STATIONARY REFLECTION AND THE FAILURE OF THE SCH

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Abstract. In this paper we prove that from large cardinals it is consistent that there is a singular strong limit cardinal v such that the singular cardinal hypothesis fails at v and every collection of fewer than $\mathrm{cf}(v)$ stationary subsets of v^+ reflects simultaneously. For $\mathrm{cf}(v) > \omega$, this situation was not previously known to be consistent. Using different methods, we reduce the upper bound on the consistency strength of this situation for $\mathrm{cf}(v) = \omega$ to below a single partially supercompact cardinal. The previous upper bound of infinitely many supercompact cardinals was due to Sharon.

§1. Introduction. We study stationary reflection at successors of singular cardinals and its connection with cardinal arithmetic. We start by recalling some basic definitions. For an ordinal δ and a set $S \subseteq \delta$, we say that S is *stationary* if it meets every closed and unbounded subset of δ. If $\{S_i \mid i \in I\}$ is a collection of stationary subsets of a regular cardinal κ, then we say that $\{S_i \mid i \in I\}$ reflects simultaneously if there is an ordinal δ such that $S_i \cap \delta$ is stationary for all $i \in I$.

The consistency of stationary reflection at the successor of singular cardinal is already complex in the context of the generalized continuum hypothesis (GCH). A theorem of Magidor [14] shows that it is consistent relative to the existence of infinitely many supercompact cardinals that every finite collection of stationary subsets of $\aleph_{\omega+1}$ reflects. Recently, the second and third authors [13] were able to show the same result from an assumption below the existence of a cardinal κ which is κ^+-supercompact. Both of these models satisfy GCH. Combining stationary reflection at the successor of a singular cardinal with the failure of SCH presents additional difficulties.

For a singular cardinal v, the singular cardinal hypothesis (SCH) at v is the assertion that if v is strong limit, then $2^v = v^+$. The failure of the singular cardinal hypothesis at a singular cardinal v is known to imply the existence of many nonreflecting objects. For instance, Foreman and Todorcevic [4] have shown that the failure of the singular cardinal hypothesis at v implies that there are two stationary subsets of $[v^+]^\omega$ which do not reflect simultaneously. This was improved by Shelah [22] to obtain a single stationary subset of $[v^+]^\omega$ which does not reflect. Reflection for stationary subsets of $[v^+]^\omega$ has a different character than reflection for stationary subsets of ordinals.

In [19, Corollary 5.4], Poveda, Rinot, and Sinapova showed that the failure of SCH at cardinal v implies the failure of simultaneous reflection for collections of

cf (v) many stationary subsets of v^+. In his PhD thesis from 2005, Sharon [21] proved that relative to the existence of infinitely many supercompact cardinals it is consistent that there is a singular cardinal v of cofinality ω such that SCH fails at v and every stationary subset of v^+ reflects. Sharon's method is a tour de force construction, which builds on Gitik's long extenders forcing [6] for ω-sequence of hypermeasurable cardinals. As such, the construction does not extend to singular cardinals of uncountable cofinalities, and the question of whether the failure of SCH at singular κ of uncountable cofinality together with stationary reflection at κ^+ is at all consistent.

This paper follows a study by the authors on stationary reflection at successors of singular cardinals at which SCH fails. This study was prompted by two other recent studies. First, the work of second and third authors in [13] on stationary reflection in Prikry forcing extensions from subcompactness assumptions. The arguments of [13] show how to examine the stationary reflection in the extension by Prikry type forcing by studying suitable iterated ultrapowers of V. This approach and method has been highly effective in our situation and we follow it here. The second study is Gitik's recent work [8] for blowing up the power of a singular cardinal using a Mitchell order increasing sequence of overlapping extenders. The new forcing machinery of [8] gives new models combining the failure of SCH with reflection properties at successors of singulars. Moreover the arguments are uniform in the choice of cofinality. For example, Gitik has shown that in a related model the tree property holds at κ^+ [7] and that for all $\delta < \kappa$ there is a stationary subset of κ^+ of ordinals of cofinality greater than δ which is not a member of $I[\kappa^+]$ [9].

Our first main theorem shows how to construct a model in which there is a singular cardinal v such that stationary reflection holds at v^+, SCH fails at v and the cofinality of v can be chosen in advance.

THEOREM 1.1. *Let $\eta < \lambda$ be regular cardinals. Suppose that there is an increasing sequence of cardinals $\langle \kappa_\alpha \mid \alpha < \eta \rangle$ with:*

(1) $\eta < \kappa_0$,
(2) *for each $\alpha < \eta$, κ_α carries a (κ_α, λ)-extender E_α and there is a supercompact cardinal between $\sup_{\beta < \alpha} \kappa_\beta$ and κ_α, and*
(3) *the sequence $\langle E_\alpha \mid \alpha < \eta \rangle$ is Mitchell order increasing and coherent.*

Then, there is a cardinal and cofinality preserving extension in which, setting $\kappa = \sup_{\alpha < \eta} \kappa_\alpha$, $2^\kappa = \lambda$ and every collection of fewer than η stationary subsets of κ^+ reflects.[1]

We have the following improvement of Gitik's result about $I[\kappa^+]$.

THEOREM 1.2. *Let $\langle \kappa_\alpha \mid \alpha < \eta \rangle$ be as in Theorem 1.1 with the exception that we only assume there is a single supercompact cardinal $\theta < \kappa_0$. There is a cardinal and cofinality preserving extension in which setting $\kappa = \sup_{\alpha < \eta} \kappa_\alpha$, we have that κ is strong limit, $2^\kappa = \lambda$ and there is a scale of length κ^+ such that the set of nongood points of cofinality less than θ is stationary.*

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[1]After proving this theorem, Gitik also gave an alternative proof for the same result (see [10]).

After a suitable preparation, the extension is obtained using the same forcing as in Theorem 1.1. Standard arguments show that in this model, the set of nongood points is stationary in cofinalities that are arbitrarily high below θ. Further, the same argument shows that we can take θ between $\sup_{\alpha<\beta}\kappa_\alpha$ and κ_β and reach the same conclusion.

These two results continue a long line of research about the interaction between the failure of the singular cardinal hypothesis and weak square-like principles [11, 18, 23–26].

We do not know whether these results can also be obtained at small singular cardinals of uncountable cofinalities. For concreteness we suggest the following question.

QUESTION 1.3. *Is it consistent that* SCH *fails* $\aleph_{\omega_1}$ *and every stationary subset of* $\aleph_{\omega_1+1}$ *reflects?*

We also give another model for stationary reflection at ν^+ where ν is a singular cardinal of cofinality ω where the singular cardinal hypothesis fails. This construction replaces the supercompactness assumption in Sharon's result and Theorem 1.1 with the weaker one of subcompactness together with hypermeasurability.

THEOREM 1.4. *Suppose that* κ *is* κ^+-Π_1^1-*subcompact and carries a* (κ, κ^{++})-*extender. There is a forcing extension in which* κ *is singular strong limit of cofinality* ω, $2^\kappa = \kappa^{++}$ *and every finite collection of stationary subsets of* κ^+ *reflects simultaneously.*

In Theorem 6.22, we will also show how to collapse cardinals and make κ in Theorem 1.4 to be $\aleph_\omega$.[2]

The construction and proof follows the lines of the second and third authors' paper [13], and the work of Merimovich [15] on generating generics for extender based forcing over iterated ultrapowers. The large cardinal assumption in the theorem is the natural combination of the assumption from [13] and an assumption sufficient to get the failure of SCH by extender based forcing. Unfortunately, we are unable to adapt the argument from the previous theorem to a singular cardinal of uncountable cofinality. We ask:

QUESTION 1.5. *It is possible to obtain the result of Theorem 1.1 without any supercompactness assumptions? Namely, is it possible to start with a single partially supercompact cardinal* κ *and force* κ *to be a strong limit singular of uncountable cofinality such that every stationary subset of* κ^+ *reflects? Is it possible to obtain that together with the failure of* SCH *at* κ?

The paper is organized as follows. In Section 2, we define Gitik's forcing from [8], which will be used in our main theorem. In Section 3, we prove that in mild generic extensions of V we can find a generic for Gitik's forcing over a suitable iterated ultrapower. In Section 4, we give the proof of the main theorem by arguing that stationary reflection holds in the generic extension of the iterated ultrapower constructed in the previous section. In Section 5, we prove Theorem 1.2. In Section 6, we give the proof of Theorem 1.4 and its variation for $\aleph_\omega$.

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[2]This theorem did not appear in the original version of this paper.

§2. Gitik's forcing. In this section we give a presentation of Gitik's forcing [8] for blowing up the power of a singular cardinal with a Mitchell order increasing sequence of extenders.

Let us start with the following definitions:

DEFINITION 2.1.   Let E_0, E_1 be (κ_0, λ_0) and (κ_1, λ_1)-extenders respectively. We say that E_0 is less than E_1 in the Mitchell order, or $E_0 \trianglelefteq E_1$, if $E_0 \in \mathrm{Ult}(V, E_1)$. We say that E_0 coheres with E_1 if $j_{E_1}(E_0) \upharpoonright \lambda_0 = E_0$.

The existence of a long Mitchell increasing and pairwise coherent sequence of extenders E_i, where E_i is a (κ_i, λ)-extender, follows from the existence of superstrong cardinal or even weaker large cardinal axioms.

Before we begin with the definition of the forcing, let us show a few basic facts about extenders and Mitchell order. We recall the notion of width from [2].

DEFINITION 2.2.   Let $k \colon M \to N$ be an elementary embedding between transitive models of set theory and let μ be an ordinal. We say the embedding k has width $\leq \mu$ if every element of N is of the form $k(f)(a)$ for some $f \in M$ and $a \in N$ such that $M \models |\mathrm{dom}(f)| \leq \mu$.

LEMMA 2.3.   *Let E be a (κ, λ)-extender and let $\leq_E$ be the Rudin–Keisler order of the extender E. Let $k \colon V \to M$ be an elementary embedding with width $\leq \kappa$. Then the set $k``\,\mathrm{dom}\,E$ is $\leq_{k(E)}$-cofinal in $\mathrm{dom}\,k(E)$.*

PROOF.   By [6], the Rudin–Keisler order $\leq_E$ is κ^+-directed.

Let $a \in \mathrm{dom}\,k(E)$. Then, by the definition of width, there is a function $f \colon \kappa \to \mathrm{dom}\,E$ and some generator b such that $k(f)(b) = a$. Let a' be a $\leq_E$ upper bound of the image of f. Then, clearly, $k(a')$ is $\leq_{k(E)}$ above a.   ⊣

LEMMA 2.4.   *Let E_0 be a (κ_0, λ)-extender and let E_1 be a (κ_1, λ)-extender, where $\kappa_0 \leq \kappa_1$. Let us assume that $E_0 \trianglelefteq E_1$. Then the following diagram commutes*:

$$
\begin{array}{ccc}
V & \xrightarrow{\;E_0\;} & M_0 \\
{\scriptstyle E_1}\big\downarrow & & \big\downarrow{\scriptstyle j_{E_0}(E_1)} \\
M_1 & \xrightarrow{\;E_0\;} & N
\end{array}
$$

where each arrow represents the ultrapower map using the indicated extender.

PROOF.   First, since $E_0 \in M_1$, all maps are internally defined and in particular, all models are well founded.

Moreover,

$$\left(j_{E_0}\right)^V \upharpoonright M_1 = \left(j_{E_0}\right)^{M_1}.$$

In order to verify this, it is sufficient to show that those maps are the same on sets of ordinals. First, let us verify that they agree of the ordinals. Indeed, let us consider the class:

$$\{(f, a) \mid f \colon \kappa_0^{<\omega} \to \mathrm{Ord}\},$$

ordered using the extender E_0:

$$(f_0, a_0) \leq (f_1, a_1) \iff j_{E_0}^V(f_0)(a_0) \leq j_{E_0}^V(f_1)(a_1).$$

Using the combinatorial definition of the extender ultrapower, we conclude that M_1 can compute this order correctly, and in particular, it computes correctly the ordertype of the elements below any constant function, or indeed below any element of the form $[f, a]$ that represents an ordinal. Indeed, as the ultrapower by E_1 is closed under κ_0-sequences, it contains the above class. Since $E_0 \trianglelefteq E_1$, M_1 contains the relevant measures and Rudin–Keisler projections.

For an arbitrary set of ordinals, y, one can easily compute whether $[f, a] \in [c_y, \{\kappa_0\}]$ using the combinatorial information of the extender.

Let us consider an element of $\mathrm{Ult}(M_1, E_0)$. By the definition, it has the form: $x = j_{E_0}(g)(a_0)$ where $g : \kappa_0^{<\omega} \to M_1$. Going backwards, we can find a function in V, f, and a generator a_1 such that for every z, $j_{E_1}(f)(z, a_1) = g(y)$ and therefore

$$x = j_{E_0}\left(j_{E_1}(f)(-, a_1)\right)(a_0) = j_{j_{E_0}(E_1)}\left(j_{E_0}(f)\right)\left(a_0, j_{E_0}(a_1)\right).$$

This element in obviously in $\mathrm{Ult}(M_0, j_{E_0}(E_1))$.

