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A CLASS OF STRONGLY STABLE OPERATOR APPROXIMATIONS

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Abstract

We show the strongly stable convergence of some non-collectively-compact approximations of compact operators. Special attention is devoted to Anselone's singularity subtraction discretization of certain singular integral operators. Numerical experiments are provided.

1. Introduction

It is well known (cf. [3]) that uniform and collectively-compact approximations of compact operators in infinite dimensional complex Banach spaces are strongly stable in a neighborhood of each nonzero eigenvalue. This means, roughly speaking, that nonzero eigenvalues are approximated with preservation of their algebraic multiplicity and maximal invariant subspaces associated with them are approximated with convergence in gap. The notion of strongly stable convergence was introduced by Chatelin in [3].

However, for singular compact integral operators, Anselone's collectively-compact theory (in [1]) does not cover some commonly used discretizations such as singularity subtraction, proposed by Anselone himself in [2].

In this work we prove that:

If T is a compact linear operator in an infinite dimensional complex Banach space X and (T_n) is a sequence of linear bounded operators in X (which need not

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be compact) such that

C1. $\forall x \in X \lim_{n \to \infty} T_n x = Tx$,

C2. $\lim_{n \to \infty} \|(T_n - T)T_n\| = 0$,

then (T_n) is a strongly stable approximation to T in a neighborhood of each nonzero eigenvalue of T.

Conditions C1 and C2 are satisfied by uniform and collectively-compact discretizations but they are also satisfied by norm perturbations of collectively compact approximations T_n of a compact T, namely, $T_n = K_n + B_n$ where B_n converges in norm to 0 and K_n is a collectively compact approximation to T.

A singularity subtraction example is discussed in Section 4.

2. Mathematical background

Let X be a complex infinite dimensional Banach space. $\|\ \|$ will denote both the norm of X and that of the algebra $\mathscr{L}(X)$, the space of linear bounded operators defined in X. 1 stands for the identity in X and z denotes $z1$ for any complex number z.

We state in this section a set of lemmas which will be needed in the proof of our main theorem, to which is devoted the next section. Lemmas 1 and 6 are well-known results, so proofs will be omitted.

LEMMA 1. *If* $(A_n) \subseteq \mathscr{L}(X)$, $A \in \mathscr{L}(X)$ *and* $\forall x \in X\ A_n x \to Ax$, *then there is a constant* $C > 0$ *and an integer* n_0 *such that* $\sup_{n > n_0} \|A_n\| \leqslant C$.

LEMMA 2. *If* $(A_n) \subseteq \mathscr{L}(X)$, $(B_n) \subseteq \mathscr{L}(X)$, $A \in \mathscr{L}(X)$, $B \in \mathscr{L}(X)$ *and* $\forall x \in X\ A_n x \to Ax$ *and* $B_n x \to Bx$, *then* $\forall x \in X\ A_n B_n x \to ABx$.

LEMMA 3. *If* $(A_n) \subseteq \mathscr{L}(X)$, $A \in \mathscr{L}(X)$, $K \in \mathscr{L}(X)$ *is compact and* $\forall x \in X\ A_n x \to Ax$, *then* $\|(A_n - A)K\| \to 0$.

LEMMA 4. *If* Γ *is a compact subset of the complex plane*, $z \in \Gamma \to U(z) \in \mathscr{L}(X)$ *is a continuous function in* Γ, $(A_n) \subseteq \mathscr{L}(X)$, $A \in \mathscr{L}(X)$ *and* $\forall x \in X$, $A_n x \to Ax$, *then* $\forall x \in X$, $\sup_{z \in \Gamma} \|(A_n - A)U(z)x\| \to 0$.

Let $T \in \mathscr{L}(X)$ be compact and $(T_n) \subseteq \mathscr{L}(X)$ such that
C1. $\forall x \in X$, $T_n x \to Tx$,
C2. $\|(T_n - T)T_n\| \to 0$.

Let λ be a nonzero eigenvalue of T isolated by a simple closed Jordan cuve Γ not enclosing 0 and lying in the resolvent set $\rho(T)$ of T. For $z \in \Gamma$ we set $R(z) = (T - z)^{-1}$. The spectral projection associated with λ is

$$P = -\frac{1}{2\pi i}\int_\Gamma R(z)\,dz \quad (i^2 = -1)$$

and it has finite rank, say m. The image space M of P is the (maximal) invariant subspace associated with λ and has dimension m.

