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REGULAR ELLIPTIC CLASSES AND THE STABLE RELATIVE TRACE FORMULA

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ABSTRACT. We study the relative trace formula of a reductive group over an algebraic number field. Following Langlands we stabilize the geometric side of the relative trace formula contributed by the elliptic regular double cosets.

1. **Introduction.** The trace formula was first introduced by Selberg for the group SL(2, $\mathbb{R}$) and was established for any reductive group G over an algebraic number field F by Arthur in a long series of papers beginning with [1]. The phenomenon of stability of the trace formula was discovered by Langlands and it arises on the spectral side as L-indistinguishability while on the geometric side it reflects the "packetting" of the conjugacy classes over F into conjugacy classes over the algebraic closure $\bar{F}$ of F (see [10] to [13]). L-indistinguishability was first studied by Labesse and Langlands. Shelstad and Langlands introduced the concept of endoscopic groups. Langlands presented in [10] the regular elliptic part of the geometric side of the stable trace formula.

In [4] Jacquet and Lai introduced the relative trace formula for GL(2)—there they took a finite Galois extension E of F and considered the problem of integrating the kernel $K(x, y)$ of the regular representation of GL$(2, E)$ over GL$(2, F)$. In the situation of the relative trace formula one replaces the usual conjugacy classes by the double cosets of $G(E)$ modulo $G(F)$. Here one can again study the problems of stability. In this paper we give a "relative" version of Langlands [10], i.e. we stabilize the geometric side of the relative trace formula contributed by the regular elliptic double cosets. The author would like to thank Professor Langlands for a conversation on the geometry of double cosets and the referee for helpful comments.

2. **Notations.** Let F be an algebraic number field and E be a finite Galois extension of F. We write $\mathbb{A}$ for the adeles of F and $\mathbb{A}_E$ for the adeles of E. If we denote an algebraic group defined over F by $\mathbb{G}$, we shall write G for its group $\mathbb{G}(F)$ of F rational points, $G_{\mathbb{A}}$ for its group of F adelic points. To simplify notations we shall not distinguish G and $\mathbb{G}$ when it is clear from the context.

Let $\mathbb{G}$ be a connected reductive algebraic group defined over F. Consider $\mathbb{G}$ as an algebraic group over E and apply to it the Weil restriction functor from E to F to obtain $\tilde{G}_{\mathbb{A}}$. For each rational character χ of $\tilde{G}_{\mathbb{A}}$ defined over F, we define the homomorphism

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$|\chi|$ by

$$|\chi|(x) = \prod_v |\chi(x_v)|_v, \ x = \prod_v x_v \in \tilde{G}.$$

The intersection of the kernel of all the $|\chi|$ is denoted by ${}^\circ\tilde{G}_{\mathbb{A}}$. The locally compact group ${}^\circ\tilde{G}_{\mathbb{A}}$ contains $\tilde{G}$ as a discrete subgroup.

We denote the cardinality of a set S by $|S|$.

3. **Double cosets.** Write the Galois group Gal(E/F) of the extension E/F as $\{\sigma_1, \dots \sigma_n\}$ with $\sigma_1 =$ identity. For any σ in Gal(E/F), if $\sigma\sigma_i = \sigma_j$ we write $\sigma(i) = j$. The group $\tilde{\mathbb{G}}$ is characterised by having rational points $\tilde{\mathbb{G}}(F) = \mathbb{G}(E)$ and action of Gal(E/F) on $\tilde{\mathbb{G}}(E) = \mathbb{G}(E) \times \cdots \times \mathbb{G}(E)$ given by: $\sigma\big((g_i)\big) = \big(\sigma(g_{\sigma^{-1}(i)})\big)$, for (g_i) in $\tilde{\mathbb{G}}(E)$. We embed $\mathbb{G}(E)$ diagonally in $\tilde{\mathbb{G}}(E)$ and consider the action of (h, g) in $\mathbb{G}(E) \times \mathbb{G}(E)$ taking (g_i) in $\tilde{\mathbb{G}}(E)$ to $(h^{-1}g_i g)$. Since the action is given by multiplication in $\mathbb{G}$ which is in turn given by polynomials in F, we get an action of $\mathbb{G} \times \mathbb{G}$ on $\tilde{\mathbb{G}}$ defined over F. Let V be the quotient variety for this action. Then V is defined over F and $V(F)$ equals the double coset space $\mathbb{G}(F)\backslash \mathbb{G}(E)/\mathbb{G}(F)$. If we choose double coset representative of a point in $V(E)$ such that it can be written as $x = \mathbb{G}(E)(g_i)\mathbb{G}(E)$ with $g_1 = 1$, then by looking at the first coordinate of the equation $h^{-1}(g_i)g = (g_i)$ we see that $h = g$ and so the isotropy subgroup $\mathbb{G} \times \mathbb{G}_{(g_i)}$ of (g_i) is the intersection of centralizers of the g_i in $\mathbb{G}$.

