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Endomorphism Algebras of Kronecker Modules Regulated by Quadratic Function Fields

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Abstract. Purely simple Kronecker modules $\mathcal{M}$, built from an algebraically closed field K, arise from a triplet (m, h, α) where m is a positive integer, $h\colon K \cup \{\infty\} \to \{\infty, 0, 1, 2, 3, \dots\}$ is a height function, and α is a K-linear functional on the space $K(X)$ of rational functions in one variable X. Every pair (h, α) comes with a polynomial f in $K(X)[Y]$ called the regulator. When the module $\mathcal{M}$ admits non-trivial endomorphisms, f must be linear or quadratic in Y. In that case $\mathcal{M}$ is purely simple if and only if f is an irreducible quadratic. Then the K-algebra End $\mathcal{M}$ embeds in the quadratic function field $K(X)[Y]/(f)$. For some height functions h of infinite support I, the search for a functional α for which (h, α) has regulator 0 comes down to having functions $\eta\colon I \to K$ such that no planar curve intersects the graph of η on a cofinite subset. If K has characterictic not 2, and the triplet (m, h, α) gives a purely-simple Kronecker module $\mathcal{M}$ having non-trivial endomorphisms, then h attains the value ∞ at least once on $K \cup \{\infty\}$ and h is finite-valued at least twice on $K \cup \{\infty\}$. Conversely all these h form part of such triplets. The proof of this result hinges on the fact that a rational function r is a perfect square in $K(X)$ if and only if r is a perfect square in the completions of $K(X)$ with respect to all of its valuations.

1 Introduction

For some time it has been our ambition to tabulate the endomorphism algebras of Kronecker modules of rank-2. We have been motivated by the intricacy of the questions that this project has spawned, as well as by the surprising commutative algebras that have emerged. For instance, we discovered that the endomorphism algebras for one class of such modules are the coordinate rings of all elliptic curves. A sample of our work can be found in [13]. A second motivation for our project is that work already done on Kronecker modules has proven useful in development of the representation theory of general classes of associative algebras, *e.g.,* [1, 5, 6, 8, 9, 17]. In this continuation of our project, we examine the role that pure simplicity plays in controlling the endomorphism algebra. Take a finite-dimensional algebra A over a field K, and an A-module M, typically infinite-dimensional. We say that a submodule P is pure in M provided P is a direct summand in any submodule N which is a finite-dimensional extension of P. Every direct summand of M is pure, but not conversely. For further discussion of the notion, see [10]. The module M is *purely*

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simple when (0) and M are its only pure submodules. This is a strong version of indecomposability. Unlike indecomposables, purely simple modules must have countable rank.

We work with an algebraically closed underlying field K. Every purely simple Kronecker module of rank-2 is infinite-dimensional over K, and as shown in [4], they can be constructed from a triple (m, h, α), where m is a positive integer, h is a height function, and α is a K-linear functional on the space $K(X)$ of rational functions in a single variable X. This construction will be presented in detail below, and denoted by $\mathbb{V}(m, h, \alpha)$. The construction, however, includes modules as well that are not purely simple.

To each pair (h, α) we attach a polynomial f in $K(X)[Y]$, either monic or zero, called the regulator of (h, α). The endomorphisms of $\mathbb{V}(m, h, \alpha)$ form a K-algebra. We show that unless f is linear or quadratic in Y, $\operatorname{End}\mathbb{V}(m, h, \alpha)$ is the trivial algebra K. When the regulator is linear, the module $\mathbb{V}(m, h, \alpha)$ has a finite-dimensional direct summand, and $\operatorname{End}\mathbb{V}(m, h, \alpha)$ is non-commutative but well understood [14, p. 1568]. When the regulator is quadratic, the story gets interesting. In this case $\operatorname{End}\mathbb{V}(m, h, \alpha)$ must be commutative, and may well be non-trivial. In the quadratic case we show that $\mathbb{V}(m, h, \alpha)$ is purely simple if and only if the regulator f of (h, α) is irreducible. We also obtain that in the case of irreducible, quadratic regulator the algebra $\operatorname{End}\mathbb{V}(m, h, \alpha)$ embeds in the quadratic function field $K(X)[Y]/(f)$. Thus purely simple Kronecker modules with non-trivial endomorphisms make a tangential connection with hyper-elliptic curves. Using regulators we identify those height functions which permit a module $\mathbb{V}(m, h, \alpha)$ to be purely simple with non-trivial endomorphisms. We also construct examples of irreducible quadratic regulators that yield trivial endomorphism algebras, and more importantly examples that yield non-trivial algebras $\operatorname{End}\mathbb{V}(m, h, \alpha)$.

In the remainder of this introduction we carefully formulate the framework for proving our results.

The Setup

Start with an algebraically closed field K. A module over the algebra $\left[\begin{smallmatrix} K & K^2 \\ 0 & K \end{smallmatrix}\right]$ is called a *Kronecker module*, but from a practical point of view it can be seen as a pair of linear transformations between a pair of K-linear spaces: $U \overset{a}{\underset{b}{\rightrightarrows}} V$. An *endomorphism* of $U \rightrightarrows V$ is a pair of K-linear maps $U \xrightarrow{\psi} U$, $V \xrightarrow{\varphi} V$ for which the diagrams commute:

$$\begin{array}{ccc} U & \xrightarrow{a} & V \\ \downarrow{\scriptstyle\psi} & & \downarrow{\scriptstyle\varphi} \\ U & \xrightarrow{a} & V \end{array} \qquad \begin{array}{ccc} U & \xrightarrow{b} & V \\ \downarrow{\scriptstyle\psi} & & \downarrow{\scriptstyle\varphi} \\ U & \xrightarrow{b} & V \end{array}.$$

In order to introduce a working model for the purely simple modules of rank-2, some technicalities are needed. These have been presented in previous papers [15]. As they are not generally familiar, we will review them now.

Let $K(X)$ be the field of rational functions in X. For each θ in K we adopt the shorthand

$$X_\theta = (X - \theta)^{-1}.$$

Every non-zero t in $K(X)$ has a unique factorization

$$t = \lambda \prod_{\theta \in K} X_\theta^{j_\theta}, \tag{1}$$

where $\lambda \in K$, $j_\theta \in \mathbb{Z}$, and all but finitely many j_θ are 0. For each θ in K, the integer j_θ is denoted by $\operatorname{ord}_\theta(t)$, and the integer $-\sum_{\theta \in K} j_\theta$ is denoted by $\operatorname{ord}_\infty(t)$. If we agree that $\operatorname{ord}_\theta(0) = -\infty$ for all θ in $\mathrm{K} \cup \{\infty\}$, and s, t are rational functions, the familiar valuation properties hold:

$$\operatorname{ord}_\theta(st) = \operatorname{ord}_\theta(s) + \operatorname{ord}_\theta(t),$$

$$\operatorname{ord}_\theta(s + t) \leq \max\{\operatorname{ord}_\theta(s), \operatorname{ord}_\theta(t)\},$$

$$\operatorname{ord}_\theta(s + t) = \operatorname{ord}_\theta(t) \ \text{ when } \operatorname{ord}_\theta(s) < \operatorname{ord}_\theta(t).$$

If $\theta \in \mathrm{K} \cup \{\infty\}$ and $\operatorname{ord}_\theta(t) > 0$, the rational function t has a *pole* at θ and its order is $\operatorname{ord}_\theta(t)$. The functions X^n and X_θ^{n+1}, where $\theta \in K$ and $0 \leq n$, form the standard basis of $K(X)$ over K. The expansion of a rational function in terms of the standard basis is known as its partial fraction expansion. For θ in K, a positive power of X_θ appears in the partial fraction expansion of t if and only if t has a pole at θ. In that case the highest power of X_θ appearing is $X_\theta^{\operatorname{ord}_\theta(t)}$. A positive power of X appears in the partial fraction expansion of t if and only if t has a pole at ∞. Then the highest power of X that appears is $X^{\operatorname{ord}_\infty(t)}$.

Any function $h\colon \mathrm{K} \cup \{\infty\} \to \{\infty, 0, 1, 2, \dots\}$ is known as a *height function*. The attached K-linear space

$$R_h = \{s \in K(X) : \operatorname{ord}_\theta(s) \leq h(\theta) \text{ for all } \theta \text{ in } \mathrm{K} \cup \{\infty\}\}$$

is called a *pole space*. Pole spaces have an intrinsic definition as well. They are the non-zero subspaces R of $K(X)$ with the property that whenever $t \in R$, then every function s, such that $\operatorname{ord}_\theta(s) \leq \max\{0, \operatorname{ord}_\theta(t)\}$ for all θ in $\mathrm{K} \cup \{\infty\}$, is also in R. Every pole space contains K. Given a pole space R_h, put

$$R_h^- = \{r \in R_h : \operatorname{ord}_\infty(r) < h(\infty)\}.$$

We see that $XR_h^- \subseteq R_h$, and R_h^- is the biggest subspace of R_h to tolerate such inclusion. The modules

$$\mathbb{F}_h = \left(R_h^- \overset{a}{\underset{b}{\rightrightarrows}} R_h, \ \text{where } a\colon r \mapsto r \text{ and } b\colon r \mapsto Xr\right)$$

are interesting because they tabulate exactly the class of all torsion-free, indecomposable, rank-1 modules, see [3]. In addition all $\mathbb{F}_h$ are purely simple. The endomorphism algebra of $\mathbb{F}_h$ is the K-algebra of rational functions t in R_h for which $tR_h \subseteq R_h$.

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This coincides with the algebra of functions having poles only at those θ in $\mathrm{K} \cup \{\infty\}$ where $h(\theta) = \infty$. We call it the *pole algebra* of h. When h assumes only the values 0 or ∞, the pole space R_h is already a pole algebra.

For a consideration of rank-2 modules we need to work with K-linear functionals $\alpha\colon K(X) \to K$. If α is such a functional and $r \in K(X)$, let $\langle \alpha, r \rangle$ denote the value in K that α takes at r. Given a functional α and a rational function r, it is shown in [7, Proposition 3.4] that there is a unique rational function $\partial_\alpha(r)$ so that $\partial_\alpha(r)(\theta)$ is defined at all θ in K where $r(\theta)$ is defined, and for all such θ

$$\partial_\alpha(r)(\theta) = \Big\langle \alpha, \frac{r - r(\theta)}{X - \theta} \Big\rangle = \langle \alpha, (r - r(\theta))X_\theta \rangle. \tag{2}$$

From this it is easy to see that the mapping $(\alpha, r) \mapsto \partial_\alpha(r)$ is K-linear in both α and r. The K-linear map $\partial_\alpha\colon K(X) \to K(X)$ will be called a *deriver*. For a functional α and a rational function r, the functional given by $t \mapsto \langle \alpha, rt \rangle$ will be denoted by $\alpha * r$. The name *deriver* is motivated by the following derivation-like property which is easy to deduce from (2):

$$\partial_\alpha(st) = s\partial_\alpha(t) + \partial_{\alpha * t}(s) \quad \text{for any functional } \alpha \text{ and rational functions } s, t. \tag{3}$$

The explicit calculation of ∂_α on the standard basis of $K(X)$ goes like this:

$$\begin{aligned} \partial_\alpha(1) &= 0 \\ \partial_\alpha(X^n) &= \langle \alpha, X^{n-1} \rangle + \langle \alpha, X^{n-2} \rangle X + \cdots + \langle \alpha, 1 \rangle X^{n-1} \\ \partial_\alpha(X_\theta^n) &= -\langle \alpha, X_\theta^n \rangle X_\theta - \langle \alpha, X_\theta^{n-1} \rangle X_\theta^2 - \cdots - \langle \alpha, X_\theta \rangle X_\theta^n, \end{aligned} \tag{4}$$

for all θ in K and all $n \geq 1$. The formulas (4) will be used often. In conjunction with the partial fraction expansion of r, they reveal that

$$\begin{aligned} \mathrm{ord}_\theta(\partial_\alpha(r)) &\leq \max\{0, \mathrm{ord}_\theta(r)\} \quad \text{for all } \theta \text{ in } K, \\ \mathrm{ord}_\infty(\partial_\alpha(r)) &< \max\{0, \mathrm{ord}_\infty(r)\}. \end{aligned}$$

Consequently every pole of $\partial_\alpha(r)$ is a pole of r. Furthermore, if R_h is a pole space, then $\partial_\alpha(R_h) \subseteq R_h^-$. In particular, derivers leave pole spaces invariant.

Now we can define the family of rank-2 modules that interest us. Although our presentation will not make it apparent, these modules will comprise exactly all extensions of finite-dimensional $\mathbb{F}_k$ by infinite-dimensional $\mathbb{F}_h$. Thus this paper is about the family of extensions of finite-dimensional torsion-free, rank-1 Kronecker modules by arbitrary torsion-free, rank-1 Kronecker modules.

Let $\mathcal{R}$ denote the important module $\mathbb{F}_h$ that goes with the biggest pole space $R_h = K(X)$. Our rank-2 modules will be presented as embedded in $\mathcal{R}^2$.

We shall be working with K-linear subspaces of the space $K(X)^2$ of pairs of rational functions. Such pairs will be written in column notation. For a positive integer m, let P_m be the space of polynomials of degree strictly less than m. This is nothing but the pole space corresponding to the height function which assumes the value 0 on K and $m - 1$ at ∞.

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Definition Given a triplet (m, h, α) where m is a positive integer, h is a height function and α is a functional, put

$$V(m,h,\alpha) = \left\{ \begin{pmatrix} r \\ s \end{pmatrix} \in K(X)^2 : r \in R_h \text{ and } \partial_\alpha(r) + s \in P_m \right\}, \tag{5}$$

$$V^-(m,h,\alpha) = \left\{ \begin{pmatrix} r \\ s \end{pmatrix} \in V : r \in R_h^- \text{ and } \partial_\alpha(r) + s \in P_{m-1} \right\}. \tag{6}$$

The computations of this paper will rely heavily on the definition of $V(m, h, \alpha)$. Observe that $XV^-(m, h, \alpha) \subseteq V(m, h, \alpha)$. