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THE THEORY OF HECKE INTEGRALS

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§ 1. Introduction

Let H denote the complex upper half-plane and let $\eta(z)$ denote Dedekind's η-function

$$\eta(z) = e^{\pi iz/12} \prod_{n=1}^{\infty} (1 - e^{2\pi inz}) , \qquad z \in H .$$

It is clear that $\eta(z) \neq 0$ for $z \in H$ and therefore, we may speak of $\log \eta(z)$, where the logarithm is taken to be the principal branch. If we set $f(z) = \log \eta(z)$, then $f(z)$ is analytic in the upper half-plane, and satisfies the following properties:

$$f(z + 1) = f(z) + \frac{\pi i}{12} \tag{1}$$

$$f\left(-\frac{1}{z}\right) = f(z) + \frac{1}{2} \log \left(\frac{z}{i}\right) \tag{2}$$

$$f(z) = \frac{\pi iz}{12} + \sum_{n=1}^{\infty} a_n e^{2\pi inz} , \qquad a_n = -\sum_{\substack{d|n \\ d>0}} \frac{1}{d} . \tag{3}$$

A slightly more technical property of $f(z)$ is that

$$f(z) = O(y^{-K}) \qquad \text{for some } K > 0,\ z = x + iy,\ y \to 0, \text{ uniformly for } x \text{ in any finite interval.} \tag{4}$$

Properties (1) and (2) together imply that if $\sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is an integral, unimodular matrix, then

$$f\left(\frac{az + b}{cz + d}\right) = f(z) + \frac{1}{2} \log \left(\frac{cz + d}{i}\right) + \pi iS\begin{pmatrix} a & b \\ c & d \end{pmatrix} , \tag{5}$$

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for suitable $S\begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Property (5) is the classical law of transformation of $\log \eta(z)$ and many proofs of it have been given by (among others) Dedekind [1], Rademacher [9, 10], Siegel [11], Weil [13] and the first author [4]. In particular, the function $S\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ can be explicitly calculated in terms of Dedekind sums.

The starting point for the present paper is the work [13] of Weil, in which a very simple proof of (2) is given by appealing to the functional equation of the Riemann zeta function and the Mellin inversion formula, which allows one to pass from the Fourier series development (3) to an appropriate zeta function (namely $\zeta(s)\zeta(s+1)$). The principle of associating a Dirichlet series to an automorphic form and, conversely, an automorphic form to an appropriate type of Dirichlet series is a timehonored procedure going back to Hecke [5]. Of course, the function $f(z)$ is not an automorphic form, but rather the logarithm of an automorphic form, so the correspondence implicit in Weil's paper [13] does not quite fit into Hecke's theory. However, this suggests that there is a correspondence between functions "similar to" $\log \eta(z)$ on the one hand, and certain Dirichlet series with analytic continuation and functional equation on the other. It is the purpose of this paper to construct such a correspondence, and to study some of the analogues of $\log \eta(z)$ so derived as well as their associated Dirichlet series.

It should be noted that there are many other classically known functions satisfying properties similar to (1)–(4). For example, if

$$\theta(z) = \sum_{n=-\infty}^{\infty} e^{-\pi i n^2 z}$$

is the usual elliptic θ-function, then $\log \theta(z)$ satisfies analogues of (1)–(4). (See Section 3 for details.) In this paper, we will construct several new classes of functions of this sort.

One of the chief interests of functions such as $\log \eta(z)$ is their deep connection with arithmetic. The transformation properties of $\log \eta(z)$ and $\log \theta(z)$ have been used by Hecke [6], Meyer [7], and Siegel [12] to study the class numbers of certain abelian extensions of real quadratic fields. In the last section of this paper, we will derive, as a consequence of our theory, a curious class number formula which seems to have a fundamental connection with the problem of determining all imaginary quadratic fields having a given class number.

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§ 2. The correspondence theorem

Let us begin by axiomatizing the properties (1)–(4) of $f(z) = \log \eta(z)$.

DEFINITION 2.1. Let $\lambda > 0$, A, B, C, γ be given complex numbers. A function $f(z)$ on $\boldsymbol{H}$ is said to be a *Hecke integral of signature* $\{\lambda, A, B, C, \gamma\}$ provided that:

(H1) $f(z + \lambda) = f(z) + 2\pi i A$

(H2) $f(-1/z) = \gamma f(z) + B \log(z/i) + C$

(H3) By (H1), $f(z) - 2\pi i A z/\lambda$ is periodic with period λ. Let us suppose that it is regular in the local uniformizing variable $e^{2\pi i z/\lambda}$. That is, we assume that $f(z)$ can be written in a convergent expansion of the form

$$f(z) = \frac{2\pi i A z}{\lambda} + \sum_{n=0}^{\infty} a_n e^{2\pi i n z/\lambda}.$$

In particular, $f(z)$ is holomorphic in $\boldsymbol{H}$.

(H4) $f(x + iy) = O(y^{-K})$ for some $K > 0$, as $y \to 0$, uniformly for x in a finite interval.

If we apply (H2) to $f(z)$ and $f(-1/z)$, we easily see that

$$(1 - \gamma)\left[f(z) + f\left(-\frac{1}{z}\right)\right] = 2C$$

$$(1 + \gamma)\left[f(z) - f\left(-\frac{1}{z}\right)\right] = -2B \log\left(\frac{z}{i}\right).$$

If $\gamma = -1$, then $B = 0$ by the second equation. If $\gamma = +1$, then $C = 0$ by the first equation. If $\gamma \neq \pm 1$, then

$$f(z) = \frac{C}{1 - \gamma} - \frac{B}{1 + \gamma} \log\left(\frac{z}{i}\right),$$

so that by (H1), we have $A = B = 0$ and thus $f(z)$ is constant. Thus, we may restrict ourselves to $\gamma = \pm 1$. If $\gamma = +1$, then $B = 0$ and if $\gamma = -1$, then $C = 0$. *Throughout this paper, we shall always assume that* $\gamma = \pm 1$.

Note that if $f(z)$ is a Hecke integral of signature $\{\lambda, A, B, C, +1\}$ then so is $f(z)+a$ for any constant a. Thus, we may normalize our Hecke integrals in case $\gamma=+1$ so that $a_0=0$. *Throughout this paper we shall always assume that Hecke integrals are so normalized.* Note that we make no corresponding normalization in case $\gamma=-1$.

It is our purpose in this section to show that there is a $1-1$ correspondence between (normalized) Hecke integrals having given signature and certain Dirichlet series having analytic continuation and functional equation. Namely, let us associate to the Hecke integral

$$f(z)=\frac{2\pi iAz}{\lambda}+\sum_{n=1}^{\infty}a_n e^{2\pi inz/\lambda}$$

the Dirichlet series

$$\phi_f(s)=\sum_{n=1}^{\infty}\frac{a_n}{n^s}\ .$$

By using (H4) and the argument of [8, p. I–4], we see that $a_n=O(n^c)$ for some $c>0$ and therefore that $\phi_f(s)$ converges in some half-plane. Let us define the function

$$\Phi_f(s)=\left(\frac{2\pi}{\lambda}\right)^{-s}\Gamma(s)\phi_f(s)\ .$$

THEOREM 2.2. *Let f satisfy* (H1)–(H4). *The function $\Phi_f(s)$ can be analytically continued to a meromorphic function in the entire s-plane. Moreover,*

$$\Phi_f(s)-\frac{B}{s^2}-\frac{C}{s}-\frac{2\pi A}{\lambda}\frac{1}{s+1}+\frac{2\pi A\gamma}{\lambda}\frac{1}{s-1}$$

is an entire function of finite genus and $\Phi_f(s)$ satisfies the functional equation

$$\Phi_f(-s)=\gamma\Phi_f(s)\ .$$

Proof. For $\sigma=\operatorname{Re}(s)$ large enough the Dirichlet series $\phi_f(s)$ is absolutely and uniformly convergent in the half-plane $\operatorname{Re}(s)\geq\sigma$ (we have used (H4), see [8, p. I–48]). Therefore, for $\operatorname{Re}(s)\geq\sigma$, we have

$$\begin{aligned}\Phi_f(s)&=\int_0^\infty\left[f(iy)-\frac{2\pi iA(iy)}{\lambda}\right]y^{s-1}dy\\&=\int_1^\infty\left[f(iy)+\frac{2\pi Ay}{\lambda}\right]y^{s-1}dy+\int_0^1\left[f(iy)+\frac{2\pi Ay}{\lambda}\right]y^{s-1}dy\ .\end{aligned}\tag{6}$$

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However,

$$\begin{aligned}\int_0^1 \left[f(iy) + \frac{2\pi Ay}{\lambda}\right] y^{s-1} dy &= \int_1^\infty \left[f(iy^{-1}) + \frac{2\pi Ay^{-1}}{\lambda}\right] y^{-s-1} dy \\ &= \int_1^\infty f(iy^{-1}) y^{-s-1} dy + \frac{2\pi A}{\lambda} \int_1^\infty y^{-s-2} dy \\ &= \int_1^\infty f(iy^{-1}) y^{-s-1} dy + \frac{2\pi A}{\lambda} \frac{1}{s+1} \\ &= \int_1^\infty [\gamma f(iy) + B \log y + C] y^{-s-1} dy + \frac{2\pi A}{\lambda} \frac{1}{s+1}\end{aligned} \tag{7}$$

