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REPRESENTATIONS OF QUADRATIC FORMS AND THEIR APPLICATION TO SELBERG'S ZETA FUNCTIONS

Dedicated to the memory of Taira Honda

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Let M and L be quadratic lattices over the maximal order of an algebraic number field. In case of dealing with representations of M by L, they sometimes assume certain indefiniteness and the condition rank L-rank $M \geq 3$. In this case, representation problems are reduced not to global but to local problems by virtue of the strong approximation theorem for rotations and of the fact that for regular quadratic spaces U, V over a non-archimedean local field there is an isometry from U to V if $\dim V - \dim U \geq 3$. On the contrary, global properties seem to be strongly concerned if we omit one of those two assumptions. As an example we prove in §1 that there is a sublattice of codimension 1 which characterizes L in a certain sense. In §2 we prove as its application that certain Selberg's zeta functions are linearly independent.

We denote by $\boldsymbol{Q}, \boldsymbol{Z}, \boldsymbol{Q}_p$ and $\boldsymbol{Z}_p$ the rational number field, the ring of rational integers, the p-adic completion of $\boldsymbol{Q}$, and the p-adic completion of $\boldsymbol{Z}$. We mean by a quadratic lattice L over $\boldsymbol{Z}$ (resp. $\boldsymbol{Z}_p$) a $\boldsymbol{Z}$ (resp. $\boldsymbol{Z}_p$)-lattice in a regular quadratic space U over $\boldsymbol{Q}$ (resp. $\boldsymbol{Q}_p$), and by definition $\operatorname{rank} L = \dim U$. For a quadratic lattice L over $\boldsymbol{Z}$ (or $\boldsymbol{Z}_p$) we denote by $Q(x)$ and $B(x, y)$ the quadratic form and the bilinear form associated with L ($2B(x, y) = Q(x + y) - Q(x) - Q(y)$), and by dL the determinant of $(B(e_i, e_j))$ where $\{e_i\}$ is a basis of L over $\boldsymbol{Z}$ (or $\boldsymbol{Z}_p$). dL is uniquely determined for a quadratic lattice L over $\boldsymbol{Z}$, and for a quadratic lattice L over $\boldsymbol{Z}_p$, dL is unique up to the squares of units in $\boldsymbol{Z}_p$. For two ordered sets $(a_1, a_2, \cdots, a_n)$, $(b_1, b_2, \cdots, b_n)$, we define the order $(a_1, a_2, \cdots, a_n) \leq (b_1, b_2, \cdots, b_n)$ by either $a_i = b_i$ for $i < k$ and $a_k < b_k$ for some $k \leq n$

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or $a_i = b_i$ for any i.

Let L be a quadratic lattice over Z_p; then L has a Jordan splitting $L = L_1 \perp L_2 \perp \cdots \perp L_k$, where L_i is a p^{a_i}-modular lattice and $a_1 < a_2 < \cdots < a_k$. We denote by $t_p(L)$ the ordered set $(\underbrace{a_1, \cdots, a_1}_{\text{rank } L_1}, \cdots, \underbrace{a_k, \cdots, a_k}_{\text{rank } L_k})$. For simplicity we denote $t_p(Z_p L)$ by $t_p(L)$ for a quadratic lattice L over Z.

§1. LEMMA. *Let L be a Z_p-lattice in a regular quadratic space U over Q_p; then L has a Z_p-submodule*) M satisfying the following conditions* 1), 2):

1) $dM \neq 0$, rank M = rank $L - 1$, *and M is a direct summand of L as a module.*

2) *Let L' be a Z_p-lattice in U containing M; then $L' = L$ if $dL' = dL$, and $t_p(L') \geq t_p(L)$.*

