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# VECTOR LYAPUNOV FUNCTIONS AND CONDITIONAL STABILITY FOR SYSTEMS OF IMPULSIVE DIFFERENTIAL-DIFFERENCE EQUATIONS

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**Abstract**

By means of piecewise continuous vector functions, which are analogues of the classical Lyapunov functions and *via* the comparison method, sufficient conditions are found for conditional stability of the zero solution of a system of impulsive differential-difference equations.

## 1. Introduction

The mathematical modelling of many real processes and phenomena in physics, biology, population dynamics, bio-technologies, control theory, *etc.*, leads to the study of differential equations whose solutions are discontinuous functions, such as the so-called impulsive differential equations.

Impulsive differential-difference equations are a natural generalization of impulsive ordinary differential equations ([2, 3, 13]). These equations adequately model processes which are characterized by jumps in state as well as by the fact that the process under consideration depends on its history at each moment of time. Such a generalization of the notion of an impulsive differential equation enables us to study different types of classical problems as well as to "control" the solvability of differential-difference equations (without impulses). For example, the scalar autonomous ordinary differential-difference equation

$$\dot{N}(t) = rN(t)\left[1 - \frac{N(t-\tau)}{K}\right], \quad t \geq 0, \tag{1}$$

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commonly known as the logistic equation with time delay $\tau$, is most frequently employed in modelling the population dynamics of a single species, where $N(t)$ is the population at time $t$, $r$ is the growth rate of the species, and $K$ is the carrying capacity of the habitat. The per-capita growth rate in (1) is a linear function of the population $N$ (and can be termed the density of the population).

Equation (1), called Hutchinson's equation [10], has been studied by many authors: see for example Cunningham [8], Gopalsamy [9], Kuang [11], Zhang and Gopalsamy [18, 19]. It can be used to describe certain control systems. Similar equations can also be used in economic studies of business cycles. One can also use such models in mathematical ecology.

If the population of a given species is regulated by some impulsive biotic and anthropogeneous factors at certain moments of time it is not reasonable to expect a regular solution. Instead, the solution must have some jumps at these moments and the jumps follow a specific pattern. An adequate mathematical model of the dynamics of the population in this case will be an impulsive differential-difference equation of the form

$$\begin{cases} \dot{N}(t) = rN(t)[1 - N(t-\tau)/K], & t > 0, t \neq t_k, \\ \Delta N(t_k) = N(t_k + 0) - N(t_k - 0) = \alpha_k(N(t_k - 0)), & t_k > 0, k = 1, 2, \dots, \end{cases} \tag{2}$$

where $0 < t_1 < t_2 < \cdots$, $N(t_k - 0)$ and $N(t_k + 0)$ are respectively the population density before and after impulsive perturbations, and $\alpha_k$ are functions which characterize the magnitude of the impulse effect at the moments $t_k$.

By means of models of type (2), it is possible to investigate one of the most important mathematical ecology problem—the problem of ecological system stability and consequently the problem of the optimal control of such systems.

A wider application of impulsive differential-difference equations in the description of a number of real processes requires the formulation of effective criteria for stability of their solutions ([1, 4, 5, 6, 7]).

In the present paper we study the conditional stability of the zero solution of an impulsive system of differential-difference equations with fixed moments of impulse effect by means of vector Lyapunov functions. The priorities of this approach are useful and well known in investigations into the stability of the solutions of differential and differential-difference equations ([12, 14, 15]).

The investigations of the present paper are carried out by virtue of piecewise continuous functions, which are analogues to the classical Lyapunov functions ([17]). Sufficient conditions are proved for conditional stability of the zero solution for a system of impulsive differential-difference equations with fixed moments of the impulse effect, by means of comparison with an impulsive vector equation and differential inequalities. A technique is applied, based on certain minimal subsets of a suitable space of piecewise continuous functions, from which the derivatives of the piecewise continuous auxiliary functions are estimated ([14, 16]).

## 2. Statement of the problem. Preliminary notes

Let $R^n$ be the $n$-dimensional Euclidean space with elements $\mathbf{x} = \operatorname{col}(x_1, \dots, x_n)$ and norm $|\mathbf{x}| = (\sum_{k=1}^n x_k^2)^{1/2}$. Let $R_+ = [0, \infty)$, $h \in R_+$ and $t_0 \in R$.

We consider the system of impulsive differential-difference equations

$$\begin{cases} \dot{x}(t) = f(t, x(t), x(t-h)), & t > t_0, t \neq \tau_k, \\ \Delta x(\tau_k) = x(\tau_k + 0) - x(\tau_k) = I_k(x(\tau_k)), & \tau_k > t_0, k = 1, 2, \dots, \end{cases} \tag{3}$$

where $f : (t_0, \infty) \times R^n \times R^n \to R^n$; $I_k : R^n \to R^n, k = 1, 2, \dots$; $t_0 \equiv \tau_0 < \tau_1 < \tau_2 < \cdots$ and $\lim_{k\to\infty} \tau_k = \infty$.