By integration by parts, we have (assuming $\operatorname{Re}(s) > 0$)

$$\int_1^\infty (\log y) y^{-s-1} dy = \frac{1}{s^2} . \tag{8}$$

Thus, by (7) and (8), we have

$$\begin{aligned}&\int_0^1 \left[f(iy) + \frac{2\pi Ay}{\lambda}\right] y^{s-1} dy \\ &\quad = \gamma \int_1^\infty \left[f(iy) + \frac{2\pi Ay}{\lambda}\right] y^{-s-1} dy + \frac{2\pi A}{\lambda} \frac{1}{s+1} + \frac{B}{s^2} \\ &\quad\quad + C \int_1^\infty y^{-s-1} dy - \gamma \int_1^\infty \frac{2\pi A}{\lambda} y^{-s} dy \\ &\quad = \int_1^\infty \left[f(iy) + \frac{2\pi Ay}{\lambda}\right] y^{-s-1} dy \\ &\quad\quad + \frac{B}{s^2} + \frac{C}{s} - \frac{2\pi A\gamma}{\lambda} \frac{1}{s-1} + \frac{2\pi A}{\lambda} \frac{1}{s+1} .\end{aligned} \tag{9}$$

Combining equations (6) and (9), we see that

$$\begin{aligned}\Phi_f(s) = \int_1^\infty \left[f(iy) + \frac{2\pi Ay}{\lambda}\right] y^{s-1} dy + \gamma \int_1^\infty \left[f(iy) + \frac{2\pi Ay}{\lambda}\right] y^{-s-1} dy \\ + \frac{B}{s^2} + \frac{C}{s} - \frac{2\pi A\gamma}{\lambda} \frac{1}{s-1} + \frac{2\pi A}{\lambda} \frac{1}{s+1} .\end{aligned} \tag{10}$$

Trivial estimates show that the integrals on the right hand side of (10) are entire functions of finite genus. This completes the proof of the analytic continuation assertion concerning $\Phi_f(s)$. Using the fact that either $\gamma = 1$, $C = 0$ or $\gamma = -1$, $B = 0$, we see that (10) immediately implies that

$$\Phi_f(-s) = \gamma\Phi_f(s) .$$

This completes the proof of Theorem 2.2.

Let $\phi(s) = \sum_{n=1}^{\infty} a_n/n^s$ be a somewhere convergent Dirichlet series, and let $\lambda > 0$, $B \geq 0$, A, C, $\gamma = \pm 1$ be given. We shall say that $\phi(s)$ *is of signature* $\{\lambda, A, B, C, \gamma\}$ provided that:

I. The function $\Phi(s) = (2\pi/\lambda)^{-s}\Gamma(s)\phi(s)$ can be analytically continued to a meromorphic function such that

$$\Phi(s) - \frac{B}{s^2} - \frac{C}{s} + \frac{2\pi A\gamma}{\lambda}\frac{1}{s-1} - \frac{2\pi A}{\lambda}\frac{1}{s+1}$$

is an entire function of finite genus.

II. $\Phi(s) = \gamma\Phi(-s)$.

Note that as before, the conditions guarantee that either $\gamma = +1$ and $C = 0$ or $\gamma = -1$ and $B = 0$.

THEOREM 2.3. *The mapping*

$$f(z) \rightarrow \Phi_f(s)$$

induces a bijection from the set of Hecke integrals of signature $\{\lambda, A, B, C, \gamma\}$ *to the set of Dirichlet series of the same signature.*

Proof. It is clear that from $\Phi_f(s)$ we can recover $f(z)$. [Recall that all Hecke integrals are assumed normalized. In case $\gamma = -1$, we can recover a_0 since $C = 2f(i)$.] Thus, the map is injective. To prove surjectivity, let us consider a Dirichlet series $\phi(s)$ of signature $\{\lambda, A, B, C, \gamma\}$. If

$$\phi(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s} ,$$

let us set

$$f(z) = \frac{2\pi i A z}{\lambda} + \sum_{n=1}^{\infty} a_n e^{2\pi i n z/\lambda} .$$

It is clear that $\Phi_f(s) = (2\pi/\lambda)^{-s}\Gamma(s)\phi(s)$, so that it suffices to show that $f(z)$ is a Hecke integral of signature $\{\lambda, A, B, C, \gamma\}$. Properties (H1) and (H3) are clear. Using customary arguments involving the Phragmen-Lindelöf theorem and Stirling's formula, we find that $\phi(\sigma + it) = O(|t|^4)$,

$|t| \to \infty$, uniformly for σ in any finite interval. Moreover, for σ sufficiently large, the Mellin inversion formula implies that

$$f(z) - \frac{2\pi i A z}{\lambda} = \frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} \Phi_f(s) \left(\frac{z}{i} \right)^{-s} ds \ .$$

Using the above estimate for $\phi(\sigma + it)$ and Stirling's formula, we see immediately that $f(z)$ satifies (H4). Thus it suffices to show that (H2) holds. Moreover, the same estimate for the integrand allows us to justify shifting the line of integration from $\mathrm{Re}\,(s) = \sigma$ to $\mathrm{Re}\,(s) = -\sigma$, yielding

$$\begin{aligned} f(z) - \frac{2\pi i A z}{\lambda} &= \frac{1}{2\pi i} \int_{-\sigma - i\infty}^{-\sigma + i\infty} \Phi_f(s) \left(\frac{z}{i} \right)^{-s} ds + \sum_{s=0,\pm 1} \mathrm{Res}\, \Phi_f(s) z^{-s} ds \\ &= \frac{\gamma}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} \Phi_f(s) \left(-\frac{z}{i}^{-1} \right)^{-s} ds + \sum_{s=0,\pm 1} \mathrm{Res}\, \Phi_f(s) \left(\frac{z}{i} \right)^{-s} ds \qquad (11) \\ &= \gamma f\left(-\frac{1}{z} \right) + \frac{2\pi i A \gamma z^{-1}}{\lambda} + \sum_{s=0,\pm 1} \mathrm{Res}\, \Phi_f(s) \left(\frac{z}{i} \right)^{-s} ds \ . \end{aligned}$$

Trivial calculations yield that

$$\begin{aligned} \mathrm{Res}_{s=1}\, \Phi_f(s) \left(\frac{z}{i} \right)^{-s} &= -\frac{2\pi A \gamma}{\lambda} \left(\frac{z}{i} \right)^{-1} \\ &= -\frac{2\pi i A \gamma z^{-1}}{\lambda} \end{aligned} \qquad (12)$$

$$\begin{aligned} \mathrm{Res}_{s=-1}\, \Phi_f(s) \left(\frac{z}{i} \right)^{-s} &= +\frac{2\pi A}{\lambda} \left(\frac{z}{i} \right) \\ &= -\frac{2\pi i A z}{\lambda} \ . \end{aligned} \qquad (13)$$

At $s = 0$, we have

$$\Phi_f(s) = \frac{B}{s^2} + \frac{C}{s} + O(1)$$

$$\left(\frac{z}{i} \right)^{-s} = e^{-s \log (z/i)} = 1 - s \log (z/i) + \cdots ,$$

so that

$$\mathrm{Res}_{s=0}\, \Phi_f(s) \left(\frac{z}{i} \right)^{-s} = -B \log \left(\frac{z}{i} \right) + C \ . \qquad (14)$$

From equations (11)–(14), we have

$$f(z) = \gamma f\left(-\frac{1}{z}\right) - B \log\left(\frac{z}{i}\right) + C \,. \tag{15}$$

Now, using the fact that either $\gamma = 1$, $C = 0$ or $\gamma = -1$, $B = 0$, we see that (H2) holds. Thus, Theorem 2.3 is completely proved.

§ 3. Examples

Let us now use the correspondence embodied in Theorems 2.2 and 2.3 to construct examples of Hecke integrals. It should be noted that our procedure in this section is precisely the reverse of the procedure introduced by Hecke in his germinal paper [5]. Hecke sought to investigate, by using the theory of automorphic functions, the properties of Dirichlet series or, more exactly, the phenomenon that certain Dirichlet series are uniquely determined by their functional equation. In this section, we will use information about various Dirichlet series to construct Hecke integrals of various signatures.

EXAMPLE 3.1. Let $f(z) = \log \eta(z)$. Then $f(z)$ is of signature $\{1, 1/24, 1/2, 0, 1\}$. The corresponding Dirichlet series is computed in [13] to be $\phi(s) = -\zeta(s)\zeta(s+1)$, where $\zeta(s)$ denotes the Riemann zeta function.

EXAMPLE 3.2 (Ogg). Let $f(z) = \log \theta(z)$. Then $f(z)$ is of signature $\{2, 0, 1/2, 0, 1\}$. The corresponding Dirichlet series is computed in [8, p. I–45] to be

$$\phi(s) = -2^{-s}\{-5 + 2(2^s + 2^{-s})\}\zeta(s)\zeta(s+1) \,.$$

Example 3.2 suggests a general procedure for constructing Hecke integrals from known ones, as the following example shows.