Proof. Firstly we assume that L is modular; then we may assume that L is unimodular without loss of generality by scaling. Let L' be a lattice in question in 2); then $dL' = dL$, $t_p(L') \geq t_p(L)$ imply that L' is also unimodular. Suppose that L has an orthogonal base, that is, $L = \perp_{i=1}^{n} Z_p v_i$. We put $M = \perp_{i=1}^{n-1} Z_p v_i$; then M satisfies 1), and M is unimodular. Hence M splits L' and $L' = M \perp aZ_p v_n$ for $a \in Q_p$. Since L' is unimodular, a is a unit. This means $L' = L$. If L does not have an orthogonal base, then $p = 2$ and $L = \perp_{i=1}^{k} Z_2[u_i, v_i]$, where $Z_2[u_i, v_i] \cong \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ for $i < k$, and $Z_2[u_k, v_k] \cong \begin{pmatrix} 2c & 1 \\ 1 & 2c \end{pmatrix}$ ($c = 0$ or 1). Put $M = \perp_{i=1}^{k-1} Z_2[u_i, v_i] \perp Z_2[u_k + v_k]$; then $Q(u_k + v_k) = 4c + 2 \neq 0$ implies $dM \neq 0$. The rest of 1) is obvious. Since a unimodular lattice $\perp_{i=1}^{k-1} Z_2[u_i, v_i]$ splits L, L' and M, we may assume $k = 1$ to prove 2). Now we have $L = Z_2[u, v]$, $M = Z_2[u + v]$, where $Q(u) = Q(v) = 2c$, $B(u, v) = 1$, and L' is a unimodular lattice containing $u + v$. Since $Q(u + v) = 2(2c + 1)$, $u + v$ is maximal in L'. Hence $L' = Z_2[u + v, au + bv]$ for some a, b in Q_2. From the assumption that L' is unimodular follows that $B(u + v, au + bv) = (a + b)(2c + 1)$ is a unit and $Q(au + bv) = 2c(a^2 + b^2) + 2ab$ is in Z_2. Put $a + b = x$; then x is a unit. $Q(au + bv) = 2(2c - 1)a^2 - 2(2c - 1)ax$

*) We mean a finitely generated module by a module for brevity in this paper.

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$+2cx^2 \in \boldsymbol{Z}_2$ implies $a \in \boldsymbol{Z}_2$. Hence we get $a, b \in \boldsymbol{Z}_2$, and $L' = L$. Coming back to general cases, let L be a quadratic lattice and $L = \perp_{i=1}^{k} L_i$, where L_i is p^{a_i}-modular and $a_1 < \cdots < a_k$. Denote by M_k a submodule of L_k which satisfies 1), 2) in case of $L = L_k$ in Lemma and put $M = \perp_{i=1}^{k-1} L_i \perp M_k$. Obviously M satisfies the condition 1). Let L' be a lattice in question in 2); then from the assumptions $t_p(L') \geq t_p(L)$, $L' \supset M$ follows that L_1 splits L, M and L' (82:15 in [2]). Hence we have only to prove the Lemma for the orthogonal complements of L_1 in L, L' and M. By induction it suffices to prove it in case of $k = 1$. This was proved firstly.

We call M a characteristic submodule of L.

THEOREM. *Let L be a $\boldsymbol{Z}$-lattice in a regular quadratic space U over $\boldsymbol{Q}$; then L has a $\boldsymbol{Z}$-submodule M satisfying the following conditions* 1), 2):

1) *$dM \neq 0$, $\operatorname{rank} M = \operatorname{rank} L - 1$, and M is a direct summand of L as a module.*

2) *Let L' be a quadratic lattice over $\boldsymbol{Z}$ in some regular quadratic space U' over $\boldsymbol{Q}$ satisfying $dL' = dL$, $\operatorname{rank} L' = \operatorname{rank} L$, $t_p(L') \geq t_p(L)$ for any prime p. If there is an isometry φ from M to L' such that $\varphi(M)$ is a direct summand of L' as a module, then L' is isometric to L.*

