

AN EXTENSION OF HENRICI THEOREM FOR THE JOINT APPROXIMATE SPECTRUM OF COMMUTING SPECTRAL OPERATORS

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Abstract

Given two m -tuples of commuting spectral operators on a Hilbert space, $T = (T_1, \dots, T_m)$ and $S = (S_1, \dots, S_m)$, an extended version of Henrici perturbation theorem is obtained for the joint approximate spectrum of S under perturbation by T . We also derive an extended version of Bauer-Fike theorem for such tuples of operators. The method used involves Clifford algebra techniques introduced by McIntosh and Pryde.

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1. Introduction

The study of tuples of commuting operators was the subject of a wide literature carrying out many resemblances with the single case. Various kinds of spectra were introduced, namely by Harte [10, 11], Taylor [21] and Coburn and Schechter [6]. Even more, Taylor [20] has introduced a functional calculus for tuples of operators, while Harte [11] established 'Spectral mappings theorems' for them. For more about joint spectral theory, the reader is referred to the survey paper of Curto [7].

Later on many authors focused on perturbations of tuples of operators. The use of analytic 'multi-functions' enabled Klimek [13] to prove the results known for single analytic perturbations. The turning point in this theory was the formulation by McIntosh and Pryde [14] of a functional calculus using tools from Clifford Analysis. Using these tools, Pryde [18] generalized the theorem of Bauer-Fike for commuting

tuples of matrices, while those of Henrici and Hoffman-Wielandt were extended by Bhatia and Bhattacharyya [3, 2]. We also note that other properties were proved by Ming [16] and Müller [17].

Our aim in this paper is to extend Henrici [19, Theorem 1.9, page 172] and Bauer-Fike [19, Theorem 1.6, page 171] theorems for tuples of commuting spectral operators defined on a complex Hilbert space. In fact, we establish bounds similar to the matrix case given by Pryde [18] and by Bhatia-Bhattacharyya [3] which are inspired by the Bauer-Fike and Henrici bounds for single matrices. This will be done using Clifford analysis.

The paper is organized as follows: Following [18], we first give some necessary tools and definitions, then we establish the Henrici theorem for tuples of operators having real spectra. At the end we consider operators with complex spectra.

2. Preliminaries

We denote by H a nontrivial complex Hilbert space and by $\mathbb{B}(H)$ the algebra of bounded linear operators on H with respect to the uniform norm. All the operators considered in this paper are in $\mathbb{B}(H)$. We will denote by \mathbb{R} , (respectively \mathbb{C}) the scalar field of real, (respectively of complex) numbers.

Let us consider an m -tuple of commuting operators: $T = (T_1, \dots, T_m)$, where $T_j \in \mathbb{B}(H)$, $T_j T_k = T_k T_j$ for $j, k = 1, 2, \dots, m$. The *joint approximate spectrum* of T is defined by:

$$\text{Sp}_{\text{ap}}(T) = \left\{ \lambda = (\lambda_1, \dots, \lambda_m) \in \mathbb{C}^m; \inf_{\|x\|=1} \left\{ \sum_{1 \leq j \leq m} \|T_j x - \lambda_j x\| \right\} = 0 \right\}.$$

For a single operator T , the approximate spectrum will be denoted by $\sigma_{\text{ap}}(T)$. The *joint compression spectrum* (or *right spectrum*) of T is defined by

$$\text{Sp}_{\text{com}}(T) = \left\{ \lambda = (\lambda_1, \dots, \lambda_m) \in \mathbb{C}^m, \sum_{1 \leq j \leq m} (T_j - \lambda_j I) \mathbb{B}(H) \neq \mathbb{B}(H) \right\}.$$

The set of *joint eigenvalues* of T which we denote by $\text{Sp}_p(T)$ is defined as follows: $\lambda = (\lambda_1, \dots, \lambda_m) \in \text{Sp}_p(T)$ if there is $x \in H, x \neq 0$ such that $T_j x = \lambda_j x$ for $j = 1, 2, \dots, m$. One then has $\text{Sp}_p(T) \subset \text{Sp}_{\text{ap}}(T)$. For a single operator T , $\text{Sp}_p(T)$ will be denoted by $\sigma_p(T)$.

Naturally the *joint spectrum* of T is $\text{Sp}(T) = \text{Sp}_{\text{ap}}(T) \cup \text{Sp}_{\text{com}}(T)$. We mention that the definitions of spectra are those of Harte [10], so the joint spectra considered in this paper are the Harte spectra. Let us emphasize that it is known [4, 7] that $\text{Sp}_{\text{ap}}(T)$ is a compact nonempty subset of \mathbb{C}^m .

