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RESEARCH ARTICLE

Periods of elliptic surfaces with $p_g = q = 1$

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Abstract
We prove that the period mapping is dominant for elliptic surfaces over an elliptic curve with 12 nodal fibers, and that its degree is larger than 1. This settles the final case of infinitesimal Torelli for a generic elliptic surface.

Contents

1. Introduction

In order to distinguish smooth projective varieties varying in a family with continuous parameters, it is often useful to integrate the holomorphic forms over topological cycles. This idea was used to great effect classically to distinguish smooth curves of a given genus $g > 0$. A modern reformulation of this problem in higher dimension asks whether the period mapping from a moduli space of varieties to an associated space of *periods* is injective, either locally or globally on the source. We will show that while the local injectivity statement is true generically, the global statement fails for an important class of elliptic surfaces.

An *elliptic surface* is a smooth, projective surface S equipped with a relatively minimal, genus one fibration $\pi\colon S \to C$ to a smooth curve and a distinguished section s. Moduli spaces $F_{g,d}$ of elliptic surfaces are indexed by two nonnegative integers, $g = g(C)$ and $d = \frac{1}{12}\chi_{\mathrm{top}}(S)$. Counted with multiplicity, there are $12d$ singular fibers. The canonical bundle of S is pulled back from a line bundle $L \otimes \omega_C$ of degree $d + 2g - 2$ on C. We henceforth assume $d > 0$ (that is, S has at least one singular fiber) so that $p_g(S) := h^0(K_S) = g + d - 1$.

In this paper, we focus on the moduli space $F := F_{1,1}$. Since $g(C) = 1$, $K_S = \pi^*L$ for a degree 1 line bundle $L = \mathcal{O}_C(p)$, and generically the fibration π has 12 singular fibers. There is a morphism

$S \to \overline{S}$ contracting ADE configurations in fibers not intersecting the section s. This contraction has a Weierstrass form [Kas77]

$$\overline{S} = \{y^2 = x^3 + ax + b\} \subset \mathbb{P}_C(L^2 \oplus L^3 \oplus \mathcal{O}),$$

where $a \in H^0(C, L^4)$ and $b \in H^0(C, L^6)$. A quick parameter count reveals that $\dim F = 1+4+6-1 = 10$ where the parameters are, respectively, the j-invariant of (C, p), the section a, the section b and the quotient by the action of $\lambda \in \mathbb{C}^*$ via $(a, b) \mapsto (\lambda^4 a, \lambda^6 b)$.

Noether's formula implies that the Hodge numbers of S are $h^{2,0}(S) = h^{1,0}(S) = 1$ and $h^{1,1}(S) = 12$. The Neron-Severi group $\mathrm{NS}(S) = H^{1,1}(S, \mathbb{C}) \cap H^2(S, \mathbb{Z})$ always contains the classes of the fiber f and section s which have intersection numbers $s^2 = -1$, $s \cdot f = 1$, $f^2 = 0$. Hence, there is a copy of the odd unimodular lattice

$$I_{1,1} \simeq \mathbb{Z}s \oplus \mathbb{Z}(s + f) \subset \mathrm{NS}(S).$$

Its orthogonal complement $\{s, f\}^\perp \subset H^2(S, \mathbb{Z})$ is an even (since $[K_S] = f$), unimodular lattice of signature $(2, 10)$, so it is isometric to $I\!I_{2,10} = H \oplus H \oplus E_8$.

Let $\Gamma := O(I\!I_{2,10})$ and define the *period domain* to be

$$\mathbb{D} := \mathbb{P}\{x \in I\!I_{2,10} \otimes \mathbb{C} \,\big|\, x \cdot x = 0,\ x \cdot \bar{x} > 0\}.$$

It is a ten-dimensional Type IV Hermitian symmetric domain. By general results of Griffiths [Gri68], there is a holomorphic *period map* $P\colon F \to \mathbb{D}/\Gamma$ sending $[S] \in F$ to the line $H^{2,0}(S) \subset \{s, f\}^\perp \otimes \mathbb{C}$. This map is only well-defined mod Γ since the isometry $\{s, f\}^\perp \to I\!I_{2,10}$ is ambiguous up to post-composition by an element of Γ. We may now state the first theorem of the paper:

Theorem 1.1. *P is dominant.*

Remark 1.2. For surfaces S with $h^{2,0}(S) \geq 2$, the associated period map *cannot* be dominant due to Griffiths transversality. The general member $S \in F_{g,d}$ satisfies $h^{2,0}(S) = 1$ only when $(g, d) = (1, 1)$ or $(g, d) = (0, 2)$. In the latter case, the surfaces under consideration are elliptic K3 surfaces. By the Torelli theorem for K3 surfaces [PSS71, LP81], the period mapping gives an isomorphism onto the corresponding period space.

A *local*, respectively *infinitesimal*, Torelli theorem verifies the local injectivity of P, respectively injectivity of dP, at some point. Such a result implies that P is generically finite onto its image. A *generic* Torelli theorem further proves that P is generically one-to-one onto its image. Finally, a *global* Torelli theorem implies that P is an embedding, or an isomorphism if the dimensions are appropriate. We prove that, unlike for K3 surfaces,

Theorem 1.3. $\deg P > 1$. *Thus, generic Torelli is false for $P\colon F \to \mathbb{D}/\Gamma$.*

Remark 1.4. By a result of Lönne [Lö2], the monodromy representation for the universal family over F is the subgroup of $O(I\!I_{2,10})$ preserving the connected component of $\mathbb{D}$, so P does not factor through $\mathbb{D}/\Gamma'$ for any subgroup $\Gamma' \subset \Gamma$.

To prove Theorem 1.1, we employ a degeneration argument, similar to Friedman's proof [Fri84] of the Torelli theorem for K3 surfaces. First degenerate the base curve C to a nodal curve C_0 formed from gluing two points on $\mathbb{P}^1$. An elliptic fibration $S \to C$ may be degenerated to an elliptic fibration $S_0 \to C_0$, and the simplest case is when the fiber over the node of C_0 is smooth. Normalizing,

$$S_0^\nu = X \to \mathbb{P}^1 = C_0^\nu$$

is an elliptic fibration with $(g, d) = (0, 1)$ – that is, a rational elliptic surface. To reconstruct S_0 from X, we glue two smooth fibers X_p and X_q for $p, q \in \mathbb{P}^1$ in such a way that a section of $X \to \mathbb{P}^1$ is glued to form a section of $S_0 \to C_0$.

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The period map for such singular surfaces S_0 does not land in $\mathbb{D}/\Gamma$, but maps into the boundary divisor Δ of a toroidal extension $\mathbb{D}/\Gamma \hookrightarrow (\mathbb{D}/\Gamma)^{\mathrm{II}}$. It suffices to prove that the boundary period map $P^{\mathrm{II}}\colon \{\text{moduli of } S_0\} \to \Delta$ is dominant. We find an explicit surface S_0 for which any deformation of its period deforms its moduli. Thus, P^{II} has at least one fiber containing a 0-dimensional component, implying dominance of P^{II}, and in turn, P.

To prove Theorem 1.3, we describe a second type of degeneration of $S \to C$, to a fibration $S_0 \to C$ (here the base stays constant) whose generic fiber is a nodal curve. We analyze the limiting period mapping for these surfaces and prove that they too map dominantly into the boundary divisor Δ. Since two different degenerations dominate the same divisor Δ, we obtain that $\deg P > 1$.

Our method of proof suggests an interesting conjecture. Each surface $S \in F$ contains two natural elliptic curves meeting at a point: the unique representative of the canonical class K_S and the marked section curve s. The degenerations we employ in the proof leave one of these curves fixed and degenerate the other to a nodal curve. Conjecture 4.5 describes a birational involution of F, which commutes with the period mapping, and swaps the roles of the two natural elliptic curves.

History of the result

In 1983, M.-H. Saito [Sai83] claimed to prove the following infinitesimal Torelli theorem for elliptic surfaces: the differential dP is injective if the j-invariant map $j\colon C \to \mathbb{P}^1_j$ is non-constant, and $h^{2,0}(S) = g + d - 1 > 0$. However, in 2019, Ikeda [Ike19] found a four-dimensional family $\mathcal{B} \subset F_{1,1}$ for which $P\big|_{\mathcal{B}}$ has three-dimensional image, despite the general member of $\mathcal{B}$ having non-constant j-map. Thus, [Sai83] has a gap, but the proof still works when ω_S is basepoint free. Observe that $\omega_S \simeq \pi^*(L \otimes \omega_C)$ is basepoint free for all $S \in F_{g,d}$ when $g > 0$ and $d > 1$, and ω_S is basepoint free for generic $S \in F_{g,d}$ when $g > 1$ and $d = 1$. The only cases where ω_S fails to be basepoint free for generic S are $(g, d) = (1, 1)$ and $(g, d) = (0, 1)$. The latter is the case of rational elliptic surfaces, where the period map is trivial.

In 2020, R. Kloosterman [Klo22] independently proved that the infinitesimal Torelli theorem holds for elliptic surfaces with non-constant j-map when $d \neq 1$, or when $d = 1$ and $h^0(C, L) = 0$. The techniques generalized those of Kiĭ [Kiĭ78] and Lieberman-Wilsker-Peters [LWP77] from the $g = 0$ case. Conversely, Kloosterman conjectured [Klo22, Conj. 6.1] that when $d = h^0(C, L) = 1$, the infinitesimal Torelli theorem is false. But this condition holds at every point of $F_{1,1}$, so our Theorem 1.1 proves that Kloosterman's conjecture is, in fact, false.

Regarding a generic Torelli theorem, Chakiris [Cha82] proved that generic Torelli holds in the $g = 0$, $d \geq 2$ case. Recently, Shepherd-Barron [SB20] has generalized these results to a higher genus base: elliptic surfaces $S \to C$ with $q = h^{1,0}(S)$ and $p_g = h^{2,0}(S)$ satisfying the bounds $4p_g > 5(q - 1)$, $p_g \geq q + 3$ also obey a generic Torelli theorem. By our Theorem 1.3, generic Torelli is false when $p_g = q = 1$. Hence, the second linear inequality $p_g \geq q + 3$ appears to be necessary for Shepherd-Barron's results to hold.

