l_n(s)^2}{f_n(s)^4}\,ds\right)$. From (4.16) and (4.17), we derive $\lim_{M \to +\infty} a(M) = 0$.

In the same way, setting $b(M) = \sup_{n \ge n_3}\left(\int_{J_n^\delta \setminus [-M, +\infty)} \frac{l_n(s)^2}{f_n(s)^4}\,ds\right)$, we get $\lim_{M \to +\infty} b(M) = 0$.

Since l_n and f_n converge uniformly to l and f on every bounded interval (and since 1 is a lower bound of f_n on J_n^δ), we conclude that

$$\int_{\mathbb{R}} \frac{l(s)^2}{f(s)^4}\,ds = \lim_{n \to +\infty} \int_{J_n^\delta} \frac{l_n(s)^2}{f_n(s)^4}\,ds = 1.$$

Hence, from (4.14),

$$\int_{\mathbb{R}} \dot{l}(s)^2 - 6\frac{l(s)^2}{f(s)^4}\,ds \le C_2 \int_{\mathbb{R}} \dot{l}(s)^2\,ds. \tag{4.18}$$

Since $l_n \in H_n^\delta$, $\int_{J_n^\delta} \frac{l_n(s)}{f_n(s)^4}\,ds = \int_{J_n^\delta} \frac{\dot{f}_n(s)}{f_n(s)^4} l_n(s)\,ds = 0$. Using inequalities (4.16) and (4.17), and the fact that l_n, f_n, $\dot{f}_n$ converge respectively to l, f, $\dot{f}$, uniformly on every bounded interval (note that $\dot{f}(s) = \frac{2s}{\sqrt{2s^2+1}}$ and that $\|\dot{f}_n\|_\infty$ is bounded), we can easily check that

$$\int_{\mathbb{R}} \frac{l(s)}{f(s)^4}\,ds = \lim_{n \to +\infty} \int_{J_n^\delta} \frac{l_n(s)}{f_n(s)^4}\,ds = 0; \int_{\mathbb{R}} \frac{\dot{f}(s)}{f(s)^4} l(s)\,ds = \lim_{n \to +\infty} \int_{J_n^\delta} \frac{\dot{f}_n(s)}{f_n(s)^4} l(s)\,ds = 0.$$

Hence $l \in H$, and inequality (4.18) contradicts Lemma 4.1, since $C_2 < \overline{C}_2$ and $\int_{\mathbb{R}} \dot{l}(s)^2\,ds > 0$.

We now prove Lemma 4.1: E is a real Hilbert space endowed with the scalar product

$$(l_1, l_2)_E = \int_{\mathbb{R}} \dot{l}_1(s)\dot{l}_2(s) + 6\frac{l_1(s)l_2(s)}{f(s)^4}\,ds.$$

Let $\|l\|_E^2 = (l, l)_E$. Let K be the endomorphism of E defined by:

$$\forall (X, Y) \in E^2 \quad (KX, Y)_E = 6\int_{\mathbb{R}} \frac{X(s)Y(s)}{f(s)^4}\,ds.$$

We can easily check that K is compact symmetric, has norm one (if X is a constant map, $KX = X$). As E is separable, it admits a base composed of eigenvectors of K.

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The eigenvalues $\lambda_1, \ldots, \lambda_p \ldots$ of K (all are of finite multiplicity) form a strictly decreasing convergent to 0 sequence of positive reals. We denote by E_{λ_i} the corresponding eigenspaces; it is obvious that $\lambda_1 = 1$ and E_{λ_1} is the set of constant functions.

We shall prove that

$$(*) \qquad \lambda_2 = \frac{1}{2} \quad \text{and} \quad E_{\lambda_2} \text{ is spanned by } (\dot f).$$

Once $(*)$ is proved, we shall be able to write that $\mathcal{H} = (E_{\lambda_1} \oplus E_{\lambda_2})^{\perp_E}$ and hence for all $X \in \mathcal{H}$ $(KX, X)_E \leq \lambda_3 \|X\|_E^2$ with $\lambda_3 < \frac{1}{2}$. We shall get for $X \in \mathcal{H}$

$$6 \int_{\mathbb{R}} \frac{X(s)^2}{f(s)^4}\, ds \leq \lambda_3 \Big(\int_{\mathbb{R}} \dot X(s)^2 + \frac{6X(s)^2}{f(s)^4}\, ds \Big),$$

hence

$$\int_{\mathbb{R}} \dot X(s)^2 ds - \int_{\mathbb{R}} \frac{6X(s)^2}{f(s)^4}\, ds \geq \left(1 - \frac{\lambda_3}{1-\lambda_3}\right) \int_{\mathbb{R}} \dot X(s)^2\, ds.$$

Set $\overline{C}_2 = 1 - \frac{\lambda_3}{1-\lambda_3}$; $\lambda_3 < \frac{1}{2}$ implies $\overline{C}_2 > 0$, and Lemma 4.1 will be proved.

