MATHEMATICAL RESULTS OF THE GENERAL PLANETARY THEORY IN RECTANGULAR COORDINATES

V.A. BRUMBERG, L.S. EVDOKIMOVA and V.I. SKRIPNICHENKO
Institute of Theoretical Astronomy, Leningrad

ABSTRACT. Mathematical construction of the general planetary theory has led to the series of two forms for the coordinates of eight major planets (excluding Pluto). The series of the first form are Poisson series where all orbital elements except the semi-major axes occur in literal shape. The series of the second form are polynomial-exponential series with respect to the time and serve to calculate the ephemerides. The arbitrary constants of the theory are related to the Keplerian elements. The terms of the zero and first degree in eccentricities and inclinations have been found in the second approximation with respect to the disturbing masses. Among those of particular interest are the resonant terms caused by the commensurabilities of the mean notations of triplets of planets.

1. INTRODUCTION

This paper summarizes the results of the general planetary theory for eight major planets based on the semi-analytical method exposed earlier (Brumberg and Chapront, 1973). The actual construction of the theory started at the Bureau des Longitudes (Paris) on a computer IBM-360-65 has been continued at the Institute of Theoretical astronomy (Leningrad) on a computer BESM-6. At present our main results are as follows:

1. literal series of the first approximation theory for all eight planets;
2. polynomial solution of the secular system for the slow variables;
3. numerical series for ephemeris calculation;
4. linear second approximation theory.

The exact meaning of these results will be explained below.

The ephemerides based on this theory are erroneous due to the neglect of higher order terms (truncation errors) and the tentative estimates of the constants of the theory (estimation errors). Therefore, for completing our theory both in mathematical and astronomical sense it is necessary to add to our series some corrections influenced by the higher orders terms and to improve the numerical values of the constants by means of comparison of our ephemerides with observations.

V. Szebehely (ed.), Dynamics of Planets and Satellites and Theories of Their Motion, 33-48.

https://doi.org/10.1017/S0252921100062084 Published online by Cambridge University Press

However, the results already obtained seem to be of interest by themselves. They may be used for comparison with the results of different planetary theories now in progress. Moreover, they enable one to estimate a size of the resulting series representing the motion of the major planets and to realize the amount of necessary calculations. That is why the presentation of our results seems to be useful, even though the final completion of our theory may be available in the future.

2. TWO FORMS OF SERIES OF THE GENERAL PLANETARY THEORY

The theory has been built in p_i, w_i-variables related to the heliocentric rectangular coordinates x_i, y_i, z_i ($i = 1, \ldots, N$; $N = 8$) as follows:

$$x_i + \sqrt{-1}\, y_i = A_i\,(1 - p_i)\,\exp \sqrt{-1}\,\lambda_i, \qquad z_i = A_i\, w_i, \tag{1}$$

$$\lambda_i = n_i t + \varepsilon_i, \qquad n_i^2 A_i^3 = k^2\,(m_0 + m_i). \tag{2}$$

Expressions for p_i, w_i may be given in two forms. The first form corresponds to the Poisson series (4N, N-1) where 4N polynomial variables have an order of planetary eccentricities and inclinations and N-1 angular arguments are differences of the mean longitudes. The coefficients of these series are real numbers depending on the adopted numerical values of mean notions n_i and masses m_i. Using numerical values of these constants dictates the semi-analytical form of our theory. Nevertheless, taking into account that all remaining parameters of the theory enter in literal form these series will be referred to as literal series of the general planetary theory.

With a large computer it would be suitable to construct these series by iterations without using the developments in powers of the disturbing masses (Brumberg, 1974). We have tested this way to recalculate the intermediate solution independent of the planetary eccentricities and inclinations (Brumberg et al., 1975a). But for the iterative solution on the whole the capcaities of BESM-6 are unsufficient and we have found the Poisson series using the expansions in powers of the disturbing masses. Explicitly these series have the form

$$p_i = \underset{o}{p_i} + \mu \underset{1}{p_i} + \mu^2 \underset{2}{p_i} + \ldots , \tag{3}$$

$$w_i = \underset{0}{w_i} + \mu \underset{1}{w_i} + \mu^2 \underset{2}{w_i} + \ldots , \tag{4}$$

μ (=0.001) being a basic small parameter. Here $\underset{o}{p_i}$ and $\underset{0}{w_i}$ represent the Poisson series (4,0) for the undisturbed motion

$$\underset{o}{p_i} = \sum_{m=1}^{\infty} \sum p_{pqrs}\, a_i^p\, \bar{a}_i^q\, b_i^r\, \bar{b}_i^s, \tag{5}$$

https://doi.org/10.1017/S0252921100062084 Published online by Cambridge University Press

$$\underset{0}{w}_i = \sum_{m=1}^{\infty} \sum w_{pqrs}\, a_i^p\, \bar{a}_i^q\, b_i^r\, \bar{b}_i^s, \tag{6}$$

where p_{qqrs}, w_{pqrs} are absolute numerical constants having the same values for all planets. Polynomial variables a_i and b_i are of order of eccentricity and inclination of planet i respectively. A bar denotes a complex conjugate quantity. Here and everywhere below the inner summation without explicit limits is performed over all non-negative values of power indices, the sum of which is equal to m. Evidently, the value of m indicates an analytical order of smallness of the corresponding term with respect to the planetary eccentricities and inclinations.