Proof. Let $\operatorname{rank} L = 2$; by scaling we may assume that a matrix $\begin{pmatrix} 2a' & b' \\ b' & 2c' \end{pmatrix}$ corresponding to L satisfies that a', b', c' are integers such that $(a', b', c') = 1$, $a' > 0$. From the classical theory we know that there is an element u in L such that $Q(u) = 2p$, where p is a prime with $(p, 2dL) = 1$. Hence L has a matrix $\begin{pmatrix} 2p & b \\ b & 2c \end{pmatrix}$ where $0 < b < p$. Let e be an integer such that $e^2 \equiv -dL \bmod p$ and $0 < e < p$. From $dL = 4pc - b^2$ follows $b = e$ or $p - e$. If there is an integer x such that $dL = 4px - e^2$, then there is no integer y satisfying $dL = 4py - (p-e)^2$. Therefore the condition $0 < b < p$ determines b uniquely. Now we put $M = \boldsymbol{Z}[u]$. If L' satisfies the condition 2), then L' has a matrix $\begin{pmatrix} 2p & b'' \\ b'' & 2c'' \end{pmatrix}$ $(b'', c'' \in \boldsymbol{Q})$, since L' contains a primitive vector u' with $Q(u') = Q(u) = 2p$. $t_q(L') \geq t_q(L)$ implies $b'', c'' \in \boldsymbol{Z}_q$ for any prime q. Hence b'', c'' are integers, and we may assume $0 < b'' < p$. As above we have $b'' = b$. Hence L' is isometric to L. Let rank L be larger than 2. By scaling

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we may assume that the scale of L is in $\boldsymbol{Z}$, and L is not negative definite. For brevity we denote $\boldsymbol{Z}_pN$ by N_p for a quadratic lattice N over $\boldsymbol{Z}$. For a prime p dividing $2dL$ we can take an element v_p in L_p such that v_p is in the orthogonal complement of a characteristic submodule of L_p. Put $Q(v_p) = u_p p^{r_p}$, where Q is the quadratic form associated with L and u_p is a unit of $\boldsymbol{Z}_p$, and $r_p \geq 0$. We take a prime q such that $(q, 2dL) = 1$ and $q \prod_{p|2dL} p^{r_p} \equiv Q(v_l) \bmod l^t$ for any prime l dividing $2dL$ and a sufficiently large fixed integer t. Put $a = q \prod_{p|2dL} p^{r_p}$; then $Q(L_p)$ contains a for a prime $p|2dL$, since $a^{-1}Q(v_p)$ is a square of a unit of $\boldsymbol{Z}_p$. If a prime p does not divide $2dL$, then L_p is unimodular and $Q(L_p) = \boldsymbol{Z}_p$ (92: 1b in [2]). Therefore from the non-negative-definiteness of L follows that $U = \boldsymbol{Q}L$ represents a by virtue of the Minkowski-Hasse theorem. Since $a^{-1}Q(v_p)$ is a square of unit of $\boldsymbol{Z}_p$ for p dividing $2dL$, we may assume that $Q(v_p) = a$ and the orthogonal complement of v_p in L_p is a characteristic submodule of L_p. We can take an element v in U such that $Q(v) = a$, and v and v_p are sufficiently near if $p|2dL$. Put $S = \{p\,;\, v \notin L_p$, and $\sigma_p v \in L_p$ for a rotation σ_p with $\operatorname{ord}_p \theta_p(\sigma_p) \equiv 1 \bmod 2\}$, where θ_p stands for the spinor norm; then $p \nmid 2dL$ if $p \in S$. We take a prime $h \neq 2$ such that $h \equiv \prod_{p \in S} p \bmod (2dL)^t$ and $\left(\dfrac{-adL}{h}\right) = 1$. Put $u = \sigma(v)$, where σ is a rotation of U whose spinor norm is $h \prod_{p \in S} p$ (101: 8 in [2]). For a prime p with $p \nmid 2hqdL$ there is a rotation σ_p such that $\sigma_p \sigma^{-1} u = \sigma_p v \in L_p$ and $\operatorname{ord}_p \theta_p(\sigma_p) \equiv 0$ or $1 \bmod 2$ according to $p \notin S$ or $p \in S$ respectively, and then $\operatorname{ord}_p \theta_p(\sigma_p \sigma^{-1}) \equiv 0 \bmod 2$. Hence there is a rotation η_p such that $\theta_p(\eta_p) = 1$, $\eta_p(u) \in L_p$ by virtue of 92: 5 in [2] for $p \nmid 2hqdL$. If $p = h$, then there is a rotation η_p such that $\eta_p(u) \in L_p$ since $Q(u) = a$ is a unit of $\boldsymbol{Z}_p$ and L_p is unimodular. Since $\eta_p(u)$ splits L_p and its orthogonal complement N_p in L_p is a unimodular lattice with $\left(\dfrac{-dN_p}{h}\right) = \left(\dfrac{-adL}{h}\right) = 1$, N_p is isotropic. Hence we may assume that the spinor norm of η_p is 1 by virtue of 55: 2a in [2]. For $p|2dL$ put $\eta_p = \sigma^{-1}$; then $\eta_p(u)$ is sufficiently near to v_p and $\theta_p(\eta_p) = 1$. By the strong approximation theorem regarding the set $\{p\,;\, \text{prime} \neq q\}$ as an indefinite set for U, there is a rotation η such that η and η_p are sufficiently near at both p dividing $2dL$ and p satisfying $u \notin L_p$ for $p \nmid 2qdL$, and $\eta L_p =$