Suppose now $\mathbb{G}^*$ is a quasi-split form of $\mathbb{G}$ given by an isomorphism $\psi : \mathbb{G} \longrightarrow \mathbb{G}^*$ defined over a finite Galois extension E. Let $\tilde{\mathbb{G}}^*$ be $R_{E/F}\mathbb{G}^*$ and V^* be the quotient variety for the action of $\mathbb{G}^* \times \mathbb{G}^*$ on $\tilde{\mathbb{G}}^*$. Then V and V^* are isomorphic over $\bar{F}$—the isomorphism Ψ being induced by ψ. If $\mathbb{G}$ is given by the inner twisting $\sigma \longmapsto a_\sigma ? a_\sigma^{-1}$ with a_σ in $\mathbb{G}^*(\bar{F})$, then the action of Galois on $\tilde{\mathbb{G}}$ is given by the following

$$(g_i) \longmapsto \big(a_\sigma \sigma(g_{\sigma^{-1}(i)})a_\sigma^{-1}\big).$$

A point v^* in $V^*(\bar{F})$ is a double coset $\mathbb{G}(\bar{F})(g_i)\mathbb{G}(\bar{F})$ and $\sigma(v^*)$ is the double coset $\mathbb{G}(\bar{F})\big(\sigma(g_{\sigma^{-1}(i)})\big)\mathbb{G}(\bar{F})$. For $v = \mathbb{G}(\bar{F})(g_i)\mathbb{G}(\bar{F})$ in $V(\bar{F})$ we get $\Psi(v)$ is $\mathbb{G}(\bar{F})(g_i)\mathbb{G}(\bar{F})$ and we deduce from the equality

$$\mathbb{G}(\bar{F})\big(a_\sigma\big(\sigma(g_{\sigma^{-1}(i)})\big)a_\sigma^{-1}\big)\mathbb{G}(\bar{F}) = \mathbb{G}(\bar{F})\big(\sigma(g_{\sigma^{-1}(i)})\big)\mathbb{G}(\bar{F})$$

the equation

$$\Psi \circ \sigma = \sigma \circ \Psi.$$

Thus V and V^* are isomorphic over F.

4. **Elliptic part.** We say that an element γ of $\tilde{G}$ is relatively regular if $\sigma(\gamma)\gamma^{-1}$ is a regular element of $\tilde{G}$ for all σ in Gal(E/F).

For a smooth compactly supported function f on $\tilde{G}_{\mathbb{A}}$, an element γ in $\tilde{G}$ and a maximal torus T of G defined over F, we put

$$\Phi_T(\gamma, f) = \iint f(x^{-1}\gamma y)\, dx\, dy$$

where the variables x ranges over $G_{\mathbb{A}}$ and y over $T_{\mathbb{A}} \backslash G_{\mathbb{A}}$. Let $\mathfrak{T}$ be a set of representatives of equivalence classes of anisotropic maximal torii $\mathbb{T}$ of $\mathbb{G}$ defined over F with respect to conjugation over F by elements of G. We write $\mathcal{T}_{\mathrm{re}}(f)$ for the regular elliptic part of the geometric side of the relative trace formula for f:

$$\mathcal{T}_{\mathrm{re}}(f) = \sum_{T \in \mathfrak{T}} \frac{\tau(T)}{\omega(T)} \sum_{T \backslash \tilde{T}}' \Phi_T(\gamma, f),$$

where $\sum'$ denotes the summation over all relatively regular elements; $\tau(T)$ is the Tamagawa number of T and $\omega(T)$ is the order of the Weyl group $\Omega_F(T, G)$. (Here in the sum $\sum_{\mathfrak{T}}$ we abused notation and wrote T for $\mathbb{T}$.)