Let $PC(t_0)$ be the class of all piecewise continuous in $(t_0 - h, t_0)$ functions $\varphi : [t_0 - h, t_0] \to R^n$ with points of discontinuity of the first kind $\theta_1, \dots, \theta_s \in (t_0 - h, t_0)$, at which they are continuous from the left.

Let $\varphi_0 \in PC(t_0)$. Denote by $x(t; t_0, \varphi_0)$ the solution of system (3) satisfying the initial conditions

$$\begin{cases} x(t; t_0, \varphi_0) = \varphi_0(t), & t_0 - h \le t \le t_0, \\ x(t_0 + 0; t_0, \varphi_0) = \varphi_0(t_0). \end{cases} \tag{4}$$

The solution $x(t) = x(t; t_0, \varphi_0)$ of the initial value problem (3),(4) is characterized by the following:

1. For $t_0 - h \le t \le t_0$ the solution $x(t)$ satisfies the initial conditions (4).
2. The solution $x(t)$ is a piecewise continuous function for $t > t_0$ with points of discontinuity of the first kind $\tau_k > t_0$, $k = 1, 2, \dots$, at which it is continuous from the left, that is, at the moments of impulse effect $\tau_k$ the following relations are valid:

$$x(\tau_k - 0) = x(\tau_k), \quad x(\tau_k + 0) = x(\tau_k) + I_k(x(\tau_k)), \quad k = 1, 2, \dots.$$

3. If for some positive integer $j$ we have $\tau_k < \tau_j + h < \tau_{k+1}$, $k = 0, 1, 2, \dots$, then in the interval $[\tau_j + h, \tau_{k+1}]$ the solution $x(t)$ of problem (3), (4) coincides with the solution of the problem

$$\begin{cases} \dot{y}(t) = f(t, y(t), x(t - h + 0)), \\ y(\tau_j + h) = x(\tau_j + h), \end{cases}$$

and if $\tau_j + h \equiv \tau_k$ for $j = 0, 1, 2, \dots$, $k = 1, 2, \dots$, then in the interval $[\tau_j + h, \tau_{k+1}]$ the solution $x(t)$ coincides with the solution of the problem

$$\begin{cases} \dot{y}(t) = f(t, y(t), x(t - h + 0)), \\ y(\tau_j + h) = x(\tau_j + h) + I_k(x(\tau_j + h)). \end{cases}$$

We now introduce some notation.

Let $\mathscr{K}$ be the class of all continuous and strictly increasing functions $a : R_+ \to R_+$, such that $a(0) = 0$; $\|\varphi\| = \sup_{s\in[t_0-h,t_0]} |\varphi(s)|$ is the norm of the function $\varphi \in PC(t_0)$; $G_k = \{(t,x) \in [t_0, \infty) \times R^n : \tau_{k-1} < t < \tau_k\}$, $k = 1, 2, \ldots$.

We also introduce the following conditions.

H1. The function $f$ is continuous in $(t_0, \infty)\times R^n \times R^n$ and $f$ is Lipschitz continuous with respect to its second and third arguments uniformly on $t \in (t_0, \infty)$.

H2. $f(t, 0, 0) = 0$, $t \in (t_0, \infty)$.

H3. The functions $I_k$ are continuous in $R^n$, $k = 1, 2, \ldots$.

H4. $I_k(0) = 0$, $k = 1, 2, \ldots$.

H5. $t_0 \equiv \tau_0 < \tau_1 < \tau_2 < \cdots$.

H6. $\lim_{k\to\infty} \tau_k = \infty$.

We define the sets

$$B_\alpha(t_0, PC(t_0)) = \{\varphi \in PC(t_0) : \|\varphi\| < \alpha\}, \qquad S(\alpha) = \{x \in R^n : |x| < \alpha\},$$
$$\overline{B}_\alpha(t_0, PC(t_0)) = \{\varphi \in PC(t_0) : \|\varphi\| \le \alpha\}, \qquad \overline{S}(\alpha) = \{x \in R^n : |x| \le \alpha\}.$$

Let $M(n-l)$, $l < n$, be a $(n-l)$-dimensional manifold in $R^n$, containing the origin. We set

$$M_{t_0}(n-l) = \{\varphi \in PC(t_0) |\, \varphi : [t_0 - h, t_0] \to M(n-l)\}.$$

DEFINITION 1. The zero solution of the system (3) is said to be:

(a) *Conditionally stable* with respect to the manifold $M(n-l)$, if for each $t_0 \in R$ and $\varepsilon > 0$ there exists a positive function $\delta = \delta(t_0, \varepsilon)$ which is continuous in $t_0$ for each fixed $\varepsilon > 0$ and such that if $\varphi_0 \in \overline{B}_\delta(t_0, PC(t_0)) \cap M_{t_0}(n-l)$, then $x(t; t_0, \varphi_0) \in S(\varepsilon)$ for $t > t_0$.

