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Certain classes of univalent functions with negative coefficients II

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Let $P^*(\alpha, \beta)$ denote the class of functions

$$f(z) = z - \sum_{n=2}^{\infty} |a_n| z^n$$

analytic and univalent in $|z| < 1$ for which

$$\left|\frac{f'(z)-1}{f'(z)+(1-2\alpha)}\right| < \beta \quad (|z| < 1) ,$$

where $\alpha \in [0, 1)$, $\beta \in (0, 1]$.

Sharp results concerning coefficients, distortion theorem and radius of convexity for the class $P^*(\alpha, \beta)$ are determined. A comparable theorem for the classes $C^*(\alpha, \beta)$ and $P^*(\alpha, \beta)$ is also obtained. Furthermore, it is shown that the class $P^*(\alpha, \beta)$ is closed under 'arithmetic mean' and 'convex linear combinations'.

1. Introduction

Let f be a function of complex variable z , regular in the unit disc Δ $\{|z| < 1\}$ and normalized by the condition $f(0) = 0$ and $f'(0) = 1$. The Taylor expansion of such a function around the origin is given by

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n .$$

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Further, we assume that f is univalent in Δ. The class S of such univalent functions has been widely studied. Much of the interest in this area has centered around determining the estimates of coefficients and also the estimates for $|f(z)|$ and $|f'(z)|$. Failure to settle Bieberbach's conjecture, namely $|a_n| \le n$, $n = 2, 3, 4, \ldots$, in its generality, led to an attempt on the part of many workers to investigate various subclasses of the class of univalent functions. One such subclass, T, is the class of functions whose non-zero coefficients, from the second on, are negative; that is, an analytic and univalent function f is in T if and only if it can be expressed as

$$f(z) = z - \sum_{n=2}^{\infty} |a_n| z^n .$$

In [2], Schild considered a subclass of T consisting of polynomials having $|z| = 1$ as radius of univalence. For this class, he obtained a necessary and sufficient condition in terms of the coefficients and with the aid of which he derived better results for certain quantities connected with the conformal mapping of univalent functions. Silverman [3] determined coefficient inequalities, distortion, and covering theorems for the subclasses $S^*(\alpha)$ and $C^*(\alpha)$ of T, classes of starlike functions of order α and convex functions of order α respectively.

A function $f \in T$ is in $P^*(\alpha, \beta)$ if and only if

$$\left|\frac{f'(z)-1}{f'(z)+(1-2\alpha)}\right| < \beta \quad (|z| < 1) , \tag{1.1}$$

for α $(0 \le \alpha < 1)$ and β $(0 < \beta \le 1)$. Two subclasses $S^*(\alpha, \beta)$ and $C^*(\alpha, \beta)$ of T, obtained by replacing $f'(z)$ with $\left(\frac{zf'(z)}{f(z)}\right)$ and $\left(1 + \frac{zf''(z)}{f'(z)}\right)$ respectively in (1.1), have been studied by the authors in [1].

In this paper, sharp results concerning coefficients, distortion theorem, and radius of convexity for the class $P^*(\alpha, \beta)$ are determined. We also obtain comparable theorems for the classes $C^*(\alpha, \beta)$ and $P^*(\alpha, \beta)$. In the last section we assert that the class $P^*(\alpha, \beta)$ is closed under 'arithmetic mean' and 'convex linear combinations'.

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2. Coefficients theorem

THEOREM 1. *A function*

$$f(z) = z - \sum_{n=2}^{\infty} |a_n| z^n$$

is in $P^*(\alpha, \beta)$ *if and only if*

$$\sum_{n=2}^{\infty} n(1+\beta)|a_n| \le 2\beta(1-\alpha) .$$

This result is sharp.