2. Type II$_b$ degenerations

Let $\pi_0\colon S_0 \to C_0$ be an elliptic fibration over an irreducible, nodal, arithmetic genus one curve C_0 with smooth fiber over the node, and $\chi_{\mathrm{top}}(S_0) = 12$. Such a fibration has a Weierstrass form $\{y^2 = 4x^3 - a_0x - b_0\}$ with $a_0 \in H^0(C_0, \mathcal{O}_{C_0}(4P_0))$ and $b_0 \in H^0(C_0, \mathcal{O}_{C_0}(6P_0))$ for some point $P_0 \in (C_0)_{\mathrm{sm}}$. See Figure 1.

Let $C_0 \hookrightarrow \mathcal{C}$ be a smoothing over $(B, 0)$ to a genus 1 curve, with smooth total space, and let $\mathcal{P}$ be an extension of P_0 to a section of $\rho\colon \mathcal{C} \to (B, 0)$. Then, for any $k > 0$, Cohomology and Base Change [Har77, III.12.11] implies that $\rho_*\mathcal{O}_{\mathcal{C}}(k\mathcal{P})$ is a rank k vector bundle over B. In particular, a_0, b_0 extend locally to sections a, b of $\rho_*\mathcal{O}_{\mathcal{C}}(4\mathcal{P})$, $\rho_*\mathcal{O}_{\mathcal{C}}(6\mathcal{P})$, and so we can smooth the elliptic fibration $S_0 \hookrightarrow \mathcal{S}$ over $(B, 0)$. The resulting total space $\mathcal{S}$ is smooth with S_0 reduced normal crossings. The double locus D is the smooth elliptic curve fibering over the node of C_0.

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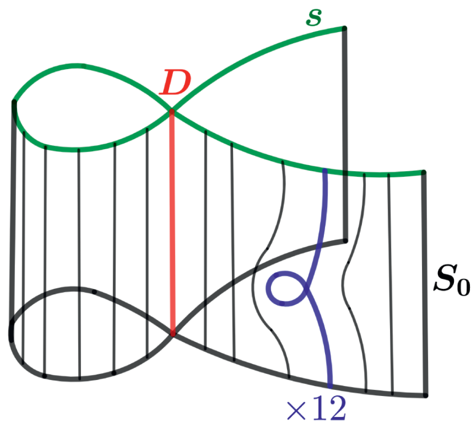

Figure 1. *A Type II_b surface S_0 with double locus D and section s.*

Definition 2.1. We call such a degeneration $\mathcal{S} \to \mathcal{C} \to (B, 0)$ a *Type II_b degeneration*, and we call the central fiber S_0 a *Type II_b elliptic surface*.

The subscript b indicates that the base degenerates. The terminology is motivated by a similar terminology in the classification of one-parameter degenerations of K3 surfaces due to Kulikov and Persson-Pinkham [Kul77, PP81]. They classify their $K_{\mathcal{S}}$-trivial, reduced normal crossing degenerations into Types I, II, III depending on the depth of the singularity stratification of S_0. Here, we instead have $K_{\mathcal{S}} = \mathcal{O}_{\mathcal{S}}(\mathcal{F})$ for a relative fiber $\mathcal{F} \to (B, 0)$.

As a reduced normal crossing degeneration, the Picard-Lefschetz transformation $T\colon H^2(S_t, \mathbb{Z}) \to H^2(S_t, \mathbb{Z})$ is unipotent and has a logarithm $N := \log T$. Furthermore, there is a formula for N which can be deduced from the Picard-Lefschetz transformation for a nodal degeneration of curves, or from [Cle69, Thm. 5.6].

Let $\gamma_t \subset C_t$ denote the vanishing 1-cycle of the node of C_0. Since the fiber over the node of C_0 is smooth, the restriction of the elliptic fibration $\pi_t\colon S_t \to C_t$ to the curve γ_t is a topologically trivial 2-torus bundle. Trivialize it, and let α, β be oriented generators of the homology of some fiber. Define $u := [\gamma_t \times \alpha] \in H^2(S_t, \mathbb{Z})$, $v := [\gamma_t \times \beta] \in H^2(S_t, \mathbb{Z})$. Then,

Proposition 2.2. $N(x) = (x \cdot u)v - (x \cdot v)u$.

Here $u, v \in \{s, f\}^\perp$ because s, f are classes of line bundles on the total space $\mathcal{S}$, and hence monodromy-invariant. So the classes u, v determine a rank 2 isotropic lattice $I := (\mathbb{Z}u \oplus \mathbb{Z}v)^{\mathrm{sat}} \subset I\!I_{2,10}$.

Let U_I be the unipotent subgroup of $\mathrm{Stab}_\Gamma(I)$ acting trivially on I and $I^\perp/I$. From the theory of toroidal compactifications [AMRT75] (see also [Loo03, Sec. 1A], [AE23, Prop. 4.16] for the case of Type IV domains), the unipotent quotient

$$\mathbb{D}/U_I \hookrightarrow A_I$$

embeds as a punctured disk bundle inside a $\mathbb{C}^*$-bundle $A_I \to I^\perp/I \otimes \mathcal{E}$. Here $\mathcal{E}$ is the universal elliptic curve over $\mathbb{C} \setminus \mathbb{R}$ whose fiber over $\tau \in \mathbb{C} \setminus \mathbb{R}$ is the elliptic curve $\mathbb{C}/\mathbb{Z} \oplus \mathbb{Z}\tau$. Since $T \in U_I$ the period map P induces a holomorphic period map $B^* \to \mathbb{D}/U_I$.

We enlarge $A_I \hookrightarrow \overline{A}_I$ to a line bundle and define $(\mathbb{D}/U_I)^{\mathrm{II}}$ as the closure of $\mathbb{D}/U_I$ in $\overline{A}_I$. This closure is a holomorphic disk bundle over $I^\perp/I \otimes \mathcal{E}$. The nilpotent orbit theorem [Sch73, Thm. 4.9] (the case at hand follows as in [Fri84, Thm. 4.2]) implies that the period map from B^* extends to a holomorphic map $P\colon (B, 0) \to (\mathbb{D}/U_I)^{\mathrm{II}}$ sending 0 into the boundary divisor $\Delta := \overline{A}_I \setminus A_I$. As the zero-section of

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the line bundle, the boundary divisor is naturally isomorphic to

$$\Delta \simeq I^{\perp}/I \otimes \mathcal{E}.$$

Note that $I^{\perp}/I$ is an even, negative-definite, unimodular lattice of rank 8, which uniquely determines it to be $I^{\perp}/I = E_8$.

There is also a direct construction of the period point $P(0) \in E_8 \otimes \mathcal{E}$ from the singular surface S_0 described as follows. Let $X \to \mathbb{P}^1$ be the rational elliptic surface normalizing $S_0 \to C_0$ and denote the section and fiber classes again by s and f. Then $\{s, f\}^{\perp} \subset H^2(X, \mathbb{Z})$ is isomorphic to E_8. Let X_p and X_q be the two elliptic fibers glued to form the double locus D of S_0. A class $\gamma \in \{s, f\}^{\perp}$ defines a line bundle $\mathcal{L}_\gamma \in \mathrm{Pic}(X)$, and we declare

$$\psi_{S_0}(\gamma) := \mathcal{L}_\gamma\big|_{X_p} \otimes \mathcal{L}_\gamma\big|_{X_q}^{-1} \in E := \mathrm{Pic}^0(X_p), \tag{2.1}$$

where we have used the gluing isomorphism $X_p \to X_q$ to form the tensor product of these two restrictions.

Then ψ_{S_0} defines a homomorphism $\psi_{S_0} \in \mathrm{Hom}(E_8, E) \simeq E_8 \otimes E$. Fixing an identification of $\{s, f\}^{\perp}$ with a fixed copy of the E_8 lattice, then deforming S_0 in moduli of Type II$_b$ surfaces, we get a local holomorphic period map

$$P^{\mathrm{II}} \colon \mathrm{Def}_{S_0} \to \mathrm{Hom}(E_8, \mathcal{E}),$$

which is identical to the extension of P coming from the nilpotent orbit theorem. The equivalence of these two definitions of the period map follows from Carlson's description [Car85] of the mixed Hodge structure on S_0; see Section 6 and Proposition 6.6. From this description of the boundary period mapping, we see the following:

1. To prove that P is dominant, it suffices to show that P^{II} is dominant from the moduli of Type II$_b$ elliptic surfaces to $\mathrm{Hom}(E_8, \mathcal{E})$.
2. On Type II$_b$ surfaces, the period map P^{II} is constructed by comparing the restriction of a line bundle in $\{s, f\}^{\perp} \subset \mathrm{Pic}(X)$ to the two glued fibers.

Observe that (1) follows from the observation at the beginning of this section that every Type II$_b$ elliptic surface is smoothable to the interior of F, so the Zariski closure of $\mathrm{im}(P) \subset (\mathbb{D}/\Gamma)^{\mathrm{II}}$ must contain $\mathrm{im}(P^{\mathrm{II}})$.

3. Dominance of the period map

Fix a smooth cubic $D \subset \mathbb{P}^2$ and let $\gamma \in PGL_3(\mathbb{C})$ be generic. Then D and $\gamma(D)$ generate a pencil of cubics with 9 distinct base points. Blowing up at the nine base points $D \cap \gamma(D) = \{p_1, \ldots, p_9\}$ of this pencil, we get a rational elliptic surface $X \to \mathbb{P}^1$, together with an isomorphism $\gamma \colon D \to \gamma(D)$ between two of its fibers. The nine blow-ups give rise to nine exceptional sections $F_1, \ldots, F_9$ of the resulting elliptic fibration. Let $t \colon D \to D$ be an arbitrary translation and consider the surface S_0 which results from gluing our two fibers of $X \to \mathbb{P}^1$ by the isomorphism

$$\gamma \circ t \colon D \to \gamma(D).$$

This construction defines a family of singular surfaces $\mathcal{S} \to U$ over a Zariski open subset $U \subset PGL_3(\mathbb{C}) \times E$ where $E := \mathrm{Pic}^0(D)$.