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L_p otherwise. Put $\eta(u) = w$; then $Q(w) = a$ and $w \in L_p$ if $p \neq q$. Since η and σ^{-1} are sufficiently near for $p \mid 2dL$ and $w = \eta\sigma(v)$ is sufficiently near to v_p for $p \mid 2dL$, hence the orthogonal complement of w in L_p is a characteristic submodule in L_p. Moreover for $p \nmid 2qdL$ L_p is unimodular and $Q(w)$ is a unit of $\boldsymbol{Z}_p$. This implies that the orthogonal complement of w is also a characteristic submodule in L_p. Put $M = \{x \in L;\, x \perp w\}$. Then a submodule M of L satisfies the condition 1) and $dM = q^r m$, where q is a prime with $q \nmid 2dL$ and a prime $p \mid 2dL$ if $p \mid m$, and $r \geq 0$, and moreover M_p is a characteristic submodule of L_p for $p \neq q$. Let L' be a quadratic lattice in question in 2). Since L' represents M and $dL' = dL$, $U' = \boldsymbol{Q}L'$ is isometric to $U = \boldsymbol{Q}L$. Hence we may assume that L' is in U and $L' \supset M$. Since M_p is a characteristic submodule of L_p for $p \neq q$, $L'_p = L_p$ for $p \neq q$. Take a basis $\{w_i\}$ of L_q such that $M_q = \mathop{\perp}\limits_{i=1}^{n-1} \boldsymbol{Z}_q w_i$ ($n = \operatorname{rank} L$) and $\operatorname{ord}_q Q(w_1) \leq \cdots \leq \operatorname{ord}_q Q(w_{n-1})$; then a matrix corresponding to L_q is

$$\begin{pmatrix} a_1 q^{r_1} & & & b_1 \\ & \ddots & 0 & \vdots \\ 0 & & a_{n-1} q^{r_{n-1}} & b_{n-1} \\ b_1 & \cdots & b_{n-1} & b_n \end{pmatrix},$$