EXAMPLE 3.3. Let $f(z)$ be a Hecke integral of signature $\{\lambda, A, B, C, \gamma\}$. Let N, M be integers such that $n | M$ for $1 \le n \le N$. Further, let $a_0, \cdots, a_N$ be any N complex numbers. The function

$$p(s) = a_0 + \sum_{n=1}^{N} a_n\left(\frac{1}{n^s} + n^s\right)$$

satisfies $p(s) = p(-s)$ and $M^{-s}p(s)$ is a Dirichlet polynomial. Since $f(z)$ is a Hecke integral of signature $\{\lambda, A, B, C, \gamma\}$, we know that the function

$$\Phi(s) = \left(\frac{2\pi}{\lambda}\right)^{-s}\Gamma(s)\phi(s) \,,$$

where $\phi(s)$ is the Dirichlet series corresponding to $f(z)$, satisfies

$$\Phi(s) = \gamma\Phi(-s) .$$

Let us set $\phi_{p,M}(s) = M^{-s}p(s)\phi(s)$. Then $\phi_{p,M}(s)$ is a Dirichlet series which converges (at least) in the half-plane of convergence of $\phi(s)$. Moreover, if we set

$$\Phi_{p,M}(s) = \left(\frac{2\pi}{\lambda M}\right)^{-s}\Gamma(s)\phi_{p,M}(s) ,$$

then

$$\Phi_{p,M}(s) = \gamma\Phi_{p,M}(-s) .$$

Moreover, it is clear that

$$\Phi_{p,M}(s) - \frac{p(0)B}{s^2} - \frac{p(0)C}{s} + \frac{2\pi A\gamma p(1)}{\lambda(s-1)} - \frac{2\pi A p(-1)}{\lambda(s+1)}$$

is entire of finite genus. Therefore, $\phi_{p,M}(s)$ is of signature $\{\lambda M, p(1)A, p(0)B, p(0)C, \gamma\}$ (note $p(1) = p(-1)$). If $f_{p,M}(z)$ is the Fourier series development corresponding to $\phi_{p,M}(s)$, then $f_{p,M}(z)$ is a Hecke integral of signature $\{\lambda M, p(1)A, p(0)B, p(0)C, \gamma\}$. Example 3.2 corresponds to the case $\phi(s) = -\zeta(s)\zeta(s+1)$, $p(s) = -5 + 2(2^s + 2^{-s})$, $M = 2$.

EXAMPLE 3.4. Let χ be a non-trivial primitive Dirichlet character defined modulo its conductor f and let $L(s, \chi)$ denote the usual L-series:

$$L(s, \chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s} , \qquad \mathrm{Re}\,(s) > 1 .$$

Let

$$\varepsilon = \varepsilon(\chi) = \begin{cases} 0 & \text{if } \chi(-1) = 1 \\ 1 & \text{if } \chi(-1) = -1 . \end{cases}$$

It is well-known (for example, see [2, p. 67]) that $L(s, \chi)$ can be analytically continued to an entire function which satisfies the functional equation

$$R(1 - s, \bar{\chi}) = \frac{i^{\varepsilon} f^{1/2}}{\tau(\chi)} R(s, \chi) ,$$

where

$$R(s, \chi) = \left(\frac{\pi}{f}\right)^{-(s+\varepsilon)/2} \Gamma\left(\frac{s+\varepsilon}{2}\right) L(s, \chi) ,$$

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$\bar{\chi}$ = the complex conjugate of χ, and $\tau(\chi)$ is the Gaussian sum defined by

$$\tau(\chi) = \sum_{m \pmod f} \chi(m) e^{2\pi i m/f} .$$

For reference, let us record a well-known fact about $\tau(\chi)$ [1, p. 350]:

$$\tau(\chi)\tau(\bar{\chi}) = \chi(-1)f . \tag{16}$$

Let us set

$$\Phi_1(s) = R(s, \chi)R(s+1, \bar{\chi}) .$$

Then

$$\begin{aligned} \Phi_1(-s) &= R(-s, \chi)R(1-s, \bar{\chi}) \\ &= R(1-(1+s), \chi)R(1-s, \bar{\chi}) \\ &= R(s+1, \bar{\chi})R(s, \chi) \cdot \frac{i^{2\varepsilon} f}{\tau(\chi)^2} \\ &= \Phi_1(s) . \end{aligned} \tag{17}$$

Using the duplication formula for the Γ-function, we readily see that

$$\Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{s+1}{2}\right) = \sqrt{\pi}\, 2^{1-s}\Gamma(s)$$

$$\Gamma\left(\frac{s+1}{2}\right)\Gamma\left(\frac{s+2}{2}\right) = \frac{s}{2}\sqrt{\pi}\, 2^{1-s}\Gamma(s) .$$

Therefore, we see that

$$\Phi_1(s) = \begin{cases} 2\sqrt{f}\left(\dfrac{2\pi}{f}\right)^{-s}\Gamma(s)L(s, \chi)L(s+1, \bar{\chi}) & (\varepsilon = 0) , \\ \dfrac{f^{3/2}}{\pi} s \left(\dfrac{2\pi}{f}\right)^{-s}\Gamma(s)L(s, \chi)L(s+1, \bar{\chi}) & (\varepsilon = 1) . \end{cases} \tag{18}$$

By combining equations (17) and (18), we see that the function

$$\Phi(s) = \left(\frac{2\pi}{f}\right)^{-s}\Gamma(s)L(s, \chi)L(s+1, \bar{\chi}) \tag{19}$$

satisfies

$$\Phi(-s) = \gamma\Phi(s) , \qquad \gamma = (-1)^{\varepsilon} = \chi(-1) . \tag{20}$$

Moreover, $\Phi(s)$ is entire if $\varepsilon = 0$ and has a simple pole at $s = 0$ if $\varepsilon = 1$. Thus, we see that $\phi(s) = L(s, \chi)L(s+1, \bar{\chi})$ has signature $\{f, 0, 0, 0, 1\}$ if $\varepsilon = 0$, and $\{f, 0, 0, C, -1\}$ if $\varepsilon = 1$, where

$$\begin{aligned} C &= \lim_{s\to 0} s\Phi(s) \\ &= L(0,\chi)L(1,\bar\chi) \\ &= L(1,\bar\chi)L(1,\bar\chi)\frac{(\pi/f)^{-1}\Gamma(1)}{(\pi/f)^{-1/2}\Gamma(1/2)}\frac{if^{1/2}}{\tau(\chi)} \quad \text{(by the functional equation for } L(s,\chi)) \\ &= L(1,\bar\chi)^2\frac{if}{\pi\tau(\chi)}\,. \end{aligned}$$

Thus we have established the following result:

THEOREM 3.1. *The Dirichlet series* $\phi(s) = L(s,\chi)L(s+1,\bar\chi)$ *has signature* $\{f,0,0,0,1\}$ *if* $\chi(-1)=1$ *and* $\{f,0,0,iL(1,\bar\chi)^2 f/\pi\tau(\chi),-1\}$ *if* $\chi(-1)=-1$.

A simple calculation shows that

$$L(s,\chi)L(s+1,\bar\chi) = \sum_{n=1}^{\infty}\left(\chi(n)\sum_{\substack{d\mid n\\ d>0}}\frac{\bar\chi^2(d)}{d}\right)\frac{1}{n^s}\,.$$

Therefore, the Hecke integral $f_\chi(z)$ corresponding to $L(s,\chi)L(s+1,\bar\chi)$ is just

$$f_\chi(z) = \sum_{n=1}^{\infty}\left(\chi(n)\sum_{\substack{d\mid n\\ d>0}}\frac{\bar\chi^2(d)}{d}\right)e^{2\pi inz/f}$$

Thus, by Theorem 2.3, we have

THEOREM 3.2. *Let* χ *be a non-trivial Dirichlet character defined modulo its conductor* f. *Then the function*

$$f_\chi(z) = \sum_{n=1}^{\infty}\left(\chi(n)\sum_{\substack{d\mid n\\ d>0}}\frac{\bar\chi^2(d)}{d}\right)e^{2\pi inz/f}$$

is a Hecke integral of signature $\{f,0,0,0,1\}$ *if* $\chi(-1)=1$ *and signature* $\{f,0,0,iL(1,\bar\chi)^2 f/\pi\tau(\chi),-1\}$ *if* $\chi(-1)=-1$.

COROLLARY 3.3. *If* $\chi(-1)=1$, *then*

$$f_\chi\left(\frac{az+b}{cz+d}\right) = f_\chi(z)$$

for all substitutions $\begin{pmatrix} a & b \\ c & d\end{pmatrix} \in G(f)$, *where* $G(f)$ *is the Hecke group generated by the substitutions* $\begin{pmatrix} 1 & f \\ 0 & 1\end{pmatrix}$ *and* $\begin{pmatrix} 0 & -1 \\ 1 & 0\end{pmatrix}$.

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Proof. By Theorem 3.2, $f_\chi(z)$ is invariant under each of the two generating substitutions for $G(f)$.

COROLLARY 3.4. *If* $\chi(-1) = -1$, *then*

$$f_\chi\left(-\frac{1}{z}\right) = -f_\chi(z) + \frac{if}{\pi\tau(\chi)}(L(1,\bar{\chi}))^2 .$$

Moreover, if $\sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G(f)$, *then*

$$f_\chi\left(\frac{az+b}{cz+d}\right) = \pm f_\chi(z) + C(\sigma) ,$$

where $C(\sigma)$ *depends only on* σ *and not on* z.

The functions $f_\chi(z)$ seem to have some arithmetic significance, which will be discussed in Section 6.