A very general surface over (γ, t) does not have a section, as there are only countably many sections of $X \to \mathbb{P}^1$; for a sufficiently general translation t, none of these will glue to a section of the singular surface. Still, for all such surfaces, there is a period homomorphism $\psi_{S_0} \colon H^2(X, \mathbb{Z}) \to E$ defined by (2.1). It descends to the rank 9 quotient $L := H^2(X, \mathbb{Z})/\mathbb{Z}f$ because $f|_D = \mathcal{O}_D$ and $f|_{\gamma(D)} = \mathcal{O}_{\gamma(D)}$.

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There is a translation action of $t \in E$ on U given by $(\gamma_0, t_0) \mapsto (\gamma_0, t_0 \circ t) =: (\gamma_0', t_0')$. It acts on the period homomorphism as follows:

$$\psi_{S_0'}(v) = \psi_{S_0}(v) + (v \cdot f)t. \tag{3.1}$$

From this formula, we deduce that the dominance of the period map for Type II$_b$ elliptic surfaces is equivalent to dominance of the more general period map

$$PGL_3(\mathbb{C}) \times E \dashrightarrow \mathrm{Hom}(L, E). \tag{3.2}$$

Consider the codimension one subtorus of $\mathrm{Hom}(L, E)$ for which $\psi_{S_0}(h) = 0 \in E$, where h is the pullback of the hyperplane class on $\mathbb{P}^2$. The inverse image of this subtorus contains, as a component, the locus of (γ, t) for which $t = 0$, because under a projective linear identification γ, we have $\gamma^*\mathcal{O}_{\gamma(D)}(1) = \mathcal{O}_D(1)$. Thus, the dominance of (3.2) is implied by the dominance of

$$PGL_3(\mathbb{C}) \dashrightarrow \mathrm{Hom}(H^2(X, \mathbb{Z})/\mathbb{Z}f + \mathbb{Z}h, E). \tag{3.3}$$

This follows because the action of $t \in E$ on $\mathrm{Hom}(L, E)$ described by (3.1) is translation by an elliptic subcurve transverse to the codimension 1 subtorus of $\mathrm{Hom}(L, E)$ appearing on the right-hand side of (3.3).

Finally, $\mathbb{Z}^9 \simeq \mathrm{span}\{F_i \,|\, i = 1, \ldots, 9\} = h^\perp$ surjects onto $H^2(X, \mathbb{Z})/\mathbb{Z}f + \mathbb{Z}h$. Pulling back the period map to this lattice, we get a map

$$\begin{aligned} PGL_3(\mathbb{C}) &\dashrightarrow \mathrm{Hom}(\mathbb{Z}^9, E)/\mathfrak{S}_9 \\ \gamma &\mapsto \{\psi_{S_0}(F_1), \ldots, \psi_{S_0}(F_9)\}. \end{aligned} \tag{3.4}$$

Here, the base points $D \cap \gamma(D)$, and hence the exceptional curves F_i, are not canonically ordered; they are permuted by the monodromy of the universal family. This is why we must quotient the target by the symmetric group $\mathfrak{S}_9$. Since $\sum_{i=1}^9 [F_i] = 3h - f$ in $H^2(X, \mathbb{Z})$, the image of the period map (3.4) lands in

$$\{(e_1, \ldots, e_9) \in E^9 \,|\, e_1 + \cdots + e_9 = 0\}/\mathfrak{S}_9 = A_8 \otimes E/W(A_8) \simeq \mathbb{P}^8.$$

The last isomorphism follows from a well-known theorem of Looijenga [Loo76]. Applying the definition of ψ_{S_0} gives a very explicit construction of (3.4):

Definition 3.1. Fix a smooth cubic $D \subset \mathbb{P}^2$. Define $E := \mathrm{Pic}^0(D)$ and let $A\colon \mathrm{Sym}^9 E \to E$ denote the addition map. For a generic $\gamma \in PGL_3(\mathbb{C})$, set $D \cap \gamma(D) = \{p_i\}_{i=1}^9$ and $q_i := \gamma^{-1}(p_i) \in D$. We define

$$\begin{aligned} \Psi\colon PGL_3(\mathbb{C}) &\dashrightarrow A^{-1}(0) \simeq \mathbb{P}^8 \\ \gamma &\mapsto \{\mathcal{O}_D(p_i - q_i)\}_{i=1}^9. \end{aligned} \tag{3.5}$$

Theorem 3.2. *The rational map Ψ from (3.5) is dominant. Thus, the period mapping for Type II$_b$ surfaces is dominant.*

Proof. Let $G \subset PGL_3(\mathbb{C})$ be the finite subgroup for which $g(D) = D$. We claim that Ψ extends, as a morphism, from U to $PGL_3(\mathbb{C}) \setminus G$. This is easy: the map Ψ extends continuously because $D \cap \gamma(D)$ is still a finite set for all $\gamma \in PGL_3(\mathbb{C}) \setminus G$. Normality of $PGL_3(\mathbb{C}) \setminus G$ implies that a continuous extension is algebraic.

We choose D and γ carefully so that the set $D \cap \gamma(D)$ has only three elements. Concretely, consider the extremal cubic pencil $X_{9111} \to \mathbb{P}^1_{[\lambda:\mu]}$ in the notation of [MP86], given by the equation

$$\lambda(x^2y + y^2z + z^2x) + \mu(xyz) = 0.$$

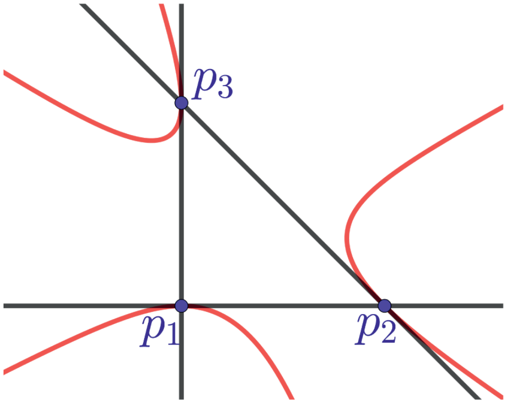

Figure 2. *The pencil generated by two cubics, shown in red and black, with set-theoretic base locus three blue points.*

See Figure 2. Let $D := D_{[\lambda:\mu]}$ be a generic fiber, and let $\gamma = \mathrm{diag}(1, \zeta_3, \zeta_3^2)$ where ζ_3 is a primitive third root of unity. Then $\gamma(D) = D_{[\zeta_3\lambda:\mu]}$, and so D and $\gamma(D)$ generate the pencil. The intersection multiset $D \cap \gamma(D)$ is $\{3p_1, 3p_2, 3p_3\}$ where

$$p_1 = [1 : 0 : 0],\ p_2 = [0 : 1 : 0],\ p_3 = [0 : 0 : 1].$$

Since this $\gamma \in PGL_3(\mathbb{C})$ fixes p_1, p_2, p_3, the period $\Psi(\gamma) = \{0, \ldots, 0\} \in \mathrm{Sym}^9 E$ vanishes. To prove that Ψ is dominant, it suffices to show that there is no small deformation $\gamma' \in PGL_3(\mathbb{C})$ of γ for which $\Psi(\gamma') = \{0, \ldots, 0\}$.

Suppose, to the contrary, that there were. Since $\Psi(\gamma') = \{0, \ldots, 0\}$, every base point in $D \cap \gamma'(D)$ is fixed by γ'. If $|D \cap \gamma'(D)| \geq 4$, then γ' must fix a line in $\mathbb{P}^2$. This is impossible for a small deformation of γ, which has isolated fixed points. Conversely, $|D \cap \gamma'(D)| \geq 3$ because each of p_1, p_2, p_3 deforms to some fixed point of γ'. Hence, γ' fixes exactly three points p'_1, p'_2, p'_3. Furthermore, $D \cap \gamma'(D) = \{3p'_1, 3p'_2, 3p'_3\}$ as a multiset, again because γ' is near γ, and the map

$$PGL_3(\mathbb{C}) \setminus G \to \mathrm{Sym}^9(D)$$

sending $\gamma' \mapsto D \cap \gamma'(D)$ with multiplicities is continuous.

Since $\mathrm{mult}_{p'_i}(D \cap \gamma'(D)) \geq 2$, we deduce that γ' preserves the tangent direction $T_{p'_i}D$ and the corresponding tangent line L'_i. Thus, $\gamma' \in PGL_3(\mathbb{C})$ fixes the point $L'_i \cap L'_j \in \mathbb{P}^2$. But, as we noted before, γ' only fixes three points (this holds not just on D but in the ambient plane $\mathbb{P}^2$). Using that γ' is a small deformation of γ, we deduce that

$$L'_1 \cap L'_2 = p'_2,\ L'_2 \cap L'_3 = p'_3,\ L'_3 \cap L'_1 = p'_1.$$

Write $p'_i = p_i + t_i$ for a translation t_i. By the addition law on a cubic, we have

$$2p'_1 = -p'_2,\ 2p'_2 = -p'_3,\ 2p'_3 = -p'_1$$

from which we can conclude that $t_1 = (-2)^3 t_1$ i.e. t_1 is 9-torsion. But since t_i are small, we conclude that $t_1 = t_2 = t_3 = 0$ and so $p'_i = p_i$.