5. **Stabilisation.** The process of stabilizing the trace formula involves two ingredients, namely the stable conjugacy of torii (Langlands [9]) and the local-global hypothesis (Langlands [10]: VI§ 4, VII§ 7). The part concerning the stable conjugacy classes of torii applies directly to the relative trace formula and will be considered first.

Let $\bar{F}$ be the separable closure of F. For a maximal F-torus T of G, the groups $\mathfrak{A}$, $\mathfrak{D}$, $\mathfrak{E}$, $\mathfrak{K}$ are defined as in Langlands [10] II§ 3. We recall their definitions for the convenience of the readers. Let G_{sc} be the simply connected covering of the derived group G_{der} of G and T_{sc} be the inverse image of T in G_{sc}. The group of coweights of a torus ? is denoted by $X_*(?)$. Then $\mathfrak{K}(T/F)$ is defined to be the group of complex characters on $X_*(T_{\mathrm{sc}})$ which are trivial on the intersection of $X_*(T_{\mathrm{sc}})$ with the lattice generated by $\{\sigma\mu - \mu \mid \sigma \in \mathrm{Gal}(\bar{F}_v/F_v), \mu \in X_*(T)\}$ for any place v of F and any extension of v to $\bar{F}$. Let K be a splitting field of T, Y be the group consisting of elements in $X_*(T_{\mathrm{sc}})$ whose norm from K to F is zero, Z be the subgroup of $X_*(T)$ generated by the elements $\sigma\mu - \mu$, $\sigma \in \mathrm{Gal}(K/F)$, $\mu \in X_*(T)$. Then $\mathfrak{E}(T/F)$ is the quotient group $Y/Y \cap Z$. Each element of $\mathfrak{K}(T/F)$ defines a character of $\mathfrak{E}(T/F)$. If g is in $\mathbb{G}(\bar{F})$, write T^g for $g^{-1}Tg$. Let $\mathfrak{A}(T/f)$ be the set of all elements g in $\mathbb{G}(\bar{F})$ such that both T^g and isomorphism $T \longrightarrow T^g : t \longmapsto g^{-1}tg$ are defined over F. Let $\mathfrak{D}(T/F)$ be the quotient $T(\bar{F}) \backslash \mathfrak{A}(T/F)/G(F)$. Two F-torii T, T' are said to be *stably conjugated* over F if there exists a g in $\mathfrak{A}(T/F)$ such that $T' = T^g$. Let $\mathfrak{T}_{\mathrm{st}}$ be a set of representatives of stable conjugacy classes of all the anisotropic maximal torii T of G defined over F.

If $\delta \in \mathfrak{D}(T/F)$ is represented by $a \in \mathfrak{A}(T)$ we put $\Phi_{T^\delta}(\gamma^\delta, f) = \Phi_{T^a}(\gamma^a, f)$. Replacing $\mathfrak{T}$ in $\mathcal{T}_{\mathrm{re}}(f)$ by $\mathfrak{T}_{\mathrm{st}}$ we get

$$\mathcal{T}_{\mathrm{re}}(f) = \sum_{T \in \mathfrak{T}_{\mathrm{st}}} \frac{\tau(T)}{\omega(T)} \sum_{T \backslash \tilde{T}}' \sum_{\delta \in \mathfrak{D}(T/F)} \Phi_{T^\delta}(\gamma^\delta, f).$$

6. **Second reduction.** Before we go on to the second reduction to κ-orbital integrals we introduce the diagrams of Langlands in three steps. The classes of L-indistinguishable representations are to be analyzed with the help of endoscopic groups H of G. The harmonic analysis on G is related to that on H by means of the transfer of orbital integrals. The diagrams are introduced to relate the tori in H to those in G.