(b) *Conditionally uniformly stable* with respect to $M(n-l)$, if the function $\delta$ in $(a)$ is independent of $t_0$.

(c) *Conditionally globally equi-attractive* with respect to $M(n-l)$, if for each $t_0 \in R$, $\alpha > 0$ and $\varepsilon > 0$ there exists a positive number $T = T(t_0, \alpha, \varepsilon)$ such that if $\varphi_0 \in \overline{B}_\alpha(t_0, PC(t_0)) \cap M_{t_0}(n-l)$, then $x(t; t_0, \varphi_0) \in S(\varepsilon)$ for $t \ge t_0 + T$.

(d) *Conditionally uniformly globally attractive* with respect to $M(n-l)$, if the number $T$ in (c) is independent of $t_0$.

(e) *Conditionally globally equi-asymptotically stable* with respect to $M(n-l)$, if it is conditionally stable and conditionally globally equi-attractive with respect to $M(n-l)$.

(f) *Conditionally uniformly globally asymptotically stable* with respect to $M(n-l)$, if it is conditionally uniformly stable and conditionally uniformly globally attractive with respect to $M(n-l)$.

REMARK 1. If $M(n-l) = R^n$, then the definitions (a)–(f) are reduced to the usual definitions of stability by Lyapunov for the zero solution of the system (3).

Together with the system (3) we shall consider the following system of impulsive ordinary differential equations

$$\begin{cases} \dot{u} = F(t, u), & t \neq \tau_k,\ t > t_0, \\ \Delta u(t_k) = B_k(u(t_k)), & k = 1, 2, \dots,\ \tau_k > t_0, \end{cases} \tag{5}$$

where $u : (t_0, \infty) \to R^m$; $F : (t_0, \infty) \times \Omega \to R^m$; $B_k : \Omega \to R^m$, $k = 1, 2, \dots$; $\Omega$ is a domain in $R^m$ containing the origin, $m \leq n$.

Let $u_0 \in R^m$. We denote by $u(t) = u(t; t_0, u_0)$ the solution of the system (3), which satisfies the initial condition $u(t_0 + 0; t_0, u_0) = u_0$ and by $J^+(t_0, u_0)$ the maximal interval of the form $[t_0, \omega)$ in which the solution $u(t) = u(t; t_0, u_0)$ is defined.

We introduce into $R^m$ a partial ordering in the following way: for the vectors $u, v \in R^m$ we shall say that $u \geq v$ if $u_i \geq v_i$ for each $i = 1, 2, \dots, m$ and $u > v$, if $u_i > v_i$ for each $i = 1, 2, \dots, m$.

DEFINTION 2. The solution $u^+ : J^+(t_0, u_0) \to R^m$ of the system (5) for which $u^+(t_0 + 0; t_0, u_0) = u_0$ is said to be a *maximal solution* if any other solution $u : (t_0, \tilde{\omega}) \to R^m$ for which $u(t_0 + 0; t_0, u_0) = u_0$ satisfies the inequality $u^+(t) \geq u(t)$ for $t \in J^+(t_0, u_0) \cap (t_0, \tilde{\omega})$.

DEFINTION 3. The function $\psi : \Omega \to R^m$ is said to be *monotone increasing* in $\Omega$ if $\psi(u) > \psi(v)$ for $u > v$ and $\psi(u) \geq \psi(v)$ for $u \geq v$, $u, v \in \Omega$.

DEFINTION 4. The function $\psi : \Omega \to R^m$ is said to be *nondecreasing* in $\Omega$ if $\psi(u) \geq \psi(v)$ for $u \geq v$, $u, v \in \Omega$.

DEFINTION 5. The function $F : (t_0, \infty) \times \Omega \to R^m$ is said to be *quasi-monotone increasing* in $(t_0, \infty) \times \Omega$ if for each pair of points $(t, u)$ and $(t, v)$ from $(t_0, \infty) \times \Omega$ and for $j \in \{1, 2, \dots, m\}$ the inequality $F_j(t, u) \geq F_j(t, v)$ holds whenever $u_j = v_j$ and $u_j \geq v_j$ for $j = 1, 2, \dots, m$, $i \neq j$, that is, for any fixed $t \in (t_0, \infty)$ and any $j \in \{1, 2, \dots, m\}$ the function $F_j(t, u)$ is nondecreasing with respect to $(u_1, u_2, \dots, u_{j-1}, u_{j+1}, \dots, u_m)$.

Let $e \in R^m$ be the vector $(1, 1, \dots, 1)$ and let $\Omega \supset \{u \in R^m : 0 \leq u \leq e\}$.