Thus, γ' fixes (p_1, p_2, p_3), implying that $\gamma' \in (\mathbb{C}^*)^2 \subset PGL_3(\mathbb{C})$ lies in the maximal torus associated to the coordinates $[x : y : z]$. Furthermore, γ' preserves the base locus scheme $D \cap \gamma'(D)$, as this is the unique subscheme of D which has length 3 at each of p_1, p_2, p_3. So γ' induces an automorphism of the pencil generated by D and $\gamma'(D)$. Since the automorphism group of a rational elliptic surface is

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discrete, and γ' is a small deformation of γ, the automorphism γ' must have order 3. But no nontrivial small deformation of $\gamma = \mathrm{diag}(1, \zeta_3, \zeta_3^2)$ *within the torus* $(\mathbb{C}^*)^2$ has order 3. This is a contradiction. □

Remark 3.3. Our original proof of Theorem 3.2 checked by computer that $d\Psi$ was nondegenerate for an explicitly chosen D and γ.

Proof of Theorem 1.1. By the discussion at the end of Section 2, P is dominant if P^{II} is. The latter follows from Theorem 3.2. □

4. Type II$_f$ degenerations

We consider in this section degenerations of $S \to C$ that keep the base C constant. These are never of Type II$_b$ because in all such degenerations, $j(C) \to \infty$.

Take a one-parameter deformation of $a, b \in H^0(C, \mathcal{O}_C(4p)), H^0(C, \mathcal{O}_C(6p))$ over $(B, 0)$ until the discriminant $4a_0^3 + 27b_0^2 = 0 \in H^0(C, \mathcal{O}_C(12p))$ vanishes identically. For instance, we can take the fiber over $0 \in B$ to be

$$y^2 = x^3 - 3r^2x + 2r^3$$

with $r \in H^0(C, \mathcal{O}_C(2p))$. The degeneration

$$\overline{\mathcal{S}} \to C \times B \to (B, 0)$$

of elliptic surfaces has a central fiber $\overline{S}_0 \to C$ whose generic fiber is irreducible nodal, with two cuspidal fibers over the zeroes of r. In particular, the normalization $\overline{S}_0^\nu := X \to C$ is the smooth $\mathbb{P}^1$-bundle $X = \mathbb{P}_C(\mathcal{O} \oplus L)$, and $\overline{S}_0$ is reconstructed from gluing a bisection D of $X \to C$, branched over the two zeroes of r. This bisection D is glued along the involution switching the two sheets of $\nu\colon D \to C$.

For future reference, note that $\mathrm{NS}(X) \simeq H^2(X, \mathbb{Z})$ is spanned by the $\mathbb{P}^1$-fiber class f and the class of the section $s_\infty = \mathbb{P}_C(\mathcal{O} \oplus 0)$, with intersection form

$$f \cdot f = 0, \quad s_\infty \cdot f = 1, \quad s_\infty \cdot s_\infty = -1,$$

and $K_X = -f - 2s_\infty$. The other natural section $s_0 = \mathbb{P}_C(0 \oplus L)$ has class $f + s_\infty$.

The bisection $D \subset X$ has genus 2, being a double cover of C branched over two points. Thus, its cohomology class is $[D] = 2f + 2s_\infty = -K_X + f = 2s_0$. Note that $[D]^2 = 4$ and $[D] \cdot K_X = -2$. The section s that is present on the smooth surfaces in the family $\mathcal{S}$ limits to s_∞, which is the unique section of X disjoint from D.

Proposition 4.1. *Generically, two singular fibers limit to each cuspidal fiber of* $\overline{S}_0$. *The limits of the remaining eight singular fibers lie over a degree* 8 *divisor in C. The only restriction on this divisor is that it is linearly equivalent to* $8p$.

Proof. Consider a deformation of the Weierstrass equation

$$y^2 = x^3 - (3r^2 + \epsilon g_4)x + (2r^3 + \epsilon g_4 r + \epsilon^2 g_6),$$

where $g_d \in H^0(C, \mathcal{O}_C(dp))$ has degree d. The discriminant $\Delta = 4a^3 + 27b^2$ is

$$\Delta = 9r^2(12rg_6 - g_4^2)\epsilon^2 + \mathcal{O}(\epsilon^3).$$

Thus, the Zariski closure of the discriminant divisor is

$$\lim_{\epsilon \to 0} \mathrm{div}(\Delta) = 2 \cdot \mathrm{div}(r) + \mathrm{div}(12rg_6 - g_4^2).$$

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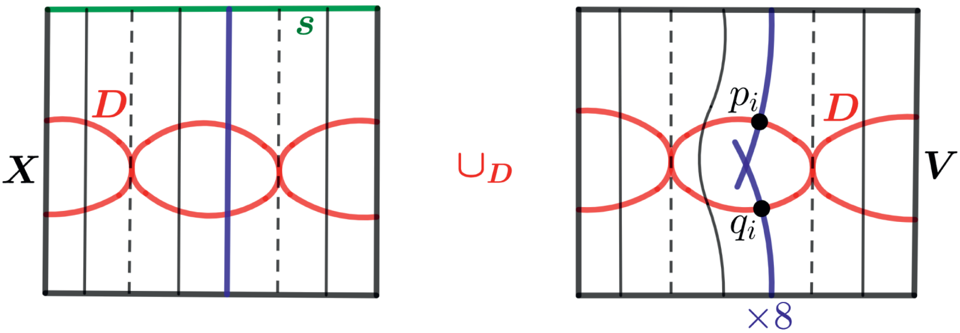

Figure 3. *A Type II_f surface $S_0 = X \cup_D V$ with the genus 2 double locus D shown in red, the section s in green, limits of 8 nodal fibers in blue, and limits of pairs of nodal fibers dashed.*

For fixed r, the sections rg_6 form a linear subspace $\mathbb{P}^5 \subset \mathbb{P}^7 = \mathbb{P}H^0(C, \mathcal{O}(8p))$ of codimension 2. The sections $g_4^2 \in \mathbb{P}H^0(C, \mathcal{O}(8p))$ are the image of the degree 2 Veronese embedding, followed by a linear projection

$$v_2 \colon \mathbb{P}^3 \hookrightarrow \mathbb{P}^9 = \mathbb{P}\mathrm{Sym}^2 H^0(C, \mathcal{O}(4p)) \dashrightarrow \mathbb{P}^7.$$

The inverse image of $\{\mathrm{div}(rg_6)\} = \mathbb{P}^5 \subset \mathbb{P}^7$ is a copy of $\mathbb{P}^7 \subset \mathbb{P}^9$ under the linear projection. Thus, the vanishing loci of linear combinations are represented geometrically as the join of the projective subvarieties $v_2(\mathbb{P}^3)$, $\mathbb{P}^7 \subset \mathbb{P}^9$. This join is all of $\mathbb{P}^9$. Thus, we can realize any divisor in $|8p|$ as $\lim_{\epsilon \to 0} \mathrm{div}(\Delta) - 2 \cdot \mathrm{div}(r)$. □

For general g_4 and g_6, the punctured family over $B \setminus 0$ has smooth total space. The threefold $\overline{\mathcal{S}}$ is a double cover branched over the vanishing locus of the cubic $x^3 - (3r^2 + \epsilon g_4)x + (2r^3 + \epsilon g_4 r + \epsilon^2 g_6)$, so it can only be singular where two of the roots of the cubic coincide. This shows that the singular locus $\overline{\mathcal{S}}_{\mathrm{sing}} \subset V(y, x - r, \epsilon)$ is contained in the singularities of the fibers of $\overline{S}_0 \to C$.

Since $\epsilon^2 \parallel \Delta$, the local equation of the double cover is generically $y^2 = u^2 + \epsilon^2$ along the nodes of $\overline{S}_0 \to C$. So the nodes form a family of A_1-singularities in $\overline{\mathcal{S}}$. At the nodes on the fibers lying over $\mathrm{div}(12rg_6 - g_4^2)$, the local equation is rather $y^2 = u^2 + v\epsilon^2$. Thus, to find a semistable model $\mathcal{S} \to (B, 0)$, we simply blow up the double locus of $\overline{S}_0$ in the total space $\overline{\mathcal{S}}$.

The resulting central fiber is $S_0 = X \cup_D V$ for a ruled surface $V \to C$, which contains D as a bisection and has 8 reducible fibers over the points in $\mathrm{div}(12rg_6 - g_4^2)$; see Figure 3. Thus, $V \sim Bl_{p_1,\ldots,p_8} X$ is deformation-equivalent to the blow-up of X at 8 points on D, with the double locus on V identified with D via the strict transform. It is only deformation-equivalent because $V \to C$ could be the projectivization of a non-split extension of L by $\mathcal{O}$. Regardless, we can identify

$$H^2(V, \mathbb{Z}) = H^2(X, \mathbb{Z}) \oplus_{i=1}^{8} \mathbb{Z}E_i$$

and $[D] = 2s_0 - [E_1] - \cdots - [E_8] = -K_V + f$.

Definition 4.2. We call the degeneration $\mathcal{S} \to C \times B \to (B, 0)$ a *Type II_f degeneration*, and we call the central fiber S_0 a *Type II_f elliptic surface*.

From Section 6 and Proposition 6.6, the mixed Hodge structure of a Type II_f surface has a period map to $E_8 \otimes \mathcal{E}$ which can be described as follows. Consider the sublattice $\{K_V, f\}^\perp \subset H^2(V, \mathbb{Z})$. This is isometric to the root lattice

$$D_8 = \{(a_1, \ldots, a_8) \in \mathbb{Z}^8 \,\big|\, a_1 + \cdots + a_8 \in 2\mathbb{Z}\}$$

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via the map $(a_1,\dots,a_8)\mapsto \sum_{i=1}^8 a_i[E_i]-\left(\frac{1}{2}\sum_{i=1}^8 a_i\right)f$. When this isometry is understood, we will refer to $\{K_V,f\}^\perp$ simply as D_8.

Let $E:=\mathrm{Pic}^0(D)/\mathrm{Pic}^0(C)$ be the Prym variety of the double cover $\nu\colon D\to C$. We define a period homomorphism

$$\begin{aligned}\psi_{S_0}\colon D_8&\to E\\ \gamma&\mapsto \mathcal{L}_\gamma\big|_D \bmod \mathrm{Pic}^0(C)\end{aligned}\tag{4.1}$$

by lifting an element $\gamma\in D_8$ to an element $\mathcal{L}_\gamma\in\mathrm{Pic}(V)$. These lifts form a $\mathrm{Pic}^0(C)$-torsor, and thus, the image of $\mathcal{L}_\gamma\big|_D\in\mathrm{Pic}^0(D)$ under the map to E is well-defined.