STEP ONE. We choose a quasi-split form $\mathbb{G}^*$ of $\mathbb{G}$, i.e. $\mathbb{G}^*$ is a connected reductive group over F with a Borel subgroup $\mathbb{B}^*$ defined over F, we fix an isomorphism $\psi: \mathbb{G} \longrightarrow \mathbb{G}^*$ defined over $\bar{F}$ such that $\psi^{-1}\sigma(\psi): \mathbb{G} \longrightarrow \mathbb{G}$ is inner for all σ in $\mathrm{Gal}(\bar{F}/F)$. We fix a maximal F-torus $\mathbb{T}^*$ in $\mathbb{B}^*$ ([10] pp. 33, 38).

STEP TWO. Take a maximal F-torus $\mathbb{T}^*$ of $\mathbb{G}^*$. Then a κ in $\mathfrak{K}(\mathbb{T}^*/F)$ defines a quasi-split group $\mathbb{H}$ over F (an endoscopic group of $\mathbb{G}$, [10], pp. 33, 21). We fix a maximal F-torus $\underline{\mathbb{T}}_H$ in the Borel subgroup over F in $\mathbb{H}$ and an isomorphism $\underline{\mathbb{T}}_H \longrightarrow \underline{\mathbb{T}}^*$ over $\bar{F}$. The data $\{\mathbb{T}^*, \kappa\}$ defines an isomorphism over F from $\mathbb{T}^*$ onto a maximal F-torus $\mathbb{T}_H$ of $\mathbb{H}$ via a diagram

$$D^*: \mathbb{T}_H \longrightarrow \underline{\mathbb{T}}_H \longrightarrow \underline{\mathbb{T}}^* \longleftarrow \mathbb{T}^*,$$

in which every arrow is an isomorphism over $\bar{F}$ ([10], pp. 147).

STEP THREE. We say that a maximal F-torus $\mathbb{T}^*$ of $\mathbb{G}^*$ lifts to $\mathbb{G}$ globally if there is a maximal F-torus $\mathbb{T}$ of $\mathbb{G}$ and a $g \in \mathbb{G}^*(\bar{F})$ such that the restriction ψ_{T,T^*} of $\mathrm{ad}(g) \circ \psi$ to $\mathbb{T}$ maps onto $\mathbb{T}^*$ and is defined over F ([10], pp. 140). In this case we have a global diagram

$$D: \mathbb{T}_H \longrightarrow \underline{\mathbb{T}}_H \longrightarrow \underline{\mathbb{T}}^* \longleftarrow \mathbb{T}^* \overset{\psi_{T,T^*}}{\longleftarrow} \mathbb{T}.$$

We say that $\mathbb{T}^*$ of $\mathbb{G}^*$ lifts to $\mathbb{G}$ locally if for every completion F_v of F there is a maximal F_v-torus $\mathbb{T}_v$ of $\mathbb{G}$ and a $g \in \mathbb{G}^*(\bar{F}_v)$ such that the restriction ψ_{T_v,T^*} of $\mathrm{ad}(g) \circ \psi$ to $\mathbb{T}_v$ maps onto $\mathbb{T}^*$ and is defined over F_v ([10], pp. 140, 38, 159, 161). In this case we have a set of local diagrams:

$$D(v): \mathbb{T}_H \longrightarrow \underline{\mathbb{T}}_H \longrightarrow \underline{\mathbb{T}}^* \longleftarrow \mathbb{T}^* \overset{\psi_{T_v,T^*}}{\longleftarrow} \mathbb{T}_v.$$