We introduce the sets

$$B(\alpha) = \{u \in R^m : 0 \leq u < \alpha e\},$$
$$B(\overline{\alpha}) = \{u \in R^m : 0 \leq u \leq \alpha e\}, \quad \alpha = \text{const} > 0,$$
$$R(m-l) = \{u = (u_1, \dots, u_m) \in R^m : u_1 = u_2 = \cdots = u_l = 0\}, \quad l < m.$$

We shall consider solutions $u(t)$ of the system (5) for which $u(t) \geq 0$ and hence the following definitions on conditional stability of the zero solution of this system will be used.

DEFINTION 6. The zero solution $u(t; 0, 0) = 0$ of the system (5) is said to be:

(a) *Conditionally stable* with respect to the manifold $R(m-l)$, if for each $t_0 \in R$ and $\varepsilon > 0$ there exists a positive function $\delta = \delta(t_0, \varepsilon)$ which is continuous in $t_0$ for each $\varepsilon > 0$ and such that if $u_0 \in B(\bar{\delta}) \cap R(m-l)$, then $u^+(t; t_0, u_0) \in B(\varepsilon)$ for $t > t_0$.

(b) *Conditionally uniformly stable* with respect to $R(m-l)$, if the function $\delta$ from $(a)$ does not depend on $t_0$.

(c) *Conditionally globally equi-attractive* with respect to $R(m-l)$, if for each $t_0 \in R$, $\alpha > 0$ and $\varepsilon > 0$ there exists a positive number $T = T(t_0, \alpha, \varepsilon)$ such that if $u_0 \in B(\overline{\alpha}) \cap R(m-l)$, then $u^+(t; t_0, u_0) \in B(\varepsilon)$ for $t \geq t_0 + T$.

(d) *Conditionally uniformly globally attractive* with respect to $R(m-l)$, if the number $T$ in $(c)$ does not depend on $t_0$.

(e) *Conditionally globally equi-asymptotically stable* with respect to $R(m-l)$, if it is conditionally stable and conditionally globally equi-attractive with respect to $R(m-l)$.

(f) *Conditionally uniformly globally asymptotically stable* with respect to $R(m-l)$, if it is conditionally uniformly stable and conditionally uniformly globally attractive with respect to $R(m-l)$.

Our attention will now turn to piecewise continuous auxiliary vector functions which are analogues of the classical Lyapunov functions ([17]).

DEFINTION 7. We say that the vector function $V : [t_0, \infty) \times R^n \to R^m$, $V = (V_1, \ldots, V_m)$, belongs to the class $V_0$ if the following conditions are fulfilled:

1. The function $V$ is continuous in $\cup_{k=1}^{\infty} G_k$, $V(t, x) \geq 0$ and $V(t, 0) = 0$ for $t \in [t_0, \infty)$.
2. The function $V$ satisfies the Lipschitz condition locally with respect to $x$ on each of the sets $G_k$.
3. For each $k = 1, 2, \ldots$ and $x \in R^n$ there exist the finite limits
$$V(\tau_k - 0, x) = \lim_{\substack{(t,x)\to(\tau_k,x)\\(t,x)\in G_k}} V(t, x), \qquad V(\tau_k + 0, x) = \lim_{\substack{(t,x)\to(\tau_k,x)\\(t,x)\in G_{k+1}}} V(t, x).$$
4. The equalities $V(\tau_k - 0, x) = V(\tau_k, x)$, $k = 1, 2, \ldots$, are valid.

Further on, we will also use the following functional classes:

$PC[[t_0, \infty), R^n] = \{x : [t_0, \infty) \to R^n : x$ is a piecewise continuous function in $(t_0, \infty)$ with points of discontinuity of the first kind where it is continuous from the left$\}$;

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$$\Omega_0 = \{x \in PC[[t_0, \infty), R^n] : V(s, x(s)) \le V(t, x(t)),\ t - h \le s \le t,\ t \ge t_0,\ V \in V_0\}.$$

Let $V \in V_0$ for $t \in (t_0, \infty)$, $t \neq \tau_k$, $k = 1, 2, \ldots$ and $x \in PC[[t_0, \infty), R^n]$. We also introduce the function

$$D_- V(t, x(t)) = \liminf_{\sigma \to 0^-} \frac{1}{\sigma}[V(t + \sigma, x(t) + \sigma f(t, x(t), x(t - h))) - V(t, x(t))].$$

LEMMA 1 ([5, 6]). *Let the following conditions hold:*

1. *Conditions* H1–H6 *are met.*

2. *The function* $F : (t_0, \infty) \times \Omega \to R^m$ *is quasi-monotone increasing in* $(t_0, \infty) \times \Omega$, *continuous in each of the sets* $(\tau_{k-1}, \tau_k] \times \Omega$, *and for* $k \in N$ *and* $v \in \Omega$ *there exists the limit*

$$\lim_{\substack{(t,u) \to (\tau_k, v) \\ t > \tau_k}} F(t, u).$$

3. *The functions* $B_k : \Omega \to R^m$ *are continuous in* $\Omega$ *and such that the functions* $\psi_k : \Omega \to R^m$, $\psi_k(u) = u + B_k(u)$, $k = 1, 2, \ldots$, *are nondecreasing in* $\Omega$.