Remark 4.3. The period point $\psi_{S_0}\in\mathrm{Hom}(D_8,E)$ determines, up to a finite isogeny, the period point in $E_8\otimes E$. The extensions of an element of $\mathrm{Hom}(D_8,E)$ to an element of $\mathrm{Hom}(E_8,E)$ are a torsor over $\mathrm{Hom}(E_8/D_8,E)=E[2]$.

Proof of Theorem 1.3. To show $\deg P>1$, it suffices to prove that the moduli of Type II_f surfaces (appearing as limits of elliptic surfaces in F) also dominate the boundary divisor Δ. This follows from Theorem 4.4 below. □

Theorem 4.4. *The period mapping for Type II_f surfaces is dominant.*

Proof. The period point ψ_{S_0} and limit mixed Hodge structure of $\mathcal{S}$ are encoded, up to a finite map, in the data $(\nu\colon D\to C,\{r_i\}_{i=1}^8)$ consisting of

1. a degree 2 map $\nu\colon D\to C$ from a genus 2 to a genus 1 curve, and
2. a multiset of 8 points $\{r_1,\dots,r_8\}\subset C$.

Let $\iota\colon D\to D$ be the involution switching the sheets of ν and let $\{p_i,q_i\}=\nu^{-1}(r_i)$. Then $\mathcal{O}_D(p_i-q_i)\in\mathrm{Pic}^0(D)$ gives, upon quotienting by $\mathrm{Pic}^0(C)$, the period

$$\psi_{S_0}(F_i-F_i')=[\mathcal{O}_D(p_i-q_i)]\in E,$$

where F_i+F_i' is a reducible fiber of the ruling $V\to C$. Ranging over the eight reducible fibers, the tuple

$$(\mathcal{O}_D(p_i-q_i)\bmod \mathrm{Pic}^0(C))_{i=1}^8\in E^8$$

encodes ψ_{S_0} up to torsion because $\bigoplus_{i=1}^8\mathbb{Z}(F_i-F_i')\subset D_8$ has finite index.

Let $\{r_9,r_{10}\}\in C$ be the branch points of ν. Then ν is determined by the monodromy representation $\rho\colon\pi_1(C\setminus\{r_9,r_{10}\},*)\to\mathbb{Z}_2$. Let $\mathrm{Prym}^2\mathcal{C}$ be the moduli space of Prym data $(C,\{r_9,r_{10}\},\rho)$ over the universal genus 1 curve $\mathcal{C}\to\mathcal{M}_1$. It is a Deligne-Mumford stack of dimension 2, one dimension for $j(C)$ and another for the element $r_9-r_{10}\in\mathrm{Pic}^0(C)$, well-defined up to sign. The data of ρ is finite.

A point $r_i\in C$ determines p_i up to switching $p_i\leftrightarrow q_i$ which acts by negation on the image of $\mathcal{O}_D(p_i-q_i)$ in E. Thus, we globally get a well-defined map

$$\begin{aligned}\Psi\colon \mathrm{Sym}^8\mathcal{C}\times_{\mathcal{M}_1}\mathrm{Prym}^2\mathcal{C}&\to\mathbb{Z}^8\otimes\mathcal{E}/\mathfrak{S}_8^\pm\\ (C,\{r_1,\dots,r_8\},\{r_9,r_{10}\},\rho)&\mapsto\{\mathcal{O}_D(p_i-q_i)\bmod\mathrm{Pic}^0(C)\}_{i=1}^8,\end{aligned}\tag{4.2}$$

where $\mathcal{E}$ is the universal elliptic curve. Since the image of each $\mathcal{O}_D(p_i-q_i)$ in E is only well-defined up to sign, and the reducible fibers of $V\to C$ are unordered, we must quotient the target by the signed permutation group $\mathfrak{S}_8^\pm$.

Observe that $\mathrm{Sym}^8\mathcal{C}\times_{\mathcal{M}_1}\mathrm{Prym}^2\mathcal{C}$ is ten-dimensional. There is a single condition ensuring that a point in the domain of Ψ arises from a degeneration of surfaces in F: If $L\to C$ is the Hodge bundle, then

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$r_9 + r_{10} \in |2L|$ and so by Proposition 4.1, $\{r_1, \dots, r_8\}, \{r_9, r_{10}\}$ can arise so long as $r_1 + \cdots + r_8 \in |8L|$ (i.e., the relation

$$r_1 + \cdots + r_8 - 4(r_9 + r_{10}) = 0 \in \mathrm{Pic}^0(C) \tag{4.3}$$

is satisfied). So the Type II_f limits of degenerations from F are described by

$$Z = \{\text{elements of } \mathrm{Sym}^8 \mathcal{C} \times_{\mathcal{M}_1} \mathrm{Prym}^2 \mathcal{C} \,\big|\, r_1 + \cdots + r_8 - 4(r_9 + r_{10}) = 0\}.$$

Our goal is to prove the dominance of the map $\Psi\big|_Z : Z \to \mathbb{Z}^8 \otimes \mathcal{E}/\mathfrak{S}_8^{\pm}$.

Fix an elliptic curve fiber E of $\mathcal{E}$, consider the point $\{0, \dots, 0\} \in \mathrm{Sym}^8 E$, and let $\ker_E(\Psi) := \Psi^{-1}(\{0, \dots, 0\})$. It suffices to prove that $Z \cap \ker_E(\Psi)$ contains, as a component, some zero-dimensional scheme. Let $L_E \subset \mathrm{Prym}^2\mathcal{C}$ be the sublocus of Prym data whose Prym variety is E. It is a curve inside the surface $\mathrm{Prym}^2\mathcal{C}$. Then, $\ker_E(\Psi)$ contains, as a component, an unramified double cover $M_E \to L_E$ on which $r = r_1 = \cdots = r_8$ and $r \in \{r_9, r_{10}\}$ because the morphism $D \to E$ sending $p \mapsto \mathcal{O}_D(p - \iota(p)) \bmod \mathrm{Pic}^0(C)$ is surjective.

The defining equation (4.3) of Z restricts to M_E to give the equation

$$4(r_9 - r_{10}) = 0 \in \mathrm{Pic}^0(C)$$

(i.e., $r_9 - r_{10} \in \mathrm{Pic}^0(C)[4]$). The locus in L_E on which $r_9 - r_{10}$ is 4-torsion is finite and nonempty. So the theorem follows. □

The proofs of Theorems 3.2 and 4.4 suggest a rather wild conjecture:

Conjecture 4.5. *$F_{1,1}$ admits a period-preserving birational involution $S \leftrightarrow S'$ for which $j(C) = j(F')$ and $j(F) = j(C')$. Here, C, C' are the bases and F, F', are the canonical fibers. Furthermore, S and S' are moduli spaces of stable vector bundles on each other of rank 2, determinant $\mathcal{O}(s)$, and $c_2 = \mathrm{pt}$. A Fourier-Mukai transform induces an isomorphism of their integral Hodge structures.*

The existence of such a birational involution would give a geometric explanation for why degenerations of Type II_b and II_f can have the same periods, even though $j(C) \to \infty$ in the former, while $j(F) \to \infty$ in the latter.

5. A family losing dimension

Let $F^{\mathrm{cusp}} \hookrightarrow F$ be the closure of the sublocus of elliptic fibrations $S \to C$ which have six cuspidal (Kodaira type II) fibers. These fibrations are isotrivial and have a Weierstrass form $y^2 = x^3 + b$ for some $b \in H^0(C, \mathcal{O}_C(6p))$. There is a fiber preserving automorphism $\sigma : S \to S$, given by

$$\sigma : (x, y) \mapsto (\zeta_3 x, -y),$$

and $\sigma^*\Omega_S = \zeta_6 \Omega_S$ acts nontrivially on the holomorphic 2-form by a primitive sixth root of unity. Furthermore, since σ preserves s and f, it defines an element $\sigma^* \in \Gamma = O(I\!I_{2,10})$ which is easily checked to fix only the origin of $I\!I_{2,10}$. So σ^* endows $I\!I_{2,10}$ with the structure of a Hermitian lattice of hyperbolic signature $(1, 5)$ over the Eisenstein integers $\mathbb{Z}[\zeta_6]$, and

$$\mathbb{B} := \mathbb{P}\{x \in I\!I_{2,10} \otimes \mathbb{C} \,\big|\, x \cdot \overline{x} > 0,\ \sigma^* x = \zeta_6 x\} \subset \mathbb{D}$$

is a Type I Hermitian symmetric subdomain (a complex ball), of dimension 5. Letting $\Gamma_0 := \{\gamma \in \Gamma \,\big|\, \gamma \circ \sigma^* = \sigma^* \circ \gamma\}$ be the group of Hermitian isometries, we get a period map to a 5-dimensional ball quotient

$$F^{\mathrm{cusp}} \to \mathbb{B}/\Gamma_0.$$

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But $\dim F^{\text{cusp}} = 1 + 5 = 6$ with parameters corresponding to $j(C)$ and the relative locations of the six cuspidal fibers. Thus, $P\big|_{F^{\text{cusp}}}$ has positive fiber dimension.