We are always interested to know whether or not an operator has eigenvalues. In case of general commuting tuples of operators we do not know so far of any answer. However, if the operators are algebraic or compact having certain property, then the answer is affirmative. We recall that an operator $T \in \mathbb{B}(H)$ is *algebraic* if there is a nontrivial polynomial P such that $P(T) = 0$.

PROPOSITION 2.1. *Let $T = (T_1, \dots, T_m)$ be a commuting tuple of compact operators. Suppose that each T_k has at least one nonzero point in its spectrum, then $\text{Sp}_p(T) \neq \emptyset$.*

PROOF. Pick $0 \neq \lambda_{1,1} \in \sigma_p(T_1)$, then there is $x \in H, x \neq 0$ with $T_1x - \lambda_{1,1}x = 0$. Now, by assumption and by the properties of compact operators, there is $\nu_2 \in \mathbb{N}^*, j_2 \in \mathbb{N}^*$ such that $(T_2 - \lambda_{j_2,2}I)^{\nu_2-1}x \neq 0$ and $(T_2 - \lambda_{j_2,2})^{\nu_2}x = 0$.

Put $u_1 = x, u_2 = (T_2 - \lambda_{j_2,2}I)^{\nu_2-1}u_1$, then $T_2u_2 - \lambda_{j_2,2}u_2 = 0$ and $T_1u_2 - \lambda_{1,1}u_2 = 0$. By induction we construct a joint eigenvector: suppose we have u_{m-1} , then $T_k u_{m-1} - \lambda_{j_k,k} u_{m-1} = 0$ for $k = 1, \dots, m-1$ with $j_1 = 1$. Then there is ν_m, j_m such that $(T_m - \lambda_{j_m,m}I)^{\nu_m-1}u_{m-1} \neq 0$ and $(T_m - \lambda_{j_m,m}I)^{\nu_m}u_{m-1} = 0$. Set

$$u_m = (T_m - \lambda_{j_m,m}I)^{\nu_m-1}u_{m-1},$$

then

$$T_m u_m - \lambda_{j_m,m} u_m = 0 = T_{m-1} u_m - \lambda_{j_{m-1},m-1} u_m = \dots = T_1 u_m - \lambda_{1,1} u_m.$$

Thereby $(\lambda_{1,1}, \dots, \lambda_{j_m,m}) \in \text{Sp}_p(T)$. □

For algebraic operators the proof is essentially the same.

Since the tool we use is Clifford analysis we now make a brief incursion into it. Let (e_1, \dots, e_m) be the canonical basis of \mathbb{R}^m . The *Clifford algebra* \mathbb{R}_m over \mathbb{R}^m is the real algebra of maximal dimension with identity 1 containing \mathbb{R} and \mathbb{R}^m as sub-algebras and such that $x^2 = -\|x\|^2$ for every $x \in \mathbb{R}^m$. It follows that $e_k^2 = -1$ and $e_k e_j = -e_j e_k$ for $k \neq j$. Now we set $e_\emptyset = 1$ and if $E = \{s_1, \dots, s_k\} \subset \{1, \dots, m\}$, define $e_E = (-1)^\sigma e_{t_1} \cdots e_{t_k} = (-1)^\sigma e_{t_1 \dots t_k}$, where $1 \leq t_1 < \dots < t_k$ and σ is the number of inversions in the permutation

$$\pi = \begin{pmatrix} t_1 & \cdots & t_k \\ s_1 & \cdots & s_k \end{pmatrix}.$$

Clearly \mathbb{R}_m has dimension 2^m and has a basis consisting of $e_E, E = \emptyset$ or $E = \{s_1, \dots, s_k\}$ with the s_j 's ordered in a natural way.

A full description of the real Clifford algebras \mathbb{R}_m for $m \leq 4$ is given in [8]. There \mathbb{R}_m is seen under many aspects, involution and conjugation are constructed over it. We recall that \mathbb{R}_1 is isomorphic to both \mathbb{C} , the matrix algebra $SO(2)$ and to $\text{Spin}(2)$, the

spinor real algebra; \mathbb{R}_2 is isomorphic to \overline{H} the quaternions algebra and to the matrix algebra $SU(2)$. For $m = 3$, \mathbb{R}_3 is isomorphic to $\overline{H} \oplus \overline{H}$ and \mathbb{R}_4 is isomorphic to $\overline{H}(2)$, the algebra of 2×2 matrices over \overline{H} . For more details the reader is referred to [8].