The next terms in (3), (4) yield the series of the first approximation theory

$$\underset{1}{p}_i = \sum_{j=1}^{N(i)} \underset{1}{p}_{ij}, \qquad \underset{1}{w}_i = \sum_{j=1}^{N(i)} \underset{1}{w}_{ij}, \tag{7}$$

where perturbations $\underset{1}{p}_{ij}$, $\underset{1}{w}_{ij}$ due to a specific couple of planets i and j represent Poisson series (8,1):

$$\underset{1}{p}_{ij} = \sum_{m=0}^{\infty} \sum \underset{1}{p}^{(ij)}_{pqrsp'q'r's'}\, a_i^p\, \bar{a}_i^q\, b_i^r\, \bar{b}_i^s\, a_j^{p'}\, \bar{a}_j^{q'}\, b_j^{r'}\, \bar{b}_j^{s'}, \tag{8}$$

$$\underset{1}{w}_{ij} = \sum_{m=1}^{\infty} \sum \underset{1}{w}^{(ij)}_{pqrsp'q'r's'}\, a_i^p\, \bar{a}_i^q\, b_i^r\, \bar{b}_i^s\, a_j^{p'}\, \bar{a}_j^{q'}\, b_j^{r'}\, \bar{b}_j^{s'}. \tag{9}$$

The coefficients of these series are exponential series with respect to the argument $\sqrt{-1}\,(\lambda_i - \lambda_j)$ with real coefficients. These coefficients contain a multiplier x_{ij} determined by the relation

$$\mu x_{ij} = m_j/(m_0 + m_i). \tag{10}$$

Therefore, if one changes the adopted values of planetary masses it is sufficient to multiply the obtained coefficients by a related correction factor (within the first approximation theory one may neglect the variations of coefficients due to the changes of semimajor axes related with the planetary masses by the third Kepler law (2)).

Series (9) is self-conjugate and does not contain pure exponentail terms with m = 0 (such terms in (8) yield the quasi-periodic intermediary).

Our work results mainly in actual constructing (8) and (9) for all couples of the major planets.

Literal series (5), (6), (8), (9) are important for the investigation of the analytical and numerical structure of the planetary inequalities. For ephemeris computation the second form of the series of the general planetary theory is more suitable. By this form only the

https://doi.org/10.1017/S0252921100062084 Published online by Cambridge University Press

mean longitudes of the planets retain their literal shape whereas all other planetary elements have numerical values. These series will be called numerical series of the general planetary theory. They are represented by the exponential series in multiples of $\sqrt{-1}\,\lambda_i$ ($i = 1, \ldots, N$), the coefficients being polynomials in powers of time. These polynomials result from the slow secular variation of the planetary elements. Their coefficients are complex numbers. Construction of the numerical series is achieved by substitution into literal series

$$a_i = \alpha_i \exp \sqrt{-1}\,\lambda_i, \qquad b_i = \beta_i \exp \sqrt{-1}\,\lambda_i \tag{11}$$

α_i, β_i being Lagrange elements satisfying the autonomous secular system of differential equations. The polynomial solution of this system is of the form (Brumberg et al., 1975b):

$$\alpha_i = \sum_{k=0}^{\infty} \alpha_k^{(i)} t^k, \qquad \beta_i = \sum_{k=0}^{\infty} \beta_k^{(i)} t^k \tag{12}$$

where $\alpha_k^{(i)}$, $\beta_k^{(i)}$ are numerical complex coefficients. $\alpha_o^{(i)}$ and $\beta_0^{(i)}$ may be regarded as arbitrary constants determining all subsequent coefficients. In fact, solution (12) represents an expansion in powers of the dimensionless quantity μnt, n being a characteristic mean planetary motion and this solution may be valid for interval of several centuries. The substitution of (11) and (12) into literal series gives numerical series described above. Due to the unsufficient memory capacity of BESM-6 we had to change here the order of summation and to deal with these series as power series, the coefficients being exponential polynomials in multiples of the mean longitudes with complex coefficients. There results

$$p_{0i} = \sum_{k=0}^{\infty} p_k^{(i)} t^k, \qquad w_{0i} = \sum_{k=0}^{\infty} w_k^{(i)} t^k, \tag{13}$$

$$p_{1ij} = \sum_{k=0}^{\infty} p_k^{(ij)} t^k, \qquad w_{1ij} = \sum_{k=0}^{\infty} w_k^{(ij)} t^k. \tag{14}$$

$p_k^{(i)}$ and $w_k^{(i)}$ in (13) are one-argument exponential series with respect to $\sqrt{-1}\,\lambda_i$ and the terms of order m in (5) and (6) lead in these series to multiples $-m, -m+2, \ldots, m-2, m$. $p_k^{(ij)}$ and $w_k^{(ij)}$ in (14) are two-argument exponential series with regard to $\sqrt{-1}\,\lambda_i$ and $\sqrt{-1}\,\lambda_j$. The terms of the power order m and multiple $\pm\sigma$ with respect to $\sqrt{-1}\,(\lambda_i - \lambda_j)$ in (8) and (9) result in (14) to multiples $\sqrt{-1}\,(k\lambda_i + l\lambda_j)$ with k and l changing in limits $\pm\sigma$ $\pm m$ provided that the sum $k+l$ may take only the values $-m, -m+2, \ldots, m-2, m$.