It seems likely that $P\big|_{F^{\text{cusp}}}$ is surjective, with generic fiber dimension 1. Regardless, this gives a second example, after Ikeda's [Ike19], proving that P is *not* a finite map, even though it is generically finite by Theorem 1.1:

Corollary 5.1. *P is not finite.*

6. Mixed Hodge Structures

MHS of a normal crossings surface

Let S_0 be a reduced normal crossings surface with smooth double locus and no triple points. Our goal in this section is to explicitly describe the mixed Hodge structure on $H^2(S_0)$. Let $S_0 = \bigcup_{i=1}^m S_i$ with the double curve $D_{ij} = S_i \cap S_j$ a smooth, possibly disconnected or empty curve for all $i < j$. Let $D := \bigcup_{i<j} D_{ij}$. The Mayer-Vietoris sequence associated to a covering of S_0 by neighborhoods of the irreducible components S_i reads

$$\bigoplus_{i=1}^m H^1(S_i) \xrightarrow{\iota^*} \bigoplus_{i<j} H^1(D_{ij}) \to H^2(S_0) \to \bigoplus_{i=1}^m H^2(S_i) \xrightarrow{\text{res}} \bigoplus_{i<j} H^2(D_{ij}). \tag{6.1}$$

Here, ι^* and res are signed restriction maps. Let $K \subset \bigoplus H^2(S_i)$ be the kernel of the morphism res – that is, $K = \{(\alpha_i \in H^2(S_i)) \,\big|\, \alpha_i \cdot D_{ij} = \alpha_j \cdot D_{ij}\}$. Define

$$J := \operatorname{coker}(\iota^*).$$

By exactness of the sequence (6.1), we obtain a short exact sequence

$$0 \to J \to H^2(S_0) \to K \to 0.$$

In fact, it is a short exact sequence of mixed Hodge structures with left-hand term J pure of weight 1, and the right-hand term K pure of weight 2.

Proposition 6.1. *If $p_g(S_i) = 0$ for all components $S_i \subset S_0$ (equivalently, K is Hodge-Tate of weight 2), then the Carlson classifying map [Car85]*

$$\phi : K \to \operatorname{Jac}(J)$$

of the extension coincides with the Abel-Jacobi map. More precisely, an element of K is a tuple $(\alpha_i \in H^2(S_i, \mathbb{Z}))$ represented by line bundles $\mathcal{L}_i$ such that for each $i < j$, we have $c_1(\mathcal{L}_i|_{D_{ij}}) - c_1(\mathcal{L}_j|_{D_{ij}}) = 0 \in H^2(D_{ij})$. Then $\phi = \pi \circ \mathrm{AJ} \circ \psi$, where

$$(\alpha_i \in H^2(S_i, \mathbb{Z})) \overset{\psi}{\mapsto} \bigoplus\nolimits_{i<j} \mathcal{L}_i|_{D_{ij}} \otimes \mathcal{L}_j|_{D_{ij}}^{-1} \in \operatorname{Pic}^0(D),$$

$\mathrm{AJ}\colon \operatorname{Pic}^0(D) \to \operatorname{Jac}(D) = \operatorname{Jac}(H^1(D))$ *is the classical Abel-Jacobi isomorphism, and* $\pi\colon \operatorname{Jac}(D) \to \operatorname{Jac}(J)$ *is the projection map.*

Proof. Following Carlson's construction, the classifying map ϕ for a weight separated extension of mixed Hodge structures is given by the composition of two splittings. First, choose a left-splitting $a : H^2(S_0) \to J$ over $\mathbb{Z}$. Next, choose a right-splitting $b : K \to F^1H^2(S_0)_{\mathbb{C}}$ over $\mathbb{C}$, which respects the Hodge filtration. The composition $a_{\mathbb{C}} \circ b : K \to J_{\mathbb{C}}$ gives the classifying map after passing to the Jacobian quotient:

$$\phi : K \to J_{\mathbb{C}}/(J_{\mathbb{Z}} + F^1 J_{\mathbb{C}}).$$

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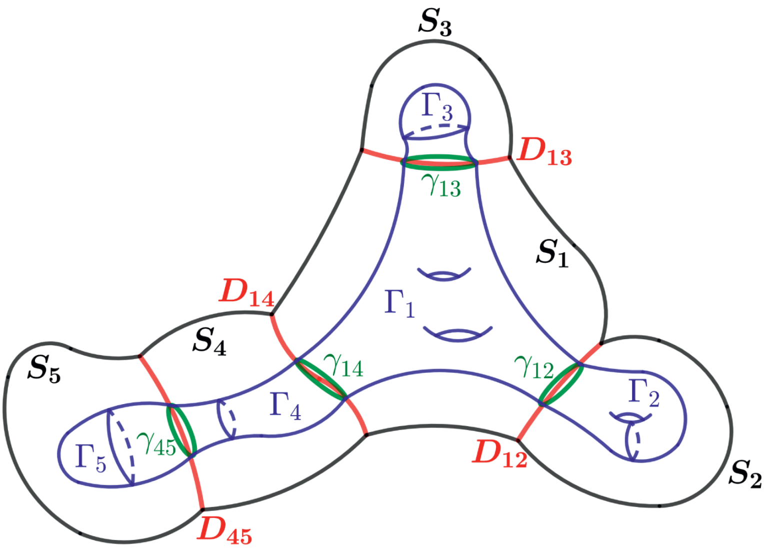

Figure 4. *Heuristic diagram of irreducible components* S_i *in black, double curves* D_{ij} *in red,* 1*-cycles* $\gamma_{ij} \subset D_{ij}$ *in green, and* 2*-cycles* $\Gamma_i \subset S_i$ *capping the* 1*-cycles in blue.*

For a, it suffices to produce a morphism on homology $\ker(\iota_*) \to H_2(S_0)$, and then use the universal coefficient theorem to give a map in the opposite direction:

$$H^2(S_0) \to H_2(S_0)^* \to \ker(i_*)^* \simeq \operatorname{coker}(\iota^*) = J.$$

To define the morphism $\ker(\iota_*) \to H_2(S_0)$, choose a basis for $\ker(\iota_*)$ at the singular chain level: tuples of 1-cycles $t_k = (\gamma_{ij}^k \in \mathcal{Z}_1(D_{ij}))$ such that for each i,

$$\sum_j \iota_*(\gamma_{ij}^k) = \partial(\Gamma_i^k) \text{ for some } \Gamma_i^k \in \mathcal{C}_2(S_i).$$

We use the convention that $\gamma_{ij} = -\gamma_{ji}$. Choosing such Γ_i^k for each t_k in the basis of $\ker(i_*)$, we construct a 2-cycle (see Figure 4),

$$T_k = \bigcup_i \Gamma_i^k \in \mathcal{Z}_2(S_0).$$

We take the 1-cycles γ_{ij}^k to be $\mathbb{Z}$-linear combinations of some fixed $2g(D_{ij})$ loops on each D_{ij}, whose union we call γ, chosen so that their complement in D_{ij} is a contractible $4g$-gon. The assignment $t_k \mapsto [T_k] \in H_2(S_0)$ then induces a splitting

$$a : H^2(S_0) \to J.$$

To construct a splitting b, we use the Čech-de Rham model of $H^2(S_0, \mathbb{C})$, and its Hodge filtration F^1. An element of $H^2(S_0, \mathbb{C})$ is represented by two tuples of differential forms:

$$(\omega_i \in \mathcal{Z}^2(S_i))_i \text{ and } (\theta_{ij} \in \mathcal{A}^1(D_{ij}))_{i<j}$$

such that for all $i < j$, we have $\omega_i|_{D_{ij}} - \omega_j|_{D_{ij}} = d\theta_{ij}$. If furthermore, $\theta_{ij} \in \mathcal{A}^{1,0}(D_{ij})$ for all $i < j$, then the element lies in $F^1H^2(S_0, \mathbb{C})$.

Given $(\alpha_i) \in K = \ker(\mathrm{res})$, we know that $\alpha_i|_{D_{ij}} - \alpha_j|_{D_{ij}} = 0 \in H^2(D_{ij})$. To define $b : K \to F^1H^2(S_0, \mathbb{C})$, select a basis for K; for each basis element $(\alpha_i) \in K$, there exists line bundles $\mathcal{L}_i$ such that $c_1(\mathcal{L}_i) = \alpha_i$. Since each S_i is projective, we may assume that the $\mathcal{L}_i \simeq \mathcal{O}_{S_i}(C_i - C_i')$, where C_i and C_i' are ample effective curves on S_i meeting each D_{ij} transversely away from γ. We take $\omega_i \in \mathcal{Z}^2(S_i)$ representing $c_1(\mathcal{L}_i)$ and supported on a small neighborhood of $C_i \cup C_i'$. Since $\omega_i|_{D_{ij}} - \omega_j|_{D_{ij}} \in \mathcal{Z}^2(D_{ij})$ integrates to 0, it has a $\overline{\partial}$-primitive $\theta_{ij} \in \mathcal{A}^{1,0}(D_{ij})$, unique up to the addition of a holomorphic one-form.

To interpret the composition $\phi = a_{\mathbb{C}} \circ b : K \to J_{\mathbb{C}}$, we will regard $J_{\mathbb{C}}$ as $\mathrm{Hom}(\ker(\iota_*), \mathbb{C})$. Then $(a_{\mathbb{C}} \circ b)(\alpha_i)$ is the unique homomorphism $\ker(\iota_*) \to \mathbb{C}$ which sends t_k to

$$\sum_{i=1}^{m} \int_{\Gamma_i^k} \omega_i + \sum_{i<j} \int_{\gamma_{ij}^k} \theta_{ij}. \tag{6.2}$$

We henceforth drop the index k as we will consider a single basis vector $t = t_k$.

We will make two simplifications in order to compare ϕ with the Abel-Jacobi map. First, the chains Γ_i can be replaced with $\Gamma_i + x_i$ for any $x_i \in \mathcal{Z}_2(S_i)$ such that the tuple of homology classes (x_i) is Poincaré dual to an element of K. By Lefschetz duality, there is a perfect pairing associated to the 4-manifold with boundary

$$I : H_2(S_i - N_\epsilon(\gamma)) \times H_2(S_i - N_\epsilon(\gamma), \partial) \to \mathbb{Z},$$

and we have $\int_{\Gamma_i} \omega_i = I(C_i - C_i', \Gamma_i) \in \mathbb{Z}$. Since (α_i) is primitive in K, one can find $x \in K$ such that

$$I(C_i - C_i', x) = -I(C_i - C_i', \Gamma_i).$$

So replacing Γ_i with $\Gamma_i + x_i$, we may assume that the first sum in (6.2) vanishes.