We define a new Hilbert space denoted by $H \otimes \mathbb{R}_m$ whose elements are of the form $\sum_S x_S \otimes e_S$, $x_S \in H$, $S \subset \{1, \dots, m\}$ endowed with the inner product: $(x, y) = \sum_{S,S'} (x_S, y_{S'})$ for $x, y \in H \otimes \mathbb{R}_m$ and the norm $\|x\|^2 = \sum_S \|x_S\|^2$. Let $\mathbb{B}(H \otimes \mathbb{R}_m)$ denote the algebra of bounded linear operators on $H \otimes \mathbb{R}_m$ together with the operator norm $\|\cdot\|$. Then $\mathbb{B}(H) \otimes \mathbb{R}_m$ may be identified with a sub-algebra of $\mathbb{B}(H \otimes \mathbb{R}_m)$ as follows:

$$\left(\sum_S T_S \otimes e_S \right) \left(\sum_{S'} x_{S'} \otimes e_{S'} \right) = \sum_{S,S'} T_S(x_{S'}),$$

so that $\mathbb{B}(H) \subset \mathbb{B}(H) \otimes \mathbb{R}_m \subset \mathbb{B}(H \otimes \mathbb{R}_m)$. For a commuting m -tuple of $\mathbb{B}(H)$, $T = (T_1, \dots, T_m)$ we define the *Clifford operator* of T by

$$\text{Cliff}(T) = i \sum_{1 \leq j \leq m} T_j \otimes e_j \in \mathbb{B}(H) \otimes \mathbb{R}_m \subset \mathbb{B}(H \otimes \mathbb{R}_m).$$

If $x \in H$, then

$$\text{Cliff}(T)x = i \sum_{1 \leq j \leq m} T_j x \otimes e_j.$$

We also define the *adjoint* of $\text{Cliff}(T)$ by

$$\text{Cliff}(T)^* = i \sum_{1 \leq j \leq m} T_j^* \otimes e_j.$$

Later on we denote by $\|\text{Cliff}(T)\|$ the operator norm of $\text{Cliff}(T)$ and by $\gamma(T)$ the spectral set

$$\gamma(T) = \left\{ \lambda = (\lambda_1, \dots, \lambda_m) \in \mathbb{R}^m, \sum_{1 \leq j \leq m} (T_j - \lambda_j I)^2 \text{ is singular} \right\}.$$

McIntosh and Pryde proved in [14] that $\gamma(T) = \text{Sp}(T) \cap \mathbb{R}^m$, and if the T_j 's have real spectra, then $\gamma(T) = \text{Sp}(T)$. In this situation $0 \notin \text{Sp}(T)$ if and only if $\text{Cliff}(T)$ is regular and

$$(\text{Cliff}(T))^{-1} = \left(\sum_{1 \leq j \leq m} T_j^2 \right)^{-1} \text{Cliff}(T).$$

We also define the *joint spectral radius* of T :

$$r(T) = \max\{|\lambda|, \lambda \in \text{Sp}(T)\}.$$

PROPOSITION 2.2. Let $T = (T_1, \dots, T_m)$ be an m -tuple of commuting bounded operators, where every T_j is normal, then

- (a) $\text{Cliff}(T)$ is normal.
- (b) $r(\text{Cliff}(T)) = r(T)$.
- (c) $\|\text{Cliff}(T)\| = r(T)$.

The proof is the same as the one of [18, Proposition 3.2], so we omit it.

We are now in a position to give the desired generalization of the results given by Pryde [18] and Bhatia and Bhattacharyya [3]. Following them we first consider perturbations of real spectra then of complex spectra.

3. Perturbations of real spectra

The class of operators in which we are interested are spectral operators:

DEFINITION 3.1 ([9, page 1939]). We say that a bounded operator T is a *spectral operator* if it is the sum $T = S + N$ of a scalar operator S and a quasi-nilpotent operator N commuting with S .

For the definition of a scalar and quasi-nilpotent operator the reader is referred to [9]. It is known that such decomposition of a spectral operator is unique and is called the canonical decomposition of T . For a single spectral operator $T = S + N$, it is known that $\sigma(T) = \sigma(S)$. We will first extend this property for tuples of spectral operators having real spectra.

PROPOSITION 3.2. Let $T = (T_1, \dots, T_m)$ be a commuting tuple of spectral operators having real spectra. For $1 \leq j \leq m$, let Λ_j be the scalar part of the canonical decomposition of T_j and N_j its quasi-nilpotent part. Set $\Lambda = (\Lambda_1, \dots, \Lambda_m)$, then $\text{Sp}(T) = \text{Sp}(\Lambda)$.