We now prove $(*)$. Let $\overline{E} = \{X \in E \mid \int \frac{X(s)}{f(s)^4}\, ds = 0\} = E_{\lambda_1}^{\perp_E}$. We have $\lambda_2 = \sup_{\|x\|_E^2 \leq 1, x \in \overline{E}} (KX, X)_E$, and $E_{\lambda_2} = \{X \in \overline{E} \mid (KX, X)_E = \lambda_2 \|X\|_E^2\}$. E_{λ_2} is the set of $X \in E$ which satisfy equation

$$(P_{\lambda_2}). \qquad \ddot X(s) = 6\left(1 - \frac{1}{\lambda_2}\right) \frac{X(s)}{f(s)^4}$$

Let $X \in E_{\lambda_2}$; define $X(s) = X(0) + Y(s)$. We get: $\int_{\mathbb{R}} \dot X(s)^2\, ds = \int_{\mathbb{R}} \dot Y(s)^2\, ds$ and

$$\begin{aligned}\int_{\mathbb{R}} \frac{X(s)^2}{f(s)^4}\, ds &= \int_{\mathbb{R}} \frac{Y(s)^2}{f(s)^4}\, ds + \int_{\mathbb{R}} \frac{X(0)^2}{f(s)^4}\, ds + 2 \int_{\mathbb{R}} \frac{\big(X(0), Y(s)\big)}{f(s)^4}\, ds \\ &= \int_{\mathbb{R}} \frac{Y(s)^2}{f(s)^4}\, ds - X(0)^2 \int_{\mathbb{R}} \frac{ds}{f(s)^4},\end{aligned}$$

because $\int_{\mathbb{R}} \frac{X(s)}{f(s)^4}\, ds = 0$.

Let $v(s) = \dot f(s) = \frac{2s}{\sqrt{2s^2+1}}$; note that, since $\ddot f(s) = \frac{2}{f(s)^3}$, $\ddot v(s) = -\frac{6v(s)}{f(s)^4}$. As X satisfies (P_{λ_2}), it is of class C^∞ on $\mathbb{R}$, as well as Y and we can write $Y(s) = v(s)z(s)$, with z of class C^∞ on $\mathbb{R}$ (because $Y(0) = 0$). Let $M \in \mathbb{R}_+^*$. We have:

$$-\int_{-M}^{M} 6\frac{Y(s)^2}{f(s)^4}\, ds = \int_{-M}^{M} \ddot v(s) v(s) z(s)\, ds,$$

$$\int_{-M}^{M} \dot Y(s)^2\, ds = \int_{-M}^{M} v(s)^2 \dot z(s)^2 + \dot v(s)^2 z(s)^2 + 2\dot v(s) v(s) \dot z(s) z(s)\, ds.$$

We get

$$\begin{aligned}\int_{-M}^{M} \dot Y(s)^2 - \frac{6Y(s)^2}{f(s)^4}\, ds &= \int_{-M}^{M} v(s)^2 \dot z(s)^2\, ds + \int_{-M}^{M} \frac{d}{ds}\big(\dot v(s) v(s) z(s)^2\big)\, ds \\ &= \int_{-M}^{M} v(s)^2 \dot z(s)^2\, ds + \big[\dot v(s) v(s) z(s)^2\big]_{-M}^{M}.\end{aligned}$$

We have $z(s)^2 = \frac{Y(s)^2}{v(s)^2} \leq \frac{|s|}{v(s)^2}\left(\int_{\mathbb{R}} |\dot{Y}(s)|^2\right)$. Hence

$$|\dot{v}(M)v(M)z(M)^2| \leq \left(\int_{\mathbb{R}} \dot{Y}(s)^2\, ds\right)\frac{M}{|v(M)|}\frac{2}{(2M^2+1)^{\frac{3}{2}}}.$$

Since $\lim_{M\to+\infty} v(M) = \sqrt{2}$, $\lim_{M\to+\infty} \dot{v}(M)v(M)z(M)^2 = 0$. In the same way $\lim_{M\to+\infty} \dot{v}(-M)v(-M)z(-M)^2 = 0$. Finally, we get

$$\int_{\mathbb{R}} \dot{X}(s)^2 - \frac{6X(s)^2}{f(s)^4}\, ds = \int_{\mathbb{R}} v(s)^2\dot{z}(s)^2\, ds + 6X(0)^2\int_{\mathbb{R}} \frac{ds}{f(s)^4}.$$

This leads to

$$\forall X \in E_{\lambda_2} \quad \|X\|_E^2 - 2(KX, X)_E \geq 0 \tag{4.19}$$

and the equality holds iff $X = \mu v$ with $\mu \in \mathbb{R}$.