Second, the primitives θ_{ij} are not closed, so the second integral does not make sense on the homology classes $[\gamma_{ij}^k]$. To remedy this, we construct smooth 1-forms $\lambda_{ij} \in \mathcal{Z}^1(D_{ij})$ supported away from γ such that $d(\theta_{ij} + \lambda_{ij}) = 0$. Let ℓ_{ij} be a smooth 1-chain on $D_{ij} \setminus \gamma$ with boundary the signed intersection points:

$$\partial \ell_{ij} = (C_i - C_i') \cap D_{ij} - (C_j - C_j') \cap D_{ij}.$$

By Lemma 6.2 below, we may produce a form λ_{ij} supported in a neighborhood of ℓ_{ij}. This allows us to write the Carlson map for our extension as

$$\phi((\alpha_i)) = \left[t \mapsto \sum_{i<j} \int_{\lambda_{ij}} (\theta_{ij} + \lambda_{ij}) \right] \in J_{\mathbb{C}}/(J_{\mathbb{Z}} + F^1 J_{\mathbb{C}}).$$

But for any $\tau \in \Omega^1(D_{ij})$, since $\theta_{ij} \in \mathcal{A}^{1,0}(D_{ij})$ we have, again by Lemma 6.2,

$$\int_{D_{ij}} (\theta_{ij} + \lambda_{ij}) \wedge \tau = \int_{D_{ij}} \lambda_{ij} \wedge \tau = \int_{\ell_{ij}} \tau.$$

Observe that the classical Abel-Jacobi map $\mathrm{AJ} \colon \mathrm{Pic}^0(D) \to \mathrm{Jac}(D)$ indeed sends $[\partial \ell_{ij}] \mapsto \int_{\ell_{ij}}$. The proposition follows. □

Now, we produce the one-form λ_{ij} with the desired properties.

Lemma 6.2. *Let C be a Riemann surface and let $\mathcal{L} = \mathcal{O}_C(q - p)$. There is a hermitian metric h on $\mathcal{L}$, a $(1,0)$-form $\theta \in \mathcal{A}^{1,0}(C)$, and a smooth 1-form λ supported in a neighborhood of a path ℓ from p to q for which*

1. $\overline{\partial}\theta = \frac{i}{2\pi}\partial\overline{\partial}\log(h)$,
2. $d\lambda = -\overline{\partial}\theta$, *and*
3. $\int \lambda \wedge \tau = \int_\ell \tau$ *for any holomorphic one-form* τ.

Proof. Let z be a chart to $\mathbb{C}$ from a neighborhood of ℓ. There exists a function $f\colon C \setminus \{p, q\} \to \mathbb{C}^*$ of the following form:

$$f = \begin{cases} \frac{z-q}{z-p} & \text{if } z \in N_{\epsilon/2}(\ell) \\ \text{smooth interpolation} & \text{if } z \in N_{\epsilon/2}(\ell)^c \cap N_\epsilon(\ell) \\ 1 & \text{if } z \notin N_\epsilon(\ell). \end{cases}$$

Such a smooth interpolation exists because $\frac{z-q}{z-p}$ has winding number zero along the boundary of $N_{\epsilon/2}(\ell)$. Let $s \in \mathrm{Mero}(C, \mathcal{L})$ be a meromorphic section with a zero at q and a pole at p. Then, there is a hermitian metric h on $\mathcal{L}$ for which $h(s, \overline{s}) = |f|^2$. The associated curvature form is $\frac{i}{2\pi}\partial\overline{\partial}\log|f|^2$, and since $c_1(\mathcal{L}) = 0$, we can find a $(1,0)$-form θ satisfying (1). Furthermore, $\lambda = -\frac{i}{2\pi}(\overline{\partial}\log(f) - \partial\log(\overline{f}))$ is a $(0,1)$-form, supported in $A := N_{\epsilon/2}(\ell)^c \cap N_\epsilon(\ell)$ and satisfying (2).

It remains to check (3). We may write $\tau = dg$ for some holomorphic function $g\colon N_\epsilon(\ell) \to \mathbb{C}$. Applying Stokes's formula and the residue formula, we have

$$\begin{aligned} \int_C \lambda \wedge \tau &= -\frac{i}{2\pi}\int_A \overline{\partial}\log(f) \wedge dg = \frac{i}{2\pi}\int_A d(ig \cdot d\log(f)) = \frac{i}{2\pi}\int_{\partial A} ig \cdot d\log(f) \\ &= -\frac{i}{2\pi}\int_{\partial N_{\epsilon/2}(\ell)} g \cdot d\log\left(\tfrac{z-q}{z-p}\right) = -\frac{i}{2\pi}(2\pi i)(g(q) - g(p)) = \int_\ell \tau. \end{aligned}$$

□

More generally, the lemma holds for any degree zero line bundle $\mathcal{O}_C(\sum(q_i - p_i))$, for a union of paths connecting each pair of points p_i to q_i by taking the product of the hermitian metrics, and sum of the corresponding θ's and λ's.

Remark 6.3. To apply Lemma 6.2 to the proof of Proposition 6.1, our forms ω_i must be such that $\omega_i|_{D_{ij}} - \omega_j|_{D_{ij}}$ is the two-form $\frac{i}{2\pi}\partial\overline{\partial}\log(h)$ supported in a neighborhood of ℓ_{ij}. This is achieved by choosing $\omega_i = \frac{i}{2\pi}\partial\overline{\partial}\log(h_i)$ for hermitian metrics on h_i on $\mathcal{L}_i$ (and similarly for j) so that $h = h_i/h_j$ is the desired hermitian metric on $\mathcal{L}_i|_{D_{ij}} \otimes \mathcal{L}_j|_{D_{ij}}^{-1}$. Note though that we must allow the two-form ω_i to be supported in a tubular neighborhood of $C_i \cup C_i' \cup \ell_{ij}$ rather than just $C_i \cup C_i'$. Since ℓ_{ij} is disjoint from γ, the argument of Lemma 6.1 is unaffected.

Clemens-Schmid sequence

Let $\mathcal{S} \to (B, 0)$ be a degeneration of projective surfaces with smooth total space and reduced normal crossings central fiber $S_0 = \bigcup_{i=1}^m S_i$ with smooth double locus. Assume, furthermore, that $p_g(S_i) = 0$ for all i.

The monodromy is unipotent by Clemens [Cle69]. So let N be the nilpotent logarithm of the monodromy operator on $H^*(S_t)$. We have the Clemens-Schmid sequence [Mor84] relating the integral cohomology of S_0 and S_t:

$$0 \to H^0(S_t) \xrightarrow{N} H^0(S_t) \to H_4(S_0) \to H^2(S_0) \to H^2(S_t) \xrightarrow{N} H^2(S_t). \tag{6.3}$$

Since the monodromy operator acts trivially on $H^0(S_t)$, the first nilpotent operator in (6.3) is identically 0. Using these two observations, the Clemens-Schmid sequence can be shortened to

$$0 \to H^0(S_t) \to H_4(S_0) \simeq \mathbb{Z}^m \to H^2(S_0) \to H^2(S_t) \xrightarrow{N} H^2(S_t). \tag{6.4}$$

The limit mixed Hodge structure $H^2(S_t)$ has a monodromy-weight filtration defined in terms of N: $\{0\} = W_0 \subset W_1 \subset W_2 \subset W_3 = H^2(S_t)$.

$$\begin{aligned} W_1H^2(S_t) &= \mathrm{im}(N); \\ W_2H^2(S_t) &= \ker(N); \\ W_3H^2(S_t) &= H^2(S_t). \end{aligned}$$

We call $\ker(N)$ the *1-truncated mixed Hodge structure*. To describe the 1-truncation explicitly, we combine (6.4) and (6.1) above at their common term $H^2(S_0)$, with Mayer-Vietoris written horizontally and Clemens-Schmid written vertically.

$$\begin{array}{ccccccccc} & & & & \mathrm{im}H_4(S_0) & \longrightarrow & \mathrm{span}\{\xi_k\} & & \\ & & & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & J & \longrightarrow & H^2(S_0) & \longrightarrow & K & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & J & \longrightarrow & \ker(N) & \longrightarrow & \Lambda & \longrightarrow & 0. \end{array}$$

Here, $\xi_k := \sum_j [D_{jk}] - [D_{kj}]$, where $[D_{jk}] \in H^2(S_j)$ and $[D_{kj}] \in H^2(S_k)$ are the fundamental classes of the double loci, and Λ is the cokernel of $J \to \ker(N)$. We have that $\xi_k = c_1(\mathcal{O}_{\mathcal{S}}(S_k)|_{S_0})$. By Proposition 6.1, we have $\xi_k \in \ker(\phi\colon K \to \mathrm{Jac}(J))$ because the line bundles $\mathcal{O}_{\mathcal{S}}(S_k)\big|_{S_i} \simeq \mathcal{O}_{\mathcal{S}}(S_k)\big|_{S_j}$ agree on the double locus. Hence, the Carlson extension homomorphism ϕ descends to a homomorphism

$$\psi_{S_0}\colon \Lambda \to \mathrm{Jac}(J)$$

encoding the 1-truncated mixed Hodge structure.

Application

In this section, we apply the general results above to the mixed Hodge structures associated to the degenerations of Type II_b and II_f, and relate their associated periods to the boundary of the toroidal extension $(\mathbb{D}/\Gamma)^{\mathrm{II}}$.

It is convenient to make an order 2 base change and resolution to the Type II_b degenerations. The effect is to normalize the first component and insert a second component isomorphic to $\mathbb{P}^1 \times E$ where E is the fiber over the node of C_0. This second component is glued to the rational elliptic surface $X \to \mathbb{P}^1$ along the two fibers X_p, X_q.