PROOF. It is known [14, Proposition 10.1] that $\text{Sp}(T) = \gamma(T)$, so $\lambda = (\lambda_1, \dots, \lambda_m) \in \text{Sp}(T)$ if and only if $0 \in \sigma\left(\sum_{1 \leq j \leq m} (T_j - \lambda_j I)^2\right)$. The latter operator is equal to

$$\sum_{1 \leq j \leq m} (\Lambda_j - \lambda_j I)^2 - 2 \sum_{1 \leq j \leq m} (\Lambda_j - \lambda_j I)N_j + \sum_{1 \leq j \leq m} N_j^2.$$

Since the operator $2 \sum_{1 \leq j \leq m} (\Lambda_j - \lambda_j I)N_j + \sum_{1 \leq j \leq m} N_j^2$ is quasi-nilpotent and commutes with the operator $\sum_{j=1}^m (\Lambda_j - \lambda_j I)^2$, then $0 \in \sigma\left(\sum_{1 \leq j \leq m} (T_j - \lambda_j I)^2\right)$ if and only if $0 \in \sigma\left(\sum_{1 \leq j \leq m} (\Lambda_j - \lambda_j I)^2\right)$ which means that $\lambda \in \gamma(S) = \text{Sp}(S)$ and this completes the proof. \square

From now on, the symbols Λ and S have the meaning specified in Proposition 3.2.

The first relevant result we are going to establish is an extension of Bauer-Fike theorem [19, page 171].

PROPOSITION 3.3. *Let $T = (T_1, \dots, T_m)$ and $S = (S_1, \dots, S_m)$ be two m -tuples of commuting operators. If $\lambda \in \text{Sp}_{\text{ap}}(S) \setminus \text{Sp}(T)$ and $A \in \mathbb{B}(H)$ is regular then*

$$\|A^{-1} [\text{Cliff}(T - \lambda I)]^{-1} A\|^{-1} \leq \|A^{-1} \text{Cliff}(T - S)A\|.$$

PROOF. By the hypothesis, there is a sequence $(x_n) \subset H$, $\|x_n\| = 1$ such that

$$\sum_{j=1}^m \|(S_j - \lambda_j I)x_n\| \rightarrow 0$$

and $\text{Cliff}(T - \lambda I)$ is regular. Now

$$\begin{aligned} \text{Cliff}(T - \lambda I)x_n &= i \sum_{j=1}^m (T_j - \lambda_j I)x_n \otimes e_j \\ &= \text{Cliff}(T - S)x_n + \text{Cliff}(S - \lambda I)x_n, \end{aligned}$$

which leads to

$$x_n = [\text{Cliff}(T - \lambda I)]^{-1} \text{Cliff}(T - S)x_n + [\text{Cliff}(T - \lambda I)]^{-1} \text{Cliff}(S - \lambda I)x_n.$$

Inserting $AA^{-1} = I$, we get

$$\begin{aligned} x_n &= AA^{-1} [\text{Cliff}(T - \lambda I)]^{-1} AA^{-1} \text{Cliff}(T - S)AA^{-1}x_n \\ &\quad + AA^{-1} [\text{Cliff}(T - \lambda I)]^{-1} \text{Cliff}(S - \lambda I)x_n \end{aligned}$$

and then

$$(3.1) \quad \|A^{-1}x_n\| \leq \|A^{-1} [\text{Cliff}(T - \lambda I)]^{-1} A\| \|A^{-1} \text{Cliff}(T - S)A\| \|A^{-1}x_n\| + \|A^{-1} [\text{Cliff}(T - \lambda I)]^{-1} \text{Cliff}(S - \lambda I)x_n\|.$$

From (3.1) follows that

$$\begin{aligned} \limsup_{n \rightarrow \infty} \|A^{-1}x_n\| &\leq \|A^{-1} [\text{Cliff}(T - \lambda I)]^{-1} A\| \\ &\quad \times \|A^{-1} \text{Cliff}(T - S)A\| \limsup_{n \rightarrow \infty} \|A^{-1}x_n\|. \end{aligned}$$

Since $\|A^{-1}x_n\| \leq \|A^{-1}\|$ and $\limsup_{n \rightarrow \infty} \|A^{-1}x_n\| > 0$ we get

$$1 \leq \|A^{-1} [\text{Cliff}(T - \lambda I)]^{-1} A\| \|A^{-1} \text{Cliff}(T - S)A\|,$$

which completes the proof. □

In the remaining part of this section we suppose that all the operators under consideration have real spectra.

Now we are going to give the version of Henrici's bound extended to our situation, generalizing thereby the result stated by Bhatia and Bhattacharyya [3, Theorem 3.2] to the case of tuples of spectral operators. This will be done without defining the measure of nonnormality as it was the case in [3].