4. *The function* $u^+ : (t_0, \infty) \to R^m$ *is the maximal solution of the system* (5) *for which* $u^+(t_0 + 0) = u_0 \in \Omega$ *and* $u^+(\tau_k + 0) \in \Omega$ *for* $\tau_k \in (t_0, \infty)$.

5. *The function* $V \in V_0$ *is such that*

$$V(t_0, \varphi_0(t_0)) \le u_0,$$
$$V(\tau_k + 0, x(\tau_k) + I_k(x(\tau_k))) \le \psi_k(V(\tau_k, x(\tau_k))), \quad k = 1, 2, \ldots,$$

*and the inequality*

$$D_- V(t, x(t)) \le F(t, V(t, x(t))), \quad t \neq \tau_k,$$

*holds true as* $t \in [t_0, \infty)$, $x \in \Omega_0$.

*Then*

$$V(t, x(t; t_0, \varphi_0)) \le u^+(t; t_0, u_0), \quad t \in (t_0, \infty).$$

## 3. Main results

THEOREM 1. *Let the following conditions hold:*

1. *Conditions* H1–H6 *are fulfilled.*

2. *The function* $F : (t_0, \infty) \times \Omega \to R^m$ *is quasi-monotone increasing in* $(t_0, \infty) \times \Omega$, *continuous in each of the sets* $(\tau_{k-1}, \tau_k] \times \Omega$, *and* $F(t, 0) = 0$ *for* $t \in (t_0, \infty)$.

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3. *For $k \in N$ and $v \in \Omega$ there exists the limit*

$$\lim_{\substack{(t,u)\to(\tau_k,v)\\ t>\tau_k}} F(t,u).$$

4. *The functions $B_k : \Omega \to R^m$ are continuous in $\Omega$, $B_k(0) = 0$ and the functions $\psi_k : \Omega \to R^m$, $\psi_k(u) = u + B_k(u)$, $k = 1, 2, \dots$, are nondecreasing in $\Omega$.*

5. *The function $V : [t_0, \infty) \times R^n \to R^m$, $m \le n$, $V = (V_1, \dots, V_m)$ belongs to the class $V_0$, $\sup_{(t_0,\infty)\times R^n} |V(t,x)| = K \le \infty$ and $\Omega = \{u \in R^m : 0 \le u < K\}$.*

6. *The set $M(n-l) = \{x \in R^n : V_k(t+0,x) \equiv 0,\ k = 1, 2, \dots, l\}$ is $(n-l)$-dimensional manifold in $R^n$, containing the origin, $l < n$.*

7. *The following inequalities are valid:*

$$a(|x|)e \le V(t,x), \quad (t,x) \in [t_0, \infty) \times R^n, \tag{6}$$

*where $a \in \mathscr{K}$;*

$$D_- V(t, x(t)) \le F(t, V(t, x(t))), \quad t \ne \tau_k,\ k = 1, 2, \dots, \tag{7}$$

*for $t \ge t_0$ and $x \in \Omega_0$;*

$$V(\tau_k + 0, x(\tau_k) + I_k(x(\tau_k))) \le \psi_k(V(\tau_k, x(\tau_k))), \quad k = 1, 2, \dots. \tag{8}$$

*Then:*

ASSERTION 1. *If the zero solution of the system* (5) *is conditionally stable with respect to the manifold $R(m-l)$, then the zero solution of the system* (3) *is conditionally stable with respect to the manifold $M(n-l)$.*

ASSERTION 2. *If the zero solution of the system* (5) *is conditionally globally equi-attractive with respect to the manifold $R(m-l)$, then the zero solution of the system* (3) *is conditionally globally equi-attractive with respect to the manifold $M(n-l)$.*

PROOF OF ASSERTION 1. Let $t_0 \in R$ and $\varepsilon > 0$ $(a(\varepsilon) < K)$ be chosen. Let the zero solution of the system (5) be conditionally stable with respect to $R(m-l)$. Then there exists a positive function $\delta_1 = \delta_1(t_0, \varepsilon)$ which is continuous in $t_0$ for given $\varepsilon$ and is such that if $u_0 \in B(\overline{\delta}_1) \cap R(m-l)$, then $u^+(t; t_0, u_0) < a(\varepsilon)e$ for $t > t_0$.

It follows from the properties of the function $V$ that there exists $\delta = \delta(t_0, \varepsilon) > 0$ such that if $\varphi_0(t_0) \in S(\overline{\delta})$ then $V(t_0, \varphi_0(t_0)) \in B(\overline{\delta}_1)$.

Let $\varphi_0 \in \overline{B}_\delta(t_0, PC(t_0)) \cap M_{t_0}(n-l)$. Then $\varphi_0(t_0) \in S(\overline{\delta})$ and therefore $V(t_0, \varphi_0(t_0)) \in B(\overline{\delta}_1)$. Moreover, $V_k(t_0, \varphi_0(t_0)) = 0$ for $k = 1, 2, \dots, l$, that is, $V(t_0, \varphi_0(t_0)) \in R(m-l)$. Thus $u^+(t; t_0, V(t_0, \varphi_0(t_0))) < a(\varepsilon)e$ for $t > t_0$.

On the other hand, if $x(t) = x(t; t_0, \varphi_0)$ is the solution of the initial problem (3) and (4), then it follows from the conditions of Theorem 1 that the function $V \in V_0$ satisfies all conditions of Lemma 1. Using this fact and (6), we arrive at

$$a(|x(t)|)e \le V(t, x(t)) \le u^+(t; t_0, V(t_0, \varphi_0(t_0))) \le a(\varepsilon)e$$

for $t > t_0$.

Hence $|x(t; t_0, \varphi_0)| < \varepsilon$ for $t > t_0$, that is, the zero solution of the system (3) is conditionally stable with respect to the manifold $M(n-l)$.

PROOF OF ASSERTION 2. Let $t_0 \in R$, $\alpha > 0$ and $\varepsilon > 0$ ($a(\varepsilon) < K$) be given.

It follows from the properties of the function $V$ that there exists $\alpha_1 = \alpha_1(t_0, \alpha) > 0$ such that if $x \in S(\overline{\alpha})$, then $V(t_0, x) \in B(\overline{\alpha}_1)$.

If the zero solution of the system (5) is conditionally globally equi-attractive with respect to $R(m-l)$, then there exists a number $T = T(t_0, \alpha, \varepsilon) > 0$ such that if $u_0 \in B(\overline{\alpha}_1) \cap R(m-l)$, then $u^+(t; t_0, u_0) < a(\varepsilon)e$ for $t \geq t_0 + T$.

Let $\varphi_0 \in \overline{B}_\alpha(t_0, PC(t_0)) \cap M_{t_0}(n-l)$. Then $\varphi_0(t_0) \in S(\overline{\alpha}) \cap M(n-l)$ and $V(t_0, \varphi_0(t_0)) \in B(\overline{\alpha}_1) \cap R(m-l)$. Therefore $u^+(t; t_0, V(t_0, \varphi_0(t_0))) < a(\varepsilon)e$ for $t \geq t_0 + T$.

If $x(t) = x(t; t_0, \varphi_0)$ is the solution of the initial problem (3) and (4), then it follows from Lemma 1 that

$$V(t, x(t)) \leq u^+(t; t_0, V(t_0, \varphi_0(t_0))), \quad t > t_0.$$

The last inequality and (6) imply the inequalities

$$a(|x(t)|)e \leq V(t, x(t)) \leq u^+(t; t_0, V(t_0, \varphi_0(t_0))) \leq a(\varepsilon)e$$

for $t \geq t_0 + T$.

Therefore $|x(t; t_0, \varphi_0)| < \varepsilon$ for $t \geq t_0 + T$, which leads to the conclusion that the zero solution of the system (3) is conditionally globally equi-attractive with respect to the manifold $M(n-l)$.

COROLLARY 1. *Let the conditions of Theorem* 1 *be fulfilled. Then, if the zero solution of the system* (5) *is conditionally globally equi-asymptotically stable with respect to the manifold* $R(m-l)$, *the zero solution of the system* (3) *is conditionally globally equi-asymptotically stable with respect to the manifold* $M(n-l)$.

THEOREM 2. *Let the conditions of Theorem* 1 *be fulfilled, and let there exist a function* $b \in \mathcal{K}$ *such that* $V(t, x) \leq b(|x|)e$ *for* $(t, x) \in [t_0, \infty) \times R^n$. *Then*:

1. *If the zero solution of the system* (5) *is conditionally uniformly stable with respect to the manifold* $R(m-l)$, *then the zero solution of the system* (3) *is conditionally uniformly stable with respect to the manifold* $M(n-l)$.
2. *If the zero solution of the system* (5) *is conditionally uniformly globally attractive with respect to the manifold* $R(m-l)$, *then the zero solution of the system* (3) *is conditionally uniformly globally attractive with respect to the manifold* $M(n-l)$.

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The proof of Theorem 2 is analogous to the proof of Theorem 1. We shall just note that in this case the function $\delta$ and the number $T$ can be chosen independently of $t_0$.