§ 4. Existence and uniqueness theorems

In this section, we shall give a survey of all possible Hecke integrals. Throughout, let $G(\lambda)$ denotes the Hecke group generated by the substitutions

$$\begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} .$$

Let us denote by $\mathscr{H}(\lambda, \gamma; A, B, C)$ the set of all Hecke integrals of signature $\{\lambda, A, B, C, \gamma\}$, except that we will not require that hypothesis (H4) be satisfied. Unfortunately, the set $\mathscr{H}(\lambda, \gamma; A, B, C)$ has very little structure. It will be a consequence of what we shall prove that if $0 < \lambda < 2$, then $\mathscr{H}(\lambda, \gamma; A, B, C)$ is either empty or consists of a single element. In order to introduce some structure, we define

$$\mathscr{H}(\lambda, \gamma) = \bigcup_{A,B,C \in \boldsymbol{C}} \mathscr{H}(\lambda, \gamma; A, B, C) .$$

It is clear that if $f_i \in \mathscr{H}(\lambda, \gamma; A_i, B_i, C_i)$ $(i = 1, 2)$, and if $\alpha_i \in \boldsymbol{C}$ $(i = 1, 2)$, then

$$\alpha_1 f_1 + \alpha_2 f_2 \in \mathscr{H}(\lambda, \gamma; \alpha_1 A_1 + \alpha_2 A_2, \alpha_1 B_1 + \alpha_2 B_2, \alpha_1 C_1 + \alpha_2 C_2) . \tag{21}$$

Therefore, we see that $\mathscr{H}(\lambda, \gamma)$ is a vector space over $\boldsymbol{C}$.

An important linear subspace of $\mathscr{H}(\lambda, \gamma)$ is the one defined by

$$\mathscr{H}_0(\lambda, \gamma) = \mathscr{H}(\lambda, \gamma; 0, 0, 0) .$$

The functions belonging to $\mathscr{H}_0(\lambda, \gamma)$ satisfy the relations

$$f(z + \lambda) = f(z)$$
$$f\left(-\frac{1}{z}\right) = \gamma f(z)$$

and can be written in a convergent Fourier series of the form

$$f(z) = \sum_{n=1}^{\infty} a_n e^{2\pi i n z/\lambda} .$$

In particular, if $f \in \mathscr{H}_0(\lambda, \gamma)$, then f^2 is an automorphic function for $G(\lambda)$ which is zero at ∞ and regular for all $z \in \boldsymbol{H}$. Thus,

$$\mathscr{H}_0(\lambda, \gamma) = \{0\} \qquad \text{if } 0 < \lambda < 2 , \tag{22}$$

since there are no non-constant automorphic functions regular at ∞ in this case. In what follows, we shall describe the subspace $\mathscr{H}_0(\lambda, \gamma)$ more closely.

The importance of the subspace $\mathscr{H}_0(\lambda, \gamma)$ derives from the relationship

$$\mathscr{H}_0(\lambda, \gamma) + \mathscr{H}(\lambda, \gamma; A, B, C) = \mathscr{H}(\lambda, \gamma; A, B, C) , \tag{23}$$

which follows immediately from (21). Therefore, to describe $\mathscr{H}(\lambda, \gamma; A, B, C)$, it suffices to describe $\mathscr{H}_0(\lambda, \gamma)$ and then to describe the elements of $\mathscr{H}(\lambda, \gamma; A, B, C)$ modulo $\mathscr{H}_0(\lambda, \gamma)$. To do this in the context of a vector space, we will instead describe $\mathscr{H}_0(\lambda, \gamma)$ and the factor space

$$\bar{\mathscr{H}}(\lambda, \gamma) = \mathscr{H}(\lambda, \gamma) / \mathscr{H}_0(\lambda, \gamma) .$$

We shall see that this quotient space has complex dimension at most 2. We shall compute this dimension as a function of λ, γ. In each case, we will give an explicit basis for $\bar{\mathscr{H}}(\lambda, \gamma)$.

It is important to recall at this point that all Hecke integrals are assumed normalized. In case $\gamma = -1$, the constant function $C/2$ is a (normalized) Hecke integral of signature $\{\lambda, 0, 0, C, -1\}$. Further, recall that if $\gamma = +1$, then $C = 0$ and if $\gamma = -1$, then $B = 0$.

PROPOSITION 4.1. *We have*

$$\dim_C \bar{\mathscr{H}}(\lambda, \gamma) \leq 2 .$$

Proof. Follows immediately from (21) and the fact that if $\gamma = 1$,

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then $C = 0$ and if $\gamma = -1$, then $B = 0$.

Let us henceforth consider the cases $\gamma = +1$ and $\gamma = -1$ separately.

Case I: $\gamma = +1$.

The basic principle in our analysis of this case is an exponential-logarithmic correspondence which reduces considerations to the study of automorphic forms for Hecke's group $G(\lambda)$. Therefore, let us review the basic definitions about such automorphic forms and set up some notation.

Let $\lambda > 0$, k be real and let $\mathscr{M}(\lambda, k)$ denote the space of all holomorphic automorphic forms of weight k and multiplier i^k for $G(\lambda)$. In other words, $f \in \mathscr{M}(\lambda, k)$ if and only if

(i) $f(z + \lambda) = f(z)$

(ii) $f(-1/z) = (z/i)^k f(z)$

(iii) f is holomorphic on $\boldsymbol{H} \cup \{i\infty\}$, where the holomorphy condition at $i\infty$ is interpreted to be with respect to the local parameter $e^{2\pi iz/\lambda}$, so that $f(z)$ possesses a convergent expansion of the form

$$f(z) = \sum_{n=0}^{\infty} a_n e^{2\pi inz/\lambda} . \tag{24}$$

It will be necessary to consider slightly more general automorphic forms, namely those which are *meromorphic at* $i\infty$. The definition is the same as above, except instead of (24), we require that f have an expansion about $i\infty$ of the form

$$f(z) = \sum_{n=N}^{\infty} a_n e^{2\pi inz/\lambda} ,$$

for some integer N. If $a_N \neq 0$, we say that f has *order* N *at* $i\infty$. Let $\mathscr{M}_\infty(\lambda, k)$ denotes the space of all automorphic forms which are meromorphic at $i\infty$.

Finally, since the automorphic forms we consider will arise from exponentiation, they will have no zeros in $\boldsymbol{H}$. Therefore, let us write $\mathscr{M}^*(\lambda, k)$ (resp. $\mathscr{M}^*_\infty(\lambda, k)$) for the set of all f in $\mathscr{M}(\lambda, k)$ (resp. $\mathscr{M}_\infty(\lambda, k)$) which have no zeros in $\boldsymbol{H}$.

If $f \in \mathscr{H}(\lambda, +1; A, B, 0)$, then the function

$$F(z) = e^{f(z)}$$

clearly satisfies the conditions

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$$F(z+\lambda)=e^{2\pi iA}F(z)$$
$$F\left(-\frac{1}{z}\right)=\left(\frac{z}{i}\right)^{B}F(z)\ .$$

Moreover, if $f(z)$ has an expansion

$$f(z)=\frac{2\pi iAz}{\lambda}+\sum_{n=1}^{\infty}a_n e^{2\pi inz/\lambda}\ ,$$

then we see that

$$F(z)=\exp\left(2\pi iAz/\lambda+a_1e^{2\pi iz/\lambda}+\cdots\right)$$
$$=e^{2\pi iAz/\lambda}\{1+b_1e^{2\pi iz/\lambda}+\cdots\}\ .$$

Therefore, we have the following result.

LEMMA 4.2. *Let A be an integer, B be real and let $f\in\mathscr{H}(\lambda,+1;A,B,0)$. Then $F(z)=\exp(f(z))$ belongs to $\mathscr{M}^*_\infty(\lambda,B)$ and has order A at $i\infty$.*

Lemma 4.2 has a converse. For if $F(z)\in\mathscr{M}^*_\infty(\lambda,B)$ has order A at $i\infty$, then $F(z)$ can be written in the form

$$F(z)=e^{2\pi iAz/\lambda}F_1(z)\ ,$$

where $F_1(z)$ is holomorphic in $\boldsymbol{H}\cup\{i\infty\}$ and has no zeros there. We may certainly write

$$F_1(z)=e^{g(z)}\ ,$$

where $g(z)$ is analytic for $z\in\boldsymbol{H}$. The function $g(z)$ is determined only up to an additive factor $2k\pi i$ $(k\in\boldsymbol{Z})$. Since $F_1(z)$ has an expansion of the form

$$F_1(z)=1+b_1e^{2\pi iz/\lambda}+\cdots\ ,$$

we see that $g(z)$ can be written

$$g(z)=2k\pi i+\sum_{n=1}^{\infty}c_ne^{2\pi inz/\lambda}\ ,\qquad k\in\boldsymbol{Z}\ .$$

Let us choose $k=0$. Then

$$\log F(z)=\frac{2\pi iAz}{\lambda}+\sum_{n=1}^{\infty}c_ne^{2\pi inz/\lambda}\ ,$$

where the logarithm denotes the principal branch. Let us set $f(z)=\log F(z)$. Then we clearly have

$$f(z+\lambda) = f(z) + \frac{2\pi i A}{\lambda}$$
$$f\left(-\frac{1}{z}\right) = f(z) + B\log\left(\frac{z}{i}\right).$$