THEOREM 3.4. *Let $T = (T_1, \dots, T_m)$ and $S = (S_1, \dots, S_m)$ be two m -tuples of commuting spectral operators. Let $T_j = \Lambda_j + N_j$ be the canonical decomposition of the T_j 's, where Λ_j is scalar and N_j is quasi-nilpotent. Then given $\beta \in \text{Sp}_{\text{ap}}(S) \setminus \text{Sp}(T)$, there is a self-adjoint bounded invertible operator B such that for*

$$K = [B \text{Cliff}(\Lambda - \beta I)B^{-1}]^{-1} B \text{Cliff}(N)B^{-1},$$

we have

$$\frac{\delta}{\|(I + K)^{-1}\|} \leq \|B \text{Cliff}(T - S)B^{-1}\|,$$

where $\delta = \min\{|\alpha - \beta|, \alpha \in \text{Sp}(T)\}$.

PROOF. Let $\beta \in \text{Sp}_{\text{ap}}(S) \setminus \text{Sp}(T)$, then by Proposition 3.2, $\beta \in \text{Sp}_{\text{ap}}(S) \setminus \text{Sp}(\Lambda)$. Now the Λ_j 's is a family of commuting scalar operators, hence by [9, Theorem 4, page 1947], there is a bounded invertible self-adjoint operator B such that $B\Lambda_j B^{-1}$ are normal, hence $B(\Lambda_j - \beta_j I)B^{-1}$ are also normal. A direct computation gives:

$$\begin{aligned} & B \text{Cliff}(T - \beta I)B^{-1} \\ &= B \text{Cliff}(\Lambda - \beta I)B^{-1} [I + [B \text{Cliff}(\Lambda - \beta I)B^{-1}]^{-1} B \text{Cliff}(N)B^{-1}]. \end{aligned}$$

Put $K = [B \text{Cliff}(\Lambda - \beta I)B^{-1}]^{-1} B \text{Cliff}(N)B^{-1}$, then K is quasi-nilpotent. Thus $I + K$ is invertible and we have

$$B[\text{Cliff}(T - \beta I)]^{-1} B^{-1} = (I + K)^{-1} B[\text{Cliff}(\Lambda - \beta I)]^{-1} B^{-1}.$$

Now put $\mu = \|B[\text{Cliff}(\Lambda - \beta I)]^{-1} B^{-1}\|^{-1}$, then we get

$$\frac{\mu}{\|(I + K)^{-1}\|} \leq \|B[\text{Cliff}(T - \beta I)]^{-1} B^{-1}\|^{-1}.$$

Hence using Proposition 3.3, we get

$$\frac{\mu}{\|(I + K)^{-1}\|} \leq \|B \text{Cliff}(T - S)B^{-1}\|.$$

It remains to prove that $\mu = \delta = \min\{|\alpha - \beta|, \alpha \in \text{Sp}(T)\}$.

Since $B(\Lambda_j - \beta_j I)B^{-1} = \tilde{\Lambda}_j$ are normal for $j = 1, \dots, m$ then by Proposition 2.2, $B \text{Cliff}(\Lambda - \beta I)B^{-1}$ is normal, and thereby $B[\text{Cliff}(\Lambda - \beta I)]^{-1}B^{-1}$ is also normal. It follows that

$$\|B[\text{Cliff}(\Lambda - \beta I)]^{-1}B^{-1}\| = r(B[\text{Cliff}(\Lambda - \beta I)]^{-1}B^{-1}).$$

Since $B[\text{Cliff}(\Lambda - \beta I)]^{-1}B^{-1}$ and $[\text{Cliff}(\Lambda - \beta I)]^{-1}$ have the same joint spectra, we get $\mu = \max\{|\xi|^{-1}, \xi \in \sigma(\text{Cliff}(\Lambda - \beta I))\}$. Observing that $\sigma(\text{Cliff}(\Lambda - \beta I)) = \gamma(\Lambda - \beta I)$, we get

$$\mu = \max\{|\xi|^{-1}, \xi \in \gamma(\Lambda - \beta I)\} = \max\{|\lambda - \beta|^{-1}, \lambda \in \gamma(\Lambda) = \text{Sp}(\Lambda)\}.$$

Recall that by Proposition 3.2, we have $\text{Sp}(\Lambda) = \text{Sp}(T)$, which implies that $\mu = \max\{|\lambda - \beta|^{-1}, \lambda \in \gamma(\Lambda) = \text{Sp}(T)\} = \delta$, which completes the proof. \square