COROLLARY 2. *Let the conditions of Theorem 2 be satisfied. Then, if the zero solution of the system* (5) *is conditionally uniformly globally asymptotically stable with respect to the manifold* $R(m-l)$, *the zero solution of the system* (3) *is conditionally uniformly globally asymptotically stable with respect to the manifold* $M(n-l)$.

## 4. Applications

**4.1. A population system of two competing species** We consider the population system of two competing species modelled by the impulsive differential-difference equations

$$\begin{cases} \dot N_1(t) = -N_1(t) + 2N_2(t) + e^{-t}N_1(t-h) + N_2(t-h)\sin t, & t \neq t_k, \\ \dot N_2(t) = 2N_1(t) - N_2(t) + N_1(t-h)\sin t + e^{-t}N_2(t-h), & t \neq t_k, \\ \Delta N_1(t) = aN_1(t) + bN_2(t), \quad t = t_k,\ k = 1,2,\dots, \\ \Delta N_2(t) = bN_1(t) + aN_2(t), \quad t = t_k,\ k = 1,2,\dots, \end{cases} \tag{9}$$

where $t > 0$, $h > 0$, $N_1 > 0$ and $N_2 > 0$ for $t > 0$,

$$a = \frac{1}{2}\left(\sqrt{1+c_1} + \sqrt{1+c_2} - 2\right), \quad b = \frac{1}{2}\left(\sqrt{1+c_1} - \sqrt{1+c_2}\right),$$

$-1 < c_1 \leq 0, -1 < c_2 \leq 0, 0 < t_1 < t_2 < \cdots$ and $\lim_{k\to\infty} t_k = \infty$. We also consider the comparison system

$$\begin{cases} \dot u(t) = 2\big(e^{-t} + \sin t + 1\big)u(t), & t \neq t_k, \\ \dot v(t) = 2\big(-3 + e^{-t} - \sin t\big)v(t), & t \neq t_k, \\ \Delta u(t_k) = c_1 u(t_k),\ \Delta v(t_k) = c_2 v(t_k), & k = 1,2,\dots, \end{cases} \tag{10}$$

where $u, v > 0$ for $t > 0$.

We will use the vector function $V(t,x,y) = \big((x+y)^2, (x-y)^2\big)^T$, $(t,x,y) \in R_+ \times R_+ \times R_+$. Then

$$\Omega_0 = \{(x,y) \in PC[R_+, R_+ \times R_+] : V(s,x(s),y(s)) \leq V(t,x(t),y(t)), \\ t-h \leq s \leq t, t \geq 0\}.$$

For $t \geq 0$ and $(N_1, N_2) \in \Omega_0$ the following inequalities are valid:

$$\begin{aligned}
&D_-V(t, N_1(t), N_2(t)) \\
&\quad \leq 2\begin{pmatrix} (N_1(t) + N_2(t))^2 \\ -3(N_1(t) - N_2(t))^2 \end{pmatrix} \\
&\quad + 2\begin{pmatrix} (N_1(t) + N_2(t))(N_1(t-h) + N_2(t-h))(e^{-t} + \sin t) \\ (N_1(t) - N_2(t))(N_1(t-h) - N_2(t-h))(e^{-t} - \sin t) \end{pmatrix} \\
&\quad \leq 2\begin{pmatrix} 1 + e^{-t} + \sin t & 0 \\ 0 & -3 + e^{-t} - \sin t \end{pmatrix} V(t, N_1(t), N_2(t)),
\end{aligned}$$

for $t \neq t_k$, $k = 1, 2, \dots$ and

$$\begin{aligned}
&V\big(t_k + 0, N_1(t_k) + aN_1(t_k) + bN_2(t_k), N_2(t_k) + bN_1(t_k) + aN_2(t_k)\big) \\
&\quad = V(t_k, N_1(t_k), N_2(t_k)) + \begin{pmatrix} c_1 & 0 \\ 0 & c_2 \end{pmatrix} V(t_k, N_1(t_k), N_2(t_k)), \quad k = 1, 2, \dots .
\end{aligned}$$

Let

$$R(3-1) = R(2) = \{(u, 0), u \geq 0\}$$

and

$$M(3-1) = M(2) = \{(N_1, N_2) \in R_+ \times R_+ : N_1 = N_2\}.$$

Since all the conditions of Theorem 1 are fulfilled and the zero solution of the system (10) is conditionally stable with respect to the manifold $R(2)$ ([2]), then the zero solution of the system (9) is conditionally stable with respect to the manifold $M(2)$.