We formally set

$$R_n(z) = (T_n - z)^{-1}, \quad P_n = -\frac{1}{2\pi i}\int_\Gamma R_n(z)\,dz, \quad M_n = \text{Image space of } P_n.$$

LEMMA 5. *$z \in \Gamma \to R(z) \in \mathscr{L}(X)$ is continuous.*

LEMMA 6. *T, P and, for $z \in \rho(T)$, $R(z)$ commute one with each other.*

LEMMA 7. *Under conditions* C1 *and* C2 *there exists an integer n_0 such that, for $n > n_0$, $R_n(z)$ exists and belongs to $\mathscr{L}(X)$. Moreover*

$$\forall x \in X\ \forall z \in \Gamma,\ R_n(z)x \to R(z)x.$$
$$\exists C_\Gamma > 0 \text{ such that } \sup_{\substack{z\in\Gamma\\ n>n_0}} \|R_n(z)\| \leqslant C_\Gamma.$$

PROOF. There exists n_0 and δ such that for $n > n_0$:

$$\forall z \in \Gamma\, \frac{1}{|z|}\|R(z)(T - T_n)T_n\| \leqslant \delta < 1$$

and

$$r_\sigma(T - T_n) < |z|,$$

where r_σ denotes spectral radius. Then, for $n > n_0$,

$$1 - \frac{1}{z}R(z)(T - T_n)T_n = R(z)\Big(1 - \frac{1}{z}(T - T_n)\Big)(T_n - z)$$

is nonsingular with inverse in $\mathscr{L}(X)$, and the same holds for $1 - \dfrac{1}{z}(T - T_n)$. Hence, for $n > n_0$ we have $z \in \rho(T_n)$ and

$$R_n(z) = \Big(1 - \frac{1}{z}R(z)(T - T_n)T_n\Big)^{-1} R(z)\Big(1 - \frac{1}{z}(T - T_n)\Big)$$
$$= \Big(1 + \sum_{k=1}^{\infty}\Big(\frac{1}{z}R(z)(T - T_n)T_n\Big)^k\Big) R(z)\Big(1 - \frac{1}{z}(T - T_n)\Big),$$

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so the constant C_Γ in the conclusion of the lemma may be chosen as

$$C_\Gamma = \frac{1}{1-\delta} \sup_{\substack{z\in\Gamma \\ n>n_0}} \left\| R(z)\left(1 - \frac{1}{z}(T - T_n)\right)\right\|$$

which is certainly finite.

Finally, for $z \in \Gamma$ and $n > n_0$:

$$R(z) - R_n(z) = R_n(z)(T_n - T)R(z),$$

so $\forall x \in X, \forall z \in \Gamma, \forall n > n_0$

$$\|(R(z) - R_n(z))x\| \leqslant C_\Gamma \|(T_n - T)R(z)x\| \to 0.$$

LEMMA 8. *Under conditions* C1 *and* C2 *there exists an integer* n_0 *such that for* $n > n_0$, P_n *is well defined, belongs to* $\mathscr{L}(X)$ *and is a projection onto* M_n. *Moreover*, $\forall x \in X$, $P_n x \to Px$.

PROOF. For the integer n_0 of Lemma 7 we have: $\forall x \in X, \forall n > n_0$,

$$\|(P_n - P)x\| = \frac{1}{2\pi}\left\| \int_\Gamma (R_n(z) - R(z))x\,dz \right\|$$
$$\leqslant C \sup_{z\in\Gamma} \|(T - T_n)R(z)x\| \to 0$$

for some constant $C > 0$.

3. The main result.

We first recall the notion of strongly stable approximation, keeping the notation of the preceding sections.

(T_n) is a strongly stable approximation to T around the eigenvalue λ if the following conditions are satisfied, in addition to the pointwise convergence C1:

SS1) For any Jordan curve Γ isolating λ and all n large enough, $R_n(z)$ is defined for $z \in \Gamma$, belongs to $\mathscr{L}(X)$ and $\forall x \in X$, $R_n(z)x \to R(z)x$.

SS2) For all n large enough, $\dim M_n = \dim M$.

We now state and prove the main result of the paper.

THEOREM. *Under conditions* C1 *and* C2, (T_n) *is a strongly stable approximation to* T *around* λ.