([10] pp. 135). The set $D = \{D(v)\}$ is referred to as a pseudoglobal diagram; we also say that D^* lifts to a pseudoglobal diagram $D = \{D(v)\}$. Two diagrams are said to be congruent if they have the same endoscopic data ([10] Lemma 7.10). Langlands ([10] pp. 135, 137) defined an invariant $\kappa\big(\varepsilon(D)\big)$ whenever D is congruent to a global diagram and in this case for $\gamma^* = (\gamma_v^*)$ in $\mathbb{T}^*(\mathbb{A})$, $\gamma_v^* = \psi_{T_v,T^*}(\gamma_v)$, γ_v in $\mathbb{T}_v(F_v)$, $\delta^* = (\delta_v^*)$ in $\mathfrak{e}(\mathbb{T}^*/\mathbb{A})$, $\delta_v^* = \psi_{T_v,T^*}\big(\delta(v)\big)$, $\delta(v)$ in $\mathfrak{E}(\mathbb{T}_v/F_v)$ and $f = \prod_v f_v$, where, for almost all v, f_v is the quotient of the characteristic function of a hyperspecial compact subgroup U_v divided by its measure, we put

$$\Phi^{\kappa}_{\mathbb{T}^*}(\gamma^*, f) = \kappa\big(\varepsilon(D)\big) \sum_{\delta^* \mathfrak{E}(\mathbb{T}^*/\mathbb{A})} \kappa(\delta^*) \prod_v \Phi_{\mathbb{T}_v^{\delta(v)}}(\gamma_v^{\delta(v)}, f_v).$$

if $\delta(v) \in \mathfrak{D}(\mathbb{T}_v/F_v)$ for all v and set it to be zero otherwise. And if D^* does not lift to a diagram D which is congruent to a global diagram we set $\Phi^{\kappa}_{\mathbb{T}^*}$ to be equal to zero. For the purpose of making this definition we want to show that for a given relatively regular γ^* there is only finite number of δ^* such that the corresponding integral

$$\Phi_{\mathbb{T}_v^{\delta(v)}}(\gamma_v^{\delta(v)}, f_v) \neq 0,$$

for all v. Suppose $\tilde{\mathbb{G}}$ splits over an unramified extension L_v of F_v, and U_v is the isotropy subgroup of the vertex p in the apartment A of the Bruhat-Tits building $\mathcal{X}$ of $\tilde{\mathbb{G}}(L_v)$.

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Assume that γ lies in $\tilde{\mathbb{T}}(F_v) \cap U_v$ and $\alpha(\gamma)$ is congruent to 1 modulo $\mathfrak{p}_v$ for no root α. Then the fixed-point set of γ in X is contained in A. Let a represent $\delta(v)$. For x, y in $G(F_v)$ we get $f_v(x^{-1}\gamma^a y)$ is 1 or 0 according as $y^{-1}\big((\sigma(\gamma^{-1}\gamma)^a\big)y$ lies in U_v or not; this in turn depends on whether $ay \cdot p$ lies in A or not. If A contains $ay \cdot p$ then $\delta(v)$ is trivial in $\mathfrak{E}(\mathbb{T}_v/F_v)$; otherwise the function $(x, y) \longmapsto f(x^{-1}\gamma^a y)$ is identically zero.

LEMMA. *(i) The value of $\Phi^{\kappa}_{\mathbb{T}^*}(\gamma^*, f)$ is independent of the choice of ψ_{T_v, T^*}.*

(ii) If $a \in \mathfrak{A}(\mathbb{T}^*/F)$, $'\mathbb{T}^* = (\mathbb{T}^*)^a$, $'\gamma^* = (\gamma^*)^a$, and if $'\kappa$ is obtained from κ by transport of structures then

$$\Phi^{'\kappa}_{'\mathbb{T}^*}('\gamma^*, f) = \Phi^{\kappa}_{\mathbb{T}^*}(\gamma^*, f).$$