**4.2. A second model** In this section we study the physical model

$$\begin{cases}
\dot{x}(t) = (1+\cos t)x(t-h) + (1-\cos t)y(t-h) + (\cos t - 1)z(t-h), & t \neq t_k, \\
\dot{y}(t) = (1-e^{-t})x(t-h) + (1+e^{-t})y(t-h) + (e^{-t} - 1)z(t-h), & t \neq t_k, \\
\dot{z}(t) = (\cos t - e^{-t})x(t-h) + (e^{-t} - \cos t)y(t-h) + (e^{-t} + \cos t)z(t-h), & t \neq t_k, \\
\Delta x(t) = a_{1k}x(t) + b_{1k}(y(t) - z(t)), \quad t = t_k,\ k = 1, 2, \dots, \\
\Delta y(t) = a_{2k}y(t) + b_{2k}(z(t) - x(t)), \quad t = t_k,\ k = 1, 2, \dots, \\
\Delta z(t) = a_{3k}z(t) + b_{3k}(x(t) - y(t)), \quad t = t_k,\ k = 1, 2, \dots,
\end{cases} \tag{11}$$

where $t \geq 0$, $h > 0$, $x, y, z \in R$ and for $i = 1, 2, 3$,

$$a_{ik} = \left(\sqrt{1 + d_{ik}} + \sqrt{1 + d_{i-1,k}} - 2\right)/2, \quad b_{ik} = \left(\sqrt{1 + d_{ik}} - \sqrt{1 + d_{i-1,k}}\right)/2$$

with $d_{0,k}$ interpreted as $d_{3k}$, $-1 < d_{ik} \leq 0$, $i = 1, 2, 3$, $k \in N$, $0 < t_1 < t_2 < \cdots$ and $\lim_{k\to\infty} t_k = \infty$.

We also consider the comparison system

$$\begin{cases} \dot{u}_1(t) = 4u_1(t), \quad \dot{u}_2(t) = 4e^{-t}u_2(t), \quad \dot{u}_3(t) = 4\cos t\, u_3(t), \quad t \neq t_k, \\ \Delta u_1(t_k) = d_{1k}u_1(t_k), \quad \Delta u_2(t_k) = d_{2k}u_2(t_k), \\ \Delta u_3(t_k) = d_{3k}u_3(t_k), \quad k = 1, 2, \dots, \end{cases} \tag{12}$$

where $u_1, u_2, u_3 > 0$ for $t > 0$.

We will use the vector function

$$V(t, x, y, z) = \big(V_1, V_2, V_3\big)^T = \big((x + y - z)^2, (-x + y + z)^2, (x - y + z)^2\big)^T.$$

Then,

$$\begin{aligned} \Omega_0 = \{(x, y, z) \in PC[R_+, R^3] : V(s, x(s), y(s), z(s)) \leq V(t, x(t), y(t), z(t)), \\ t - h \leq s \leq t, \ t \geq 0\}. \end{aligned}$$

For $t \geq 0$ and $(x, y, z) \in \Omega_0$ the following inequalities are valid:

$$\begin{aligned} &D_- V(t, x(t), y(t), z(t)) \\ &= 4 \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{-t} & 0 \\ 0 & 0 & \cos t \end{pmatrix} \begin{pmatrix} [x(t)+y(t)-z(t)] & [x(t-h)+y(t-h)-z(t-h)] \\ [-x(t)+y(t)+z(t)] & [-x(t-h)+y(t-h)+z(t-h)] \\ [x(t)-y(t)+z(t)] & [x(t-h)-y(t-h)+z(t-h)] \end{pmatrix} \\ &\leq 4 \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{-t} & 0 \\ 0 & 0 & \cos t \end{pmatrix} V(t, x(t), y(t), z(t)) \end{aligned}$$

for $t \neq t_k$, $k = 1, 2, \dots$ and

$$\begin{aligned} &V\big(t_k + 0,\ x(t_k) + a_{1k}x(t_k) + b_{1k}(y(t_k) - z(t_k)), \\ &\qquad y(t_k) + a_{2k}y(t_k) + b_{2k}(z(t_k) - x(t_k)), z(t_k) + a_{3k}z(t_k) + b_{3k}(x(t_k) - y(t_k))\big) \\ &\quad = V(t_k, x(t_k), y(t_k), z(t_k)) + \begin{pmatrix} d_{1k} & 0 & 0 \\ 0 & d_{2k} & 0 \\ 0 & 0 & d_{3k} \end{pmatrix} V(t_k, x(t_k), y(t_k), z(t_k)), \end{aligned}$$

where $k = 1, 2, \dots$. Let

$$R(3 - 1) = R(2) = \{(0, u_2, u_3) \in R^3 : u_2 \geq 0, \ u_3 \geq 0\}$$

and

$$M(3 - 1) = M(2) = \{(x, y, z) \in R^3 : x + y = z\}.$$

Since all the conditions of Theorem 1 are fulfilled and the zero solution of the system (12) is conditionally stable with respect to the manifold $R(2)$ ([2]), then the zero solution of the system (11) is conditionally stable with respect to the manifold $M(2)$.

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