On the other hand, if $y \in \mathrm{Ult}(M_0, j_{E_0}(E_1))$ then there is some generator $a_1' \in \mathrm{dom}\, j_{E_0}(E_1)$ and a function g such that $y = j_{j_{E_0}(E_1)}(g)(a_1')$. By Lemma 2.3, we may assume that $a_1' = j_{E_0}(a_1)$. Let f be a function in V and let a_0 be some generator such that $g = j_{E_0}(f)(a_0)$, then we have

$$\begin{aligned}
y &= j_{j_{E_0}(E_1)}\left(j_{E_0}(f)(a_0, -)\right)(j_{E_0}(a_1)) \\
&= j_{j_{E_0}(E_1)}\left(j_{E_0}\left(f(-, a_1)\right)(a_0)\right) \\
&= j_{E_0}\left(j_{E_1}(f)(-, a_1)\right)(a_0),
\end{aligned}$$

as wanted. ⊣

Let $\langle \kappa_\alpha \mid \alpha < \eta \rangle$ be a sequence of cardinals as in Theorem 1.1. Following Merimovich (see for example [15]), we can assume that the extenders E_α are of the form $\langle E_\alpha(d) \mid d \in [\lambda]^{\kappa_\alpha} \rangle$ where for $X \subseteq \{f \mid \mathrm{dom}\, f \subseteq d, \text{range}\, f \subseteq \kappa_\alpha, |f| < \kappa_\alpha\}$, $X \in E_\alpha(d)$ if and only if $\{(j_{E_\alpha}(\xi), \xi) \mid \xi \in d\} \in j_{E_\alpha}(X)$.

For $d \in [\lambda]^{\kappa_\alpha}$ with $\kappa_\alpha \in d$, it is easy to see that the measure $E_\alpha(d)$ concentrates on a set of order preserving functions v with $\kappa \in \mathrm{dom}(v)$. So we may assume that the elements of every measure one set mentioned below are of this form. We also fix functions $\langle \ell_\alpha \mid \alpha < \eta \rangle$ so that for every $\alpha < \eta$, $j_{E_\alpha}(\ell_\alpha)(\kappa_\alpha) = \lambda$. The existence of such functions can always be arranged by a simple preparatory forcing.

Using the set $d \in [\lambda]^{\kappa_\alpha}$ to index the extenders E_α has the advantage that the projection maps from $E_\alpha(d')$ and $E_\alpha(d)$ for $d \subseteq d'$ can be made very explicit. The measure $E_\alpha(d')$ concentrates on partial functions from d' to κ_α with domain smaller than κ_α. Thus, the map $v \to v \restriction d$ is a projection from $E_\alpha(d')$ to $E_\alpha(d)$.

For every two cardinals $\mu < \lambda$, let $\mathcal{A}(\lambda, \mu)$ be the poset consisting of partial functions $f : \lambda \to \mu$ with $|f| \leq \mu$ and $\mu \in \mathrm{dom}(f)$. Therefore $\mathcal{A}(\lambda, \mu)$ is isomorphic to Cohen forcing for adding λ many subsets of μ^+.

We let $\mathbb{P}$ be Gitik's forcing from [8] defined from the sequence of extenders $\langle E_\alpha \mid \alpha < \eta \rangle$. We give a compact description of the forcing. A condition $p \in \mathbb{P}$ is a sequence $\langle p_\alpha \mid \alpha < \eta \rangle$ such that there is a finite set $s^p \subseteq \eta$ such that for each $\alpha < \eta$, $p_\alpha = (f_\alpha, \lambda_\alpha)$ if $\alpha \in s^p$, and $p_\alpha = (f_\alpha, A_\alpha)$ otherwise, and the following conditions hold.

(1) $f_\alpha \in \mathcal{A}(\lambda_{\alpha^*}, \kappa_\alpha)$ where α^* is the next element of s^p above α if it exists and $f_\alpha \in \mathcal{A}(\lambda, \kappa_\alpha)$ otherwise.

(2) For all $\alpha \in s^p$, λ_α is a cardinal and $\sup_{\beta<\alpha} \kappa_\beta < \lambda_\alpha < \kappa_\alpha$.

(3) For all $\alpha \in \eta \setminus s^p$, if $\alpha > \max(s^p)$, then $A_\alpha \in E_\alpha(\mathrm{dom}(f_\alpha))$, otherwise if α^* is the least element of s^p above α, then $A_\alpha \in E_\alpha(\mathrm{dom}(f_\alpha))$.

(4) For $\alpha \notin s^p$, $f_\alpha^p(\kappa_\alpha) = 0$.[3]

(5) The sequence $\langle \mathrm{dom}(f_\alpha) \mid \alpha < \eta \rangle$ is increasing.

We adopt the convention of adding a superscript f_α^p, A_α^p, etc. to indicate that each component belongs to p. When the value of λ_{α^*} might behave non-trivially, we will add it as a third coordinate to the pairs p_α, where $\alpha \notin s^p$. We call $\eta \setminus s^p$ the *pure part* of p and s^p the *non-pure part* of p.

We briefly sketch the notion of extension. A condition p is a direct extension of q if $s^p = s^q$ and for all α, f_α^p is stronger than f_α^q and A_α^p projects to a subset of A_α^q using the natural Rudin–Keisler projection from $E_\alpha(\mathrm{dom}\, f_\alpha^p)$ to $E_\alpha(\mathrm{dom}\, f_\alpha^q)$.

Let us describe now the one point extension. Suppose that $v \in A_\beta^p$. We let $q = p \frown v$ be the condition with $s^q = s^p \cup \{\beta\}$ and the following.

(1) $f_\beta^q = f_\beta^p \frown v$ is the overwriting of f_β^p by v, that is,

$$(f_\beta^p \frown v)(\tau) = \begin{cases} v(\tau), & \text{if } \tau \in \mathrm{dom}(v), \\ f_\beta^p(\tau), & \text{otherwise.} \end{cases}$$

(2) $\lambda_\beta^q = \ell_\beta(v(\kappa_\beta))$.

(3) For $\alpha \in [\max(s^p) \cap \beta, \beta)$, $f_\alpha^q = f_\alpha^p \circ v^{-1}$ and $A_\alpha^q = \{\xi \circ v^{-1} \mid \xi \in A_\alpha^p\}$ if applicable.

The following analysis of dense open sets of $\mathbb{P}$ was established in [8].

LEMMA 2.5. *For every condition $p \in \mathbb{P}$ and dense open set $D \subseteq \mathbb{P}$, there are $p^* \geq^* p$ and a finite subset $\{\alpha_0, \ldots, \alpha_{m-1}\}$ of the pure part of p, such that for every sequence $\vec{v} = \langle v_{\alpha_0}, \ldots, v_{\alpha_{m-1}} \rangle \in \prod_{i<m} A_{\alpha_i}^{p^*}$, $p^* \frown \vec{v} \in D$.*

For limit $\delta \leq \eta$, we define $\bar{\kappa}_\delta = \sup_{\alpha<\delta} \kappa_\alpha$. Gitik used the previous lemma to prove:

THEOREM 2.6. *In the generic extension by $\mathbb{P}$, cardinals and cofinalities are preserved and for every limit $\delta \leq \eta$, the singular cardinal hypothesis fails at $\bar{\kappa}_\delta$.*

For use later, we define $\vec{\mathcal{A}}$ to be the full-support product $\prod_{\alpha<\eta} \mathcal{A}(\lambda, \kappa_\alpha)$, and similarly, for each $\beta < \eta$, $\vec{\mathcal{A}}_{\geq \beta} = \prod_{\beta \leq \alpha < \eta} \mathcal{A}(\lambda, \kappa_\alpha)$. We aim to show that if $\vec{H} = \langle H(\alpha) \mid \alpha < \eta \rangle$ is $\vec{\mathcal{A}}$ generic over V, then there is a generic for the image of $\mathbb{P}$ in a suitable iterated ultrapower.

§3. The iterated ultrapower M_η and the generic filter G^*. We consider the following iterated ultrapower

$$\langle M_\alpha, j_{\alpha,\beta} \mid \alpha \leq \beta \leq \eta \rangle$$

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[3]This gives a clean way to distinguish between Cohen functions associated with members of s^p and those which are not.

by the extenders in $\vec{E}$. The iteration is defined by induction of α. Let $M_0 = V$. For every $\alpha < \eta$, given that $j_{0,\alpha} \colon M_0 \to M_\alpha$ has been defined, we take $E_\alpha^\alpha = j_{0,\alpha}(E_\alpha)$, and set $j_{\alpha,\alpha+1} \colon M_\alpha \to M_{\alpha+1} \cong \mathrm{Ult}(M_\alpha, E_\alpha^\alpha)$. At limit stage $\delta \le \eta$ we take M_δ to be the direct limit of the system $\langle M_\alpha, j_{\alpha,\beta} \colon \alpha \le \beta < \delta \rangle$, and $j_{\alpha,\delta}$ to be the limit maps.

For every $\beta \le \eta$, we shall abbreviate and write j_β for $j_{0,\beta}$. For a given $\beta \le \eta$ we shall denote $j_\beta(\kappa_\alpha)$, $j_\beta(E_\alpha)$, $j_\beta(\lambda)$ by κ_α^β, E_α^β, and λ^β respectively. Similarly, we will denote $j_\beta(x) = x^\beta$ for objects $x \in V$ whose images along the iteration will be significant for our construction.

LEMMA 3.1. *Let $\beta \le \eta$. For every $x \in M_\beta$ there are $\beta_0 < \beta_1 < \cdots < \beta_{l-1}$ below β, ordinals $\tau_0, \ldots, \tau_{l-1}$ below λ, and $f \colon \prod_{i<l} \kappa_{\beta_i} \to V$, such that*

$$x = j_\beta(f)\left(j_{\beta_0}(\tau_0), j_{\beta_1}(\tau_1), \ldots, j_{\beta_{l-1}}(\tau_{l-1})\right).$$

PROOF. It is immediate from the definition of the iteration that for every $x \in M_\beta$ there are $\beta_0 < \cdots < \beta_{l-1}$ below β, finite subsets of ordinals $a_0, \ldots, a_{l-1}$, with each $a_i \in [\lambda^{\beta_i}]^{<\omega}$ a generator of $E_{\beta_i}^{\beta_i}$, and a function $g \colon \prod_{i<l} \kappa_{\beta_i}^{<\omega} \to V$ so that

$$x = j_{0,\beta}(g)(j_{\beta_0+1,\beta}(a_0), \ldots, j_{\beta_{l-1}+1,\beta}(a_{l-1})).$$

Note that for each $i < l$, $j_{\beta_i+1,\beta}(a_i) = a_i$, because

$$\mathrm{crit}(j_{\beta_i+1,\beta}) = \kappa_{\beta_i+1}^{\beta_i+1} > \lambda^{\beta_i}.$$

It follows that

$$x = j_\beta(g)(a_0, a_1, \ldots, a_{l-1}).$$

We may further assume that each $a_i < \lambda^{\beta_i}$ is an ordinal. Finally, we claim that we may replace each a_i with an ordinal of the form $j_{\beta_i}(\tau_i)$, for some $\tau_i < \lambda$. By Lemma 2.3, in M_{β_i}, the extender $E_{\beta_i}^{\beta_i} = j_{\beta_i}(E_{\beta_i})$ is generated by a subset of its measures, which are of the form $j_{\beta_i}(E_{\beta_i}(\tau_i)) = E_{\beta_i}^{\beta_i}(j_{\beta_i}(\tau_i))$. Namely, for every $a_i \in \lambda^{\beta_i}$ there exists some $\tau_i \in \lambda$ such that $a_i \le^{RK}_{E_{\beta_i}^{\beta_i}} j_{\beta_i}(\tau_i)$.

In particular

$$a_i = j_{\beta_i}(h)(b_1, \ldots, b_k) \le^{RK}_{E_{\beta_i}^{\beta_i}} j_{\beta_i}(\tau_i).$$

The claim follows. ⊣

Having defined the iterated ultrapower M_η we proceed to introduce relevant conditions in $j_\eta(\mathbb{P})$. Let $p = \langle p_\alpha \mid \alpha < \eta \rangle$ be a pure condition in $\mathbb{P}$, that is, a condition with $s^p = \emptyset$. For every $\alpha < \eta$, we set $p_\alpha = \langle f_\alpha, A_\alpha, \lambda \rangle$, where $f_\alpha \in A(\lambda, \kappa_\alpha)$ and $A_\alpha \in E_\alpha(d_\alpha)$ where $d_\alpha = \mathrm{dom}(f_\alpha) \in [\lambda]^{\le \kappa_\alpha}$.

Since $j_{\alpha,\alpha+1} = j_{E_\alpha^\alpha}^{M_\alpha}$ it follows that the function

$$v_{\alpha,\alpha+1}^{p^\alpha} := (j_{\alpha,\alpha+1} \restriction d_\alpha^\alpha)^{-1}$$

belongs to $j_{\alpha,\alpha+1}(A_\alpha^\alpha) = A_\alpha^{\alpha+1}$, which is the measure one set of the α-th component of $j_{\alpha+1}(p)$. By applying $j_{\alpha+1,\eta}$ to $j_{\alpha+1}(p)$, we conclude that the function

$$v_{\alpha,\eta}^{p^\alpha} := j_{\alpha+1,\eta}(v_{\alpha,\alpha+1}^{p^\alpha}) = (j_{\alpha,\eta} \restriction d_\alpha^\alpha)^{-1}$$

belongs to $j_\eta(A_\alpha)$, and thus $j_\eta(p)$ has an one point extension at the α-th coordinate of the form

$$j_\eta(p)^\frown \langle v_{\alpha,\eta}^{p^\alpha} \rangle.$$

It is clear that for every finite sequence $\alpha_0 < \cdots < \alpha_{m-1}$ of ordinals below η, and pure condition $p \in \mathbb{P}$, we have $j_\eta(p)^\frown \langle v_{\alpha_0,\eta}^{p^{\alpha_0}}, v_{\alpha_1,\eta}^{p^{\alpha_1}}, \ldots, v_{\alpha_{m-1},\eta}^{p^{\alpha_{m-1}}} \rangle$ is an extension of $j_\eta(p)$ in $j_\eta(\mathbb{P})$. It is also clear that if $p, q \in \mathbb{P}$ are two pure conditions so that for all $\alpha < \eta$, f_α^p, f_α^q are compatible, then for every two sequences $\alpha_0 < \cdots < \alpha_{m-1}$ and $\beta_0 < \cdots < \beta_{l-1}$ of ordinals below η, the conditions

$$j_\eta(p)^\frown \langle v_{\alpha_0,\eta}^{p^{\alpha_0}}, v_{\alpha_1,\eta}^{p^{\alpha_1}}, \ldots, v_{\alpha_{m-1},\eta}^{p^{\alpha_{m-1}}} \rangle$$

and

$$j_\eta(q)^\frown \langle v_{\beta_0,\eta}^{q^{\beta_0}}, v_{\beta_1,\eta}^{q^{\beta_1}}, \ldots, v_{\beta_{l-1},\eta}^{q^{\beta_{l-1}}} \rangle$$

are compatible in $j_\eta(\mathbb{P})$. We remark that in the assumption that the sequence of extenders $\langle E_\alpha \mid \alpha < \eta \rangle$ is Mitchell order increasing is used in order to be able to permute the order in which the extensions are done, using a sequence of applications of Lemma 2.4.

Fix $\vec{H} = \langle H(\alpha) \mid \alpha < \eta \rangle$ a V-generic for $\vec{\mathcal{A}} = \prod_{\alpha < \eta} \mathcal{A}(\lambda, \kappa_\alpha)$.

DEFINITION 3.2.   Define $G^* \subset j_\eta(\mathbb{P})$ to be the filter generated by all conditions

$$j_\eta(p)^\frown \langle v_{\alpha_0,\eta}^{p^{\alpha_0}}, v_{\alpha_1,\eta}^{p^{\alpha_1}}, \ldots, v_{\alpha_{m-1},\eta}^{p^{\alpha_{m-1}}} \rangle,$$

where $p \in \mathbb{P}$ satisfy that $f_\alpha^p \in H(\alpha)$ for all $\alpha < \eta$, $m < \omega$, and $\alpha_0 < \cdots < \alpha_{m-1}$ are ordinals below η.

Our first goal is to prove that G^* generates a $j_\eta(\mathbb{P})$ generic filter over M_η.

PROPOSITION 3.3. G_η^* is $j_\eta(\mathbb{P})$ generic over M_η.

Before moving to the proof of the proposition, we discuss finite subiterates of M_η.

3.1. Finite sub-iterations N^F. The model M_η can be seen as a directed limit of all its finite subiterates, N^F, $F \in [\eta]^{<\omega}$. Given a finite set $F = \{\beta_0, \ldots, \beta_{l-1}\} \in [\eta]^{<\omega}$, we define its associated iteration $\langle N_i^F, i_{m,n}^F \mid m \le n \le l \rangle$ by $N_0^F = V$, and $i_{n,n+1}^F \colon N_n^F \to N_{n+1}^F \cong \mathrm{Ult}(N_n^F, i_n^F(E_{\beta_n}))$. Since for the moment we handle a single finite set at a time, we will suppress the mention of the finite set F and refer only to the iteration as $i_{m,n} \colon N_m \to N_n$ for $m \le n \le l$ and as usual we set $i_n = i_{0,n}$ for $n \le l$.

The proof of Lemma 3.1 shows that the elements of N_l are of the form $i_l(f)(\tau_0, i_1(\tau_1), \ldots, i_{l-1}(\tau_{l-1}))$. Let $k \colon N_l \to M_\eta$ be the usual factor map defined by

$$k(i_l(f)(\tau_0, i_1(\tau_1), \ldots, i_{l-1}(\tau_{l-1}))) = j_\eta(f)(j_{\beta_0}(\tau_0), j_{\beta_1}(\tau_1), \ldots, j_{\beta_{l-1}}(\tau_{l-1})).$$

It is routine to verify that k is well defined, elementary, and $j_\eta = k \circ i_l$.

Moreover, following this explicit description of k, it is straightforward to verify that $k \colon N_l \to M_\eta$ is the resulting iterated ultrapower limit embedding, associated with the iteration of N_l by the sequence

$$\langle i_{0,n_\alpha}(E_\alpha) \mid \alpha \in \eta \setminus \{\beta_0, \ldots \beta_{l-1}\} \rangle,$$

where n_α is the minimal $n < l$ for which $\beta_n \geq \alpha$, if exists, and $n_\alpha = l$ otherwise. This again uses the Mitchell order assumption of the sequence of extenders in order to get the desired commutativity.

Next, we observe that our assignment of generators $v^{p^\alpha}_{\alpha,\eta}$, $\alpha < \eta$, to $j_\eta(p)$ of conditions $p \in \mathbb{P}$, in Definition 3.2, can be defined at the level of the finite subiterates.

Indeed, for a condition $p \in \mathbb{P}$, $n < l$, we temporarily define

$$v^{i_n(p)}_{n,n+1} = (i_{n,n+1} \restriction d)^{-1} = \{(i_{n,n+1}(\tau), \tau) \mid \tau \in d\},$$

where $d = \mathrm{dom}(f^{i_n(p)})_{\beta_n}$. We let $v^{i_n(p)}_{n,l} = i_{n+1,l}(v^{i_n(p)}_{n,n+1})$ be the natural push forward to N_l. As with the conditions $j_\eta(p)$, we have that

$$i_l(p) {}^\frown \langle v^{i_0(p)}_{0,l}, \ldots, v^{i_{l-1}(p)}_{l-1,l} \rangle$$

is a valid extension of $i_l(p)$ in $i_l(\mathbb{P})$.

To record this definition (which depends on F), we make a few permanent definitions which explicitly mention F as well as objects from the discussion above:

DEFINITION 3.4. Fix $i^F = i_l$ and $k^F = k$ and the model $N^F = N_l$. Denote the condition defined above by $i^F(p) {}^\frown \vec{v}^{p,F}$ and refer to it as the *natural non-pure extension of $i^F(p)$*.

Using the description of $k^F : N^F \to M_\eta$, it is straightforward to apply Łoś's Theorem and show that $k^F(\vec{v}^{p,F}) = \langle v^{p^{\beta_n}}_{\beta_n,\eta} \mid n < l \rangle$. We conclude that

$$k^F\left(i^F(p) {}^\frown \vec{v}^{p,F}\right) = j_\eta(p) {}^\frown \langle v^{p^{\beta_0}}_{\beta_0,\eta}, \ldots, v^{p^{\beta_{l-1}}}_{\beta_{l-1},\eta} \rangle.$$

Finally, we note that the forcing $\vec{\mathcal{A}} = \prod_{\alpha < \eta} \mathcal{A}(\lambda, \kappa_\alpha)$ has a natural factorization, associated with F. Setting $\beta_{-1} = 0$ and $\beta_l = \eta$, we have

$$\vec{\mathcal{A}} = \prod_{n \leq l} \vec{\mathcal{A}} \restriction [\beta_{n-1}, \beta_n),$$

where for each $n \leq l$,

$$\vec{\mathcal{A}} \restriction [\beta_{n-1}, \beta_n) = \prod_{\beta_{n-1} \leq \alpha < \beta_n} \mathcal{A}(\lambda, \kappa_\alpha).$$

For each $0 \leq n \leq l$, we define

$$\vec{\mathcal{A}}^F_n = i_n(\vec{\mathcal{A}} \restriction [\beta_{n-1}, \beta_n))$$

and denote the resulting product by $\vec{\mathcal{A}}^F = \prod_{n \leq l} \vec{\mathcal{A}}^F_n$. Suppose that $\vec{H} \subseteq \vec{\mathcal{A}}$ is a V-generic filter. For each $n \leq l$, the fact $\vec{\mathcal{A}} \restriction [\beta_{n-1}, \beta_n)$ is a $\kappa^+_{\beta_{n-1}}$-closed forcing guarantees that $i_n``\vec{H} \restriction [\beta_{n-1}, \beta_n)) \subseteq \vec{\mathcal{A}}^F_n$ forms a generic filter over N_n, hence also $N_l = N^F$.[4] For each n, we denote the resulting N_l generic for $\vec{\mathcal{A}}^F_n$ by $\vec{J}^F_n$.

This is the crucial part: the main effect of taking a non-pure extension of a condition is the reflection of its Cohen part downwards. This reflection is exactly moving from $\vec{H}$ to $\vec{J}^F_n$.

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[4]This is true by [2, Proposition 15.1], since the width of the embedding is smaller than its distributivity.

It is clear that the product $\prod_{0 \le n \le l} \vec{J}_n^F$ is $\vec{\mathcal{A}}^F$ generic over N_l. We note that for each pure $p \in \mathbb{P}$ with $\vec{f}^p \in \vec{H}$, if $\vec{f}$ is the Cohen part of the natural non-pure extension of $i^F(p)$, then $\vec{f} \in \vec{\mathcal{A}}^F$. Moreover the collection of such $\vec{f}$ forms an $\vec{\mathcal{A}}^F$-generic filter over N^F which is obtained by modifying $\prod_{0 \le n \le l} \vec{J}_n^F$ on the coordinates β_n for $n < l$ by taking the natural non-pure extensions of $i^F(p)$ for $p \in \vec{J}^F$ (see Definition 3.4).

For each individual modified filter, its genericity follows from Woodin's surgery argument [5].[5] The fact that the product remains generic follows by induction on n from applications of Easton's lemma, using the chain condition of the forcing notions with smaller indexes and the closure of the components with larger indexes. We denote by $\vec{H}^F$ the resulting N^F-generic filter over $\vec{\mathcal{A}}^F$.

REMARK 3.5. G^* can be constructed in $N^F[\vec{H}^F]$ in the same way that it was constructed in V using the fact that M_η can be described as an iterated ultrapower of N^F and starting with conditions q such that $s^q = F$ and $\vec{f}^q \in \vec{H}^F$.

We turn to the proof of Proposition 3.3.

PROOF. It is clear that $G^* \subseteq j_\eta(\mathbb{P})$ is a filter. We verify that G^* meets every dense open set $D \subseteq j_\eta(\mathbb{P})$ in M_η. Since M_η is the direct limit of its finite subiterates. There are $F = \{\beta_0, \dots \beta_{l-1}\} \in [\eta]^{<\omega}$ and $\bar{D}$ such that $k^F(\bar{D}) = D$.

Let p' be the natural non-pure extension of $i^F(p)$ for some p with $\vec{f}^p \in \vec{H}$. Now by its definition $\vec{f}^{p'} \in \vec{H}^F$. Appealing to Lemma 2.5 and the genericity of $\vec{H}^F$, we conclude that there exists a direct extension $p^* \ge^* p'$, with $\langle f_\alpha^{p^*} \mid \alpha < \eta \rangle \in \vec{H}^F$ and a finite set $\{\alpha_0, \dots, \alpha_{m-1}\} \subset (\eta \setminus F)$, such that $p^* \frown \vec{v} \in \bar{D}$ for every $\vec{v} = \langle v_{\alpha_0}, \dots, v_{\alpha_{m-1}} \rangle \in \prod_{i<m} A_{\alpha_i}^{p^*}$. By the elementarity of k^F, $k^F(p^*) \frown \vec{v} \in D$ for every $\vec{v} \in \prod_{i<m} k^F(A^{p^*})_{\alpha_i}$.

In particular,

$$k^F(p^*) \frown \langle v_{\alpha_0, \eta}^{(p^*)^{\alpha_0}}, \dots, v_{\alpha_{m-1}, \eta}^{(p^*)^{\alpha_{m-1}}} \rangle \in \mathcal{D}.$$

It remains to verify that the last condition belongs to G^*. This is immediate from Remark 3.5 and the fact that $\vec{f}^{p^*} \in \vec{H}^F$.   ⊣

PROPOSITION 3.6. $M_\eta[G^*]$ is closed under $\kappa_0 = \text{crit}(j_\eta)$ sequences of its elements in $V[\vec{H}]$.

PROOF. Let $\langle x_\mu \mid \mu < \kappa_0 \rangle$ be a sequence of elements in $M_\eta[G^*]$. Since $M_\eta[G^*]$ is a model of ZFC, we may assume that all x_μ are ordinals. By Lemma 3.1, for each x_μ is of the form

$$x_\mu = j_\eta(g_\mu)(j_{\beta_0^\mu}(\tau_0^\mu), \dots, j_{\beta_{l^\mu-1}^\mu}(\tau_{l^\mu-1}^\mu))$$

for some finite sequences $\beta_0^\mu < \dots, \beta_{l^\mu-1}^\mu < \eta$ and $\tau_0^\mu, \dots, \tau_{l^\mu-1}^\mu < \lambda$. Moreover, since the Rudin–Keisler order $\le_{E_{\beta_i^\mu}}^{RK}$ is κ_0^+-directed, we may assume that there exists some $\tau^* < \lambda$ such that $\tau_i^\mu = \tau^*$ for all $\mu < \kappa_0$ and $i < l^\mu$. Hence for $\langle x_\mu \mid \mu < \kappa_0 \rangle$ to be

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[5]Indeed, for every set $a \in N^F$ of cardinality $\le \iota_k(\kappa_{\beta_k})$, we modify the generic at most at κ_{β_k} many coordinates.

a member of $M_\eta[G^*]$, it suffices to verify that the sequences $\langle j_\eta(g_\mu) \mid \mu < \kappa_0 \rangle$ and $\langle j_\alpha(\tau^*) \mid \alpha < \eta \rangle$ belong to $M_\eta[G^*]$.

The first sequence already belongs to M_η as $\mathrm{crit}(j_\eta) = \kappa_0$ implies that it is just $j_\eta(\vec{g}) \upharpoonright \kappa_0$. The latter sequence $\langle j_\alpha(\tau^*) \mid \alpha < \eta \rangle$ can be recovered from G^* as follows. It follows from a simple density argument that for every $\alpha < \eta$, there exists some $q \in G^*$ such that:

(1) α is a non-pure coordinate of q, and
(2) $j_\eta(\tau^*) \in \mathrm{dom}(f_\alpha^q)$.

It is also clear that two conditions $q, q' \in G^*$ of this form must satisfy $f_\alpha^q(j_\eta(\tau^*)) = f_\alpha^{q'}(j_\eta(\tau^*))$. We may therefore define in $M_\eta[G^*]$ a function $t_{j_\eta(\tau^*)} \colon \eta \to \lambda^\eta$ by $t_{j_\eta(\tau^*)}(\alpha) = f_\alpha^q(j_\eta(\tau^*))$ for some condition $q \in G^*$ as above. We claim that $t_{j_\eta(\tau^*)}(\alpha) = j_\alpha(\tau^*)$ for all $\alpha < \eta$. Indeed, for every $\alpha < \eta$, there exists a condition $p \in \mathbb{P}$ with $\langle f_\alpha^p \mid \alpha < \eta \rangle \in \vec{H}$ so that $\tau^* \in \mathrm{dom}(f_\alpha^p)$, and clearly, the condition $q = j_\eta(p) ^\frown \langle v_{\alpha,\eta}^{p^\alpha} \rangle \in G^*$ has $j_\eta(\tau^*) \in \mathrm{dom}(f_\alpha^q)$. But $j_\eta(\tau^*) \in \mathrm{dom}(v_{\alpha,\eta}^{p^\alpha}) = j_{\alpha+1,\eta}``j_\alpha(\mathrm{dom}(f_\alpha^p))$ and $v_{\alpha,\eta}^{p^\alpha} = (j_{\alpha,\eta} \upharpoonright d_\alpha^\alpha)^{-1}$; thus, it follows that

$$f_\alpha^q(j_\eta(\tau^*)) = v_{\alpha,\eta}^{p^\alpha}(j_\eta(\tau^*)) = j_\alpha(\tau^*). \hspace{2em} \dashv$$

Fix some ordinal $\beta < \eta$. The forcing $\vec{\mathcal{A}}$ naturally breaks into the product $\vec{\mathcal{A}} \upharpoonright \beta \times \vec{\mathcal{A}}_{\geq\beta}$, and we observe that:

(1) The latter part $\vec{\mathcal{A}}_{\geq\beta}$ is κ_β^+-closed. Therefore, if $\vec{H}_{\geq\beta} \subset \vec{\mathcal{A}}_{\geq\beta}$ is V-generic, then by [2, Proposition 15.1] its pointwise image $j_\beta``\vec{H}_{\geq\beta}$ generates an M_β generic filter for the forcing

$$j_\beta(\vec{\mathcal{A}}_{\geq\beta}) = \prod_{\beta \leq \alpha < \eta} \mathcal{A}(\lambda^\beta, \kappa_\alpha^\beta)^{M_\beta}.$$

We denote this generic by $\vec{H}_{\geq\beta}^\beta$.

(2) Using the same arguments as above, a V-generic filter $\vec{H}_\beta \subset \vec{\mathcal{A}} \upharpoonright \beta$ generates an M_β-generic filter G_β^* for the $j_\beta(\mathbb{P}_{\vec{E} \upharpoonright \beta})$.

We conclude that in the model $V[\vec{H}]$, we can form the generic extension $M_\beta[G_\beta^* \times \vec{H}_{\geq\beta}^\beta]$ of M_β, with respect to the product $j_\beta(\mathbb{P}_{\vec{E} \upharpoonright \beta}) \times j_\beta(\vec{\mathcal{A}}_{\geq\beta})$.

We have the following.

PROPOSITION 3.7. *For each $\beta < \eta$, the model $M_\beta[G_\beta^* \times \vec{H}_{\geq\beta}^\beta]$ can compute $M_\eta[G_\eta^*]$ and therefore*

$$\bigcap_{\beta<\eta} M_\beta[G_\beta^* \times \vec{H}_{\geq\beta}^\beta] \supseteq M_\eta[G^*].$$

It seems likely that equality holds, $\bigcap_{\beta<\eta} M_\beta[G_\beta^* \times \vec{H}_{\geq\beta}^\beta] = M_\eta[G^*]$, but we will not need that in our argument.

§4. Stationary reflection in $M_\eta[G_\eta^*]$. We assume that for every $\alpha < \eta$, there is a Laver indestructible supercompact cardinal θ_α such that $\sup_{\beta<\alpha} \kappa_\beta < \theta_\alpha < \kappa_\alpha$.

We prove the following which completes the proof of Theorem 1.1.

THEOREM 4.1.   *In $M_\eta[G^*]$, every collection of fewer than η many stationary subsets of $j_\eta(\bar\kappa_\eta^+)$ reflects.*

We start by proving a stationary reflection fact that will be used as an intermediate step in the proof.

CLAIM 4.2.   *For every $\alpha < \eta$, every collection of fewer than η many stationary subsets of $j_\alpha(\bar\kappa_\eta^+)$ with cofinalities bounded by $\sup_{\beta<\alpha} \kappa_\beta^\alpha$ reflects in $M_\alpha[G_\alpha^* \times \vec H_{\geq\alpha}^\alpha]$.*

PROOF.   Let T_i for $i < \mu$ be such a collection of stationary sets. By elementarity, $j_\alpha(\theta_\alpha)$ is an indestructible supercompact cardinal between $\sup_{\beta<\alpha} \kappa_\beta^\alpha$ and κ_α^α. Recall that $\vec H_{\geq\alpha}^\alpha$ is generic for $(\kappa_\alpha^\alpha)^+$-directed closed forcing. By the indestructibility of $j_\alpha(\theta_\alpha)$, there is a $j_\alpha(\bar\kappa_\eta^+)$-supercompact embedding $k \colon M_\alpha[\vec H_{\geq\alpha}^\alpha] \to N$. Further, G_α^* is generic for $j_\alpha(\bar\kappa_\alpha^{++})$-cc forcing and $j_\alpha(\bar\kappa_\alpha^{++}) < j_\alpha(\theta_\alpha)$. By standard arguments, we can extend k to include $M_\alpha[G_\alpha^* \times \vec H_{\geq\alpha}^\alpha]$ in the extension by a $j_\alpha(\bar\kappa_\alpha^{++})$-cc forcing. Each T_i remains stationary in this extension, so $\{k(T_i) \mid i < \mu\}$ reflects at $\sup k\text{``} j_\alpha(\bar\kappa_\eta^+)$. It follows that $\{T_i \mid i < \mu\}$ reflects in $M_\alpha[G_\alpha^* \times \vec H_{\geq\alpha}^\alpha]$.   ⊣

We begin the proof of Theorem 4.1. Suppose that for each $i < \mu$, $S_i \in M_\eta[G^*]$ is a stationary subset of $j_\eta(\bar\kappa_\eta^+)$. We can assume that for all i, the cofinality of each ordinal in S_i is some fixed γ_i. It follows that there is some $\bar\alpha < \eta$ such that $\sup_{i<\mu} \gamma_i < \sup_{\beta<\bar\alpha} \kappa_\beta^{\bar\alpha}$. For each α in the interval $[\bar\alpha, \eta)$, let $T_i^\alpha = \{\delta < j_\alpha(\bar\kappa_\eta^+) \mid j_{\alpha,\eta}(\delta) \in S_i\}$.

CLAIM 4.3.   *For $\alpha \geq \bar\alpha$, if $\{T_i^\alpha \mid i < \mu\}$ reflects at an ordinal of cofinality less than κ_α^α in $M_\alpha[G_\alpha^* \times \vec H_{\geq\alpha}^\alpha]$, then $\{S_i \mid i < \mu\}$ reflects in $M_\eta[G^*]$.*

PROOF.   Let $\delta < j_\alpha(\bar\kappa_\eta^+)$ with $\mathrm{cf}(\delta) < \kappa_\alpha^\alpha$ be a common reflection point of the collection $\{T_i^\alpha \mid i < \mu\}$. We claim that each S_i reflects at $j_{\alpha,\eta}(\delta)$. Let $D \subseteq j_{\alpha,\eta}(\delta)$ be club in $M_\eta[G^*]$, of order type $\mathrm{cf}(\delta) = \mathrm{cf}(j_{\alpha,\eta}(\delta))$.

Since $\mathrm{crit}\, j_{\alpha,\eta} > \mathrm{cf}(\delta)$, $j_{\alpha,\eta}$ is continuous at δ and $E = \{\gamma < \delta \mid j_{\alpha,\eta}(\gamma) \in D\}$ is a club in δ. Note that $E \in M_\alpha[G_\alpha^* \times \vec H_{\geq\alpha}^\alpha]$. Since T_i^α reflects at δ, $T_i^\alpha \cap E \neq \emptyset$ and hence $S_i \cap D \neq \emptyset$.   ⊣

Combining the previous two claims if for some $\alpha \geq \bar\alpha$, $\{T_i^\alpha \mid i < \mu\}$ consists of stationary sets, then $\{S_i \mid i < \mu\}$ reflects. So we assume for a contradiction that for each $\alpha \geq \bar\alpha$, there are $i_\alpha < \mu$ and a club $C_\alpha \in M_\alpha[G_\alpha^* \times \vec H_{\geq\alpha}^\alpha]$ such that $T_{i_\alpha}^\alpha \cap C_\alpha = \emptyset$. We fix $J \subseteq \eta$ unbounded and $i^* < \mu$ such that for all $\alpha \in J$, $i_\alpha = i^*$.

Let I_η be the ideal of bounded subsets of η. For $\alpha \leq \eta$, let $\vec H^\alpha / I_\eta$ be the generic for $j_\alpha(\vec{\mathbb{A}})/I_\eta$ derived from $j_\alpha\text{``}\vec H$. Recall that $\vec H$ is generic for $\vec{\mathbb{A}}$ which is a product of Cohen posets, so this makes sense.

CLAIM 4.4.   *For $\alpha \geq \bar\alpha$, there is a club subset of C_α in $M_\alpha[\vec H^\alpha / I_\eta]$.*

PROOF.   We start by showing that for all $\beta > \alpha$ below η there is a club subset of C_α in $M_\alpha[\vec H^\alpha \restriction [\beta, \eta)]$.

To see this, note that $M_\alpha[G_\alpha^* \times \vec H_{\geq\alpha}^\alpha]$ is $(\bar\kappa_\beta^\alpha)^{++}$-cc extension of $M_\alpha[\vec H^\alpha \restriction [\beta, \eta)]$ and $\bar\kappa_\beta^\alpha < j_\alpha(\bar\kappa_\eta)$. Indeed, this forcing decomposes into a part contributing G_α^* which have small cardinality and the product of forcing notions $j_\alpha(\mathcal{A}(\lambda, \kappa_\gamma))$ for $\gamma < \beta$, whose product has $(\bar\kappa_\beta^\alpha)^{++}$-cc. For the moment we let $\mathbb{Q}_\beta$ denote the $(\bar\kappa_\beta^\alpha)^{++}$-cc poset used in order to move from $M_\alpha[\vec H^\alpha \restriction [\beta, \eta)]$ to $M_\alpha[G_\alpha^* \times \vec H_{\geq\alpha}^\alpha]$.

We set take $C_{\alpha,\beta}$ to be the set of closure points of the function assigning each γ to the supremum of the set

$$\{\gamma^* \mid \exists q \in \mathbb{Q}_\beta, \ q \Vdash \check{\gamma}^* = \min(\dot{C}_\alpha \setminus (\check{\gamma}+1))\},$$

as computed in the model $M_\alpha[\vec{H}^\alpha \restriction [\beta,\eta]]$. Clearly this is a club. Further if $\beta < \beta'$, then $C_{\alpha,\beta'} \subseteq C_{\alpha,\beta}$.

Since the clubs are decreasing, $\bigcap_{\beta>\alpha} C_{\alpha,\beta}$ is definable in $M_\alpha[\vec{H}^\alpha/I_\eta]$ as the set of ordinals γ such that for some condition $\vec{a} \in j_\alpha(\mathbb{A})$, $\vec{a}/I_\eta \in \vec{H}^\alpha/I_\eta$ and for all sufficiently large β, $\vec{a} \restriction [\beta,\eta)$ forces $\gamma \in C_{\alpha,\beta}$. So $\bigcap_{\beta>\alpha} C_{\alpha,\beta}$ is as required for the claim.   ⊣

Let $\dot{D}_\alpha$ be an $\vec{H}^\alpha/I_\eta$-name for the club from the previous lemma.

LEMMA 4.5. $\vec{H}^\eta/I_\eta \in M_\eta[G^*]$.

PROOF. Let $p \in j_\eta(\vec{\mathbb{A}})$. We will show how to decide in $M_\eta[G^*]$ whether $p \in \vec{H}^\eta/I_\eta$ or not.