Numerical series (13) and (14) may be used for the computation of the planetary ephemerides.

3. ARBITRARY CONSTANTS OF THE THEORY

Arbitrary constants of our theory are represented by n_i, ε_i, $\alpha_o^{(i)}$, $\beta_o^{(i)}$ $(i = 1, \ldots, N)$. Taken along with masses m_i these quantities should be determined by comparison of the computed coordinates with the results of observations. The mean motions n_i may be supposed to be known from observations sufficiently accurately and their numerical values are fixed in our theory once and for all. For initial evaluation of ε_i, $\alpha_o^{(i)}$, $\beta_0^{(i)}$ they should be related with the analogous quantities of the classicaal Keplerian expansions. This problem was treated earlier (Brumberg and Chapront, 1973) up to the terms of the seventh degree in eccentricities and inclinations by means of the straightforward comparison of (5), (6) with the classical expansions. We give below a general algorithm to obtain this relationship up to any degree of accuracy. The actual realization of this algorithm on BESM-6 has been performed with the aid of the Poisson series processor by Dasenbrock (Dasenbrock, 1973).

Omitting the subscript i rewrite (5), (6) in the form

$$p = -\frac{1}{2} a + \frac{3}{2} \bar{a} + \delta p(a, \bar{a}, b, \bar{b}), \tag{15}$$

$$w = b + \bar{b} + \delta w(a, \bar{a}, b, \bar{b}), \tag{16}$$

δp, δw denoting the series in powers of a, $\bar{a}$, b, $\bar{b}$ starting with the second degree terms. With the aid of the Keplerian processor (Brunberg and Isakovich, 1975) we find the Keplerian power expansions

$$(A \exp \sqrt{-1}\, \Lambda)^{-1}(x + \sqrt{-1}\, y) = 1 + \frac{1}{2} K - \frac{3}{2} \bar{K} + S(K, \bar{K}, L, \bar{L}), \tag{17}$$

$$A^{-1} z = L + \bar{L} + T(K, \bar{K}, L, \bar{L}), \tag{18}$$

where S and T are series in powers of K, $\bar{K}$, L, $\bar{L}$ staring with the second degree terms while these variables themselves are

$$K + e \exp \sqrt{-1}\, (\Lambda-\pi), \quad L + \frac{1}{2\sqrt{-1}} \sin I \exp \sqrt{-1}\, (\Lambda-\Omega), \tag{19}$$

$$\Lambda = nt + E. \tag{20}$$

e, I, π, Ω, E represent classical Keplerian elements, i.e. eccentricity, inclination, longitude of the perihelion, longitude of the ascending node and mean longitude at the epoch. In order to relate a, b, ε with K, L, E put

$$a = K + F(K, \bar{K}, L, \bar{L}), \tag{21}$$

$$b = L + G(K, \bar{K}, L, \bar{L}), \tag{22}$$

$$\exp \sqrt{-1}\ (\varepsilon - E) = 1 + H(K, \bar{K}, L, \bar{L}), \tag{23}$$

F, G, H being unknown power series with respect to the indicated variables. Substitution of (15)-(18), (21)-(23) into (1) results in relations

$$\frac{1}{2} F - \frac{3}{2} \bar{F} + H = \Phi, \tag{24}$$

$$G + \bar{G} = \Psi, \tag{25}$$

enabling one to determine F, G, H by iterations. Here we have

$$\Phi = S + (1 + H)\delta p + (-\frac{1}{2} K + \frac{3}{2} \bar{K})H + (-\frac{1}{2} F + \frac{3}{2} \bar{F})H, \tag{26}$$

$$\Psi = T - \delta w. \tag{27}$$

It is easy to show that F and G-series contain only the forms of odd degree in K, $\bar{K}$, L, $\bar{L}$ (starintg with the third degree terms) while H-series consists of the even degree forms only (starting with the fourth degree terms). Moreover, with p, q, r, s denoting the powers of K, $\bar{K}$, L, $\bar{L}$ the combination p-q+r-s is always unity for every term of F and G and zero for every term of H. Taking all this into account we can easily separate the variables in Equation (24). The most difficult operation of this algorithm is to substitute (21), (22) into series for δp, δw. To do this the Dasenbrock's system was supplemented with a special subroutine NTAYLR permitting to expand a function of several variables, represented by a Poisson series, in powers of variations of these variables, which are represented by Poisson series too. In the result one obtains the Poisson series expressed in new variables.

Thus having found F, G, H up to the terms of some degree with respect of K, $\bar{K}$, L, $\bar{L}$ we substitute (21), (22) into (26), (27) and obtain the expressions for Φ, Ψ accurate to one degree more. This leads to the more accurate expressions for F, G, H. Let us note that the structure of (26), (72) shows a way of modification of the algorithm for finding the series (5), (6) (Brumberg and Chapront, 1973) so as to have identically F = G = H = 0.