(4.19) implies that $\lambda_2 \leq \frac{1}{2}$; since $v \in \overline{E}$ and $(Kv, v)_E = \frac{1}{2}|v|_E^2$, $\lambda_2 = \frac{1}{2}$ and E_{λ_2} is spanned by v. This completes the proof of $(*)$, and of Lemma 4.1.

PROOF OF LEMMA 2.6. We have $U''\big(x_n(t)\big)W.W = g'\big(h_n(t)\big)\nabla^2 h\big(x_n(t)\big)W.W + g''\big(h_n(t)\big)\Big(\nabla h\big(x_n(t)\big), W\Big)^2$.

$$\begin{aligned}\int_{I_n^\delta} -\epsilon_n U''\big(x_n(t)\big)W.W\, dt = \Big(\int_{I_n^\delta} \epsilon_n g'\big(h_n(t)\big)\, dt\Big)(C_n W, W)&\\ + \int_{I_n^\delta} \epsilon_n g''\big(h_n(t)\big)\Big(\nabla h\big(x_n(t)\big), W\Big)^2 dt&\\ + \int_{I_n^\delta} \epsilon_n g'\big(h_n(t)\big)\Big(\nabla^2 h\big(x_n(t)\big) - C_n\Big)W.W\, dt.&\end{aligned} \tag{4.20}$$

$\nabla^2 h$, defined in $\Omega \cap V$, where V is a closed neighbourhood of $\partial\Omega$, can be continuously extended to $\overline{\Omega} \cap V$, and hence is uniformly continuous; since $C_n = \nabla^2 h\big(x_n(t_n)\big)$, the last term of the sum in (4.20) is bounded by $\int_{I_n^\delta} \epsilon_n \big|g'\big(h_n(t)\big)\big|\eta\big(\|x_n(t) - x_n(t_n)\|\big)|W|^2$, with $\lim_{s\to 0^+} \eta(s) = 0$; from Lemma 2.3 $\int_{I_n^\delta} \epsilon_n \big|g'\big(h_n(t)\big)\big|\, dt \leq C_1$; $\|\dot{x}_n\|_{L^\infty}$ is bounded and $\|x_n(t) - x_n(t_n)\| \leq \|\dot{x}_n\|_\infty |t - t_n|$ hence the last term of the sum in (4.20) can be written as $r(\delta)|W|^2$.

We have $\Big(\nabla h\big(x_n(t)\big), W\Big) = \Big(\nabla h\big(x_n(t)\big) - \nabla h\big(x_n(t_n)\big), W\Big)$ (since $W \in F_n$). Thus we can write $\Big|\Big(\nabla h\big(x_n(t)\big), W\Big)\Big| \leq K_{11}|t - t_n|\,|W|$ (K_{11} being a constant). Hence

$$\Big|\int_{I_n^\delta} \epsilon_n g''\big(h_n(t)\big)\Big(\nabla h\big(x_n(t)\big), W\Big)^2 dt\Big| \leq K_{11}^2|W|^2\int_{I_n^\delta} \epsilon_n g''\big(h_n(t)\big)|t - t_n|^2\, dt.$$

By Lemma 2.4, this term can be written as $r(\delta, n)|W|^2$. Finally, by Lemma 2.3 we can conclude

$$\int -\epsilon_n U''\big(x_n(t)\big)W.W = (C_n W, W)\cos\theta\, 2\sqrt{2E} + \big(r(\delta, n) + s(\delta, n)\big)|W|^2.$$

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Let us now prove the second point of the lemma; we have

$$\int_{I_n^\delta} -\epsilon_n U''\big(x_n(t)\big)n_n.n_n\,dt = \int_{I_n^\delta} -\epsilon_n g''\big(h_n(t)\big)\Big(\nabla h\big(x_n(t)\big), n_n\Big)^2 dt + \int_{I_n^\delta} -\epsilon_n g'\big(h_n(t)\big)\nabla^2 h\big(x_n(t)\big)n_n.n_n\,dt \tag{4.21}$$