In case where the operators are self-adjoint we have $\text{Sp}_{\text{ap}}(T) = \text{Sp}(T)$. Indeed, if $\lambda = (\lambda_1, \dots, \lambda_m) \notin \text{Sp}_{\text{ap}}(T)$ then there is S_1, \dots, S_m in $\mathbb{B}(H)$ such that $(T_1 - \lambda_1 I)S_1 + \dots + (T_m - \lambda_m I)S_m = I$, hence $S_1^*(T_1 - \lambda_1 I) + \dots + S_m^*(T_m - \lambda_m I) = I$, thus $\lambda \notin \text{Sp}(T)$. On the other hand, every self-adjoint operator is a spectral operator whose quasi-nilpotent part is zero. Thus, in Theorem 3.4 we have $K = 0$ and we can chose $B = I$. Applying Theorem 3.4 to both T and S (by symmetry) where T and S are tuples of commuting self-adjoint operators, we get

$$\min\{|\alpha - \beta|, \alpha \in \text{Sp}(T)\} \leq \|\text{Cliff}(T - S)\|$$

for every $\beta \in \text{Sp}(S)$ and

$$\min\{|\alpha - \beta|, \alpha \in \text{Sp}(S)\} \leq \|\text{Cliff}(T - S)\|$$

for every $\beta \in \text{Sp}(T)$. We have thus proved the following:

PROPOSITION 3.5. *Let T and S be two m -tuples of commuting self-adjoint operators and let $\Delta(\text{Sp}(T), \text{Sp}(S))$ be the Hausdorff distance between $\text{Sp}(T)$ and $\text{Sp}(S)$. Then:*

$$(3.2) \quad \Delta(\text{Sp}(T), \text{Sp}(S)) \leq \|\text{Cliff}(T - S)\|.$$

EXAMPLE 1. Let us give an example were the bound (3.2) is attained. Let P be a nontrivial self-adjoint projection. Take $T_1 = P, T_2 = I - P, T = (T_1, T_2)$ and $S = (0, 0)$. Then $\text{Sp}(T) \subset \{(0, 0), (0, 1), (1, 0), (1, 1)\}$. Since $T_1^2 + T_2^2 = (T_1 - I)^2 + (T_2 - I)^2 = I$, we conclude that the points $(0, 0)$ and $(1, 1)$ do not belong to $\text{Sp}(T)$. On the other hand, we have

$$T_1^2 + (T_2 - I)^2 = 2P, \quad \text{and} \quad (T_1 - I)^2 + T_2^2 = 2(I - P),$$

which yields $\text{Sp}(T) = \{(0, 1), (1, 0)\}$. Whence, $\Delta(\text{Sp}(T), \text{Sp}(S)) = 1 = \|\text{Cliff}(T)\|$.

To visualize how Theorem 3.4 extends Henrici’s theorem, known for tuples of matrices [3], we will consider a special case of operators. From the point of view of spectral properties the algebraic operators may be the ‘nearest’ operators to matrices.

PROPOSITION 3.6. *Let $T = (T_1, \dots, T_m)$ and $S = (S_1, \dots, S_m)$ be two m -tuples of commuting (respectively) algebraic and spectral operators. Let $T_j = \Lambda_j + N_j$ be the canonical decomposition of the T_j ’s where Λ_j is scalar and N_j is nilpotent. Then given $\beta \in \text{Sp}_{\text{ap}}(S) \setminus \text{Sp}(T)$, there is a self-adjoint bounded invertible operator B and an integer $\nu \in \mathbb{N}$ such that*

$$\frac{\delta}{\sum_{0 \leq k \leq \nu-1} (\|B \text{Cliff}(N) B^{-1}\|^k / \delta^k)} \leq \|B \text{Cliff}(T - S) B^{-1}\|,$$

where $\delta = \min\{|\alpha - \beta|, \alpha \in \text{Sp}(T)\}$.

REMARK. According to the proof of Theorem 3.4, if T and S are tuples of commuting matrices then B can be chosen unitary and we thus get Henrici theorem for tuples of matrices [3, Theorem 3.2], with respect to the operator norm. If moreover T is a single normal matrix then we get the Bauer-Fike theorem [1].

PROOF. Let $\beta \in \text{Sp}_{\text{ap}}(S) \setminus \text{Sp}(T)$, denote by K the operator

$$K = [B \text{Cliff}(\Lambda - \beta I) B^{-1}]^{-1} B \text{Cliff}(N) B^{-1}.$$

Then $K = \sum_{k,p=1}^m \tilde{T}_k \tilde{N}_p \otimes e_k e_p$, where

$$\tilde{T}_k = \left(\sum_{j=1}^m (B \Lambda_j B^{-1} - \beta_j I)^2 \right)^{-1} (B \Lambda_k B^{-1} - \beta_k I)$$

and $\tilde{N}_p = B N_p B^{-1}$. Hence \tilde{N}_p is nilpotent and $\tilde{T}_k \tilde{N}_p = \tilde{T}_k \tilde{N}_p$ for $k, p = 1, \dots, m$ (this is because $\Lambda_k \tilde{N}_p = \tilde{N}_p \Lambda_k$). It follows that $\tilde{T}_k \tilde{N}_p \otimes e_k e_p$ is nilpotent for $k = 1, 2, \dots, m$ and therefore K is nilpotent: there is $\nu \in \mathbb{N}$ such that $K^\nu = 0$. Thus $(I + K)^{-1} = \sum_{k=0}^{\nu-1} (-1)^k K^k$, which gives as in the last proof