We have obtained the series (5), (6) and the developments for F, G, H accurate to m = 11 inclusively. From this we deduce immediately the expansions of α-k, β-l in powers of k, $\bar{k}$, l, $\bar{l}$ where α, β are our arbitrary constants α_0, β_0 and k, l represent the Lagrange elements

$$k = e \exp(-\sqrt{-1}\ \pi),\ l = \frac{1}{2\sqrt{-1}} \sin I \exp(-\sqrt{-1}\ \Omega). \tag{28}$$

In virtue of the structure of H the expansion (23) has the same form expressed in K, $\bar{K}$, L, $\bar{L}$ as well as in k, $\bar{k}$, l, $\bar{l}$. Inverting these developments we find expressions of k-α, l-β in powers of α, $\bar{\alpha}$, β, $\bar{\beta}$. This is achieved by means of a special subroutine INVERS yielding the power series inversion

$$X = Y + P(Y) \tag{29}$$

in form of

$$Y = X + Q(X). \tag{30}$$

The algorithm of inversion is based on a iterative relation

$$Q(X) = - P(X + Q) \tag{31}$$

which again calls for the use of the subroutine NTAYLR. All functions occurring in (29)-(31) are vectors of arbitrary dimension.

In this manner all these expansions give a complete solution for the problem of relationship between our constants and classical Keplerian elements. This opens a way for improvement of our constants with the aid of usual methods based on Keplerian elements, i.e. to improve at first these elements and then to return to our constants.

4. CONSTRUCTION OF THE LITERAL SERIES

To begin with, we describe some technical characteristics of the computer system employed by us.

Calculations have been performed on BESM-6 with a software provided by monitoring system DUBNA. The unit of the memory capacity on BESM-6 is a page, consisting of 1024 machine words with 48 bits in each. In employing this computer system a program together with the induced system subroutines and tables of data may occupy no more than 32 pages of the operating storage. Therefore to store intermediate results we had to use magnetic drums with the mean access time of about 0.02 sec. The final results have been written on magnetic tapes, each tape containing no more than 512 pages (zones). 40 bits of the machine word are designed for a mantissa of a number and thus the single precision is adequate for our calculations. Our programs were written in FORTRAN and only several subroutines for the magnetic storage exchange were formulated in an assembler.

Computation of the series (5), (6) common to all planets is a subject of a separate program. Along with this the program gives some auxiliary developments common to all planets as well. All these expansions have been obtained with the aid of a set of subroutines for manipulation with four-argument power series. The series were calculated up to the eighth degree terms inclusively. The coefficients of these series ordered in a tabular manner were stored on a magnetic tape. The series (5) is the longest one and up to the terms of the eight degree inclusively it contains 254 terms. All these series occupy 30 zones on a tape. Limiting up to the seventh degree terms this information demands only 15 zones.

The second program is designed to compute the series (8), (9) representing the disturbing action of one planet on the other. All 56 possible combinations of couples of planets (excluding Pluto) have been considered. The accuracy of calculations is given first of all by

https://doi.org/10.1017/S0252921100062084 Published online by Cambridge University Press

the maximal degree of terms to be retained in power expansions. Depending on the magnitude of the mutual perturbations this maximal degree specific for each couple of planets has been chosen from 5 to 7. Only for the Jupiter-Saturn case we have computed some terms of the eighth degree. The maximal degree of the retained terms for each couple of planets is presented in Table I where i, j are referred to disturbed and disturbing planets respectively and the tabular value for i = j corresponds to the undisturbed motion.

Table I
Maximal degree of terms in Series (8), (9)

i \ j	1	2	3	4	5	6	7	8
1	8	7	7	5	7	5	5	5
2	6	8	6	5	6	5	5	5
3	5	6	8	6	6	5	5	5
4	5	7	7	8	7	6	5	5
5	5	5	5	5	8	8	7	6
6	5	5	5	5	7	8	7	6
7	5	5	5	5	7	7	7	7
8	5	5	5	5	6	6	7	8

As mentioned above, the coefficients of (8), (9) are exponential series in multiples of the difference between mean longitudes of disturbed and disturbing planets:

$$ {}_1P^{(ij)}_{pqrsp'q'r's'} = \sum_{k=-\infty}^{\infty} P_k \exp \sqrt{-1}\; k(\lambda_i - \lambda_j), $$

$$ {}_1W^{(ji)}_{pqrsp'q'r's'} = \sum_{k=-\infty}^{\Sigma} W_k \exp \sqrt{-1}\, k(\lambda_i - \lambda_j). $$

The number of terms retained in these series is the second accuracy characteristics of the presentation of perturbations in our theory. For all couples of planets in calculating the perturbations of zero, first and second degrees with respect to the polynomial variables we have taken into account the exponential terms with index k varying from -23 to +23. For degrees between 3 and 7 the range of k was from -11 to +11. For the eighth degree only the terms with k ranging from -5 to +5 were retained. Such a fixed stepping changing of the exponential terms number was chosen instead of a more logical smooth ranging due to some technical peculiarities of our tape storage system.