where a_i is a unit of $\boldsymbol{Z}_q$ and $0 \leq r_1 \leq \cdots \leq r_{n-1}$. Since the determinant of this matrix is a unit of $\boldsymbol{Z}_q$, we see easily $r_1 = \cdots = r_{n-2} = 0$. By taking $w_n - \sum_{i=1}^{n-2} a_i^{-1} b_i w_i$ instead of w_n, we may assume that $b_1 = \cdots = b_{n-2} = 0$ in the matrix. Then $N_q = \boldsymbol{Z}_q[w_{n-1}, w_n]$ is unimodular and $-dN_q = b_{n-1}^2 - a_{n-1} b_n q^{r_{n-1}}$. If $r_{n-1} \geq 1$, then b_{n-1} is a unit, and $-dN_q$ is a square of a unit of $\boldsymbol{Z}_q$. If $r_{n-1} = 0$, then M_q is unimodular. Hence L_q has a basis $z_1, \cdots, z_n$ such that $z_i \perp \boldsymbol{Z}_q[z_{n-1}, z_n]$ for $i \leq n-2$, $Q(z_{n-1}) = Q(z_n) = 0$, $B(z_{n-1}, z_n) = 1$ and $M_q = \boldsymbol{Z}_q[z_1, \cdots, z_{n-2}, z_{n-1} + u_q q^r z_n]$, where u_q is a unit. Since L'_q is unimodular and contains M_q primitively, we get $L'_q = \boldsymbol{Z}_q[z_1, \cdots, z_{n-2}] \perp K_q$, where K_q is unimodular and $z_{n-1} + u_q q^r z_n$ is primitive in K_q. Put $K_q = \boldsymbol{Z}_q[z_{n-1} + u_q q^r z_n, c z_{n-1} + d z_n]$ ($c, d \in \boldsymbol{Q}_q$); then $Q(c z_{n-1} + d z_n) \in \boldsymbol{Z}_q$, $B(z_{n-1} + u_q q^r z_n, c z_{n-1} + d z_n)$ is a unit, if $r > 0$. If $r = 0$, then $c, d \in \boldsymbol{Z}_q$. Hence we have $K_q = \boldsymbol{Z}_q[z_{n-1}, z_n]$ or $\boldsymbol{Z}_q[q^{-r} z_{n-1}, q^r z_n]$. From $w \perp M$ and $(z_{n-1} - u_q q^r z_n) \perp M_q$ follows that two symmetries $\tau_w, \tau_{z_{n-1} - u_q q^r z_n}$ are equal. Therefore we see $\boldsymbol{Z}_q[q^{-r} z_{n-1}, q^r z_n] = \tau_w \boldsymbol{Z}_q[z_{n-1}, z_n]$.

Thus we get $L' = L$ or $\tau_w L$ since $L'_p = \tau_w L'_p = L_p$ for $p \neq q$, and $L'_q = L_q$ or $\tau_w L_q$. This completes our proof.

For brevity we call M a characteristic submodule of L.

Remark. Our proof shows:

Let the scale of L be in $\boldsymbol{Z}$ and $\operatorname{rank} L \geq 3$; if a direct summand M of L satisfies

1) M_p is a characteristic submodule of L_p if $p \mid 2dL$,

2) $dM = q^r m$, where q is a prime with $q \nmid 2dL$, $r \geq 0$ and $p \mid 2dL$ if a prime p divides m,

then M is a characteristic submodule of L.

If we can take $r = 0$ or 1, then $\varphi(M)$ is a direct summand of L' as a module in the assertion 2) if $\varphi(M)$ is a submodule in L'. If rank $L \neq 3$ and L is indefinite, then we can easily show to take $r = 1$. In definite cases analytic methods will be required.

§2. Let S, T be $n \times n$ rational symmetric matrices. We say that S, T are equivalent if and only if there is an element U in $GL(n, \boldsymbol{Z})$ such that $S[U] = T$. For a rational symmetric matrix $S = (s_{ij})$ we define a quadratic lattice $L = \boldsymbol{Z}[e_1, \cdots, e_n]$ by $B(e_i, e_j) = s_{ij}$. L is called the quadratic lattice corresponding to S. Then $dL = |S|$.