$$B [\text{Cliff}(T - \beta I)]^{-1} B^{-1} = (I + K)^{-1} B [\text{Cliff}(\Lambda - \beta I)]^{-1} B^{-1};$$

this leads to the following inequality

$$\begin{aligned} \|B[\text{Cliff}(T - \beta I)]^{-1} B^{-1}\| &\leq \sum_{k=0}^{\nu-1} \|B[\text{Cliff}(\Lambda - \beta I)]^{-1} B^{-1}\|^k \\ &\quad \times \|B \text{Cliff}(N) B^{-1}\|^k \|B[\text{Cliff}(\Lambda - \beta I)]^{-1} B^{-1}\| \\ &= \delta^{-1} \sum_{k=0}^{\nu-1} \frac{\|B \text{Cliff}(N) B^{-1}\|^k}{\delta^k}. \end{aligned}$$

Then using Proposition 3.3 we get

$$\delta \left(\sum_{k=0}^{v-1} \frac{\|B \text{Cliff}(N) B^{-1}\|^k}{\delta^k} \right)^{-1} \leq \|B \text{Cliff}(T - S) B^{-1}\|. \quad \square$$

4. Perturbations of tuples with complex spectra

Following Pryde [18], to remove the assumption of real spectra, we shall use partitions of tuples of operators. We shall also need the ‘spectral mapping theorem’ for both the joint spectrum and the joint approximate spectrum. We first recall the definition of a partition [18].

DEFINITION 4.1. We say that an m -tuple $T = (T_1, \dots, T_m)$ has an *admissible partition* if each T_j can be written $T_j = T_{1j} + iT_{2j}$ where T_{1j}, T_{2j} have real spectra and $\pi(T) = (T_{11}, \dots, T_{1m}, T_{21}, \dots, T_{2m})$ is a $2m$ -tuple of commuting operators; $\pi(T)$ is called an *admissible partition* of T .

It was proved by Pryde and McIntosh [14, Theorem 10.8], that if $T = (T_1, \dots, T_m)$ is an m -tuple of commuting operators having an admissible partition and if

$$(4.1) \quad P : \mathbb{C}^{2m} \rightarrow \mathbb{C}^m, \quad P(\lambda_1, \dots, \lambda_{2m}) = (\lambda_1 + i\lambda_{m+1}, \dots, \lambda_m + i\lambda_{2m}),$$

then

$$(4.2) \quad \text{Sp}(T) = \text{Sp}(P(\pi(T))) = P(\text{Sp}(\pi(T))) = P(\gamma(\pi(T))).$$

This means that if $\lambda \in \text{Sp}(T)$, then $(\text{Re}(\lambda), \text{Im}(\lambda)) \in \text{Sp}(\pi(T))$, where $\text{Re}(\beta), \text{Im}(\beta)$ are respectively the real and the imaginary parts of β .

It is also known [15] that every tuple of commuting spectral operators has an admissible partition. Thus using this results one can carry over Proposition 3.2 to arbitrary tuples consisting of commuting spectral operators.

Taking this into account, in order to pass from the ‘real case’ to the ‘complex case’, we just need a ‘spectral mapping theorem’ for the joint approximate spectrum. Namely (see [5, Theorem 1, page 319] or [7, Corollary 3.7, page 43]):

Let $P : \mathbb{C}^m \rightarrow \mathbb{C}^n$ be a polynomial and T an m -tuple of commuting operators, then

$$(4.3) \quad P(\text{Sp}_{\text{ap}}(T)) = \text{Sp}_{\text{ap}}(P(T)).$$

It follows in particular, that if P is the polynomial defined by (4.1) and if $\lambda \in \text{Sp}_{\text{ap}}(T)$, then $(\text{Re}(\lambda), \text{Im}(\lambda)) \in \text{Sp}_{\text{ap}}(\pi(T))$.

We are now able to establish some perturbation theorems for tuples having complex spectra. We adopt the following notation: If $\beta = (\beta_1, \dots, \beta_m) \in \mathbb{C}^m$ we put

$$\tilde{\beta} = (\text{Re}(\beta), \text{Im}(\beta)) = (\text{Re}(\beta_1), \dots, \text{Re}(\beta_m), \text{Im}(\beta_1), \dots, \text{Im}(\beta_m)) \in \mathbb{C}^{2m}.$$