As a function of the degree m we give in Table II a number NP of m-degree power terms in the series for p-coordinate and a number NT of numerical coefficients P_k in the same series. The corresponding numbers for w-coordinate are equal or less than those for p-series.

In the described program we have dealt with a set of subroutines to manipulate with Poisson series having eight polynomial variables and one exponential argument.

Table II
Number of terms in Series (8), (9) and amount of storage pages necessary for computing the perturbations of a given degree

m	1	2	3	4	5	6	7	8
NP	4	20	60	170	396	868	1716	3235
NT	188	940	2820	7990	18612	40796	80652	152045
M	17	19	23	32	58	107	128	143

The terms of series (8), (9) have been calculated subsequently in increased order with respect to the power variables. In the same order they were transposed on the magnetic storage. In calculating the m-degree terms one needs to use just as all terms of lower degrees (from zero to m-1) so also the four-argument power expansions related to the undisturbed motion. Besides this, it is necessary to provide storage for the series of right-hand members and for the series of the m-degree perturbations themselves. All this information for each couple of planets been handled at the given moment is stored on magnetic drums in form of the tables of coefficients. In the operative storage we have only the series being in immediate processing at the given moment. In Table II we indicate a number M of the storage pages on magnetic drums necessary for computation of perturbations of different degrees m. A relatively slight increase of the necessary storage when passing from m=6 to m=7 is explained by the fact that the seventh degree terms are transposed only on tape without using drums since usually they are not employed in the subsequent calculations. In the case of the eight degree terms calculations the coefficients of the necessary series (from m=0 to m=7) are transposed from tape on drum but only for the exponential index ranging from -5 to +5.

We give below separately the amounts of processor and commercial time needed for the calculation of m-degree perturbations (for one couple of disturbed and disturbing planets):

degree	1	2	3	4	5	6	7	8
processor time	-	-	-	-	-	-	2h	4h 40m
commercial time	12s	26s	1m	5m	25m	1h35m	6h	14h

Significant distinction between the values of the second and third lines for m=7 and m=8 is due to the extensive information exchange with the magnetic drum. But this did not adversely affect the computer charge in virtue of the multiprogram regime of our computer system.

The actual calculations for all planets have been performed by subsequent steps with provision for starting solution with the results of the uncompleted previous step. Along with the coefficients P_k, W_k of the resulting series we have stored on tape the coefficients of the

https://doi.org/10.1017/S0252921100062084 Published online by Cambridge University Press

right-hand members of the secular system (11). Depending on the maximal degree of power terms (five or seven) the number of these coefficients for each couple of planets is 240 or 940 respectively. These coefficients were used further in a separate program to compute secular perturbations in form of (12).

5. CONSTRUCTION OF THE NUMERICAL SERIES

The obtained series (5), (6), (8), (9) have been used in the third program. This program may carry out the following operations:

1. For any given couple of planets to read the coefficients of the series stored on tape.
2. Instead of power variables a_i, b_i to substitute into the literal series the expressions (11), (12) resulted from the solution of the secular system. By this substitution the series (5), (6), (8), (9) transform to (13) or (14) relatively.
3. To convert the polynomial-exponential series (13), (14) to the polynomial-trigonometric series for the rectangular coordinates of the planets. The series of such form are well suited for comparison with the results of other theories presented in classical shape with trigonometric and power terms.
4. Based on (13), (14) to calculate for given moments of time the tabular values of p, w and of the rectangular coordinates of any planet in the arbitrary fixed heliocentric coordinate system.

Depending on given initial values this program permits to take into account the total perturbations of the given planet from all others or alternatively the perturbations from each planet separately. In addition one may change the maximal power degree of the terms retained in (13) and (14). It is possible to estimate a contribution due to the terms of any fixed degree.

We have controlled our results just as in performing the calculations themselves so also by comparison of our final series for rectangular coordinates with those obtained by G.A. Krasinsky in elaborating a general planetary theory using the method of von Zeipel. (Krasinsky, 1973). In performing our calculations an ideal control is the absence in the series (5), (6), (8), (9) of critical terms with associated zero divisors. This type of control was of much use particularly in testing our programs.

As to comparison of our series with the results of G.A. Krasinsky we have stated their coincidence under the same accuracy limitations. In the cases when our accuracy is lower than that of G.A. Krasinsky's theory we have observed discrepancies mainly in the eighth and ninth decimals. Only in the Mercury case where our restriction by the terms of the seventh and eight degrees is clearly unadequate the discrepancy has attained the magnitude of the sixth decimal.

To appreciate a size of the resulting numerical series of the type (13), (14) we give in Table III the numbers of terms in the series for p-coordinate with coefficients to less than 10^{-8} radians by absolute value. Here i and j correspond to the disturbed and disturbing planets respectively and k denotes the degree of t in (13), (14). Just as in

Table III
Number of terms in Series (13), (14) for p-coordinate with $\varepsilon = 10^{-8}$ radians

i	1				2			3				4				5			6					7				8		
j k	0	1	2	3	0	1	2	0	1	2	3	0	1	2	3	0	1	2	0	1	2	3	4	0	1	2	3	0	1	2
1	17	15	12	6	7	2																								
2	60	12	1		7	6	5	31	6			19	7	2		2			2					1				1		
3	22	2			43	8		7	6	5	3	50	26	8		2			2					2				2		
4					9	3		36	8			11	11	10	6	1			1					1				1		
5	26	4			16	3		25	6			50	19	2		9	8		92	59	34	19		24	3			7	2	
6	8				6	2		7	2			25	6			109	76	32	10	10	8	6	3	48	22	7		13	3	
7												5				16			48	22	7			10	8	5	2	59	14	4
8												1				5			17	2				94	28	5		7	5	2

Table I the value i=j is associated to the undisturbed motion. The indicated numbers are related to calculations with the maximal degree defined by Table I. The time t in (13), (14) is reckoned in millenia.

6. THE LINEAR THEORY OF THE SECOND ORDER

So far we have considered only the first two terms in series (3) and (4). However the main parts of $\underset{2}{p}_{i}$ and $\underset{2}{w}_{i}$ have been found too. These terms may be expressed as follows:

$$\underset{2}{p}_{i} = \sum_{j=1}^{N(i)} \underset{2}{p}_{ij} + \frac{1}{2} \sum_{j=1}^{N(i)} \sum_{k=1}^{N(i,j)} \underset{2}{p}_{ijk} \tag{32}$$

$$\underset{2}{w}_{i} = \sum_{j=1}^{N(i)} \underset{2}{w}_{ij} + \frac{1}{2} \sum_{j=1}^{N(i)} \sum_{k=1}^{N(i,j)} \underset{2}{w}_{ijk}. \tag{33}$$

$\underset{2}{p}_{ij}$ and $\underset{2}{w}_{ij}$ are represented by the Poisson series of the type (8,1) in the same manner as (8), (9). Being symmetric in j and k $\underset{2}{p}_{ijk}$ and $\underset{2}{w}_{ijk}$ are the Poisson articles (12, 2) where the variables a, $\bar{a}$, b, $\bar{b}$ for planets i, j, k enter as polynomial variables and the differences $\lambda_i - \lambda_j$ and $\lambda_i - \lambda_k$ are the trigonometric arguments. We have obtained the initial terms of the Poisson series (32), (33) with value m=0 (the quasi-periodic intermediate solution) and m=1 (the inequalities of the first degree in eccentricities and inclinations). The terms of $\underset{2}{p}_{ij}$ and $\underset{2}{w}_{ij}$ are not of much interest since they are analogous to the terms of $\underset{1}{p}_{ij}$ and $\underset{1}{w}_{ij}$ but make a significantly lesser contribution to the general solution (3), (4). Contributions of $\underset{2}{p}_{ijk}$ and $\underset{2}{w}_{ijk}$ are more essential due to the appearance of the resonance terms caused by the close commensurabilities between mean motions of the triplets of planets. These terms are analytically of the second order with respect to the planetary masses but numerically they are comparable with the first order terms. In the paper (Brumberg et al., 1975a) a detailed analysis of these terms in the quasi-periodic intermediate solution (m=0) is presented and it is established that many of such terms are omitted in the classical planetary theories*. Similar analysis may be carried out for the terms of the first degree in eccentricities and inclinations (m=1). It is convenient to write these terms in the following manner:

$$\underset{2}{p}_{ijk} = c(i,j,k,0)a_i + d(i,j,k,0)\bar{a}_i + c(i,j,k,1)a_j +$$

---

*In the paper cited there are some discrepancies with our basic results obtained by the series expansion method. The reason lies in the unsufficient accurate computation of the right-hand members by the iteration method. We are indebted to M. Luc Duriez for this remark.

$$d(i,j,k,1)\bar{a}_j + c(i,k,j,1)a_k + d(i,k,j,1)\bar{a}_k, \tag{34}$$

$$w_{2ijk} = f(i,j,k,0)b_i + \bar{f}(i,j,k,0)\bar{b}_i + f(i,j,k,1)b_j + \bar{f}(i,j,k,1)\bar{b}_j + f(i,k,j,1)b_k + \bar{f}(i,k,j,1)\bar{b}_k. \tag{35}$$

Here $c(i,j,k,\Delta)$, $d(i,j,k,\Delta)$, $f(i,j,k,\Delta)$ ($\Delta=0$, $\Delta=1$) are exponential series in two arguments: $\sqrt{-1}\ (\lambda_i - \lambda_j)$ and $\sqrt{-1}\ (\lambda_i - \lambda_k)$. These series have been constructed in the range from -7 to +7 in each argument. This construction has revealed many resonance terms caused by the close commensurabilities between mean motions of some triplets of planets. The terms are regarded to be resonant if the ratio $(sn)/n_i$ approaches to zero or ±1 (s is the N-vector of indices of the mean longitudes λ_1, ..., λ_n, n is the N-vector of planetary mean motions).