LEMMA. *Let S_i be positive definite rational matrices with $|S_i| = d$ and* rank $= n$, *and suppose that they are not equivalent. Put $\theta(Z, S_i) = \sum e^{\pi i \operatorname{tr}(S_i[G]Z)}$, where G runs over $M_{n,n-1}(\boldsymbol{Z})$, and $Z^{(n-1)} = {}^tZ$,* $\operatorname{Im} Z > 0$; *then $\theta(Z, S_i)$ are linearly independent.*

Proof. Obviously we may assume that S_i is integral. Denote by L_i the quadratic lattice corresponding to S_i; then $dL_i = d$. Put $\theta(Z, S_i) = \sum a_i(T) e^{\pi i \operatorname{tr}(TZ)}$. For $|T| \neq 0$, $a_i(T)$ is the number of isometries from the quadratic lattice corresponding to T to L_i. Suppose that $\theta(Z, S_i)$ are linearly dependent and $\sum c_i \theta(Z, S_i) = 0$ with each $c_i \neq 0$. Let $p_1, p_2, \cdots, p_t$ be all primes dividing $2d$, and A_1 be the set of L_i whose $t_{p_1}(L_i)$ is minimal in the set $\{t_{p_1}(L_i)\}$. Inductively we define the set A_{k+1} as follows; A_{k+1} is the set of L_i whose $t_{p_{k+1}}(L_i)$ is minimal in $\{t_{p_{k+1}}(L_i); L_i \in A_k\}$. For L_i in A_t we take a characteristic submodule M_i such that $(M_i)_p$ is a characteristic submodule of $(L_i)_p$ if $p \mid 2d$, and $dM_i = q_i^{r_i} m_i$, where q_i is a prime with $q_i \nmid 2d$, and $p \mid 2d$ if a prime p divides m_i (Proof of

Theorem in § 1). Put $r = \min_{\substack{L_i \in A_t \\ M_i}} r_i$, and for some L_i in A_t and its characteristic submodule M_i we have $dM_i = q_i^r m_i$. If there is an isometry σ from M_i to L_j, then σ is extended to the isometry from $\boldsymbol{Q}L_i$ to $\boldsymbol{Q}L_j$, and $\sigma^{-1}(L_j)_{p_1} \supset (M_i)_{p_1}$. By definition of a characteristic submodule we get $\sigma^{-1}(L_j)_{p_1} = (L_i)_{p_1}$ and L_j is in A_1. Inductively we obtain $\sigma^{-1}(L_j)_p = (L_i)_p$ for $p \mid 2d$ and L_j is in A_t. Suppose that $\sigma(M_i)$ is not a direct summand of L_j; then there is a direct summand N of L_j such that $N \supsetneqq \sigma(M_i)$ and $\operatorname{rank} N = \operatorname{rank} M_i$. From $\sigma^{-1}(L_j)_p = (L_i)_p$ for $p \mid 2d$ follows $(M_i)_p \subset \sigma^{-1}(N)_p \subset (L_i)_p$. Hence we have $(M_i)_p = \sigma^{-1}(N)_p$ for $p \mid 2d$ since $(M_i)_p$ is a direct summand of $(L_i)_p$, and $dN = q_i^{r'} m_i$, $r' \leq r - 2 < r$. Hence N is a characteristic submodule of L_j if $n \geq 3$ (Remark in § 1). This contradicts the minimality of r. In case of $n = 2$, from the classical theory we can take $r_i = r = 1$. Hence $r' < 0$ is a contradiction. Therefore $\sigma(M_i)$ is a direct summand of L_j. Hence L_j is isometric to L_i by virtue of Theorem in § 1. This means that we have $a_i(T) \neq 0$ and $a_j(T) = 0$ if $j \neq i$ for the matrix corresponding to M_i. This contradicts $c_i \neq 0$.