Proof. Let $|z| = 1$. Then

$$\begin{aligned} |f'(z)-1| - \beta|f'(z)+(1-2\alpha)| &= \left|- \sum_{n=2}^{\infty} n|a_n|z^{n-1}\right| - \beta\left|2(1-\alpha) - \sum_{n=2}^{\infty} n|a_n|z^{n-1}\right| \\ &\le \sum_{n=2}^{\infty} n(1+\beta)|a_n| - 2\beta(1-\alpha) \\ &\le 0 \text{ , by hypothesis.} \end{aligned}$$

Hence, by the maximum modulus theorem, $f \in P^*(\alpha, \beta)$.

For the converse, assume that

$$\left|\frac{f'(z)-1}{f'(z)+(1-2\alpha)}\right| = \left|\left\{- \sum_{n=2}^{\infty} n|a_n|z^{n-1}\right\} \Big/ \left\{2(1-\alpha) - \sum_{n=2}^{\infty} n|a_n|z^{n-1}\right\}\right| < \beta \quad (|z| < 1) .$$

Since $|\mathrm{Re}(z)| \le |z|$ for all z , we have

$$\mathrm{Re}\left\{\left\{\sum_{n=2}^{\infty} n|a_n|z^{n-1}\right\} \Big/ \left\{2(1-\alpha) - \sum_{n=2}^{\infty} n|a_n|z^{n-1}\right\}\right\} < \beta . \tag{2.1}$$

Choose values of z on the real axis so that $f'(z)$ is real. Upon clearing the denominator in (2.1) and letting $z \to 1$ through real values, we obtain

$$\sum_{n=2}^{\infty} n|a_n| \le 2\beta(1-\alpha) - \beta \sum_{n=2}^{\infty} n|a_n| .$$

This gives the required condition.

The result is sharp, the extremal function being

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$$f(z) = z - [2\beta(1-\alpha)/n(1+\beta)]z^n .$$

3. Distortion theorem

THEOREM 2. *If* $f \in P^*(\alpha, \beta)$ *, then*

$$r - \frac{\beta(1-\alpha)}{1+\beta} r^2 \le |f(z)| \le r + \frac{\beta(1-\alpha)}{1+\beta} r^2 \quad (|z| = r) , \tag{3.1}$$

and

$$1 - \frac{2\beta(1-\alpha)}{1+\beta} r \le |f'(z)| \le 1 + \frac{2\beta(1-\alpha)}{1+\beta} r \quad (|z| = r) . \tag{3.2}$$

Proof. In view of Theorem 1, we have

$$\sum_{n=2}^{\infty} |a_n| \le \frac{\beta(1-\alpha)}{1+\beta} .$$

Hence

$$\begin{aligned} |f(z)| &\le r + r^2 \sum_{n=2}^{\infty} |a_n| \\ &\le r + \frac{\beta(1-\alpha)}{1+\beta} r^2 \end{aligned}$$

and

$$\begin{aligned} |f(z)| &\ge r - r^2 \sum_{n=2}^{\infty} |a_n| \\ &\ge r - \frac{\beta(1-\alpha)}{1+\beta} r^2 . \end{aligned}$$

Thus (3.1) follows.

Also

$$\begin{aligned} |f'(z)| &\le 1 + r \sum_{n=2}^{\infty} n|a_n| \\ &\le 1 + \frac{2\beta(1-\alpha)}{1+\beta} r \end{aligned}$$

and

$$|f'(z)| \ge 1 - \frac{2\beta(1-\alpha)}{1+\beta} r .$$

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This completes the proof of the theorem.

REMARK. The bounds in (3.1) and (3.2) are sharp since the equalities are attained for the function

$$f(z) = z - \frac{\beta(1-\alpha)}{1+\beta} z^2 \quad (z = \pm r) .$$

THEOREM 3. *Let* $f \in P^*(\alpha, \beta)$. *Then the disc* $|z| < 1$ *is mapped onto a domain that contains the disc* $|w| < \frac{1+\alpha\beta}{1+\beta}$. *The result is sharp with extremal function* $z - \frac{\beta(1-\alpha)}{1+\beta} z^2$.

THEOREM 4. *If* $f \in C^*(\alpha, \beta)$, *then* $f \in P^*\left(\frac{1}{3-2\alpha}, \frac{1}{3-2\beta}\right)$. *This result is sharp with the extremal function*

$$f(z) = z - [\beta(1-\alpha)/1+\beta(3-2\alpha)]z^2 .$$

Proof. In view of Theorem 1, it is sufficient to show that

$$\sum n(2-\beta)|a_n| \le \frac{2(1-\alpha)}{3-2\alpha} .$$

But $f \in C^*(\alpha, \beta)$ implies, [1],

$$\sum_{n=2}^{\infty} n\{(n-1)+\beta(n+1-2\alpha)\}|a_n| \le 2\beta(1-\alpha) .$$

Also, we note that

$$\frac{n\{(n-1)+\beta(n+1-2\alpha)\}}{2\beta(1-\alpha)} \ge \frac{n(2-\beta)(3-2\alpha)}{2(1-\alpha)} ,$$

since

$$(n-1) + (n+1-2\alpha) \ge \beta(2-\beta)(3-2\alpha) \quad (n = 2, 3, 4, \ldots) .$$

Hence the result follows.

THEOREM 5. *If* $f \in P^*(\alpha, \beta)$, *then* f *is convex in the disc* $|z| < r = r(\alpha, \beta)$, *where*

$$r(\alpha, \beta) = \inf_n \left(\frac{1+\beta}{2\beta n(1-\alpha)}\right)^{1/(n-1)} \quad (n = 2, 3, \ldots) .$$

This result is sharp, the extremal function being of the form

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$$f(z) = z - [2\beta(1-\alpha)/n(1+\beta)]z^n .$$

Proof. It suffices to show that $\left|\frac{zf''}{f'}\right| \le 1$ for $|z| \le 1$. First, we note that

$$\left|\frac{zf''}{f'}\right| \le \left\{\sum_{n=2}^{\infty} n(n-1)|a_n||z|^{n-1}\right\} / \left\{1 - \sum_{n=2}^{\infty} n|a_n||z|^{n-1}\right\} .$$

Thus the result follows if

$$\sum_{n=2}^{\infty} n(n-1)|a_n||z|^{n-1} \le 1 - \sum_{n=2}^{\infty} n|a_n||z|^{n-1}$$

which is equivalent to

$$\sum_{n=2}^{\infty} n^2|a_n||z|^{n-1} \le 1 .$$

But by Theorem 1,

$$\sum_{n=2}^{\infty} n(1+\beta)|a_n| \le 2\beta(1-\alpha) .$$

Hence f is convex if

$$n^2|z|^{n-1} \le \frac{n(1+\beta)}{2\beta(1-\alpha)} , \quad n = 2, 3, \ldots ;$$

that is,

$$|z| \le \left\{\frac{1+\beta}{2\beta n(1-\alpha)}\right\}^{1/(n-1)} , \quad n = 2, 3, \ldots .$$

This completes the proof.

4.

In this section we assert that the class $P^*(\alpha, \beta)$ is closed under 'arithmetic mean' and 'convex linear combinations'.

THEOREM 6. *If*

$$f(z) = z - \sum_{n=2}^{\infty} |a_n|z^n$$

and

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$$g(z) = z - \sum_{n=2}^{\infty} |b_n| z^n$$

are in $P^*(\alpha, \beta)$, *then*

$$h(z) = z - \tfrac{1}{2} \sum_{n=2}^{\infty} |a_n + b_n| z^n$$

is also in $P^*(\alpha, \beta)$.

THEOREM 7. *Let*

$$f_1(z) = z , \quad f_n(z) = \frac{2\beta(1-\alpha)}{n(1+\beta)} z^n \quad (n = 2, 3, \ldots) .$$

Then $f \in P^*(\alpha, \beta)$ *if and only if it can be expressed in the form*

$$f(z) = \sum_{n=1}^{\infty} \lambda_n f_n(z) ,$$

where $\lambda_n \geq 0$, *and*

$$\sum_{n=1}^{\infty} \lambda_n = 1 .$$

Proofs of Theorems 6 and 7 follow on the lines of the proofs of Theorems 8 and 9 in [1]. The details are omitted.

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