It is of interest to compare our results with the classical theories of Newcomb and Hill. The paper (Newcomb, 1895) is devoted to the investigation of the long period inequalities of the second order in the mean longitudes of the four inner planets. Among them we are interested in the terms with the arguments

$$s\lambda = s_1\lambda_1 + \dots + s_N\lambda_N$$

for which

$$s_1 + \dots + s_N = 0, \pm 1, \pm 2.$$

There are six such arguments:

$$\lambda_1 - 5\lambda_2 + 4\lambda_3,\ 3\lambda_2 - 7\lambda_3 + 4\lambda_4,\ \lambda_3 - 2\lambda_4 + \lambda_5,$$

$$4\lambda_3 - 8\lambda_4 + 3\lambda_5,\ 3\lambda_3 - 6\lambda_4 + 2\lambda_5,\ 5\lambda_3 - 10\lambda_4 + 4\lambda_5.$$

According to Newcomb the perturbations in the motion of Mercury, Venus and the Earth associated with the first argument are negligible small while those due to the terms with the third argument are perceptible only in the orbit of Mars. Perturbations caused by the terms with the fourth argument are taken into account in the orbits of the Earth and Mars. The inequality of Le Verrier related to the second argument is present in the orbits of Venus, the Earth and Mars. The perturbations themselves are determined by a complicated artificial method. As possible arguments Newcomb indicates the fifth and sixth arguments but he does not examine the related perturbations. In the theory of the motion of Mars Newcomb (Newcomb, 1898a) takes into account the fifth argument and the argument $\lambda_2 - 2\lambda_4 + 2\lambda_6$.

The tables of perturbations of Jupiter elaborated by Hill (Hill, 1898a) present only one argument with three planetary mean motions; $3\lambda_5 - 6\lambda_6 + 3\lambda_7$. For the perturbations of Saturn Hill (Hill, 1898b) gives the terms related to four arguments of the type considered:

$-2\lambda_5 + 5\lambda_6 - 3\lambda_7$, $-\lambda_5 + 2\lambda_6 - \lambda_7$, $\lambda_5 - 3\lambda_6 + 2\lambda_7$, $\lambda_5 - 5\lambda_6 + 4\lambda_7$. In the Newcomb's theory for the motion of Uranus (Newcomb, 1898b) the following arguments occur: $-2\lambda_5 + 6\lambda_6 - 4\lambda_7$, $\lambda_5 - 4\lambda_6 + 3\lambda_7$, $2\lambda_5 - 7\lambda_6 + 5\lambda_7$, $\lambda_5 - 3\lambda_6 + 2\lambda_7$, $-\lambda_5 + 2\lambda_6 - \lambda_7$, $-\lambda_5 - \lambda_6 + 2\lambda_7$. At last, there are no such terms in the Newcomb's theory for the motion of Neptune (Newcomb, 1898c).

In our theory all terms entering into the classical theories are presented but along with them we have revealed a set of terms having numerically the same or even greater magnitude. In Table IV we give for illustration a list of arguments of the series for $\mu^2 c(i,j,k,\Delta)$ with a restricting condition that the absolute magnitudes of the corresponding coefficients are no less than 10^{-7} radians. An asterisk indicates the presence of the related term in the corresponding classical theory of Newcomb or Hill. It is to be noted that there is a full agreement in arguments for the motion of Mars between our results and the theory of Clemence (Clemence, 1961).

Table IV
Main triple arguments of the linear theory

planets	indices					planets	indices				
i j k	s_i	s_j	s_k	sn/n_i		i j k	s_i	s_j	s_k	sn/n_i	
2 3 4	3	-7	4	0.002	*	4 5 6	-3	4	-1	-2.430	
	-4	6	-2	-0.963		4 5 7	-2	7	-5	-1.002	
2 3 5	-4	5	-1	-0.976		5 6 7	-3	6	-3	-1.007	*
	3	-5	2	0.028			-2	3	-1	-0.933	
2 5 6	-2	4	-2	-1.834			-3	5	-2	-1.269	
3 2 4	5	-3	-2	-0.940			-4	6	-2	-1.866	
	-7	3	4	0.003	*		-1	3	-2	-0.074	
3 2 5	9	-6	-3	-1.006			-4	5	-1	-2.128	
	4	-3	-1	-0.961			-3	4	-1	-1.530	
3 4 5	-4	6	-2	-0.978	*		-3	2	1	-2.053	
3 5 6	-2	4	-2	-1.731		5 6 8	-3	5	-2	-1.130	
	-2	3	-1	-1.781			-4	5	-1	-2.058	
	-1	-1	2	-1.016		5 7 8	-1	-1	2	-0.997	
	-2	-1	3	-1.984			-2	4	-2	-1.579	
4 2 3	4	3	-7	0.006	*	6 5 7	5	-2	-3	-1.018	*
4 3 5	7	-4	-3	-0.999			4	-2	-2	-1.668	
	5	-3	-2	-0.959	*		-4	1	3	-0.465	
	3	-2	-1	-0.920			-5	1	4	-1.114	*
	9	-5	-4	-1.038			-8	2	6	-0.930	
4 3 6	-3	1	2	-0.991			2	-1	-1	-0.834	*
4 5 6	-2	4	-2	-1.493			3	-2	-1	-2.317	
	-1	-1	2	-1.031			-3	1	2	0.184	*
	-2	3	-1	-1.588			5	-3	-2	-3.151	
	-3	5	-2	-2.335			8	-3	-5	-1.203	
	-2	-1	3	-1.967			7	-3	-4	-1.852	
	-2	5	-3	-1.399			6	-2	-4	-0.369	
	-1	3	-2	-0.652			-6	1	5	-1.763	
	-2	1	1	-1.778			6	-3	-3	-2.502	

Table IV (Continued)

planets indices					planets indices				
i j k	s_i	s_j	s_k	sn/s_i	i j k	s_i	s_j	s_k	sn/n_i
6 5 7	10	-4	-6	-2.037	7 6 8	3	-1	-2	-0.872
6 5 8	-4	1	3	-0.980		4	-2	-2	-2.724
	10	-4	-6	-1.006		-1	-1	2	-2.832
	4	-2	-2	-1.324		-5	1	4	-0.109
	6	-3	-3	-1.986		-7	1	6	-1.089
	3	-2	-1	-2.145		1	-3	2	-6.536
	-3	1	2	-0.159		5	-3	-2	-4.576
	2	-1	-1	-0.662		5	-2	-3	-2.234
	7	-3	-4	-1.165		2	-4	2	-8.388
6 7 8	-2	4	-2	-0.955		7	-2	-5	-1.253
	-1	-1	2	-0.993		4	-1	-3	-0.382
	-3	5	-2	-1.604		6	-2	-4	-1.743
	-4	6	-2	-2.254		2	-1	-1	-1.362
	-2	3	-1	-1.127		8	-2	-6	-0.763
	-1	3	-2	-0.306		6	-4	-2	-6.428
	-3	1	2	-2.292		3	-5	2	-10.241
	-5	7	-2	-2.903		-6	1	5	-0.599
	-2	1	1	-1.471		3	-2	-1	-3.214
	-3	4	-1	-1.776		7	-5	-2	-8.280
	-4	7	-3	-2.082		4	-6	2	-12.093
7 5 6	3	1	-4	-1.326 *		-8	1	7	-1.579
	-4	-2	6	-1.053 *	8 5 6	2	1	-3	-0.890
	5	2	-7	-0.799 *		4	2	-6	-1.781
	4	1	-5	-3.178 *		-1	-1	3	0.890
	2	1	-3	0.526 *	8 7 6	5	1	-6	-1.174
7 5 8	-1	-1	2	-7.063		6	1	-7	-2.136
	4	-2	-2	-11.185		4	1	-5	-0.213
	3	-1	-2	-5.102		3	-2	-1	-10.150

7. CONCLUSION

The results exposed here summarize our work in mathematical constructing the general planetary theory.

Taking into account a rather small operative storage capacity of BESM-6 computer accessible for us it is not suitable to compute by the same method inequalities of higher degree in eccentricities and inclinations and of higher order with respect to the planetary masses. But independent of an employed computer it is more advantageous to improve a semi-analytical theory by numerical iterative methods. Our work will proceed on this line.

For investigation of the long-term evolution of planetary motions the polynomial solution of the secular system used here is invalid and should be replaced by a pure trigonometric solution. This solution will be obtained in the nearest future.

As far as comparison with observations and determination of constants are concerned, the following solution seems to be reasonable. Considering the coincidence of our results with those of Krasinsky's theory it is of no use to determine from observations the constants for both theories separately. It is more simple to do this for one theory and then to use the algorithm of the relationship between constants exposed in Section 3.

Finally, we should like to express our sincere gratitude to Dr J. Chapront and his colleagues at the Bureau des Longitudes who collaborated with us and did so much in the initial stage of this work.

REFERENCES

Brumberg, V.A.: 1974, in Y. Kozai (ed.), The Stability of the Solar System and of Small Stellar Systems, p. 139. Reidel, Dordrecht.
Brumberg, V.A., Evdokimova, L.S., and Skripnichenko, V.I.: 1975a, Astron. J. (USSR) 52, 420. (in Russian)
Brumberg, V.A., Evdokimova, L.S., and Skripnichenko, V.I.: 1975b, Celes. Mech. 11, 131.
Brumberg, V.A. and Chapront, J.: 1973, Celes. Mech. 8, 335.
Brumberg, V.A. and Isakovich, L.A.: 1975, Algorithms of Celestial Mechanics No. 4. Inst. Theoret. Astron. (Leningrad). (in Russian).
Clemence, G.M.: 1961, Astron. Papers 16, pt. 2, 261.
Dasenbrock, R.R.: 1973, Naval Research Lab. Rep. No. 7564.
Hill, G.W.: 1898a, Astron. Papers 7, pt. 3, 287.
Hill, G.W.: 1989b, Astron. Papers 7, pt. 4, 417.
Krasinsky, G.A.: 1973, in N.S. Samoylova-Yakhontova (ed.), Minor Planets, Ch. 5, 6. Nauka, Moscow. (in Russian)
Newcomb, S.: 1895, Astron. Papers 5, pt. 2, 49.
Newcomb, S.: 1898a, Astron. Papers 6, pt. 4, 383.
Newcomb, S.: 1898b, Astron. Papers 7, pt. 1, 1.
Newcomb, S.: 1898c, Astron. Papers 7, pt. 2, 145.