Thus, we see that $f(z) = \log F(z \in \mathscr{H}(\lambda, +1; A, B, 0)$. To summarize:

LEMMA 4.3. *Let A be an integer, B real and let $F \in \mathscr{M}^*_\infty(\lambda, B)$ have order A at $i\infty$. Then $f(z) = \log F(z)$ belongs to $\mathscr{H}(\lambda, +1; A, B, 0)$.*

Let us set

$$\mathscr{M}^*_\infty(\lambda) = \bigcup_{k\in \boldsymbol{Z}} \mathscr{M}^*_\infty(\lambda, k).$$

It is clear that under the operation of multiplication of functions, $\mathscr{M}^*_\infty(\lambda)$ is an abelian group. From Lemma 4.3, we have a homomorphism

$$\mathscr{M}^*_\infty(\lambda) \to \mathscr{H}(\lambda, 1)$$
$$F \mapsto \log F$$

from $\mathscr{M}^*_\infty(\lambda)$ into the additive group of the vector space $\mathscr{H}(\lambda, 1)$. A function F is mapped into $\mathscr{H}_0(\lambda, 1)$ if and only if F has weight 0 and order 0 at $i\infty$. That is, the preimage of $\mathscr{H}_0(\lambda, 1)$ is $\mathscr{M}^*(\lambda, 0)$. Thus, we have an induced injective homomorphism of abelian groups

$$\mathscr{M}^*_\infty(\lambda)/\mathscr{M}^*(\lambda, 0) \to \mathscr{H}(\lambda, 1)/\mathscr{H}_0(\lambda, 1) = \overline{\mathscr{H}}(\lambda, 1).$$

THEOREM 4.4. *The image of the injection*

$$\mathscr{M}^*_\infty(\lambda)/\mathscr{M}^*(\lambda, 0) \to \overline{\mathscr{H}}(\lambda, 1)$$

spans $\overline{\mathscr{H}}(\lambda, 1)$.

Proof. The image clearly contains all images modulo $\mathscr{H}_0(\lambda, 1)$ of functions f belonging to $\mathscr{H}(\lambda, 1; A, B, 0)$, A, B integral. These functions form a full lattice in $\overline{\mathscr{H}}(\lambda, 1)$ and thus they span $\overline{\mathscr{H}}(\lambda, 1)$.

Let us now describe $\overline{\mathscr{H}}(\lambda, 1)$. We consider separately the cases $0 < \lambda < 2$ and $\lambda \geq 2$.

THEOREM 4.5. *Assume that $0 < \lambda < 2$. Then $\mathscr{H}_0(\lambda, 1) = \{0\}$ and $\overline{\mathscr{H}}(\lambda, 1) = \mathscr{H}(\lambda, 1)$. Furthermore, we have:*

(a) *If λ is not of the form $2\cos(\pi/q)$, q an integer ≥ 3, then $\mathscr{H}(\lambda, \gamma) = \{0\}$.*

(b) *If* $\lambda = 2\cos(\pi/q)$, *q an integer* ≥ 3, *then* $\dim_C \mathscr{H}(\lambda, 1) = 1$.

Proof. If $f \in \mathscr{H}_0(\lambda, 1)$, then f is an automorphic function for $G(\lambda)$, which is regular everywhere, including $i\infty$. Since $0 < \lambda < 2$, this implies that f is constant. However, since f is a normalized Hecke integral, we see that $f = 0$. Thus $\mathscr{H}_0(\lambda, 1) = \{0\}$ and $\bar{\mathscr{H}}(\lambda, 1) = \mathscr{H}(\lambda, 1)$.

(a) If $\lambda \neq 2\cos(\pi/q)$, $q \geq 3$, then it is well-known [8, p. I–23] that $\mathscr{M}(\lambda, k) = \{0\}$ for all $k > 0$, $\mathscr{M}(\lambda, 0) = \boldsymbol{C}$. Thus, $\mathscr{M}^*_\infty(\lambda, k) = \emptyset$ for $k \neq 0$, $\mathscr{M}^*_\infty(\lambda, 0) = \boldsymbol{C}$. Therefore $\mathscr{M}^*_\infty(\lambda) = \boldsymbol{C}^*$ and $\bar{\mathscr{H}}(\lambda, 1) = \mathscr{H}(\lambda, 1)$ is spanned by constants. But since Hecke integrals are assumed normalized, this implies that $\mathscr{H}(\lambda, 1) = \{0\}$.

(b) Assume that $\lambda = 2\cos(\pi/q)$, q an integer ≥ 3. If $k > 0$, then it was proved by Hecke [8, p. I–23] that $\dim_C \mathscr{M}(\lambda, k) = 0$ unless $k = 4m/(q-2)$ for some positive integer m. In this case

$$\dim_C \mathscr{M}\left(\lambda, \frac{4m}{q-2}\right) = 1 + \left[\frac{m}{q}\right]. \tag{25}$$

Moreover, if $f \in \mathscr{M}(\lambda, 4m/(q-2))$ is not identically zero, then f has m/q zeros in a fundamental domain for $G(\lambda)$ (including any zero at $i\infty$ and counting zeros with proper multiplicities). Therefore, since the forms in $\mathscr{M}^*(\lambda)$ have no zeros in $\boldsymbol{H}$, we see that if $f \in \mathscr{M}^*(\lambda)$, then $f \in \mathscr{M}^*(\lambda, 0) = \boldsymbol{C}^\times$. Hecke showed [8, p. I–20] that there exists a function $f_\infty \in \mathscr{M}(\lambda, 4q/(q-2))$ with a simple zero at $i\infty$ and no other zeros. Now if $f \in \mathscr{M}^*_\infty(\lambda)$, then either $f \in \mathscr{M}(\lambda, 4m/(q-2))$ or $f^{-1} \in \mathscr{M}(\lambda, 4m/(q-2))$ for some positive integer m, so that f has order $\pm m/q$ at $i\infty$. In particular, $q \mid m$ and

$$f / f_\infty^{\pm m/q} \in \mathscr{M}(\lambda, 0) = \boldsymbol{C}^\times .$$

Thus, we see that f_∞ generates $\mathscr{M}^*_\infty(\lambda)$, so that by Theorem 4.4, we see that $\{\log f_\infty\}$ is a basis of $\bar{\mathscr{H}}(\lambda, 1) = \mathscr{H}(\lambda, 1)$.

As a consequence of the proof, we obtain:

COROLLARY 4.6. *Let* $\lambda = 2\cos(\pi/q)$, *q an integer* ≥ 3. *Then*

$$\mathscr{H}(\lambda, 1) = \boldsymbol{C} \log f_\infty$$

and $\log f_\infty$ *is a Hecke integral of signature* $\{\lambda, 1, 4q/(q-2), 0, +1\}$. *Thus, if* $0 < \lambda < 2$, *there exists a Hecke integral of signature* $\{\lambda, A, B, C, +1\}$ *if and only if* $\lambda = 2\cos(\pi/q)$ *for some integer* $q \geq 3$, $B = 4Aq/(q-2)$,

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$C = 0$. *In this case there is a unique Hecke integral and it equals* $A \log f_\infty$.

For the sake of completeness, let us review the construction of f_∞: Let $\mathscr{B}^*(\lambda)$ denote the domain defined by the inequalities $-\lambda/2 \leq \operatorname{Re}(z) \leq 0$, $|z| \geq 1$, $\operatorname{Im}(z) > 0$. Let τ_0 denote the left hand corner of this region. By the Riemann mapping theorem, there exists a function $g_\lambda(z)$ mapping the interior of $\mathscr{B}^*(\lambda)$ conformally onto the open upper half-plane and such that $g_\lambda(\tau_0) = 0$, $g_\lambda(i) = 1$, $g_\lambda(\infty) = \infty$. (If $\lambda = 1$, then $g_\lambda(z) = J(z)$, the elliptic modular invariant.) Then

$$f_\infty(z) = \left\{\frac{g'(z)^{2q}}{g(z)^{2q-2}(g(z) - 1)^q}\right\}^{1/(q-2)} .$$

(For details, see [8, pp. I–20–22].)

Let us now proceed to the case $\lambda \geq 2$.

PROPOSITION 4.7. *If* $\lambda \geq 2$, *then* $\dim \mathscr{H}_0(\lambda, 1) = \infty$.

Proof. If $\lambda > 2$, then we can construct $f_n \in \mathscr{H}_0(\lambda, 1)$ for each $n > 0$ such that f_n has a pole at $z = -i$. (See [8, p. I–11] for details.) The functions f_n are clearly linearly independent over $\boldsymbol{C}$.

If $\lambda = 2$, then the fundamental domain for $G(2)$ has two independent cusps whose representatives can be taken as $i\infty$ and 1. Moreover, the Riemann surface for $G(2)$ has genus 0 so that there exists a function $f_n \in \mathscr{H}_0(2, 1)$ with a pole of order n at $z = 1$ (in the appropriate uniformizing parameter). The functions f_n are linearly independent over $\boldsymbol{C}$.

THEOREM 4.8. *If* $\lambda \geq 2$, *then* $\dim_{\boldsymbol{C}} \mathscr{H}(\lambda, 1) = 2$.