THEOREM 4.2. *Let $T = (T_1, \dots, T_m)$ and $S = (S_1, \dots, S_m)$ be two m -tuples of commuting spectral operators. Let $\pi(T)$ be an admissible partition of T , say $\pi(T) = (T_{11}, \dots, T_{1m}, T_{21}, \dots, T_{2m})$ and let $T_{kj} = \Lambda_{kj} + N_{kj}$, $k, j = 1, 2, \dots, m$, be the canonical decomposition of the T_{kj} . Set $\pi(S)$ an admissible partition of S , $\pi(S) = (\Lambda_{11}, \dots, \Lambda_{1m}, \Lambda_{21}, \dots, \Lambda_{2m})$ and $\pi(N) = (N_{11}, \dots, N_{1m}, N_{21}, \dots, N_{2m})$. Given $\beta \in \text{Sp}_{\text{ap}}(S) \setminus \text{Sp}(T)$, there is a self-adjoint bounded invertible operator B such that for $K = [B \text{Cliff}(\pi(S) - \tilde{\beta}I)B^{-1}]^{-1} B \text{Cliff}(\pi(N))B^{-1}$, we have*

$$\frac{\delta}{\|(I + K)^{-1}\|} \leq \|B \text{Cliff}(\pi(T) - \pi(S))B^{-1}\|,$$

where $\delta = \min\{|\alpha - \beta|, \alpha \in \text{Sp}(T)\}$.

PROOF. By (4.2) and (4.3), one get that if $\beta \in \text{Sp}_{\text{ap}}(S) \setminus \text{Sp}(T)$ then $\tilde{\beta} \in \text{Sp}_{\text{ap}}(\pi(S)) \setminus \text{Sp}(\pi(T))$. Now applying Theorem 3.4, we get the result. □

Using Theorem 4.2, we are going to carry over [18, Corollary 6.4] for tuples of commuting normal operators.

PROPOSITION 4.3. *Let $T = (T_1, \dots, T_m)$ and $S = (S_1, \dots, S_m)$ be two m -tuples of commuting normal operators whose compression spectra are invariant under the map $z \mapsto \bar{z}$. Let $\pi(T)$, respectively $\pi(S)$, be an admissible partition of T , respectively of S , and $\pi(S^*)$ the $2m$ -tuple of operators composed by the adjoints of the operators corresponding to $\pi(S)$. Denote by $\Delta(\text{Sp}(T), \text{Sp}(S))$ the Hausdorff distance between $\text{Sp}(T)$ and $\text{Sp}(S)$. Then*

$$\Delta(\text{Sp}(T), \text{Sp}(S)) \leq \max(\|\text{Cliff}(\pi(S) - \pi(T))\|, \|\text{Cliff}(\pi(S^*) - \pi(T))\|).$$

PROOF. Applying Theorem 4.2 to $S^* = (S_1^*, \dots, S_m^*)$ with the partition $\pi(S^*)$, we get

$$\max_{\beta \in \text{Sp}_{\text{ap}}(S^*)} \min_{\alpha \in \text{Sp}(T)} |\alpha - \beta| \leq \|\text{Cliff}(\pi(S^*) - \pi(T))\|.$$

Now the approximate spectrum of S^* is nothing else but the complex conjugate of the compression spectrum of S ; this leads to

$$\max_{\beta \in \text{Sp}(S)} \min_{\alpha \in \text{Sp}(T)} |\alpha - \beta| \leq \max(\|\text{Cliff}(\pi(S) - \pi(T))\|, \|\text{Cliff}(\pi(S^*) - \pi(T))\|).$$

Interchanging S and T , we get

$$(4.4) \quad \Delta(\text{Sp}(T), \text{Sp}(S)) \leq \max(\|\text{Cliff}(\pi(S) - \pi(T))\|, \|\text{Cliff}(\pi(S^*) - \pi(T))\|, \|\text{Cliff}(\pi(T^*) - \pi(S))\|).$$

Now by the definition of the adjoint of the Clifford operator we have

$$\text{Cliff}(\pi(S^*) - \pi(T)) = (\text{Cliff}(\pi(S) - \pi(T^*)))^*.$$

Thereby the right-hand term in (4.4) is equal to

$$\max(\|\text{Cliff}(\pi(S) - \pi(T))\|, \|\text{Cliff}(\pi(S^*) - \pi(T))\|),$$

which finishes the proof. \square

By similar arguments to those used in the proof of Theorem 4.2, one can prove the following result which is an extension of [18, Theorem 6]:

PROPOSITION 4.4. *Under the assumptions of Theorem 4.2, if*

$$\epsilon = \|\text{Cliff}(\pi(T) - \pi(S))\| \|B\| \|B^{-1}\|,$$

then $\text{Sp}_{\text{ap}}(T) \subset \bigcup_{\beta \in \text{Sp}(S)} D_{\epsilon}(\beta)$, where $D_{\epsilon}(\beta)$ is the disc centered at β with radius ϵ .

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