By Lemma 2.3, the second term of the sum (4.21) is bounded by a constant. Since $n_n = \nabla h\big(x_n(t_n)\big)$ and $|\dot{x}_n|_\infty$ is bounded, the following inequality holds: $\Big(\nabla h\big(x_n(t)\big), n_n\Big)^2 \geq 1 - K_{12}|t-t_n| \geq 1/2$ when $|t-t_n| \leq 1/(2K_{12})$. Setting $u_n^\delta = \frac{1}{2}\int_{I_n^\delta} \epsilon_n g''\big(h_n(t)\big)\,dt$ and using (2.1) we get

$$\int_{I_n^\delta} -\epsilon_n g''\big(h_n(t)\big)\Big(\nabla h\big(x_n(t)\big), n_n\Big)^2 dt \leq -u_n^\delta + K_{13}.$$

Hence we have: $\int_{I_n^\delta} -\epsilon_n U''\big(x_n(t)\big)n_n.n_n\,dt \leq -u_n^\delta + C_3$, where C_3 is a constant (by Lemma 2.4, $\lim_{n\to+\infty} u_n^\delta = +\infty$).

$$\int_{I_n^\delta} -\epsilon_n U''\big(x_n(t)\big)W.n_n\,dt = \int_{I_n^\delta} -\epsilon_n g'\big(h_n(t)\big)\nabla^2 h\big(x_n(t)\big)n_n.W\,dt + \int_{I_n^\delta} -\epsilon_n g''\big(h_n(t)\big)\Big(\nabla h\big(x_n(t)\big), W\Big)\Big(\nabla h\big(x_n(t)\big), n_n\Big)\,dt. \tag{4.22}$$

By Lemma 2.3, the first term of the sum in (4.22) is bounded by $K_{14}|W|$ (K_{14} being a constant).

We have already seen that, since $W \in F_n$, $\Big|\Big(\nabla h\big(x_n(t)\big), W\Big)\Big| \leq K_{11}|t-t_n|\,|W|$. Hence

$$\begin{aligned}
&\Big|\int_{I_n^\delta} -\epsilon_n U''\big(x_n(t)\big)W.n_n\,dt\Big| \\
&\quad\leq K_{14}|W| + K_{11}\int_{I_n^\delta} \epsilon_n g''\big(h_n(t)\big)|t-t_n|\,dt|W| \\
&\quad\leq |W|\Big(K_{14} + K_{11}\Big(\int_{I_n^\delta} \epsilon_n g''\big(h_n(t)\big)(t-t_n)^2\,dt\Big)^{\frac{1}{2}}\Big(\int_{I_n^\delta} \epsilon_n g''\big(h_n(t)\big)\,dt\Big)^{\frac{1}{2}}\Big) \\
&\quad\leq |W|\Big(C_3 + r(\delta, n)\sqrt{u_n^\delta}\Big),
\end{aligned}$$

by Lemma 2.4 (provided C_3 has been chosen large enough).

PROOF OF LEMMA 2.7. (i) $\mu(t_n) = 0$ hence $|\mu(t)| \leq \sqrt{|t-t_n|}\Big(\int_{I_n^\delta} \dot{\mu}(t)^2\,dt\Big)^{\frac{1}{2}} \leq \sqrt{|t-t_n|}\,|\mu|_1$. Hence, by Lemma 2.3,

$$\Big|\int_{I_n^\delta} \epsilon_n g'\big(h_n(t)\big)\nabla^2 h\big(x_n(t)\big)\mu(t).\mu(t)\,dt\Big| \leq \delta C_1(\sup_\Omega |\nabla^2 h|)\,|\mu|_1^2.$$

$\Big(\nabla h\big(x_n(t_n)\big), \mu(t)\Big) = 0$ (since for all $t \in I_n^\delta$ $\mu(t) \in F_n$) and hence

$$\Big|\Big(\nabla h\big(x_n(t)\big), \mu(t)\Big)\Big| \leq (\sup_\Omega |\nabla^2 h|)\|\dot{x}_n\|_\infty|t-t_n|\,|\mu(t)|.$$

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$\Big(\nabla h\big(x_n(t)\big), \mu(t)\Big)^2 \leq K_{15}|t-t_n|^3|\mu|_1^2$. Hence, by Lemma 2.4,

$$\int_{I_n^\delta} \epsilon_n g''\big(h_n(t)\big)\Big(\nabla h\big(x_n(t)\big), \mu(t)\Big)^2\, dt = r(\delta, n)|\mu|_1^2, \tag{4.23}$$

which proves (i).

(ii) Since $\int_{I_n^\delta} g''\big(h_n(t)\big)\lambda(t)\, dt = 0$ and $g''(h) > 0$, λ vanishes somewhere in I_n^δ (because λ is continuous), hence $|\lambda|_{L^\infty(I_n^\delta)} \leq \sqrt{2\delta}\,|\lambda|_1$ (we also have $|\mu|_\infty \leq \sqrt{\delta}\,|\mu|_1$). Hence by Lemma 2.3, $\left|\int_{I_n^\delta} \epsilon_n g'\big(h_n(t)\big)\nabla^2 h\big(x_n(t)\big)\mu(t).n_n\lambda(t)\, dt\right|$ can be written as $r(\delta)|\mu|_1\,|\lambda|_1$; on the other hand

$$\begin{aligned}&\int_{I_n^\delta} \epsilon_n g''\big(h_n(t)\big)\left|\Big(\nabla h\big(x_n(t)\big), \mu(t)\Big)\right|\left|\Big(\nabla h\big(x_n(t)\big), n_n\Big)\right||\lambda(t)|\, dt\\ &\quad\leq \left(\int_{I_n^\delta} \epsilon_n g''\big(h_n(t)\big)\Big(\nabla h\big(x_n(t)\big), \mu(t)\Big)^2 dt\right)^{1/2}\left(\int_{I_n^\delta} \epsilon_n g''\big(h_n(t)\big)\lambda(t)^2\, dt\right)^{1/2}.\end{aligned}$$

From (4.23) and Lemma 2.5, we get estimate (ii) (we must add $s(\delta, n)$ in the estimate to take into account the fact that the inequality in Lemma 2.5 holds when $n \geq n(\delta)$).

(iii) follows immediately from Lemma 2.3 and the inequality $|\lambda|_\infty^2 \leq 2\delta|\lambda|_1^2$.

(iv) We have: $|\lambda|_\infty \leq \sqrt{2\delta}|\lambda|_1$ and $|\mu|_\infty \leq \sqrt{\delta}|\mu|_1$, hence

$$|l|_\infty \leq \sqrt{2\delta}\,|l|_1. \tag{4.24}$$

$\int_{I_n^\delta} -\epsilon_n U''\big(x_n(t)\big)W.l(t)\, dt = A + B$, with $A = \int_{I_n^\delta} -\epsilon_n g'\big(h_n(t)\big)\nabla^2 h\big(x_n(t)\big)W.l(t)\, dt$ and $B = \int_{I_n^\delta} -\epsilon_n g''\big(h_n(t)\big)\Big(\nabla h\big(x_n(t)\big), W\Big)\Big(\nabla h\big(x_n(t)\big), l(t)\Big)\, dt$.

It follows from (4.24) and Lemma 2.3 that $A = r(\delta)|W|\,|l|_1$. Moreover, we previously saw that $\left|\Big(\nabla h\big(x_n(t)\big), W\Big)\right| \leq K_{11}|t-t_n|\,|W|$. Hence $|B| \leq K_{11}\int_{I_n^\delta} \epsilon_n g''\big(h_n(t)\big)|t-t_n|\left|\Big(\nabla h\big(x_n(t)\big), l(t)\Big)\right| dt|W|$. In addition

$$\left|\Big(\nabla h\big(x_n(t)\big), l(t)\Big)\right| \leq \left|\Big(\nabla h\big(x_n(t)\big), \mu(t)\Big)\right| + |\lambda(t)| \leq \sqrt{K_{15}}\,|t-t_n|^{\frac{3}{2}}|\mu|_1 + |\lambda(t)|.$$

Therefore, by Lemma 2.4, $|B| \leq r(\delta, n)|W|\,|\mu|_1 + D|W|$, with $D = K_{11}\int_{I_n^\delta} \epsilon_n g''\big(h_n(t)\big)|t-t_n|\,|\lambda(t)|\, dt$.

As in the proof of (ii), using the Cauchy-Schwarz inequality and Lemma 2.5, we get $D = r(\delta, n) + s(\delta, n)|\lambda|_1$. As a conclusion $|B| = \big(r(\delta, n) + s(\delta, n)\big)|W|\,|l|_1$, which proves (iv).

5. **Proof of Theorem 2.** We shall assume that $0 \in \Omega$ and that $U \in C^2(\Omega, \mathbb{R})$ satisfies $U \geq 0$, $U \equiv 0$ in a neighbourhood of 0 and $U = 1/h^2$ near $\partial\Omega$. Let $\epsilon \in (0, 1)$ be fixed.