Proof. If $\lambda = 2$, let us observe that $\log \eta(z)$ and $\log \theta(z)$ have signatures $\{2, \frac{1}{6}, \frac{1}{2}, 0, 1\}$ and $\{2, 0, \frac{1}{2}, 0, 1\}$, so that it is clear that $\{\log \eta(z), \log \theta(z)\}$ forms a basis of $\mathscr{H}(\lambda, 1)$. Thus, assume $\lambda > 2$. Hecke (see [5, p. 673] or [8, p. I–11]) constructs an automorphic form $F_1(z)$ belonging to $\mathscr{M}(\lambda, k)$ for arbitrary $k > 0$. An inspection of Hecke's construction shows that $F_1(z)$ does not vanish for $z \in \boldsymbol{H}$, $F_1(i\infty) = 0$. If

$$F_1(z) = e^{2\pi i \alpha z} + \cdots , \qquad \alpha > 0 ,$$

we see that $f_1(z) = \log F_1(z)$ has signature $\{\lambda, \alpha, k, 0, +1\}$. If $F_2(z)$ is a similar automorphic form giving rise to a Hecke integral of signature $\{\lambda, \alpha, 2k, 0, +1\}$, we see that $f_2(z) = \log F_2(z) - \log F_1(z)$ has signature

$\{\lambda, 0, k, 0, +1\}$. It is immediate that $f_1(z)$ and $f_2(z)$ are linearly independent modulo $\mathscr{H}_0(\lambda, 1)$. Thus, $\dim_C \bar{\mathscr{H}}(\lambda, 1) = 2$.

The interpretation of the fact that $\dim_C \bar{\mathscr{H}}(\lambda, 1) = 2$ is that there exist Hecke integrals corresponding to every possible pair of values (A, B). By Proposition 4.7 and Eq. (23), we conclude

COROLLARY 4.9. *Let $\lambda \geq 2$, A, B arbitrary. Then there exist infinitely many Hecke integrals of signature $\{\lambda, A, B, 0, +1\}$.*

Case II: $\gamma = -1$.

First, let us observe that in this case, all the constant functions belong to $\mathscr{H}(\lambda, -1)$. Therefore, if we set

$$\mathscr{H}_1(\lambda, -1) = \bigcup_{A \in C} \mathscr{H}(\lambda, -1; A, 0, 0) ,$$

then we see that

$$\mathscr{H}(\lambda, -1) = C \oplus \mathscr{H}_0(\lambda, -1)$$
$$\bar{\mathscr{H}}(\lambda, -1) = C \oplus \bar{\mathscr{H}}_1(\lambda, -1) , \qquad \bar{\mathscr{H}}_1(\lambda, -1) = \mathscr{H}(\lambda, -1)/\mathscr{H}_0(\lambda, -1) .$$

Thus, we may proceed in this case by describing $\mathscr{H}_0(\lambda, -1)$ and $\bar{\mathscr{H}}_1(\lambda, -1)$, By Proposition 4.1, we know that

$$\dim_C \bar{\mathscr{H}}_1(\lambda, -1) \leq 1 . \tag{26}$$

Let us begin our analysis by describing our replacement for the exponential-logarithmic correspondence. Suppose that $f \in \mathscr{H}_1(\lambda, -1)$. Then $f \in \mathscr{H}(\lambda, A, 0, 0, -1)$ for some A, so that f satisfies

$$f(z + \lambda) = f(z) + A$$
$$f\left(-\frac{1}{z}\right) = -f(z) .$$

Therefore, the function $F(z) = f'(z)$ must satisfy

$$F(z + \lambda) = F(z)$$
$$F\left(-\frac{1}{z}\right) = \left(\frac{z}{i}\right)^2 F(z) .$$

Moreover, $F(z)$ has a Fourier expansion of the form

$$F(z) = \frac{2\pi i A}{\lambda} + \sum_{n=1}^{\infty} b_n e^{2\pi i n z/\lambda} .$$

In other words, $F(i\infty) = 2\pi i A/\lambda$. Therefore, we see that $F(z)$ is an automorphic form belonging to $\mathscr{M}(\lambda, 2)$ for which $F(i\infty) = 2\pi i A/\lambda$.

Conversely, if we are given $F \in \mathscr{M}(\lambda, 2)$ such that $F(i\infty) = 2\pi i A/\lambda$, we see that

$$f(z) = \int_i^z F(w)dw$$

is a Hecke integral of signature $\{\lambda, A, 0, 0, -1\}$. Thus, we have proved:

LEMMA 4.10. *The mapping*

$$f \mapsto f'$$

is a bijection from $\mathscr{H}(\lambda, A, 0, 0, -1)$ *to the set of all automorphic forms* $F(z)$ *in* $\mathscr{M}(\lambda, 2)$ *for which* $F(i\infty) = 2\pi A/\lambda$.

Letting A vary over $\boldsymbol{C}$, we have

PROPOSITION 4.11. *The mapping*

$$f \to f'$$

is a bijection from $\mathscr{H}_1(\lambda, -1)$ *to* $\mathscr{M}(\lambda, 2)$.

Let

$$S(\lambda, k) = \{f \in \mathscr{M}(\lambda, k) \,|\, f(i\infty) = 0\} .$$

Then Lemma 4.10 implies that the mapping $f \mapsto f'$ carries $\mathscr{H}_1(\lambda, -1)$ into $S(\lambda, 2)$. Therefore, we have

THEOREM 4.12. *The mapping* $f \mapsto f'$ *induces a surjective isomorphism of complex vector spaces:*

$$\bar{\mathscr{H}}_1(\lambda, -1) = \mathscr{H}_1(\lambda, -1)/\mathscr{H}_0(\lambda, -1) \to \mathscr{M}(\lambda, 2)/S(\lambda, 2) .$$

In particular, $\dim \bar{\mathscr{H}}_1(\lambda, -1) = 1$ *if and only if there exists an automorphic form in* $\mathscr{M}(\lambda, 2)$ *which does not vanish at* $i\infty$.

As with the case $\gamma = +1$, let us consider the cases $0 < \lambda < 2$ and $\lambda \geq 2$ separately.

Let us first assume that $0 < \lambda < 2$. Then (22) implies that $\mathscr{H}_0(\lambda, -1) = \{0\}$, so that

$$\mathscr{H}_1(\lambda, -1) = \bar{\mathscr{H}}_1(\lambda, -1) , \tag{27}$$

$$\mathscr{H}(\lambda, -1) = \boldsymbol{C} \oplus \mathscr{H}_1(\lambda, -1) , \tag{28}$$

THEOREM 4.13. *Assume that* $0 < \lambda < 2$.

(a) *If* λ *is not of the form* $2\cos(\pi/q)$ *for some integer* $q \geq 3$, *then* $\mathscr{H}(\lambda, -1) = \boldsymbol{C}$.

(b) *If* $\lambda = 2\cos(\pi/q)$ *for some integer* $q \geq 3$, *then*

$$\dim_{\boldsymbol{C}} \mathscr{H}(\lambda, -1) = \begin{cases} 2 & \textit{if } q \textit{ is even} \\ 1 & \textit{if } q \textit{ is odd.} \end{cases}$$

Proof. (a) As in the case $\gamma = +1$, we deduce that if $\lambda \neq 2\cos(\pi/q)$, then $\mathscr{M}(\lambda, 2) = \{0\}$, so that by Theorem 4.12, we see that $\bar{\mathscr{H}}_1(\lambda, -1) = \{0\}$. Thus, by (27) and (28), we see that $\mathscr{H}(\lambda, -1) = \boldsymbol{C}$.

(b) If $\lambda = 2\cos(\pi/q)$, then $\dim_{\boldsymbol{C}} \mathscr{M}(\lambda, k) = 0$ unless $k = 4m/(q-2)$ for some positive integer m [8, p. I–23]. In this case, we have

$$\dim_{\boldsymbol{C}} \mathscr{M}(\lambda, 4m/(q-2)) = 1 + \left[\frac{m}{q}\right]$$

$$\dim_{\boldsymbol{C}} S(\lambda, 4m/(q-2)) = \left[\frac{m}{q}\right].$$

Thus, we see that $\mathscr{M}(\lambda, 2) \neq \{0\}$ if and only if $2 = 4m/(q-2)$ for some m, i.e. q is even. But then, by the above formulas, we see that $\dim_{\boldsymbol{C}} \mathscr{M}(\lambda, 2) = \dim_{\boldsymbol{C}} S(\lambda, 2) + 1$, so that by Theorem 4.12, we see that

$$\dim_{\boldsymbol{C}} \bar{\mathscr{H}}_1(\lambda, -1) = 1,$$

so we are done by (27) and (28).

In case $\dim_{\boldsymbol{C}} \mathscr{H}_1(\lambda, -1) = 1$, we can arrive at an explicit basis as follows: Hecke [8, p. I–20] constructs a function f_0 belonging to $\mathscr{M}(\lambda, 4/(q-2))$ having a simple zero at $z = e^{\pi i/q}$ and no others in $\boldsymbol{H} \cup \{i\infty\}$. Assume that q is even. Then, since $f_0^{(q-2)/2} \in \mathscr{M}(\lambda, 2)$, we see that

$$f_\lambda(z) = \int_i^z f_0^{(q-2)/2}(u)du$$

belongs to $\mathscr{H}_1(\lambda, -1)$. Therefore, we have

COROLLARY 4.14. *Assume that* $0 < \lambda < 2$. *Then* $\mathscr{H}(\lambda, -1) = \boldsymbol{C}$ *unless* $\lambda = 2\cos(\pi/q)$ *for some even integer* $q \geq 3$, *in which case*

$$\mathscr{H}(\lambda, -1) = \boldsymbol{C} \oplus \boldsymbol{C} f_\lambda .$$

Next, let us assume that $\lambda \geq 2$. Parallel to Proposition 4.7, we have:

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PROPOSITION 4.15. *If* $\lambda \geq 2$, *then* $\dim_C \mathscr{H}_0(\lambda, -1) = \infty$.