Remark 1. Put $\theta_p(Z, S_i) = \sum e^{\pi i \operatorname{tr}(S_i[G]Z)}$, where G runs over primitive matrices in $M_{n,n-1}(\boldsymbol{Z})$. The proof of Lemma states that $\theta_p(Z, S_i)$ are linearly independent.

Remark 2. Let the class number of even integral positive definite quadratic forms over $\boldsymbol{Z}$ with det $= 1$, rank $= 8k$ be $h(8k)$. Then we have $h(8k)$ linearly independent Siegel modular forms with weight $4k$, degree $8k - 1$ defined by $\theta(Z, S_i)$ as above. The dimension of the space spanned by the corresponding Dirichlet series $\sum_{\{T\}>0} \dfrac{a(T)}{\varepsilon(T)\,|T|^s}$, where $a(T) = \#\{X \in M_{8k,8k-1}(\boldsymbol{Z})\,;\, S_i[X] = T\}$ and T runs over the representatives of equivalence classes of positive definite integral matrices, and $\varepsilon(T) =$ the order of the group of units of T, is equal to the dimension of the space spanned by the Epstein zeta functions of S_i by Theorem 4 in p. 298 in [1] and it ammounts to $[k/3] + 1$, since the space of elliptic modular forms with weight $4k$ is spanned by theta functions, and its dimension is $[k/3] + 1$. Numerically we know $h(8) = 1$, $h(16) = 2$, $h(24) = 24$, $h(32) > 8 \cdot 10^8$.

Let $S = (s_{ij})$ be a positive definite real matrix with rank $= n$. L denotes a $\boldsymbol{Z}$-lattice $\boldsymbol{Z}[e_1, \cdots, e_n]$ which has an inner product defined by $B(e_i, e_j) = s_{ij}$. For a submodule $M = \boldsymbol{Z}[f_1, \cdots, f_m]$ of L we denote $\det(B(f_i, f_j))$ by dM. Denote by $z_1, \cdots, z_{n-1}$ a system of $n - 1$ complex variables and by $s_1, \cdots, s_n$ a system of n complex variables, the two being related by the equations

$$z_k = s_{k+1} - s_k + \tfrac{1}{2}\,, \qquad k = 1, 2, \cdots, n-1\,.$$

Now the Selberg's zeta function is defined by

$$\zeta^*(S\,; s_1, s_2, \cdots, s_n) = \sum (dL_{n-1})^{-z_{n-1}} \cdot (dL_{n-2})^{-z_{n-2}} \cdots (dL_1)^{-z_1}\,,$$

where L_k runs over direct summands of L_{k+1} with $\operatorname{rank} L_k = k$ and $L_n = L$. This is absolutely convergent for $\operatorname{Re} z_k > 1$ $(1 \leq k < n)$ and satisfies certain functional equations (Theorem 1. p. 263 in [1]). Our aim in this section is to prove

THEOREM. *Let S_i be positive definite rational matrices with* rank $= n$, $|S_i| = d$. *If they are not equivalent with each other, then $\zeta^*(S_i\,; s_1, \cdots, s_n)$ are linearly independent as functions of $s_1, \cdots, s_n$. Especially the Selberg's zeta function is a complete analytic class invariant.*

Proof. Theorem is equivalent to the linear independence of $\theta_p(Z, S_i)$ by the Mellin transform in case of $n = 2$ and it is true by virtue of Remark 1. Suppose that Theorem is true for $n-1$ but false for n; then there are positive definite rational matrices S_i with rank $= n$, $|S_i| = d'$ such that

$$\sum a_i \zeta^*(S_i\,; s_1, \cdots, s_n) = 0\,,$$

where S_i are not equivalent with each other. Put

$$u(Y) = u_{1,2,\cdots,n-2}(Y) = |Y|^{\sum_{k=1}^{n-2} kz_k/(n-1)} \sum_U \prod_{k=1}^{n-2} |Y[U]_k|^{-z_k}\,,$$