LEMMA 5.1. *There is $\alpha > 0$ and for all $k \in \mathbb{N}$ there is β_k independent of ϵ such that J_ϵ has at least one critical point x_ϵ^k of Morse index $i_\epsilon(x_\epsilon^k)$ and nullity $m_\epsilon(x_\epsilon^k)$ satisfying $N + 1 + 2k - \big(m_\epsilon(x_\epsilon^k) - 1\big) \leq i_\epsilon(x_\epsilon^k) \leq N + 1 + 2k$.*

We first prove that Lemma 5.1 implies Theorem 2. Let $k \in [0, q] \cap \mathbb{N}$. Let x^k_ϵ be the critical point given by Lemma 5.1. $J_\epsilon(x^k_\epsilon)$ is bounded hence, by a result in [3], there is a sequence ϵ_n converging to 0 such that $x^k_{\epsilon_n}$ converges in $\overline{\Lambda}$ to a bounce trajectory x^k with at most $N+1+2k \leq N+1+2q$ bounce points. Hence x^k is not a grazing trajectory, and we can apply Theorem 1. For n large enough $i_{\epsilon_n}(x^k_{\epsilon_n}) + \big(m_{\epsilon_n}(x^k_{\epsilon_n}) - 1\big) \leq \bar{i}(x^k) + \overline{m}(x^k) + p(x^k)$ and $i_{\epsilon_n}(x^k_{\epsilon_n}) \geq \bar{i}(x^k) + p(x^k)$, where $p(x^k)$ is the number of bounce instants of x^k. Hence $\bar{i}(x^k) + p(x^k) \leq N+1+2k \leq \bar{i}(x^k) + \overline{m}(x^k) + p(x^k)$ and the proof of Theorem 2 is over.

Before proving Lemma 5.1 we enumerate some useful properties of J_ϵ. We shall use the notations E for $H^1(S_1; \mathbb{R}^N)$ and $\| \, \|$ for the H^1 norm. We have:

(P1) J_ϵ is invariant by the S^1-action defined on E by $\theta.x = x(\theta + .)$. We shall denote by E_0 the set of the fixed points for this action: this is an N-dimensional subspace of E. Let $F = E_0^\perp$.

(P2) There are $\rho > 0$ and $\alpha > 0$ (both independent of ϵ) such that $S = \{x \in F \mid \|x\| = \rho\} \subset \Lambda$ and $\operatorname{Inf}_S J_\epsilon > \alpha$.

We shall denote by $(F_k)_{k\geq 1}$ some sequence of subspaces of F such that $F_k \subset F_{k+1}$, F_k is S^1-invariant and $\dim F_k = 2k$. Let $E_k = F_k + E_0$.

(P3) For all $k \in \mathbb{N}$ there is β_k independent of ϵ such that on $E_{k+1} \cap \Lambda$ $J_\epsilon \leq \beta_k$.

(P4) If $x_n \in \Lambda$ and $x_n \longrightarrow x \in \partial\Lambda$ then $J_\epsilon(x_n) \longrightarrow -\infty$.

(P5) J_ϵ satisfies the Palais-Smale condition on Λ.

(P2), (P4), and (P5) are proved in [3]. Since Λ is bounded in L^∞, Λ is bounded in E_{k+1} because E_{k+1} is finite dimensional. (P3) is now a consequence of $J_\epsilon(x) \leq 1/2\|x\|^2$.

We shall use S^1 equivariant cohomology over rational coefficients. Let $S^\infty \longrightarrow CP^\infty$ denote the universal principal S^1-bundle. If $A \subset E$ and $B \subset A$ are S^1-invariant we set $H^*_{S^1}(A, B) = H^*\big((A \times S^\infty)/S^1, (B \times S^\infty)/S^1\big)$.

From now we shall abbreviate $J_\epsilon = J$. By Sard's lemma we can assume that α and β_k given by (P2) and (P3) are not critical values of J. Let $J^c = \{x \in \Lambda | J(x) \leq c\}$. We have:

LEMMA 5.2. $H^{N+1+2k}_{S^1}(J^{\beta_k}, J^\alpha) \neq 0$.

PROOF. We shall use the following facts which are proved in [10] (S is defined in (P2)):

- The projection $p\colon (S \times S^\infty)/S^1 \longrightarrow CP^\infty$ induces an isomorphism $p^*\colon H^*(CP^\infty) \longrightarrow H^*_{S^1}(S)$, where $H^*(CP^\infty)$ is the polynomial algebra over $\mathbb{Q}$ generated by ω of degree two. We set $\overline{\omega} = p^*(\omega)$.
- Let $S_{k+1} = F_{k+1} \cap S$ and $i_k\colon S_{k+1} \longrightarrow S$ denote the inclusion map. Then $i_k^*(\overline{\omega}^k) \neq 0$ in $H^{2k}_{S^1}(S_{k+1})$.