PROOF. By Lemma 7, (T_n) satisfies SS1. Hence, we only need to prove that there is an integer n_0 such that for $n > n_0$:

$$\dim M_n = \dim M.$$

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Let $\gamma(M, M_n)$ denote the gap between M and M_n, then (cf. [3]):

$$\gamma(M, M_n) \leqslant \max\{\|(P - P_n)P\|, \|(P - P_n)P_n\|\}$$

and

$$\gamma(M, M_n) < 1 \text{ implies } \dim M_n = \dim M,$$

since M is finite dimensional.

By Lemmas 3 and 8, since P is of finite rank and hence compact,

$$\|(P_n - P)P\| \to 0.$$

We shall prove that $\|(P_n - P)P_n\| \to 0$. Since $R_n(z)$ and P_n commute:

$$\|(P_n - P)P_n\| = \frac{1}{2\pi}\left\|\int_\Gamma R(z)(T - T_n)R_n(z)P_n\,dz\right\| \leqslant C\|(T - T_n)P_n\|$$

for some constant $C > 0$ and n large enough. It suffices now to prove that $\|(T - T_n)P_n\| \to 0$. We consider the decomposition

$$(T - T_n)P_n = (T - T_n)P_nP + (T - T_n)P_n(1 - P).$$

By Lemma 2: $\forall x \in X$ $(T_n - T)P_nx \to 0$, so by Lemma 3: $\|(T_n - T)P_nP\| \to 0$. Now $(T - T_n)P_n(1 - P) = (T - T_n)P_n(P_n - P)$ and $P_n - P = -1/2\pi i\int_\Gamma (R_n(z) - R(z))\,dz = -1/2\pi i\int_\Gamma R_n(z)(T - T_n)R(z)\,dz$, so there is a constant $C > 0$ such that for n large enough:

$$\|(T_n - T)P_n(1 - P)\| \leqslant C\left\|\int_\Gamma (T_n - T)P_nR_n(z)(T - T_n)R(z)\,dz\right\|$$

$$\leqslant C\left(\left\|\int_\Gamma (T_n - T)P_nR_n(z)T_nR(z)\,dz\right\| + \left\|\int_\Gamma (T_n - T)P_nR_n(z)R(z)\,dzT\right\|\right)$$

$$\leqslant C'\|(T_n - T)T_n\| + C''\left\|(T_n - T)P_n\int_\Gamma R_n(z)R(z)\,dzT\right\|$$

for some constants $C' > 0$ and $C'' > 0$. It remains to prove that

$$\left\|(T_n - T)P_n\int_\Gamma R_n(z)R(z)\,dzT\right\| \to 0$$

We define

$$V_n = \int_\Gamma R_n(z)R(z)\,dz, \qquad V = \int_\Gamma R(z)^2\,dz.$$

For n large enough, $V_n \in \mathscr{L}(X)$ and $\forall x \in X$

$$\|(V_n - V)x\| \leqslant C\sup_{z\in\Gamma}\|(T_n - T)R(z)^2x\|$$

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with some constant $C > 0$. Lemma 4 shows that $\forall x \in X$, $V_n x \to Vx$ and Lemma 2 implies that $\forall x \in X$, $(T_n - T)P_n V_n x \to 0$. Finally, with Lemma 3 we conclude $\|(T_n - T)P_n V_n T\| \to 0$ and the proof is complete.

4. A singularity subtraction example

We consider the integral operator

$$(Tx)(t) = \int_0^1 k(t,s)x(s)\,ds$$

defined in the space $X = C[0,1]$ of complex valued continuous functions on $[0,1]$ normed by the uniform convergence norm.

The eigenproblem

$$Tx = \lambda x \quad x \neq 0$$

may be rearranged in the form

$$\int_0^1 k(t,s)(x(s) - x(t))\,ds + \int_0^1 k(t,s)x(t)\,dt = \lambda x(t).$$

Let us approximate the first integral in the following way: First replace the kernel k by an approximation k_n, where convergence holds in the sense of [2]; secondly, apply a quadrature rule with weights ω_{nj} and knots t_{nj} satisfying the conditions specified in [2]. This leads to the problem

$$\sum_{j=1}^{n} \omega_{nj} k_n(t, t_{nj})\big(x(t_{nj}) - x(t)\big) + \int_0^1 k(t,s)x(t)\,ds = \lambda x(t)$$

which is the eigenproblem of the operator $T_n \in \mathscr{L}(X)$ defined by

$$(T_n x)(t) = \sum_{j=1}^{n} \omega_{nj} k_n(t, t_{nj})\big(x(t_{nj}) - x(t)\big) + \int_0^1 k(t,s)x(t)\,ds.$$

This operator is called the singularity subtraction approximation to T. As is proved in [2], under suitable conditions on $(k_n, \omega_{nj}, t_{nj})$ operator T_n satisfies C1 and C2.