Now, fix $q \in G^*$ with $\operatorname{dom} q(\alpha) \supseteq \operatorname{dom} p(\alpha)$. Let us claim that $p \in \vec{H}/I_\eta$ if and only if there is β such that $f^q(\gamma) \supseteq p(\gamma)$ for all $\gamma \in [\beta,\eta)$.

Indeed, without loss of generality, $q = j_\eta(\vec{q})^\frown\langle v_0,\dots,v_{m-1}\rangle$ for $\vec{q} \in \vec{H}$ and $v_0,\dots,v_{m-1}$ generators.

Thus, for every β which is larger than the index of the measures of $v_0,\dots,v_{m-1}$ $f^{\vec{q}(\beta)} \in H(\beta)$ and thus after applying j_η we conclude that the condition which is defined by $\langle f^q(\gamma) \mid \beta \leq \gamma < \eta\rangle$ belongs to $\vec{H}^\eta \restriction [\beta,\eta)$. Since it is compatible with p, $p \in \vec{H}^\eta/I_\eta$.

In the other direction, one can take β large enough so that $p \restriction [\beta,\eta) \in \vec{H}^\eta \restriction [\beta,\eta)$ and then $p \restriction [\beta,\eta)$ is the Cohen part of a condition in G^*.   ⊣

CLAIM 4.6. $\langle j_{\alpha,\eta}(\dot{D}_\alpha)_{\vec{H}^\eta/I_\eta} \mid \bar{\alpha} \leq \alpha < \eta\rangle \in M_\eta[G^*]$.

PROOF. By the previous lemma, $\vec{H}^\eta/I_\eta \in M_\eta[G^*]$. Further by Proposition 3.6, $M_\eta[G^*]$ is closed under η-sequences. So the sequence of names for clubs $\langle j_{\alpha,\eta}(\dot{D}_\alpha) \mid \alpha < \eta\rangle \in M_\eta[G^*]$.   ⊣

To get a contradiction it is enough to show that $\bigcap_{\bar{\alpha}\leq\alpha<\eta} j_\alpha(\dot{D}_\alpha)_{\vec{H}^\eta/I_\eta} \cap S_{i^*} = \emptyset$. Suppose that there is some δ in the intersection. We can find some $\alpha \in J$ and $\bar{\delta}$ such that $j_{\alpha,\eta}(\bar{\delta}) = \delta$. However, by the definitions of D_α and $T^\alpha_{i^*}$, we must have that $\bar{\delta} \in C_\alpha \cap T^\alpha_{i^*}$, a contradiction.

This completes the proof of Theorem 1.1.

§5. Bad scales. In this section we give the proof of Theorem 1.2. For this theorem we work with the forcing $\mathbb{P}$ as before and assume that there is an indestructibly supercompact cardinal $\theta < \kappa_0$. Working in V, let $\vec{f}$ be a scale of length $\bar{\kappa}^+_\eta$ in $\prod_{\alpha<\eta} \kappa^+_\alpha$. As before let $\vec{H}$ be generic for $\vec{\mathcal{A}}$ over V and let G^* be the $j_\eta(\mathbb{P})$ generic over M_η defined above.

LEMMA 5.1. *In $M_\eta[G^*]$, $j_\eta(\vec{f})$ is a scale of length $j_\eta(\bar{\kappa}^+_\eta)$ in $\prod_{\alpha<\eta} j(\kappa_\alpha)^+$.*

Proof. Let $g \in \left(\prod_{\alpha < \eta} j(\kappa_\alpha)^+ \right) \cap M_\eta[G^*]$. Clearly $g \in V[\vec{H}]$. Since each κ_α^+ is a continuity point of j_η, we can find an ordinal $\gamma_\alpha < \kappa_\alpha^+$ such that $j_\eta(\gamma_\alpha) > g(\alpha)$.

By the distributivity of $\vec{\mathbb{A}}$, the sequence $\tilde{g} = \langle \gamma_\alpha \mid \alpha < \eta \rangle$ belongs to V. Pick (in V) an ordinal ζ such that f_ζ dominates $\tilde{g}$. Then, $j_\eta(f_\zeta) = j_\eta(f)_{j_\eta(\zeta)}$ dominates g. ⊣

Lemma 5.2. *For $\delta < \kappa_\eta^+$ with $\eta < \mathrm{cf}(\delta) < \kappa_0$, if $j_\eta(\delta)$ is a good point for $j(\vec{f})$ in $M_\eta[G^*]$, then δ is a good point for $\vec{f}$ in $V[\vec{H}]$.*

Proof. Suppose that $A \subseteq j_\eta(\delta)$ and $\alpha^* < \eta$ witness that $j_\eta(\delta)$ is good for $\vec{f}$ in $M_\eta[G^*]$. Note that $j_\eta``\delta$ is cofinal in $j(\delta)$. By thinning $A$ if necessary we let $B \subseteq j_\eta``\delta$ be an unbounded set such that each element $\gamma \in B$ has a greatest element of A less than or equal to it. For each γ in B, let α_γ be such that for all $\alpha \geq \alpha_\gamma$,

$$ j(\vec{f})_{\max(A \cap (\gamma+1))}(\alpha) \leq j(\vec{f})_\gamma(\alpha) < j(\vec{f})_{\min(A \setminus (\gamma+1))}(\alpha). $$

Let B' be an unbounded subset of B on which α_γ is fixed. Then B' witnesses that $j_\eta(\delta)$ is good. Since $B' \subseteq j_\eta``\delta$, we have that $\{\gamma < \delta \mid j_\eta(\gamma) \in B'\}$ witnesses that δ is good for $\vec{f}$ in $V[\vec{H}]$. ⊣

In $V[\vec{H}]$, let $S = \{\delta < \bar{\kappa}_\eta^+ \mid \delta$ is nongood for $\vec{f}$ and $\eta < \mathrm{cf}(\delta) < \theta\}$. We claim that S is stationary. Let $k \colon V[\vec{H}] \to N$ be an elementary embedding witnessing that θ is $\bar{\kappa}_\eta^+$-supercompact in $V[\vec{H}]$. Standard arguments show that $\sup k``\bar{\kappa}_\eta^+$ is a nongood point for $k(\vec{f})$. It follows that $S$ is stationary, since $\sup k``\bar{\kappa}_\eta^+ \in k(C)$ every club $C \subseteq \bar{\kappa}_\eta^+$ in $V[\vec{H}]$.

Now suppose that in $M_\eta[G^*]$, there is a club D of good points for $j_\eta(\vec{f})$. In $V[\vec{H}]$, let $C = \{\delta < \bar{\kappa}_\eta^+ \mid j_\eta(\delta) \in D\}$. By the previous lemma, C is a $< \kappa_0$-club consisting of good points for $\vec{f}$. However, $S \cap C$ is nonempty, a contradiction.

§6. Countable cofinality. Our goal in this section is to show that given the assumptions of Theorem 1.4, there exists a forcing extension which adds a generic filter for the extender based Prikry forcing by a (κ, κ^{++})-extender (in particular, forces $\mathrm{cf}(\kappa) = \omega$ and $2^\kappa = \kappa^{++}$) and satisfies that every stationary subset of κ^+ reflects.

A necessary step for obtaining the latter is to add a club $D \subseteq \kappa^+$ disjoint from $S_\kappa^{\kappa^+}$, the set of ordinals $\alpha < \kappa^+$ of cofinality κ in the ground model. In [13], the second and third authors address the situation for adding a single Prikry sequence to κ. It is shown that under the subcompactness assumption of κ, there is a Prikry-type forcing which both singularizes κ and adds a club D as above, without generating new nonreflecting stationary sets.

An additional remarkable aspect of the argument of [13] is that it presents a fertile framework in which the arguments address an extension $N_\omega[\mathcal{H}]$ of an iterated ultrapower N_ω of V, and assert directly that stationary reflection holds in $N_\omega[\mathcal{H}]$ without having to specify the poset by which $\mathcal{H}$ is added.

Let us briefly describe the key ingredients of the construction of [13], to serve as a reference for our arguments in the context of the failure of SCH. In [13], one starts from a normal measure U on κ in V, and consider the ω-iterated

ultrapower by U, given by

$$N_0 = V, i_{n,n+1} \colon N_n \to N_{n+1} \cong \mathrm{Ult}(N_n, i_n(U)),$$

and the direct limit embedding $i_\omega \colon V \to N_\omega$. It is well known that the sequence of critical points $\langle \kappa_n \mid n < \omega \rangle$, $\kappa_{n+1} = i_{n,n+1}(\kappa_n)$ is Prikry generic over N_ω and that $N_\omega[\langle \kappa_n \mid n < \omega \rangle] = \bigcap_n N_n$ [1, 3].

Let $\mathbb{Q}$ be the Prikry name for the forcing for adding a disjoint club from $(S_\kappa^{\kappa^+})^V = \kappa^+ \cap \mathrm{Cof}^V(\kappa)$. The forcing $\mathbb{Q}_\omega = i_\omega(\mathbb{Q})^{\langle \kappa_n \mid n < \omega \rangle} \in N_\omega[\langle \kappa_n \mid n < \omega \rangle]$ is isomorphic to the forcing for adding a κ^+-Cohen set over V, and likewise, to adding a κ_n^+-Cohen set over N_n, for each $n < \omega$.

Moreover, taking a $\mathbb{Q}_\omega$-generic filter H over V, we have that both H and $i_{0,1}``H$ generate mutually generic filters over $N_1$ for $i_1(\mathbb{Q}_\omega) = \mathbb{Q}_\omega$. More generally, for each $n$, the sequence $H_0^n, \dots, H_n^n$, where each $H_k^n$ is generated by $i_{k,n}``H \subset i_n(\mathbb{Q}_\omega) = \mathbb{Q}_\omega$ for $1 \le k \le n$, are mutually generic filters for $\mathbb{Q}_\omega$ over N_n. With this choice of "shifts" of H, we obtain that for each $n < k$, the standard iterated ultrapower map $i_{n,k} \colon N_n \to N_k$ extends to $i_{n,k}^* \colon N_n[\langle H_0^n, \dots, H_n^n \rangle] \to N_k[\langle H_0^k, \dots, H_n^k \rangle] \subseteq N_k[\langle H_0^k, \dots, H_k^k \rangle]$. For each n, the sequence $\langle H_0^n, \dots, H_n^n \rangle$ is denoted by $\mathcal{H}_n$. The final extension $N_\omega[\mathcal{H}]$ of N_ω is given by the sequence $\mathcal{H} = \langle H_n^\omega \mid n < \omega \rangle$, where H_n^ω is the filter generated by $i_{n,\omega}``H$, which achieves the critical equality

$$N_\omega[\mathcal{H}] = \bigcap_n N_n[\mathcal{H}_n].$$

From this equality it follows at once that:

(1) $N_\omega[\mathcal{H}]$ is closed under its κ-sequences;
(2) $\kappa_\omega = i_\omega(\kappa)$ is singular in $N_\omega[\mathcal{H}]$, as $\langle \kappa_n \mid n < \omega \rangle$ belongs to each $N_n[\mathcal{H}_n]$; and
(3) $H \in N_\omega[\mathcal{H}]$, as $H = H_n^n$ for all $n < \omega$.

Therefore every stationary subset S of κ_ω^+ in $N_\omega[\mathcal{H}]$ can be assumed to concentrate at some cofinality $\rho < \kappa_\omega$. Say for simplicity that $\rho < \kappa_0$, one shows that S reflects in $N_\omega[\mathcal{H}]$ by examining its pull backs $S_n = i_{n,\omega}^{-1}(S) \subseteq \kappa_n^+$. If we have that $\kappa_n = i_n(\kappa)$ is Π_1^1-subcompact in the Cohen extension $N_n[\mathcal{H}_n]$ of N_n, then we have that if S_n is stationary then it must reflect at some $\delta < \kappa_n^+$ of cofinality $\delta < \kappa_n$. Using $i_{n,\omega}$, we conclude from this that S reflects at $i_{n,\omega}(\delta)$ in $N_\omega[\mathcal{H}]$.

To rule out the other option, of having all $S_n \subseteq \kappa_n^+$ being nonstationary in $N_n[\mathcal{H}_n]$, one takes witnessing disjoint clubs $C_n \subseteq \kappa_n^+$ and uses the fact that for each $n < \omega$, $i_{n,\omega} \colon N_n \to N_\omega$ extends to $i_{n,\omega}^* \colon N_n[\mathcal{H}_n] \to N_\omega[i_{n,\omega}``\mathcal{H}_n] \subset N_\omega[\mathcal{H}]$. This allows us to show that the club $D_n = i_{n,\omega}^*(C_n) \subseteq \kappa_\omega^+$ belongs to $N_\omega[\mathcal{H}]$ for each n. Since $N_\omega[\mathcal{H}]$ is closed under its κ sequences, it computes $\langle D_n \mid n < \omega \rangle$ correctly and thus also $D = \bigcap_n D_n$, that would have to be disjoint from S, a contradiction.

We turn now to the new construction and prove Theorem 1.4. Let V' be a model which contains a κ^+-Π_1^1-subcompact cardinal κ, which also carries a (κ, κ^{++})-extender.

Recall that κ is κ^+-Π_1^1-subcompact if for every set $A \subseteq H(\kappa^+)$ and every Π_1^1-statement Φ such that $\langle H(\kappa^+), \in, A \rangle \models \Phi$, there are $\rho < \kappa$, $B \subseteq H(\rho^+)$, and an elementary embedding $j \colon \langle H(\rho^+), \in, B \rangle \to \langle H(\kappa^+), \in, A \rangle$ with $cp(j) = \rho$, such that $\langle H(\rho^+), \in, B \rangle \models \Phi$.