Following [3] we set $\Delta^c = J^c \cup (E \backslash \Lambda)$ (of course Δ^c is S^1-invariant). By (P4) $\overline{E\backslash\Lambda} \subset \operatorname{int}(\Delta^c)$ hence, by the excision property, $H^{N+1+2k}_{S^1}(J^{\beta_k}, J^\alpha) = H^{N+1+2k}_{S^1}(\Delta^{\beta_k}, \Delta^\alpha)$.

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Let R be large enough so that $\forall x \in \Lambda \cap E_{k+1}\ \|x\| < R$. Let $D = \{x \in E \mid \|x\| = R$; or $(x \in E_0$ and $\|x\| \leq R)\}$ and $D_{k+1} = D \cap E_{k+1}$.

Let $j_1^*: H^*_{S^1}(D, E_0 \cap D) \longrightarrow H^*_{S^1}(D_{k+1}, E_0 \cap D)$ and $j_2^*: H^*_{S^1}(E\backslash S, E_0) \longrightarrow H^*_{S^1}(D, E_0 \cap D)$ be induced by the inclusions $D_{k+1} \subset D \subset E\backslash S$. We shall prove

$$\exists \gamma \in H^{N+2k}_{S^1}(E\backslash S, E_0) \quad j_1^* \circ j_2^*(\gamma) \neq 0. \tag{5.1}$$

First it is easy to define a continuous map $H: [0,1] \times (E\backslash S) \longrightarrow E\backslash S$ with the following properties: (i) $H(0,.) = \mathrm{Id}$; (ii) $H([0,1] \times E_0) \subset E_0$; (iii) $H(t,.)$ is S^1-equivariant for all t; (iv) $H(t,.)_{|D} = \mathrm{Id}_D$; (v) $H(1, E\backslash S) \subset D$.

Hence j_2^* is an isomorphism. Now let $\delta > 0$ be small. Let $G = \{x_0 + y \in (E_0 + F) \cap D \mid \|y\| = \delta\}$ and $\tilde{D} = \{x_0 + y \in (E_0 + F) \cap D \mid \|y\| \geq \delta\}$.We denote $E_{k+1} \cap \tilde{D}$ by $\tilde{D}_{k+1}$. Using the excision property we can easily check that there is a commutative diagram

$$\begin{array}{ccc} H^{N+2k}_{S^1}(D, E_0 \cap D) & \xrightarrow{j_1^*} & H^{N+2k}_{S^1}(D_{k+1}, E_0 \cap D) \\ {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle b} \\ H^{N+2k}_{S^1}(\tilde{D}, G) & \xrightarrow{j_1^*} & H^{N+2k}_{S^1}(\tilde{D}_k, G \cap D_k) \end{array} \tag{5.2}$$

where a and b are isomorphisms.

Let $B_N = \{x \in E_0 \mid \|x_0\|^2 \leq R^2 - \delta^2\}$ and $g: \tilde{D} \longrightarrow B_N \times S$ be defined by $g(x_0 + y) = (x_0, \rho y / \|y\|)$. It is clear that g is a S^1-equivariant homeomorphism (the S^1-action being defined on $B_N \times S$ by $\theta.(x_0, y) = (x_0, \theta.y)$). Moreover $g(G) = \partial B_N \times S$ and $g(\tilde{D}_{k+1}) = B_N \times S_{k+1}$. Hence we have the following commutative diagram:

(5.3)

$$\begin{array}{ccccc} H^N(B_N, \partial B_N) \otimes H^{2k}_{S^1}(S) & \simeq & H^{N+2k}_{S^1}(B_N \times S, \partial B_N \times S) & \xrightarrow{g^*} & H^{N+2k}_{S^1}(\tilde{D}, G) \\ {\scriptstyle \mathrm{Id} \otimes i^*_{k+1}}\downarrow & & \downarrow & & \downarrow{\scriptstyle j_1^*} \\ H^N(B_N, \partial B_N) \otimes H^{2k}_{S^1}(S_{k+1}) & \simeq & H^{N+2k}_{S^1}(B_N \times S_{k+1}, \partial B_N \times S_{k+1}) & \xrightarrow{g^*} & H^{N+2k}_{S^1}(\tilde{D}_{k+1}, G \cap E_{k+1}) \end{array}$$