PROOF. (i) We can change ψ_{T_v, T^*} by $\mathrm{ad}(h_v)$ with h_v in $\mathfrak{A}(\mathbb{T}^*/F_v)$. With respect to $\psi_{T_v^{h_v}, T^*}$, we consider the sum

$$\sum \kappa\big((h_v^{-1}\delta(v)\big) \prod_v \Phi_{(\mathbb{T}_v^{h_v})^{h_v^{-1}\delta(v)}}\big(\gamma(v)^{h_v h_v^{-1}\delta(v)}, f_v\big).$$

which is equal to

$$\prod_v \kappa_v(h_v^{-1}) \sum_{\delta^* \in \mathfrak{E}(\mathbb{T}^*/\mathbb{A})} \kappa(\delta^*) \prod_v \Phi_{\mathbb{T}_v^{\delta(v)}}(\gamma_v^{\delta(v)}, f_v).$$

And for the diagram D' associated to $\psi_{T_v^{h_v}, T^*}$ we have

$$\kappa\big(\varepsilon(D')\big) = \kappa\big(\varepsilon(D)\big) \prod_v \kappa_v(h_v).$$

(ii) Suppose the diagram

$$'D(v)\colon \mathbb{T}_h \longrightarrow \underline{\mathbb{T}}_H \longrightarrow \underline{\mathbb{T}}^* \longleftarrow {'\mathbb{T}^*} \overset{'\psi}{\longleftarrow} \mathbb{T}_v.$$

is obtained from $D(v)$ by using $'\psi$ which is the restriction of $\psi \circ \mathrm{ad}(g) \circ \mathrm{ad}(a^{-1})$. Since for the give a we have $\prod_v \kappa_v(a) = 1$, the invariance follows.

Let $\phi_T\colon \mathfrak{E}(\mathbb{T}/F) \longrightarrow \mathfrak{E}(\mathbb{T}/\mathbb{A})$ be the natural homomorphism. Write $\iota(F, \mathbb{T}^*)$ for the number $|\ker \phi_{T^*}| \cdot |\mathfrak{K}(\mathbb{T}^*/F)|^{-1}$.

LEMMA. *Suppose $\gamma^* \in \tilde{T}^*$ is relatively regular. Let $e(T^*)$ be 1 if T^* lifts to T globally and zero otherwise. Then*

$$\iota(F, T^*) \sum_{\kappa \in \mathfrak{K}(T^*/F)} \Phi^{\kappa}_{T^*}(\gamma^*, f) = e(T^*) \sum_{\delta \in \mathfrak{D}(T/F)} \phi_{T^\delta}(\gamma^\delta, f),$$

with $\gamma^* = \psi_{T,T^*}(\gamma)$.

PROOF. (i) Suppose $e(T^*) = 1$. Then κ defines a global diagram D and $\kappa\big(\varepsilon(D)\big) = 1$. Also $\mathfrak{E}(T/\mathbb{A}) = \mathfrak{E}(T^*/\mathbb{A})$. By the Tate-Nakayama theorem δ is in the kernel of all κ

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in $\mathfrak{K}(T^*/F)$ if and only if δ is in the image of ϕ_T and this is equivalent to δ lies in the image of $\mathfrak{D}(T/F)$ under ϕ_T. Therefore

$$\sum_{\kappa}\Phi^{\kappa}_{T^*}(\gamma^*,f) = |\ker\phi_{T^*}|^{-1}\cdot|\mathfrak{K}(\mathbb{T}^*/F)|\cdot\sum_{\delta}\Phi_{\delta}(\gamma^{\delta},f).$$

(ii) Suppose $e(T^*)=0$. It suffices to show that for any given κ we have

$$\sum_{\kappa^\circ\in\mathfrak{K}^\circ(T^*/F)}\Phi^{\kappa\kappa^\circ}_{T^*}(\gamma^*,f)=0$$

(here $\mathfrak{K}^\circ$ is defined in [10] p. 135). Furthermore we can assume that the pseudodiagram $D=D(\kappa\kappa^\circ)$ defined by $\kappa\kappa^\circ$, T and ψ_{T_v,T^*} is congruent to a global diagram. Since $\kappa^\circ(\delta^*)=1$ for δ^* in $\mathfrak{E}(T^*/\mathbb{A})$, the sum is equal to

$$\sum_{\kappa^\circ}\kappa\kappa^\circ\Big(\varepsilon\big(D(\kappa\kappa^\circ)\big)\Big)\sum_{\delta^*}\kappa(\delta^*)\prod_{v}\Phi_{T^{\delta(v)}}(\gamma_v^{\delta(v)},f)$$

and the result follows from the fact that $e(T^*)=0$ implies that

$$\sum_{\kappa^\circ}\kappa\kappa^\circ\Big(\varepsilon\big(D(\kappa\kappa^\circ)\big)\Big)=0.$$

At this point the second reduction step is immediate; namely, we get

$$\mathcal{T}_{\mathrm{re}}(f)=\sum_{T^*\in\mathfrak{T}^*_{\mathrm{st}}}\frac{\tau(T^*)}{\omega(T^*)}\sum_{T^*\backslash\tilde{T}^*}{}'\,\iota(F,T^*)\sum_{\kappa\in\mathfrak{K}(T^*/F)}\Phi^{\kappa}_{T^*}(\gamma^*,f).$$

7. **Third reduction.** After the second reduction we are in the diagram D^*. The questions are (1) the transfer of integrals to the endoscopic group H and (2) the grouping of the data $\{T^*,\gamma^*,\kappa\}$ according to the endoscopic data.

When κ is identity we write Φ^{st}_T for the κ-integral Φ^{κ}_T. To deal with (1) we combine the global hypothesis ([10] p. 49) with the relative analogue of the fundamental lemma into the following

TRANSFER HYPOTHESIS. Given a diagram D^* defined by a couple T^*, κ and a smooth compactly supported function $f=\prod f_v$ on $\tilde{G}_{\mathbb{A}}$, there is a smooth compactly supported function $f^H=\prod f^H_v$ on $\tilde{H}_{\mathbb{A}}$ such that f^H is equal to zero if D^* is not congruent to a global diagram and if D^* is congruent to a global diagram we have

$$\Phi^{\kappa}_{T^*}(\gamma^*,f)=\prod_v\Phi^{\mathrm{st}}_{T_H}(\gamma,f^H_v).$$

To simplify notation we write $\langle s,H\rangle$ for the class $\mathfrak{s}=(s,{}^LH^\circ,\ldots)$ of endoscopic data ([10] p. 19); $\Lambda(H)$ for the automorphism group Λ of $\mathfrak{s}$ ([10] p. 164) and $\mathcal{E}$ for the set of equivalence classes of elliptic endoscopic data. Roughly speaking Langlands ([10] VIII § 3) showed that there is a m to n correspondence between averaged sums over $\{(T^*,\gamma^*,\kappa)\}$ and those over $\{(\langle s,H\rangle,T_H,\gamma)\}$; with $m=\omega(T^*)$ and $n=\omega(T_H)\cdot|\Lambda(H)|$. The arguments do not involve integrals. The same arguments apply here to give the third

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reduction to stable integrals, namely $\mathcal{T}_{\rm re}(f)$ is the sum over the classes $\langle s, H\rangle$ of elliptic endoscopic data of the sum over the set $\mathfrak{T}_{\rm st}(H)$ of H of

$$\sum_{T_H\backslash \tilde{T}_H}{}' \frac{\tau(T_H)\cdot\iota(F,T^*)}{\omega(T_H)\cdot|\Lambda(H)|}\cdot\Phi^{\rm st}_{T_H}(\gamma,f^H)$$

We substitute the invariant $\iota(G,H)=\iota(F,T^*)\{\iota(F,T_H)|\Lambda(H)|\}^{-1}$ and write

$$\mathfrak{S}\mathfrak{T}_{\rm re}(f^H)=\sum_{\mathfrak{T}_{\rm st}(H)}\sum_{T_H\backslash \tilde{T}_H}{}' \frac{\tau(T_H)}{\omega(T_H)}\cdot\iota(F,T_H)\cdot\Phi^{\rm st}_{T_H}(\gamma,f^H);$$

then we obtain the required formula

$$\mathcal{T}_{\rm re}(f)=\sum_{\mathcal{E}}\iota(G,H)\cdot\mathfrak{S}\mathfrak{T}_{\rm re}(f^H).$$

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