Proof. See [8, p. I–13] and the proof of Proposition 4.7.

PROPOSITION 4.16. *If* $\lambda \geq 2$, *then* $\dim_C \mathscr{H}_1(\lambda, -1) = 1$.

Proof. If $\lambda = 2$, then $\theta(z)^4 \in \mathscr{M}(2, 2)$, $\theta(i\infty)^4 \neq 0$. Therefore, if we set

$$f(z) = \int_i^z \theta^4(u)du ,$$

then $\mathscr{H}_1(2, -1) = Cf$.

Assume $\lambda > 2$. If $F(z) \in \mathscr{M}(\lambda, 2)$, $F(i\infty) = 1$ (for the existence of such an F, see [8, p. I–12]), then

$$f(z) = \int_i^z F(u)du$$

belongs to $\mathscr{H}_1(\lambda, -1)$ and $\mathscr{H}_1(\lambda, -1) = Cf$.

Let us summarize the case $\gamma = -1$ in terms of signatures of Hecke integrals. The interpretation of the fact $\dim_C \mathscr{H}_1(\lambda, -1) = 1$ is that there exist Hecke integrals corresponding to every pair (A, C). By Proposition 4.15 and (22), we see that if $\lambda \geq 2$, then there always exist infinitely many Hecke integrals of each signature $\{\lambda, A, 0, C, -1\}$. Thus, we may summarize our results:

THEOREM 4.17. (a) *If* $0 < \lambda < 2$, *then there exists at most one Hecke integral of signature* $\{\lambda, A, 0, C, -1\}$. *The signatures for which Hecke integrals exist are* $\{\lambda, 0, 0, C, -1\}$, *except in the case* $\lambda = 2\cos(\pi/q)$, q *an even integer* ≥ 3, *in which case all signatures* $\{\lambda, A, 0, C, -1\}$ *have corresponding integrals.* (b) *If* $\lambda \geq 2$, *then there exist infinitely many Hecke integrals of signature* $\{\lambda, A, 0, C, -1\}$.

§ 5. A curious class number formula

Let us now derive a curious formula involving $f_\chi(z)$. This formula will lead to a previously-unobserved formula for $L(1, \chi)$ in terms of $\log \eta(z)$. Throughout this section, let χ be a real, odd character defined modulo its conductor $\mathfrak{f}$.

From the definition of $f_\chi(z)$, we have

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$$
\begin{aligned}
f_\chi(z) &= \sum_{m=1}^{\infty} \chi(m)\Big(\sum_{\substack{d|m\\ d>0}} \frac{1}{d}\Big) q^m\,, \qquad q = e^{2\pi i z/f} \\
&= \sum_{r\ (\mathrm{mod}\ f)} \chi(r) \sum_{\substack{m=1\\ m\equiv r\ (\mathrm{mod}\ f)}}^{\infty} \Big(\sum_{\substack{d|m\\ d>0}} \frac{1}{d}\Big) q^m \\
&= \frac{1}{f} \sum_{r\ (\mathrm{mod}\ f)} \chi(r) \sum_{t\ (\mathrm{mod}\ f)} \sum_{m=1}^{\infty} e^{2\pi i t(m-r)/f} \Big(\sum_{d|m} \frac{1}{d}\Big) q^m \\
&= \frac{1}{f} \sum_{\substack{r\ (\mathrm{mod}\ f)\\ t\ (\mathrm{mod}\ f)}} \chi(r) e^{-2\pi i t r/f} \sum_{m=1}^{\infty} \Big(\sum_{d|m} \frac{1}{d}\Big) (q e^{2\pi i t/f})^m \\
&= \frac{1}{f} \sum_{\substack{r\ (\mathrm{mod}\ f)\\ t\ (\mathrm{mod}\ f)}} \chi(r) e^{-2\pi i t r/f} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{m} q_t^{nm}\,, \qquad q_t = q e^{2\pi i t/f} \\
&= -\frac{1}{f} \sum_{\substack{t\ (\mathrm{mod}\ f)\\ r\ (\mathrm{mod}\ f)}} \chi(r) e^{-2\pi i t r/f} \sum_{n=1}^{\infty} \log(1 - q_t^n) \\
&= -\frac{1}{f} \sum_{\substack{t\ (\mathrm{mod}\ f)\\ r\ (\mathrm{mod}\ f)}} \chi(r) e^{-2\pi i t r/f} \log\Big(\prod_{n=1}^{\infty} (1 - q_t^n)\Big) \\
&= -\frac{1}{f} \sum_{t\ (\mathrm{mod}\ f)} \log\Big(\prod_{n=1}^{\infty} (1 - q_t^n)\Big) \sum_{r\ (\mathrm{mod}\ f)} \chi(r) e^{-2\pi i t r/f} \\
&= -\frac{\tau(\chi)}{f} \sum_{t\ (\mathrm{mod}\ f)} \chi(-t) \log\Big(\prod_{n=1}^{\infty} (1 - q_t^n)\Big) \\
&= \frac{\tau(\chi)}{f} \sum_{f\ (\mathrm{mod}\ f)} \chi(t) \log\Big(\prod_{n=1}^{\infty} (1 - q_t^n)\Big)
\end{aligned}
$$

Let us set $z_t = (t + z)/f$. Then

$$
\begin{aligned}
\log\Big(\prod_{n=1}^{\infty} (1 - q_t^n)\Big) &= \log \eta(z_t) - \frac{1}{12} \log q_t \\
&= \log \eta(z_t) - \frac{\pi i (t + z)}{6f}\,.
\end{aligned}
$$

Therefore

$$
\begin{aligned}
f_\chi(z) &= \frac{\tau(\chi)}{f} \sum_{t\ (\mathrm{mod}\ f)} \chi(t) \Big\{\log \eta(z_t) - \frac{\pi i t}{6f} - \frac{\pi i z}{6f}\Big\} \\
&= \frac{\tau(\chi)}{f} \sum_{t=0}^{f-1} \chi(t) \log \eta(z_t) - \frac{\pi i \tau(\chi)}{6f^2} \sum_{t=0}^{f-1} t\chi(t)\,.
\end{aligned}
$$

However, [1, p. 336] it is well-known that

$$
\frac{\pi i \tau(\chi)}{f^2} \sum_{t=0}^{f-1} t\chi(t) = L(1, \chi)\,.
$$

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Therefore, we finally have

THEOREM 5.1. *Let* $z_t = (z + t)/f$. *Then*

$$f_\chi(z) = \frac{\tau(\chi)}{f} \sum_{t=0}^{f-1} \chi(t) \log \eta(z_t) - \frac{1}{6} L(1, \chi) .$$

To see the utility of this formula, let us recall from Theorem 3.3 that

$$f_\chi\left(-\frac{1}{z}\right) = -f_\chi(z) + \frac{if}{\pi\tau(\chi)} L(1, \bar{\chi})^2 .$$

Setting $z = i$ in this formula, and using the fact that χ is real, we see that

$$f_\chi(i) = \frac{if}{2\pi\tau(\chi)} L(1, \chi)^2 .$$

Therefore, from Theorem 5.1, we deduce that

$$\frac{if}{2\pi\tau(\chi)} L(1, \chi)^2 = \frac{\tau(\chi)}{f} \sum_{t=0}^{f-1} \chi(t) \log \eta\left(\frac{i+t}{f}\right) - \frac{1}{6} L(1, \chi) . \qquad (30)$$

Since χ is a primitive, odd character, we have [1, p. 350] that

$$\tau(\chi) = i\sqrt{f} .$$

Therefore, we may rewrite equation (30) as

$$\left(\frac{f^{1/2}L(1, \chi)}{\pi}\right)^2 - \frac{1}{3}\left(\frac{f^{1/2}L(1, \chi)}{\pi}\right) - \frac{2i}{\pi} \sum_{t=0}^{f-1} \chi(t) \log \eta\left(\frac{i+t}{f}\right) = 0 . \qquad (31)$$

We rewrite the last term in a slightly more enlightening from by replacing t by $f - t$ and averaging the two sums to get

$$\begin{aligned} \frac{2i}{\pi f^{1/2}} \sum_{t=0}^{f-1} \chi(t) \log \eta\left(\frac{i+t}{f}\right) &= \frac{i}{\pi} \sum_{t=0}^{f-1} \chi(t) \left\{\log \eta\left(\frac{i+t}{f}\right) - \log \eta\left(\frac{i+f-t}{f}\right)\right\} \\ &= \frac{i}{\pi} \sum_{t=0}^{f-1} \chi(t) \left\{\log \eta\left(\frac{i+t}{f}\right) - \log \eta\left(\frac{i-t}{f}\right)\right\} \\ &\quad - \frac{i}{\pi^2} \sum_{t=0}^{f-1} \frac{\pi i}{12} \chi(t) \\ &= \frac{i}{\pi} \sum_{t=0}^{f-1} \chi(t) \left\{\log \eta\left(\frac{i+t}{f}\right) - \log \eta\left(\left(\overline{\frac{i+t}{f}}\right)\right)\right\} \end{aligned}$$