where Y is positive definite and of rank $= n-1$, U runs over the factor set $GL(n-1, Z)\Big/\left\{\begin{pmatrix} * & & * \\ & \ddots & \\ 0 & & * \end{pmatrix} \in GL(n-1, Z)\right\}$, $Y = \begin{pmatrix} \overbrace{Y_k}^{k} & * \\ * & * \end{pmatrix}$ and $z_k = s_{k+1} - s_k + \frac{1}{2}$. This is a Größen-character in the sense of §10 in [1]. We define a function $R_i(s)$ by

$$R_i(s) = \int_F \theta_p(iY, S_i)\, |Y|^s\, u(Y^{-1}) dv\,,$$

where F is the Minkowski's domain of reduced matrices in the space of all positive definite matrices with rank $= n-1$ and $dv = |Y|^{-n/2} \prod_{s \le t} dy_{st}$. Putting

$$\theta_p(iY, S_i) = \sum_{T>0} a_i(T) e^{-2\pi \operatorname{tr}(TY)} ,$$

$$\begin{aligned} R_i(s) &= \int_F \sum_{T>0} a_i(T) e^{-2\pi \operatorname{tr}(TY)} |Y|^s u(Y^{-1}) dv \\ &= \sum \frac{a_i(T)}{\varepsilon(T)} \int_{Y^{(n-1)}>0} e^{-2\pi \operatorname{tr}(TY)} |Y|^s u(Y^{-1}) dv , \end{aligned}$$

where T runs over representatives of equivalence classes of positive definite rational matrices of rank $= n-1$,

$$= \pi^{(n-1)(n-2)/4} (2\pi)^{(1-n)s} \prod_{k=1}^{n-1} \Gamma(s - c_k) \sum \frac{a_i(T)}{\varepsilon(T)} |T|^{-s} u(T) ,$$

where c_k is a certain complex number (p. 94 in [1]), thus we get

$$R_i(s) = \pi^{(n-1)(n-2)/4} (2\pi)^{(1-n)s} \prod_{k=1}^{n-1} \Gamma(s - c_k) \zeta^*(S_i ; s'_1, \cdots, s'_n) ,$$

where s'_i is defined by

$$z_k = s'_{k+1} - s'_k + \tfrac{1}{2} \; (k < n-1) , \qquad -s + \sum_{k=1}^{n-2} k z_k/(n-1) = s'_n - s'_{n-1} + \tfrac{1}{2} .$$

Hence our assumption implies $\sum a_i R_i(s) = 0$. On the other hand, from Remark 1 follows that $\sum a_i \theta_p(iY, S_i) = \sum a(T) e^{-2\pi \operatorname{tr}(TY)}$ is not zero. This yields that there is a T_0 such that $a(T_0) \neq 0$. Regarding $\sum a_i R_i(s)$ as Dirichlet series with respect to s, we obtain

$$\sum \frac{a(T)}{\varepsilon(T)} u(T) = 0 ,$$

where T runs over representatives of classes with $|T| = |T_0|$. This contradicts our assumption since $|T|^{-\sum_{k=1}^{n-2} k z_k/(n-1)} u(T)$ is by definition the Selberg's zeta function of positive definite matrix $T^{(n-1)}$.

COROLLARY. Let $f(Z) = \sum a(T) e^{2\pi i \operatorname{tr}(TZ)}$ be a Siegel modular form of degree n. If the corresponding Dirichlet series

$$\sum_{\{T\}>0} \frac{a(T) u(T)}{\varepsilon(T) |T|^s}$$

with a Größen-character $u(T)$ as $|T|^{\sum_{k=1}^{n-1} kz_j/n} \zeta^*(T; s_1, s_2, \cdots, s_n)$ is zero as a function of $s, s, \cdots, s_n$, then $a(T) = 0$ for $T > 0$.

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