After the base change and resolution, we have that in both II_b and II_f degenerations, the central fiber S_0 has two irreducible components and reduced normal crossings: $S_0 = S_1 \cup_D S_2$. The double locus D is a disjoint union of two copies of the same elliptic curve E in Type II_b and a connected, smooth genus 2 curve in Type II_f. Let $D_1 \subset S_1$ and $D_2 \subset S_2$ denote the double locus restricted to each component.

In both cases, the divisor D admits a natural involution ι, and the image of the first map ι^* in (6.1) is the $(+1)$-eigenspace of this involution on $H^1(D)$. The image of the restriction map res in (6.1) is a rank 1 subgroup of $H^2(D) \simeq H_0(D)$, so the Mayer-Vietoris sequence takes the form

$$0 \to H^1(D)^- \to H^2(S_0) \to H^2(S_1) \oplus H^2(S_2) \xrightarrow{\mathrm{res}} \mathbb{Z} \to 0. \qquad (6.5)$$

Case $\mathbf{II_b}$. The component S_1 is a rational elliptic surface X, with $D_1 = X_p \cup X_q$ a pair of isomorphic elliptic curve fibers. The component S_2 is simply $\mathbb{P}^1 \times E$ with $D_2 = \{0, \infty\} \times E$. The involution on D swaps the two isomorphic components. Note that since $[X_p] = [X_q] \in H^2(S_1)$, and similarly for S_2, the two restriction maps $H^2(S_i) \to H^2(D) \simeq H^2(E)^{\oplus 2}$ have the same image – namely, the diagonal.

Case II$_\mathbf{f}$. The component S_1 is an elliptic ruled surface $X \simeq \mathbb{P}_C(\mathcal{O} \oplus L)$, with D_1 a genus 2 bisection of class $2s_0 = 2(s_\infty + f)$. The component S_2 is the blow-up of (a deformation of) S_1 at 8 points along D_1 with D_2 the proper transform of D_1 in the blow-up. The class of D_2 is $2s_0 - \sum e_i$. The involution on D is induced by the double cover map $\nu\colon D \to C$ which comes from the ruling of X. Since D is irreducible, $H^2(D) \simeq \mathbb{Z}$.

In both cases, the Jacobian $\mathrm{Jac}(H^1(D)^-) = E$ is an elliptic curve. In Type II$_b$, it is $\mathrm{Jac}(E)$, where E is either of the double curves, while in Type II$_f$, it is the Prym variety of the double cover map $\nu\colon D \to C$. Thus, the mixed Hodge structure on $H^2(S_0)$ is encoded by a Carlson extension map $\phi \in \mathrm{Hom}(K, E)$. By the previous subsection, this extension homomorphism descends to $\psi_{S_0} \in \mathrm{Hom}(\Lambda, E)$, where

$$\Lambda = K/\mathrm{span}\{\xi_1, \xi_2\} = \ker(H^2(S_1) \oplus H^2(S_2) \xrightarrow{\text{res}} \mathbb{Z})/\mathbb{Z}(D_1, -D_2).$$

There is a symmetric bilinear form on $H^2(S_0)$. Let

$$p\colon H^2(S_0) \to H^2(S_1) \oplus H^2(S_2) \xrightarrow{\text{PD}} H_2(S_1) \oplus H_2(S_2) \to H_2(S_0)$$

be restriction, followed by the Poincaré duality, followed by inclusion. Then define $\alpha \cdot \beta := \langle \alpha, p(\beta)\rangle$ on $H^2(S_0)$. The map $H^2(S_0) \to H^2(S_t)$ respects the bilinear forms on the source, and target and the bilinear form descends to $K = \ker(\mathrm{res})$.

By Poincaré duality and the Hodge index theorem, $H^2(S_1) \oplus H^2(S_2)$ is a unimodular lattice of signature $(2, 10)$, and it is odd since at least one summand contains (-1)-curves. Since $D_1^2 + D_2^2 = 0$, the lattice vector $(D_1, -D_2)$ is isotropic, and its orthogonal complement is precisely $\ker(\mathrm{res})$. Hence, the lattice Λ is unimodular of signature $(1, 9)$.

Our degenerating families are polarized by $\mathbb{Z}s \oplus \mathbb{Z}(s + f) \subset H^2(S_t)$. The monodromy operator fixes these curve classes, and hence, we have a copy of $I_{1,1} \subset \ker(N)$. So s, f extend over the singular fiber by (6.3). They can be represented in K as follows: (s, s), $(f, 0)$ for Type II$_b$ and $(s_\infty, 0)$, (f, f) for Type II$_f$, respectively. In both cases, they span a sublattice of Λ isometric to $I_{1,1}$ whose orthogonal complement we call $\Lambda_0 \subset \Lambda$. We also have $\Lambda_0 \simeq \Lambda/I_{1,1}$ canonically.

Proposition 6.4. *The lattice Λ_0 is isometric to E_8 in both cases.*

Proof. Note that Λ_0 is unimodular of signature $(0, 8)$, so it suffices to check that it is even. The orthogonal complement of $\{s, f\}$ in $\ker(N)$ is even because $f = K_{S_t}$ and $x \cdot x \equiv x \cdot K_{S_t} \bmod 2$ for any $x \in H^2(S_t)$. Hence, its image Λ_0 is even because $\ker(N) \to \Lambda$ preserves the intersection form. □

Remark 6.5. The lattice Λ_0 can be described more directly using one irreducible component (only up to finite index in the Type II$_f$ case). For Type II$_b$, the sublattice $\{s, f\}^\perp \subset H^2(S_1)$ lies in K and is even, unimodular of signature $(0, 8)$. So it maps isometrically to $\Lambda_0 \simeq E_8$. For II$_f$, the sublattice $\{D_2, f\}^\perp \subset H^2(S_2)$ lies in K and so maps isometrically to an index two sublattice $D_8 \subset \Lambda_0 \simeq E_8$.

We summarize the results of this section in the following proposition:

Proposition 6.6. *Let $\mathcal{S} \to (B, 0)$ be a degeneration of Type II$_b$ or Type II$_f$. Let $K = \ker(H^2(S_1) \oplus H^2(S_2) \to H^2(D))$ be the kernel of signed restriction, and let $\Lambda := K/\mathbb{Z}(D_1, -D_2)$ and $\Lambda_0 = \{s, f\}^\perp \subset \Lambda$. Let E be Pic^0 of either double curve in Type II$_b$ and the Prym variety $\mathrm{Pic}^0(D)/\mathrm{Pic}^0(C)$ in Type II$_f$.*

The Carlson extension class $\phi \in \mathrm{Hom}(K, E)$ describing the mixed Hodge structure on S_0 descends to $\mathrm{Hom}(\Lambda, E)$, and so determines the 1-truncated limit mixed Hodge structure of the degeneration. This homomorphism further descends to a period point $\psi_{S_0} \in \mathrm{Hom}(\Lambda_0, E)$ where $\Lambda_0 \simeq E_8$. Explicitly.

(II$_b$) *The period point ψ_{S_0} given by the map sending $\mathcal{L} \in \{s, f\}^\perp \subset \mathrm{Pic}(S_1)$ to $\mathcal{L}\big|_{X_p} \otimes \mathcal{L}\big|_{X_q}^{-1} \in E$.*

(II$_f$) *The period point ψ_{S_0} is determined up to 2-torsion by the map sending $c_1(\mathcal{L}) \in \{D, f\}^\perp \subset H^2(S_2)$ to $\mathcal{L}\big|_D \in \mathrm{Pic}^0(D)/\mathrm{Pic}^0(C) = E$.*

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A. Appendix: Compact moduli

KSBA theory [KSB88, Ale96, Kol23] gives a general method for constructing compact moduli spaces of pairs (X, B), consisting of a projective variety X and a $\mathbb{Q}$-Weil divisor B, which form a so-called *stable slc pair*:

1. the pair (X, B) has semi-log canonical singularities,
2. $K_X + B$ is $\mathbb{Q}$-Cartier and ample.

In the case at hand, the pair $(\overline{S}, \epsilon s)$ satisfies these conditions, where $S \to \overline{S}$ is the contraction to the Weierstrass form. The paper [AB21] of Ascher and Bejleri with an appendix by Inchiostro studies the corresponding compactification by stable slc pairs $F \hookrightarrow \overline{F}^W$. Every degeneration with generic fiber in F has a unique limit in $\overline{F}^W$ called the *stable model*.

No information is lost when considering Type II_b degenerations because the stable model $\overline{S}_0$ uniquely determines S_0: It is the resolution of ADE configurations in fibers. However, for Type II_f degenerations, most period information is lost: the stable model $\overline{S}_0$ is the gluing of $\mathbb{P}_C(\mathcal{O} \oplus L)$ along the bisection D. Thus, the locus in $\overline{F}^W$ corresponding to Type II_f degenerations has dimension 2, remembering only the genus 2 double cover $\nu\colon D \to C$.

To record more period information, we can instead choose a different divisor on the general surface $S \in F$. Let $R := s + \sum_{i=1}^{12} f_i$, where f_i are the singular fibers of $S \to C$, counted with multiplicity. Because $(\overline{S}, \epsilon R)$ is a stable slc pair, we may again compactify the moduli space of such pairs using KSBA theory: $F \hookrightarrow \overline{F}^R$, where $\overline{F}^R$ is the closure of the pairs $\{(\overline{S}, \epsilon R) \,|\, S \in F\}$ in moduli of all stable slc pairs. Up to a finite map, $\overline{F}^R$ remembers the period information of a Type II_f degeneration (and this is still so for Type II_b surfaces).

Thus, it is possible that the normalization of $\overline{F}^R$ actually dominates a toroidal compactification of $\mathbb{D}/\Gamma$. An analogous result for elliptic K3 surfaces $(g, d) = (0, 2)$ holds by [ABE22]. We leave this as a conjecture:

Conjecture A.1. *There is a morphism* $(\overline{F}^R)^\nu \to \overline{\mathbb{D}/\Gamma}^{\mathfrak{F}}$ *to some toroidal compactification, for an appropriately chosen fan* $\mathfrak{F}$.

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