where g^* is an isomorphism. Let σ_N generate $H^N(B_N, \partial B_N)$. $(\mathrm{Id} \otimes i_k^*)(\sigma_N \otimes \overline{\omega}^k) = \sigma_N \otimes i_k^*(\overline{\omega}^k) \neq 0$. Hence combining (5.2) and (5.3), gives the existence of $\beta \in H^{N+2k}_{S^1}(D, E_0 \cap D)$ such that $j_1^*(\beta) \neq 0$. Since j_2^* is an isomorphism we derive (5.1). Now R was chosen such that $D_{k+1} \subset \Delta^\alpha$. Moreover $\Delta^\alpha \subset E\backslash S$. Let

$$H^{N+2k}_{S^1}(E\backslash S, E_0) \xrightarrow{r^*} H^{N+2k}_{S^1}(\Delta^\alpha, E_0) \xrightarrow{s^*} H^{N+2k}_{S^1}(D, E_0 \cap D)$$

be induced by these inclusions. Set $\overline{\gamma} = r^*(\gamma)$. Since $s^* \circ r^* = j_1^* \circ j_2^*$, we must have $s^*(\overline{\gamma}) \neq 0$.

We now consider the exact sequence

$$H^{N+2k}_{S^1}(\Delta^{\beta_k}, E_0) \xrightarrow{i^*} H^{N+2k}_{S^1}(\Delta^{\alpha}, E_0) \xrightarrow{d} H^{N+2k+1}_{S^1}(\Delta^{\beta_k}, \Delta^{\alpha})$$

and we prove that $d(\overline{\gamma}) \neq 0$, which obviously implies Lemma 5.2. Arguing by contradiction we assume that $\overline{\gamma} \in \operatorname{Ker} d = \operatorname{Im} i^*$. Then we can write $\overline{\gamma} = i^*(\eta)$ with $\eta \in H^{N+2k}_{S^1}(\Delta^{\beta_k}, E_0)$. We have $(i \circ s)^*(\eta) = s^*(\overline{\gamma}) \neq 0$, where $i \circ s$ is the inclusion map $(D_{k+1}, E_0 \cap D) \subset (\Delta^{\beta_k}, E_0)$. Now we have $(D_{k+1}, E_0 \cap D) \subset (E_{k+1}, E_0) \subset (\Delta^{\beta_k}, E_0)$, with $H^*_{S^1}(E_{k+1}, E_0) = 0$. Hence it is clear that $(i \circ s)^* = 0$, a contradiction. This completes the proof of Lemma 5.2.

We now explain why Lemma 5.2 implies Lemma 5.1. We have assumed that α and β_k are not critical values of J. Let K denote the set of critical points of J in $J^{\beta_k} \backslash J^{\alpha}$. Since J satisfies (PS), K is compact; moreover $\sup_K J < \beta_k$ and $\inf_K J > \alpha$. We could also easily check that $J''(x)$ is Fredholm for all $x \in J^{\beta_k} \backslash J^{\alpha}$. Using the Marino-Prodi perturbation method ([11]) and a result stated in [12] we derive that there are $\delta_n > 0 \longrightarrow 0$ and a sequence (g_n) of S^1-equivariant and C^2 functionals defined on Λ such that:

(i) $g_n(x) = J(x)$ outside $K^{\delta_n} = \{x \in \Lambda \mid d(x, K) \leq \delta_n\}$;

(ii) $|g_n - J|_{C^2} \longrightarrow 0$;

(iii) the critical S^1-orbits of g_n in $J^{\beta_k} \backslash J^{\alpha}$ are non-degenerate;

(iv) g_n satisfies (PS);

(v) On K^{δ_n} $\alpha < g_n < \beta_k$ (hence $\{x \in \Lambda \mid g_n(x) \leq c\} = J^c$ for $c = \alpha$ or $c = \beta_k$).

Since $H^{N+2k+1}_{S^1}(\{g \leq \beta_k\}, \{g \leq \alpha\}) = H^{N+2k+1}_{S^1}(J^{\beta_k}, J^{\alpha}) \neq 0$, by S^1-equivariant Morse theory (see for example [9] or [12] for more details), g_n has at least one non-degenerate critical S^1-orbit $\big(x^n(\theta + .)\big)$ in $J^{\beta_k} \backslash J^{\alpha}$ of Morse index $N + 1 + 2k$. From (i) and (ii), since K is compact there is a subsequence of x^n which converges to $x \in J^{\beta_k} \backslash J^{\alpha}$, critical point of J of Morse index satisfying the desired estimates.

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