Let V be obtained from V' by an Easton-support iteration of products $\mathrm{Add}(\alpha^+, \alpha^{++})$ for inaccessible $\alpha \leq \kappa$.

By [13, Lemma 42], κ remains κ^+-Π_1^1-subcompact in V and even in a further extension by $\mathrm{Add}(\kappa^+, \kappa^{++})$. Moreover, by standard arguments it is routine to verify that κ still carries a (κ, κ^{++})-extender in V. We note that as a consequence of κ^+-Π_1^1-subcompactness in V, simultaneous reflection holds for collections of fewer than κ many stationary subsets of $S_{<\kappa}^{\kappa^+}$. Further this property is indestructible under $\mathrm{Add}(\kappa^+, \kappa^{++})$.

Working in V, let E be a (κ, κ^{++})-extender. Let

$$\langle j_{m,n} \colon M_m \to M_n \mid m \leq n \leq \omega \rangle$$

be the iteration by E and let

$$\langle i_{m,n} \colon N_m \to N_m \mid m \leq n \leq \omega \rangle$$

be the iteration by the normal measure E_κ where $V = M_0 = N_0$. We write j_n for $j_{0,n}$ and i_n for $i_{0,n}$.

We describe a generic extension of M_ω in which $j_\omega(\kappa^+)$ satisfies the conclusion of the theorem. It follows by elementarity that there is such an extension of V. We are able to isolate the forcing used, but this is not required in the proof.

We start by constructing a generic object for $j_\omega(\mathbb{P}_E)$ over M_ω, where $\mathbb{P}_E$ is the extender based forcing of Merimovich. Although for the most part, we will refer to Merimovich's arguments in [15], our presentation follows a more up-do-date presentation of the forcing, given by Merimovich in [16, Section 2].

We recall that conditions $p \in \mathbb{P}_E$ are pairs of the form $p = \langle f, T \rangle$, where $f \colon d \to [\kappa]^{<\omega}$ is a partial function from κ^{++} to $[\kappa]^{<\omega}$ with domain $d \in [\kappa^{++}]^{\leq \kappa}$ with $\kappa \in d$, and T is tree of height ω, whose splitting sets are all measure one with respect to a measure $E(d)$ on V_κ, derived from the extender E. The generator of $E(d)$ is the function $\mathrm{mc}(d) = \{\langle j(\alpha), \alpha \rangle \mid \alpha \in d\}$. Therefore, a typical node in the tree is an increasing sequence of functions $\langle v_0, \ldots, v_{k-1} \rangle$ where each v_i is a partial, order preserving function $v_i \colon \mathrm{dom}(v_i) \to \kappa$, with $\kappa \in \mathrm{dom}(v_i)$ and $|v_i| = v_i(\kappa)$. The sequence $\langle v_0, \ldots, v_{k-1} \rangle$ is increasing in the sense that $v_i(\kappa) < v_{i+1}(\kappa)$ for all i. When extending conditions $p = \langle f, T \rangle \in \mathbb{P}_E$ we are allowed to:

(i) extend f in as Cohen conditions (namely, add points $\gamma < \kappa^{++}$ to $\mathrm{dom}(f)$ and arbitrarily choose $f(\gamma) \in [\kappa]^{<\omega}$), and modify the tree parts;

(ii) shrink the tree T; and

(iii) choose a point $\langle v \rangle \in \mathrm{succ}_\emptyset(T)$ to extend $p = \langle f, T \rangle$ to $p_{\langle v \rangle} = \langle f_{\langle v \rangle}, T_{\langle v \rangle} \rangle$, where $f_{\langle v \rangle}$ is defined by $\mathrm{dom}(f_{\langle v \rangle}) = \mathrm{dom}(f)$ and

$$f_{\langle v \rangle}(\alpha) = \begin{cases} f(\alpha) \cup \{v(\alpha)\}, & \text{if } \alpha \in \mathrm{dom}(v) \text{ and } v(\alpha) > \max(f(\alpha)), \\ f(\alpha), & \text{otherwise.} \end{cases}$$

Any extension of p is obtained by finite combination of (i)–(iii), and the direct extensions of p are those which are obtained by (i) and (ii). The poset $\mathbb{P}_E^*$ is the suborder of $\mathbb{P}_E$ whose extension consists only of the Cohen type extension (i). Clearly, $\mathbb{P}_E^*$ is isomorphic to $\mathrm{Add}(\kappa^+, \kappa^{++})$.

We turn to describe the construction of a $j_\omega(\mathbb{P}_E)$ generic following [15]. Let G_0 be $\mathbb{P}_E^*$ generic. We work by induction to define G_n for $n < \omega$. Suppose that we

have defined $G_n \subseteq j_n(\mathbb{P}_E^*)$ for some $n < \omega$. First, let G_{n+1}' be the closure of the set $j_{n,n+1}``G_n$. Then, we take G_{n+1} to be obtained from G_{n+1}' by adding the ordinal α to G_{n+1}' at coordinate $j_{n,n+1}(\alpha)$, for all $\alpha < j_n(\kappa^{++})$.

CLAIM 6.1. G_{n+1} is M_{n+1}-generic for $j_{n+1}(\mathbb{P}_E^*)$.

PROOF. This is a straightforward application of Woodin's surgery argument [5], so we only sketch the proof. Let D be a dense open subset of $j_{n+1}(\mathbb{P}_E^*)$. Let E be the set of all f in D such that all $j_n(\kappa)$ sized modifications of f are in D. E is dense using the closure of $j_{n+1}(\mathbb{P}_E^*)$. Now the fact that G_{n+1}' meets E implies that G_{n+1} meets D, since each condition in G_{n+1} is a $j_n(\kappa)$ sized modification of one in G_{n+1}'. ⊣

We make a few remarks.

REMARK 6.2. For each n, G_n can be obtained directly from the upwards closure of $j_n``G_0$ by combining the alterations used in construction of G_i for $i < n$.

Since the proof of the previous claim can be repeated in any suitably closed forcing extension, we have the following.

REMARK 6.3. If H is generic for $j_n(\kappa^+)$-closed forcing and mutually generic to the upwards closure of $j_n``G_0$, then H and G_n are mutually generic.

Let G_ω be the $j_\omega(\mathbb{P}_E)$-generic obtained from G_0 as in [15].

CLAIM 6.4. $M_\omega[G_\omega]$ is closed under κ-sequences.

PROOF. It is enough to show that $M_\omega[G_\omega]$ is closed under κ-sequences of ordinals. Let $\langle \gamma_\delta \mid \delta < \kappa \rangle$ be a sequence of ordinals. We can assume that each γ_δ is of the form $j_\omega(g_\delta)(\alpha_\delta, j(\alpha_\delta), \dots j_{n_\delta-1}(\alpha_\delta))$ for some $g_\delta : [\kappa]^{n_\delta} \to \kappa^{++}$ and $\alpha_\delta < \kappa^{++}$. We refer the reader to [15, Corollary 2.6] for a proof. Since $\langle j_\omega(g_\delta) \mid \delta < \kappa \rangle = j_\omega(\langle g_\delta \mid \delta < \kappa \rangle) \restriction \kappa \in M_\omega$, it is enough to show that $\langle (\alpha_\delta, \dots j_{n_\delta-1}(\alpha_\delta)) \mid \delta < \kappa \rangle \in M_\omega[G_\omega]$. To see this note that $\{\alpha_\delta \mid \delta < \kappa\} = \text{dom}(f)$ for some $f \in G_0$ and that the sequence $f(\alpha_\delta) \frown (\alpha_\delta, \dots j_{n_\delta-1}(\alpha_\delta))$ is an initial segment of the ω-sequence with index $j_\omega(\alpha_\delta)$ in G_ω. This is enough to compute $\langle (\alpha_\delta, \dots j_{n_\delta-1}(\alpha_\delta)) \mid \delta < \kappa \rangle$, since $j_\omega``f \in M_\omega$. ⊣

The following lemma is proved in [12, Section 4].

LEMMA 6.5. $\bigcap_{n<\omega} M_n[G_n] = M_\omega[G_\omega]$.

Since we will need to prove a stronger version ahead, let us sketch the proof of this lemma.

PROOF. First, the inclusion $\bigcap_{n<\omega} M_n[G_n] \supseteq M_\omega[G_\omega]$ is clear.
As in the case of the standard Bukovský–Dehornoy Theorem, in order to prove the other direction we would like to take a set of ordinals $X \in M_\omega[G_\omega]$ and consider the sets

$$Y_n = \{\alpha \mid j_{n,\omega}(\alpha) \in X\} \in M_n[G_n].$$

Note that the filter generated by $j_{n,\omega}``G_n$ is not a member of $M_\omega[G_\omega]$ and thus we must be more careful about the way we push Y_n into $M_\omega[G_\omega]$.

Let us pick for each n a name $\dot{y}_n \in M_n$ for Y_n. As the sequence of names $\langle j_{n,\omega}(\dot{y}_n) \mid n < \omega \rangle$ is a member of $M_\omega[G_\omega]$ (using its closure under countable sequence), we

must verify that we can evaluate correctly, without access to the filters $j_{n,\omega}(G_n)$, whether an ordinal γ belongs to a tail of sets $j_{n,\omega}(\dot{y}_n)_{j_{n,\omega}(G_n)}$.

This is done using Merimovich's genericity criteria [17]. Indeed, for $a \subseteq j_\omega(\kappa^{++})$ set of ordinals of cardinality $j_\omega(\kappa)$, which is the set of ordinals of an elementary substructure of a sufficiently large initial segment of M_ω, one can consider a tree Prikry type forcing using $j_\omega(E)(a)$, and verify that there is a generic sequence $\langle t_n \mid n < \omega \rangle$ for it in $M_\omega[G_\omega]$ such that, $\exists q \in j_\omega(\mathbb{P}_E^*)$ such that for all n, $q^\frown \langle t_0, \dots, t_{n-1} \rangle \in G_\omega$. Moreover, this sequence is unique up to shifts and finite modifications (see [17, Lemma 3.4])—which is the crux of this argument.

As $j_\omega(\kappa) \in a$ and we may assume that the Prikry sequence at this coordinate is the standard one, one can fix the shift and identify the sequence $\langle j_\omega(p)^\frown \langle v_0, \dots, v_{n-1} \rangle \mid n < \omega \rangle$ up to a finite error.

In particular, by taking a to be large enough, we conclude that there is $p \in G_0$ such that for all large n,

$$j_\omega(p)^\frown \langle v_0, \dots, v_{n-1} \rangle \Vdash \check{\gamma} \in j_{n,\omega}(\dot{y}_n)$$

if and only if $\gamma \in X$.

Therefore, $\gamma \in X$ if and only if there is a condition $q \in G_\omega$ with large domain a and a Prikry generic $\langle t_n \mid n < \omega \rangle$ such that the length of $q(j_\omega(\kappa))$ is zero and for all large n

$$q^\frown \langle v_0, \dots, v_{n-1} \rangle \Vdash \check{\gamma} \in j_{n,\omega}(\dot{y}_n). \qquad \dashv$$

We are now ready to describe the generic extension of M_ω. We recall some of the basic ideas from [13]. Let $\dot{\mathbb{Q}}$ be the canonical name in the Prikry forcing defined from E_κ for the poset to shoot a club disjoint from the set of $\alpha < \kappa^+$ such that $\mathrm{cf}^V(\alpha) = \kappa$. Let $\mathbb{Q}_\omega$ be the forcing $i_\omega(\dot{\mathbb{Q}})$ as interpreted by the critical sequence $\langle i_n(\kappa) \mid n < \omega \rangle$. By the argument following Claim 39 of [13], $\mathbb{Q}_\omega$ is equivalent to the forcing $\mathrm{Add}(\kappa^+, 1)$ in V. In fact, for every $n < \omega$, in N_n it is equivalent to $\mathrm{Add}(i_n(\kappa^+), 1)$.

Let $k_n \colon N_n \to M_n$ be the natural embedding given by

$$k_n(i_n(f)(\kappa, \dots i_{n-1}(\kappa))) = j_n(f)(\kappa, \dots j_{n-1}(\kappa)).$$

We discussed the notion of a *width* of an elementary embedding in the previous section. We recall a fundamental result concerning lifting embeddings and their widths (see [2]).

LEMMA 6.6. *Suppose that $k \colon N \to M$ has width μ, $\mathbb{Q} \in N$ is a μ^+-distributive forcing, and $H \subseteq \mathbb{Q}$ is generic over N, then $k``H \subset k(\mathbb{Q})$ generates a generic filter $\langle k``H \rangle$ for M.*

CLAIM 6.7. *For all $1 \le n < \omega$, k_n has width $\le (i_{n-1}(\kappa)^{++})^{N_n}$.*

PROOF. Let $x = j_n(g)(\alpha, j(\alpha), \dots j_{n-1}(\alpha))$ be an element of M_n, $g \colon [\kappa]^n \to V$ and $\alpha < \kappa^{++}$. Consider $h = i_n(g) \restriction (i_{n-1}(\kappa)^{++})^{N_n}$. Then

$$k(h)(\alpha, j(\alpha), \dots j_{n-1}(\alpha)) = x.$$

Indeed, the domain of $k(h)$ is large enough for this evaluation to make sense. $\dashv$

We are now ready to define the analogs of $\mathcal{H}$ and $\mathcal{H}_n$ for our situation. Recall that for subset X of some poset, we write $< X >$ for the upwards closure in that poset. It will be clear from the context which poset we are working with, when taking this upwards closure.

Let H be generic for $\mathbb{Q}_\omega$ over $V[G_0]$ and recall that $\mathbb{Q}_\omega = i_n(\mathbb{Q}_\omega)$ is also a member of N_n for every n, and is $i_n(\kappa^+)$-closed. We note that clearly, $i_n(\kappa^+) \geq (i_{n-1}(\kappa)^{++})^{N_n} \geq \mathrm{width}(k_n)$.

Let

$$\mathcal{H}_n = \langle <j_{m,n} \circ k_m "H> \mid m \leq n\rangle$$

and

$$\mathcal{H} = \langle <j_{m,\omega} \circ k_m "H> \mid m < \omega\rangle.$$

Note that $\mathcal{H}_n$ is a subset of $\prod_{m \leq n} j_{m,n}(k_m(\mathbb{Q}_\omega))$. We do not yet know that it is generic.

We prove a sequence of claims about $\mathcal{H}_n$ and $\mathcal{H}$. The first is straightforward.

CLAIM 6.8. *For all $n < \omega$,*

$$\mathcal{H}_{n+1} = <j_{n,n+1} "\mathcal{H}_n> \frown <k_{n+1} "H>.$$

Let $\bar{\mathcal{H}}_n = \langle <i_{m,n} "H> \mid m \leq n\rangle$.

CLAIM 6.9. $\mathcal{H}_n = <k_n "\bar{\mathcal{H}}_n>$.

This is immediate from the following. For $m \leq n$, we have

$$<k_n "<i_{m,n} "H>> = <j_{m,n} "<k_m "H>>$$

since $k_n \circ i_{m,n} = j_{m,n} \circ k_m$. In particular, since $\bar{\mathcal{H}}_n$ is generic for $i_n(\kappa^+)$-closed forcing over N_n and k_n has width less than $i_n(\kappa)$, $\mathcal{H}_n$ is generic over M_n.

CLAIM 6.10. $\mathcal{H}_n$ *is mutually generic with G_n over M_n.*

PROOF. Since G_0 and H are mutually generic over V, we can repeat the argument from Claim 18 of [13] to see that $\bar{\mathcal{H}}_n$ is generic over the model $N_n[<i_n "G_0>]$. By the previous claim, $<j_n "G_0>$ and $\mathcal{H}_n$ are mutually generic over M_n. By Remark 6.3, $\mathcal{H}_n$ is mutually generic with G_n. ⊣

LEMMA 6.11. $M_\omega[G_\omega][\mathcal{H}]$ *is closed under κ-sequences.*

PROOF. It is enough to show that every κ-sequence of ordinals from the larger model $M_\omega[G_\omega][\mathcal{H}]$ is in $M_\omega[G_\omega]$. Let $\vec{\alpha}$ be such a sequence. By construction, $\vec{\alpha} \in V[G_0][H]$ and hence in $V[G_0]$. However, $M_\omega[G_\omega]$ is closed under κ-sequences in $V[G_0]$, so $\vec{\alpha} \in M_\omega[G_\omega]$, as required. ⊣

LEMMA 6.12. $\bigcap_{n<\omega} M_n[G_n][\mathcal{H}_n] = M_\omega[G_\omega][\mathcal{H}]$.

PROOF. The proof is similar to the proof of Lemma 6.5. First, note that the embedding $j_{n,\omega} \colon M_n \to M_\omega$ lifts to an elementary embedding

$$j_{n,\omega}^* \colon M_n[\mathcal{H}_n] \to M_\omega[<j_{n,\omega} "\mathcal{H}_n>],$$

and $M_\omega[<j_{n,\omega} "\mathcal{H}_n>] \subseteq M_\omega[\mathcal{H}]$. Further, the construction of G_ω is the same whether we start in M_n or $M_n[\mathcal{H}_n]$.

It is immediate that $\bigcap_{n<\omega} M_n[G_n][\mathcal{H}_n] \supseteq M_\omega[G_\omega][\mathcal{H}]$. For the other inclusion, we work as before and suppose that $x \in \bigcap_{n<\omega} M_n[G_n][\mathcal{H}_n]$ is a set of ordinals. For each n, we can define $y_n = \{\gamma \mid j_{n,\omega}(\gamma) \in x\}$. Continuing as before, but working in $M_n[\mathcal{H}_n]$, we have that $y_n = \dot{y}_n^{G_n}$ for some name $\dot{y}_n$.

By the previous lemma, $M_\omega[G_\omega][\mathcal{H}]$ has the sequence $\langle j_{n,\omega}^*(\dot{x}_n^*) \mid n < \omega \rangle$. As in the proof of Lemma 6.5, we will now argue that for every suitable set of ordinals $a \subseteq j_\omega(\kappa^{++})$ of cardinality $j_\omega(\kappa)$, there is a unique (up to finite shift and finite modification) sequence $\langle t_n \mid n < \omega \rangle$ which is generic for the Prikry forcing for the measure $j_\omega(E)(a)$ and satisfies that $q^\frown \langle t_n \mid n < \omega \rangle \in G_\omega$ for all n.

Since there is $p \in G$ such that for $j_\omega(p)$ the sequence $\langle v_n \upharpoonright a \mid n < \omega \rangle$ will be generic for the corresponding forcing, we conclude (by uniqueness) that for every such sequence, for all large n,

$$q^\frown \langle t_n \mid n < \omega \rangle \in j_{n,\omega}(G_n)$$

assume that the length of q at $j_\omega(\kappa)$ is zero, so there is no shift relative to the indexing of the G_n-s.

For each ordinal γ one can take a large enough so that for all n, the truth value of $\gamma \in j_{n,\omega}(\dot{y}_n)$ is decided by $q^\frown \langle t_i \mid i < n \rangle$. Therefore, $\gamma \in x$ if and only if there is $q \in G_\omega$ and $\langle t_n \mid n < \omega \rangle$ Prikry generic for dom q as above such that for all large n,

$$q^\frown \langle t_i \mid i < n \rangle \Vdash \check{\gamma} \in j_{n,\omega}(\dot{y}_n). \qquad \dashv$$

Let $k_\omega \colon N_\omega \to M_\omega$ be the natural elementary embedding. Note that k_ω naturally extends to an embedding between the two Prikry generic extensions,

$$k_\omega^* \colon N_\omega[\langle i_m(\kappa) \mid m < \omega \rangle] \longrightarrow M_\omega[\langle j_m(\kappa) \mid m < \omega \rangle].$$

Similarly, for each $n < \omega$ we define $W_{n,\omega}$ to be the limit ultrapower obtained by starting in M_n and iterating the measure $j_n(E_\kappa)$, the normal measure of $j_n(E)$. As with N_ω, denote the critical points of the iteration by $j_n(E_\kappa)$ and its images by $\langle \kappa_m^n \rangle_{m<\omega}$, and let $i_\omega^n \colon V \to W_{n,\omega}$ denote the composition of the finite ultrapower map $j \colon V \to M_n$ with the latter infinite iteration map by the normal measures. Therefore elements $y_n \in W_{n,\omega}$ are of the form $i_\omega^n(f)(\alpha_0, \dots, \alpha_{n-1}, \kappa_0^n, \dots, \kappa_{m-1}^n)$ for some $n, m < \omega$, $f \colon \kappa^{n+m} \to V$ in V, and $\alpha_\ell \in j_\ell(\kappa^{++})$ for each $\ell < n$.

The structures $W_{n,\omega}$ are naturally connected via maps $k_{n,r} \colon W_{n,\omega} \to W_{r,\omega}$ for $n \le r < \omega$, given by

$$k_{n,r}(i_\omega^n(f)(\alpha_0, \dots, \alpha_{n-1}, \kappa_0^n, \dots, \kappa_{m-1}^n))$$
$$= i_\omega^r(f)(\alpha_0, \dots, \alpha_{n-1}, \kappa_0^r, \dots, \kappa_{m-1}^r).$$

It is straightforward to verify that the limit of the directed system

$$\{W_{n,\omega}, k_{n,r} \mid n \le r < \omega\}$$

is M_ω, and the direct limit maps $k_{n,\omega} \colon W_{n,\omega} \to M_\omega$, which are defined by

$$k_{n,r}\left(i_\omega^n(f)(\alpha_0, \dots, \alpha_{n-1}, \kappa_0^n, \dots, \kappa_{m-1}^n)\right)$$
$$= j_\omega(f)(\alpha_0, \dots, \alpha_{n-1}, j_n(\kappa), j_{n+1}(\kappa), \dots, j_{n+m-1}(\kappa))$$

naturally extend to the generic extensions by the suitable Prikry sequences. Indeed, denote $W_{n,\omega}[\langle j_m(\kappa) \mid m < n\rangle^\frown \langle i_m(j_n(\kappa)) \mid m < \omega\rangle]$ by $W_{n,\omega}^*$, for each $n < \omega$. Then $k_{n,\omega}$ lifts to

$$k_{n,\omega}^* \colon W_{n,\omega}^* \longrightarrow M_\omega[\langle j_m(\kappa) \mid m < \omega\rangle].$$

Finally, we note that the following implies that $M_\omega[\langle j_m(\kappa) \mid m < \omega\rangle]$ is the direct limit of the system of Prikry generic extensions

$$\langle W_{n,\omega}^*, k_{n,k}^* \mid n \leq k < \omega\rangle.$$

CLAIM 6.13. $<k_\omega^*\text{``}H> \in M_\omega[G_\omega][\mathcal{H}]$ is generic for $j_\omega(\dot{\mathbb{Q}})^{\langle j_n(\kappa) \mid n < \omega\rangle}$ over $M_\omega[\langle j_n(\kappa) \mid n < \omega\rangle]$.

PROOF. Let $D \in M_\omega[\langle j_n(\kappa) \mid n < \omega\rangle]$ be a dense open subset of the forcing $j_\omega(\dot{\mathbb{Q}})^{\langle j_n(\kappa) \mid n < \omega\rangle}$. Then there are $n < \omega$ and $\bar{D} \in W_{n,\omega}^*$ such that $k_{n,\omega}^*(\bar{D}) = D$. It follows from our arguments above that $< k_n\text{``}H >$ is generic for $k_n(\mathbb{Q}_\omega)$ over $M_n$. Now $k_n(\mathbb{Q}_\omega) \in W_{n,\omega}^*$ and $\bar{D}$ is a dense subset of it. So $< k_n\text{``}H > \cap \bar{D}$ is nonempty. Let $q \in H$ be such that $k_n(q) \in \bar{D}$. It follows that $k_\omega^*(q) = k_{n,\omega}^*(k_n^*(q)) \in D$. So we have shown that $< k_\omega^*\text{``}H >$ is generic over $M_\omega[\langle j_n(\kappa) \mid n < \omega\rangle]$. It remains to show that it is a member of $M_\omega[G_\omega][\mathcal{H}]$. Since $< k_\omega^*\text{``}H >=< k_{n,\omega}^*\text{``} < k_n^*\text{``}H >>$, we have that $< k_\omega^*\text{``}H > \in M_n[\mathcal{H}_n]$ for all $n < \omega$. By Claim 6.12, it follows that $< k_\omega^*\text{``}H > \in M_\omega[G_\omega][\mathcal{H}]$.   ⊣

To complete the proof of Theorem 1.4 we need a finer control of the relationship between $\mathbb{P}$ and $\mathbb{P}^*$ names. To this end we make some definitions.

DEFINITION 6.14. Let $f \in \mathbb{P}^*$, α be an ordinal and let $\dot{C}$ be a $\mathbb{P}^*$-name for a club subset of κ^+. We say that f *stably forces* $\check{\alpha} \in \dot{C}$ ($f \Vdash_{\mathbb{P}^*}^s \check{\alpha} \in \dot{C}$) if every alteration of f on fewer than κ many coordinates forces $\check{\alpha} \in \dot{C}$.

We have the following straightforward claims.

CLAIM 6.15. *If g extends f in $\mathbb{P}^*$ and $f \Vdash^s \check{\alpha} \in \dot{C}$, then $g \Vdash^s \check{\alpha} \in \dot{C}$.*

CLAIM 6.16. *If $f \Vdash^s \check{\alpha} \in \dot{C}$, then for every finite sequence $\vec{v}$ from some tree associated with f, $f \frown \vec{v} \Vdash^s \check{\alpha} \in \dot{C}$. If in addition $f = f' \frown \vec{v}$ for some finite sequence $\vec{v}$, then $f' \Vdash^s \check{\alpha} \in \dot{C}$.*

We define $\dot{C}^s = \{(\check{\alpha}, f) \mid f \Vdash^s \check{\alpha} \in \dot{C}\}$. Clearly it is forced that $\dot{C}^s \subseteq \dot{C}$. It is also straightforward to see that $\dot{C}^s$ is forced to be closed.

CLAIM 6.17. *$\dot{C}^s$ is forced to be unbounded in κ^+.*

PROOF. Fix some $f \in \mathbb{P}^*$ and $\alpha_0 < \kappa^+$. Take some sufficiently large θ and some $N \prec H_\theta$ of size κ such that $\mathbb{P}^*, f, \dot{C}, \alpha_0 \in N$ and $^{<\kappa}N \subseteq N$. Since $\mathbb{P}^*$ is κ^+-closed we can find a $(\mathbb{P}^*, N)$-generic condition $f^* \geq f$ with $\mathrm{dom}(f^*) = N \cap \kappa^{++}$. Let $\alpha = \sup(N \cap \kappa^{++})$.

Let f' be any condition obtained by altering f^* on a set of size less than κ. Since $^{<\kappa}N \subseteq N$, the alteration is a member of N. Hence a standard argument shows that f' is also $(\mathbb{P}^*, N)$-generic and so $f' \Vdash \check{\alpha} \in \dot{C}$.   ⊣

The name $\dot{C}^s$ behaves well when translated to a $\mathbb{P}$-name. Let

$$\dot{E} = \{(\check{\alpha}, p) \mid f^p \Vdash^s \check{\alpha} \in \dot{C}\}.$$

CLAIM 6.18. $\dot{E}$ is forced by $\mathbb{P}$ to be a club subset of κ^+.

PROOF. By the arguments of Claim 6.17, $\dot{E}$ is unbounded. Let us show first that $\dot{E}$ is forces to be closed.

Let $p = \langle f, T \rangle$ be a condition that forces that δ is an accumulation point of $\dot{E}$. Therefore, for every $\alpha < \delta$ there are $\vec{v} = \langle v_0, \ldots, v_{n-1} \rangle \in T$, $p' = \langle f', T' \rangle \leq^* p \frown \vec{v}$, and $\gamma < \delta$ such that $f' \Vdash^s \gamma \in \dot{C}$. Clearly, f' is a Cohen extension of $f \frown \vec{v}$. Denote by f^* the Cohen extension of f for which $f' = f^* \frown \vec{v}$. Since $f' \Vdash^s \gamma \in \dot{C}$, $f^* \Vdash^s \gamma \in \dot{C}$ as well. It follows that the condition $p^* = \langle f^*, T^* \rangle$, $T^* = \pi^{-1}_{\mathrm{dom}(f^*),\mathrm{dom}(f)}(T)$ is a direct extension of p and forces "$\check{\gamma} \in \dot{E}$".

Using the fact that the Cohen extension order on the function f is κ^+-closed and $\mathrm{cf}(\delta) \leq \kappa$, we can repeat this process and construct a sequence of Cohen extensions $\langle f_i \mid i \leq \mathrm{cf}(\delta) \rangle$ of f, and an increasing sequence $\langle \gamma_i \mid i < \mathrm{cf}(\delta) \rangle$ cofinal in δ, such that $f_i \Vdash^s \check{\gamma}_i \in \dot{C}$. Let $\bar{f} = f_{\mathrm{cf}(\delta)}$. Since $\dot{C}$ is a Cohen name of a club, $\bar{f} \Vdash^s \check{\delta} \in \dot{C}$. Setting $\bar{T} = \pi^{-1}_{\mathrm{dom}(\bar{f}),\mathrm{dom}(f)}(T)$ and $\bar{p} = \langle \bar{f}, \bar{T} \rangle$, we have that $\bar{p}$ is a direct extension of p, and forces $\check{\delta} \in \dot{E}$.   ⊣

REMARK 6.19. The proofs of the previous two claims work for any $\mathbb{P}^*$-name for a club $\dot{C}$ in any κ^+-closed generic extension.

We are now ready to finish the proof of Theorem 1.4.

CLAIM 6.20. In $M_\omega[G_\omega][\mathcal{H}]$, every finite collection of stationary subsets of $j_\omega(\kappa^+)$ reflects at a common point.

PROOF. Let S_i for $i < m$ be a sequence of stationary sets in $M_\omega[G_\omega][\mathcal{H}]$. By Claim 6.13, $<k_\omega``H>$ is a club in $j_\omega(\kappa^+)$ disjoint from $S^{j_\omega(\kappa^+)}_{j_\omega(\kappa)}$ as computed in $M_\omega$ and $<k_\omega``H> \in M_\omega[G_\omega][\mathcal{H}]$. So we can assume that each S_i concentrates on a fixed cofinality below $j_\omega(\kappa)$. For simplicity we assume that the cofinalities of the S_i are bounded below κ.

Let T^i_n be the set of α such that $j_{n,\omega}(\alpha) \in S_i$. For a fixed $n < \omega$, T^i_n for $i < m$ is definable in $M_n[G_n][\mathcal{H}_n]$. By the indestructibility of stationary reflection at $j_n(\kappa^+)$ in M_n, if each T^i_n for $i < m$ is stationary, then they reflect at a common point. Suppose that δ is this common reflection point. Then there are sets $A_i \in M_n$ for $i < k$ with $\mathrm{ot}(A) = \mathrm{cf}(\delta)$ such that A_i is stationary in δ and $A_i \subseteq T^i_n$. It follows that for each $i < m$, $j_{n,\omega}(A_i) = j_{n,\omega}``A_i \subseteq S_i$ and hence the collection of S_i reflect at $j(\delta)$.

We assume for the sake of a contradiction that for each $n < \omega$ at least one of the T^i_n is nonstationary. Let $\dot{C}_n$ be a name whose interpretation by G_n and $\mathcal{H}_n$, C_n, is a club disjoint from some $T^{i_n}_n$. By the arguments above and Remark 6.19, we can work in $M_n[\mathcal{H}_n]$ and translate $\dot{C}_n$ to a $j_n(\mathbb{P})$-name $\dot{E}_n$ for a club subset of $j_n(\kappa^+)$. By the construction of $\dot{E}_n$, we have that for $\alpha < j_n(\kappa^+)$, $j_{n,\omega}(\alpha)$ is in $j_{n,\omega}(\dot{E}_n)^{G_\omega}$ if and only if it is in $j_{n,\omega}(\dot{C}_n)^{<j_{n,\omega}``G_n>}$.

By the closure of $M_\omega[G_\omega]$ under ω-sequences, $\langle j_{n,\omega}(\dot{E}_n) \mid n < \omega \rangle$ is in $M_\omega[G_\omega]$. Hence we can interpret it using G_ω and $j_{n,\omega}``\mathcal{H}_n$. Let D_n denote the resulting club.

Let $i^* < m$ be such that $i^* = i_n$ for infinitely many n. We claim that $\bigcap_{n<\omega} D_n$ is disjoint from S_{i^*}. Otherwise, we have $j_{n,\omega}(\alpha) \in D_n \cap S_{i^*}$ for some n such that $i^* = i_n$. It follows that $\alpha \in C_n \cap T_n^{i_n}$, a contradiction. $\dashv$

6.1. Down to $\aleph_\omega$. In the paper [20] it is shown that it is consistent relative to infinitely many supercompact cardinals, that stationary reflection at $\aleph_{\omega+1}$ together with the failure of SCH at $\aleph_\omega$.

Let us show how to obtain the same result in our case, starting with the assumption of κ^+-Π_1^1-subcompact that carries a (κ, κ^{++})-extender, as in the previous part.

THEOREM 6.21. *Let κ be a κ^+-Π_1^1-subcompact cardinal and let us assume that it carries a (κ, κ^{++})-extender. Then, there is a forcing extension in which $\kappa = \aleph_\omega$, $2^{\aleph_\omega} = \aleph_{\omega+2}$, GCH holds below $\aleph_\omega$ and every finite collection of stationary subsets of $\aleph_{\omega+1}$ reflects simultaneously.*

The proof of this theorem is essentially the same as the proof of Theorem 1.4. We follow the argument as in the previous subsection and explain the differences.

Let us prepare the universe so that GCH holds and the subcompactness of κ is indestructible under Cohen forcing $\mathrm{Add}(\kappa^+, \kappa^{++})$. Let $j_E \colon V \to M_1$ be the extender ultrapower embedding. Let us assume moreover that the approachability ideal is always maximal. Namely, for every regular uncountable cardinal ρ, $\rho \in I[\rho]$

We define the iterations $\langle j_{n,m} \colon M_n \to M_m \mid n < m \leq \omega \rangle$, and $\langle \iota_{n,m} \colon N_n \to N_m \mid n < m \leq \omega \rangle$ as before.

By standard arguments (see for example [2, Lemma 8.5 and Proposition 15.1], and use Claim 6.7 for the computation of the width), there is an M_1-generic filter $K \subseteq \mathrm{Col}(\kappa^{+3}, < j_E(\kappa))$. Moreover, by elementarity for every n, the filter $j_n(K)$ is M_{n+1}-generic. Fix $K_0 \subseteq \mathrm{Col}(\omega_4, < \kappa)$ a V-generic and let

$$\mathcal{K} = \langle K_0 \rangle^\frown \langle j_n(K) \mid n < \omega \rangle.$$

Let us define the filters $G_n, \mathcal{H}_n$ and the models M_n for $n \leq \omega$ exactly as before and let us consider the model $M_\omega[G_\omega][\mathcal{H}][\mathcal{K}]$. Let $K_n = \mathcal{K} \restriction n + 1$.

In [12], it is shown that the parallel of Lemma 6.5, namely $M_\omega[G_\omega][\mathcal{K}] = \bigcap M_n[G_n][K_n]$, holds.

The next point in the proof in which we need to modify the argument slightly is Lemma 6.12, in which we need to show

$$M_\omega[G_\omega][\mathcal{K}][\mathcal{H}] = \bigcap M_n[G_n][K_n][\mathcal{H}_n].$$

As there are elementary embeddings

$$j_{n,\omega} \colon M_n[\mathcal{H}_n] \to M_\omega[< j_{n,\omega}``\mathcal{H}_n >] \subseteq M_\omega[\mathcal{H}],$$

the argument of [12] for extender-based Prikry forcing with interleaved collapses works without further modifications.

The claims for the existence of a generic for the forcing that shoots a club disjoint from $j_\omega(S_\kappa^{\kappa^+})$, Claim 6.13, remains the same. We are now ready for the proof of the main theorem of this subsection.

THEOREM 6.22. *In $M_\omega[G_\omega][\mathcal{H}][\mathcal{K}]$ SCH fails at $j_\omega(\kappa)$ which is $\aleph_\omega$ and every finite collection of stationary subsets of $j_\omega(\kappa^+)$ reflects simultaneously.*

PROOF. Let us verify first that the only cardinals which are collapsed are the cardinals in the intervals $[\omega_4, \kappa) \cup [\kappa^{+3}, j_1(\kappa)) \cup \cdots [j_n(\kappa^{+3}), j_{n+1}(\kappa)) \cup \cdots$.

As mentioned above,

$$M_\omega[G_\omega][\mathcal{H}][\mathcal{K}] = \bigcap M_n[G_n][\mathcal{H}_n][K_n]$$

and in particular, every M_ω-cardinal below $j_\omega(\kappa)$ which is collapsed in the extension must be collapsed in $M_n[G_n][\mathcal{H}_n][K_n]$ for all large n. For cardinals above $j_\omega(\kappa)$ one can use the continuity of the elementary embeddings in order to show that they do not change cofinality. Therefore, the arguments for the failure of SCH remains the same.

Let us prove the stationary reflection. For simplicity, let us show that every stationary subset of $j_\omega(\kappa^+)$ reflects in this model. The general case is similar.

There are two components in the proof which are slightly different: First, given a stationary set $S \in M_\omega[G_\omega][\mathcal{H}][\mathcal{K}]$, one can pull it to a set $T_n \in M_n[G_n][\mathcal{H}_n][K_n]$ exactly as before and check whether it is stationary there. Note that the forcing adding K_n is of cardinality $j_n(\kappa)$. So, if T_n is stationary in $M_n[G_n][\mathcal{H}_n][K_n]$ then there is a single condition $q \in K_n$ and stationary set $T_n' \in M_n[G_n][\mathcal{H}_n]$ such that $q \Vdash T_n' \subseteq T_n$.

Then, by the $j_n(\kappa^+)$-subcompactness of $j_n(\kappa)$ in $M_n[G_n][\mathcal{H}_n]$ one can conclude that T_n' has a reflection point δ of cofinality $< j_n(\kappa)$ and above $j_{n-1}(\kappa)^{+++}$ (see [13] for details). This is where the assumption about the approachability plays a role: we conclude that this stationary subset of δ remains stationary in $M_n[G_n][\mathcal{H}_n][\mathcal{K}_n]$. So, there is a stationary subset of δ of order type $\operatorname{cf} \delta$ in M_n, $T_n'' \subseteq T_n \cap \delta$. Since $j_{n,\omega}(T_n'') = j_{n,\omega}$ " T_n'', we conclude that $M_\omega[G_\omega][\mathcal{H}][K_n] \models j_{n,\omega}(T_n'') \subseteq S$.

By downwards absoluteness of stationarity and continuity of the embedding $j_{n,\omega}$, $j_{n,\omega}(T_n'')$ is stationary at $j_{n,\omega}(\delta)$ in $M_\omega[G_\omega][\mathcal{H}][K_n]$ and thus S reflects at $j_{n,\omega}(\delta)$ in this model.

To summarize, if there is n such that T_n is stationary, then S reflects in $M_\omega[G_\omega][\mathcal{H}][\mathcal{K}]$.

Otherwise, for all n, T_n is non-stationary. In this case, for each n there is a club name for a club $\dot{C}_n$ disjoint from T_n and we would like to push them (or subclubs of them) to $M_\omega[G_\omega][\mathcal{H}][\mathcal{K}]$.

For every n, the forcing introducing K_n is $j_n(\kappa)$-cc, and therefore, for every club $C_n \in M_n[G_n][\mathcal{H}_n][K_n]$ there is a subclub $C_n' \in M_n[G_n][\mathcal{H}_n]$. We are now in exactly the same situation of the proof of Theorem 1.4: we obtain for each n a stable name $\dot{E}_n$ which is forced to be a subclub of C_n' and thus there is in $M_\omega[G_\omega][\mathcal{H}]$ a subclub of the set $\bigcap j_{n,\omega}(\dot{E}_n)$ which is forced to be disjoint from S. ⊣

REMARK 6.23. The same method can be used to obtain arbitrary finite gap in the failure of SCH.

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