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$$= \frac{i}{\pi} \sum_{t=0}^{f-1} \chi(t) \left\{ \log \eta\left(\frac{i+t}{f}\right) - \overline{\log \eta\left(\frac{i+t}{f}\right)} \right\}$$
$$= -\frac{2}{\pi} \sum_{t=0}^{f-1} \chi(t) \operatorname{Im} \log \eta\left(\frac{i+t}{f}\right) .$$

Thus, (31) can be rewritten as

$$\left(\frac{f^{1/2}L(1,\chi)}{\pi}\right)^2 - \frac{1}{3}\left(\frac{f^{1/2}L(1,\chi)}{\pi}\right) + \frac{2}{\pi} \operatorname{Im}\left(\sum_{t=0}^{f-1} \chi(t) \log \eta\left(\frac{i+t}{f}\right)\right) = 0 . \quad (32)$$

Let K_f be the quadratic field corresponding to χ via class field theory. Since χ is odd of conductor f, we know that $K = \boldsymbol{Q}(\sqrt{-f})$. Let h denote the class number of K_f. Let us assume that $|f| > 4$. Then it is well-known [2, p. 51] that

$$h_f = \frac{f^{1/2}L(1,\chi)}{\pi} .$$

Therefore, we deduce that h_f is a zero of the quadratic equation

$$x^2 - \frac{1}{3}x + \frac{2}{\pi} \operatorname{Im}\left(\sum_{t=0}^{f-1} \chi(t) \log \eta\left(\frac{i+t}{f}\right)\right) = 0 .$$

Let us set

$$S = \frac{2}{\pi} \operatorname{Im}\left(\sum_{t=0}^{f-1} \chi(t) \log \eta\left(\frac{i+t}{f}\right)\right) .$$

Then

$$h_f = \frac{\frac{1}{3} \pm \sqrt{\frac{1}{9} - 4S}}{2} .$$

However, since $h_f \geq 1$, the plus sign must prevail and we have

$$h_f = \tfrac{1}{6}(1 + \sqrt{1 - 36S})$$
$$= \frac{1}{6}\left(1 + \sqrt{1 - \frac{72}{\pi} \operatorname{Im}\left(\sum_{t=0}^{f-1} \chi(t) \log \eta\left(\frac{i+t}{f}\right)\right)}\right) .$$

THEOREM 5.2. *Let f be the discriminant of an imaginary quadratic field, $f > 4$, h_f the class number of $\boldsymbol{Q}(\sqrt{f})$. Then*

$$h_f = \frac{1}{6}\left(1 + \sqrt{1 - \frac{72}{\pi} \operatorname{Im}\left(\sum_{t=0}^{f-1} \chi(t) \log \eta\left(\frac{i+t}{f}\right)\right)}\right) .$$

Theorem 5.2 is something of a curiosity. However, let us conclude

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this paper with a few comments concerning what seems to us the deeper significance of the function $f_\chi(z)$. From the transformation properties of $f_\chi(z)$, we have

$$2f_\chi(i) = \frac{if}{\pi\tau(\chi)} L(1, \bar{\chi})^2 . \tag{35}$$

Let us assume that χ is real. Then Siegel's theorem [2, p. 130] asserts that for any $\varepsilon > 0$, there exists a constant $c = c(\varepsilon)$ such that

$$L(1, \chi) > \frac{c}{f^\varepsilon} .$$

Therefore, since $\tau(\chi) = if^{1/2}$, we see that for any $\varepsilon > 0$, there exists a constant c_1 such that

$$f_\chi(i) > c_1 f^{1/2-\varepsilon} . \tag{36}$$

Unfortunately, the constant in Siegel's theorem cannot be determined effectively. Suppose, however, that instead of (36), we are able to prove the much weaker assertion

$$f_\chi(i) > c_2$$

for some absolute, effectively determined positive constant c_2. Then the relation (35) implies that

$$L(1, \chi) > \frac{c_3}{f^{1/4}} ; \tag{37}$$

the Dirichlet formula for $L(1, \chi)$ asserts that if h_f is the class number of the imaginary quadratic field $\boldsymbol{Q}(\sqrt{-f})$, then

$$L(1, \chi) = \frac{2\pi h_f}{w_f \sqrt{f}} ,$$

where $w_f = 4$ if $f = 4$, $= 6$ if $f = 3$, $= 2$ otherwise. The inequality (37) would then imply that

$$h_f > c_4 f^{1/4} ,$$

an inequality which suffices to determine all imaginary quadratic fields having a given class number. Thus, it would seem that it is of major significance to determine an effectively computable lower bound for $f_\chi(i)$. Let us state this formally:

PROBLEM. Determine an effectively computable absolute lower bound for the series

$$\sum_{n=1}^{\infty}\left(\chi(n)\sum_{d|n}\frac{1}{d}\right)e^{-2\pi n/f} .$$

To get a more concrete idea of what is involved in this problem, let us convert the above series into a series with positive terms.

First note that

$$\log\eta\left(\frac{z}{f}\right) = -\sum_{m=1}^{\infty}\left(\sum_{d|m}\frac{1}{d}\right)e^{2\pi imz/f} + \frac{\pi iz}{12f}$$

so that

$$\begin{aligned} f_\chi(z) - \log\eta\left(\frac{z}{f}\right) &= \sum_{m=1}^{\infty}(1+\chi(m))\left(\sum_{d|m}\frac{1}{d}\right)e^{2\pi imz/f} - \frac{\pi iz}{12f} \\ &= 2\sum_{\substack{m=1\\ \chi(m)=+1}}^{\infty}\left(\sum_{d|m}\frac{1}{d}\right)e^{2\pi imz/f} \\ &\quad + \sum_{\substack{m=1\\ (m,f)>1}}^{\infty}\left(\sum_{d|m}\frac{1}{d}\right)e^{2\pi imz/f} - \frac{\pi iz}{12f} . \end{aligned}$$

Setting $z = i$, we find that

$$\begin{aligned} f_\chi(i) - \log\eta\left(\frac{i}{f}\right) &= 2\sum_{\substack{m=1\\ \chi(m)=+1}}^{\infty}\left(\sum_{d|m}\frac{1}{d}\right)e^{-2\pi m/f} \\ &\quad + \sum_{\substack{m=1\\ (m,f)>1}}^{\infty}\left(\sum_{d|m}\frac{1}{d}\right)e^{2\pi m/f} + \frac{\pi}{12f} \\ &= \sum_{m=1}^{\infty}\varepsilon(m)e^{-2\pi m/f} + \frac{\pi}{12f} , \end{aligned}$$

where

$$\varepsilon(m) = \begin{cases} 2\sum_{d|m}\frac{1}{d} & \text{if } \chi(m)=1 \\ \sum_{d|m}\frac{1}{d} & \text{if } (m,f)>1 \\ 0 & \text{otherwise} . \end{cases}$$

However, from the transformation law for $\log\eta(z)$, we see that

$$\log\eta\left(\frac{i}{f}\right) = \log\eta(if) + \frac{1}{2}\log\left(\frac{1}{f}\right) .$$

Moreover,

$$\log \eta(if) = \frac{\pi i}{12}\cdot if - \sum_{m=1}^{\infty}\left(\sum_{d|m}\frac{1}{d}\right)e^{-2\pi mf} .$$

Therefore, we see that

$$f_\chi(i) = -\frac{\pi f}{12} - \sum_{m=1}^{\infty}\left(\sum_{d|m}\frac{1}{d}\right)e^{-2\pi mf} + \frac{1}{2}\log f + \sum_{m=1}^{\infty}\varepsilon(m)e^{-2\pi m/f} + \frac{\pi}{12f} .$$

Therefore, as $f \to \infty$, we deduce the following asymptotic formula for $f_\chi(i)$:

$$f_\chi(i) = \sum_{m=1}^{\infty}\varepsilon(m)e^{-2\pi m/f} - \frac{\pi f}{12} + \frac{1}{2}\log f + O(1) , \qquad f \to \infty . \tag{38}$$

The infinite series on the right hand side has only positive terms.

By using the Siegel theorem, as well as the Polya-Vinogradov theorem, which asserts that

$$L(1, \chi) < 2 \log f ,$$

we see that for any $\varepsilon > 0$,

$$c(\varepsilon)f^{1/2} < f_\chi(i) < \frac{4}{\pi}f^{1/2}\log^2 f .$$

Therefore, we deduce the following estimate from (38):

$$c_1(\varepsilon)f^{1/2-\varepsilon} < \sum_{m=1}^{\infty}\varepsilon(m)e^{-2\pi m/f} - \frac{\pi f}{12} < c_2 f^{1/2}\log^2 f , \tag{39}$$

where c_2 is effective, but $c_1(\varepsilon)$ is not. Conversely, if it were possible to establish (39) effectively, then this would immediately lead to an effective proof of Siegel's theorem. In fact, insofar as it is desired only to determine effectively all imaginary quadratic fields having given class number, it suffices to establish any bound of form

$$\sum_{m=1}^{\infty}\varepsilon(m)e^{-2\pi m/f} - \frac{\pi f}{12} > \lambda(f) ,$$

where $\lambda(f) \to \infty$ as $f \to \infty$.

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