The eigenvalues of T_n may be computed by solving

$$(T_n x)(t_{ni}) = \lambda_n x(t_{ni}), \quad i = 1, 2, \ldots, n,$$

which leads to a matrix eigenproblem.

More precisely, we define the matrix $A_n = (a_{ij})$ by

$$a_{ij} = \begin{cases} \omega_{nj} k_n(t_{ni}, t_{nj}) & \text{if } j \neq i \\ \omega_{ni} k_n(t_{ni}, t_{ni}) - \varepsilon_n(t_{ni}) & \text{if } j = i, \end{cases}$$

where

$$\varepsilon_n(t) = \sum_{j=1}^{n} \omega_{nj} k_n(t, t_{nj}) - \int_0^1 k(t,s)\,ds.$$

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If $u_n \in \mathbf{C}^n$ and $\lambda_n \in \mathbf{C}$ satisfy

$$A_n u_n = \lambda_n u_n, \quad \max_j |u_{nj}| = 1, \qquad \lambda_n \neq 0,$$

then λ_n is an eigenvalue of T_n and

$$\phi_n(t) = \frac{\sum_{j=1}^{n} \omega_{nj} k_n(t, t_{nj}) u_{nj}}{\lambda_n + \varepsilon_n(t)} \qquad (0 \leqslant t \leqslant 1)$$

is an associated eigenfunction.

In the case of an eigenvalue λ with algebraic multiplicity m there will be, for n large enough, a group of m close eigenvalues $\lambda_{n,1}, \lambda_{n,2}, \ldots, \lambda_{n,m}$ (maybe some of them repeated) such that their arithmetic mean, say $\hat{\lambda}_n$, is an approximation to λ of the order of $\|(T - T_n)P\|$ (see [3]).

Numerical computations are done with

$$k(t,s) = \ln(1 - \cos 2\pi(t - s)) \qquad t \neq s$$

$$k_n(t,s) = \begin{cases} k(t,s) & \text{if } \frac{1}{4n} \leqslant |t - s| \leqslant 1 - \frac{1}{4n} \\ \ln\left(1 - \cos\frac{\pi}{2n}\right) & \text{otherwise} \end{cases}$$

and the trapezoidal rule as quadrature formula.

When $n = 100$, matrix A_n provides approximations of the order of 10^{-4} at nodal points t_{nj} to the exact linearly independent eigenfunctions

$$\phi(t) = -0.343778 \cos(6\pi t - 0.353927),$$
$$\psi(t) = 0.372441 \sin(6\pi t - 0.293559)$$

associated with the exact eigenvalue $-1/3$ that, in turn, is approximated by -0.333356, which is the arithmetic mean of two close eigenvalues of A_n that differ by less than 10^{-5}.

Table 3.1 shows relative errors of several eigenvalue approximations.

TABLE 3.1

Exact Eigenvalue	Multiplicity	Approximate Eigenvalues and Relative Errors			
		$n = 10$		$n = 100$	
λ	m	$\hat{\lambda}_n$	Relative Error	$\hat{\lambda}_n$	Relative Error
-1	2	-0.994	0.60%	-1.000002	0.0002%
$-\ln 2$	1	-0.684	1.32%	-0.693154	0.0010%
$-1/2$	2	-0.505	1.00%	-0.500010	0.0020%
$-1/3$	2	-0.357	7.10%	-0.333356	0.0068%

References

[1] P. M. Anselone, *Collectively Compact Operator Approximation Theory* (Prentice-Hall, Englewood Cliffs, New Jersey, 1971).
[2] P. M. Anselone, "Singularity Subtraction in the Numerical Solution of Integral Equations", *J. Austral. Math. Soc. Ser. B* 22 (1981), 408–418.
[3] F. Chatelin, *Spectral Approximation of Linear Operators* (Academic Press, New York, 1983).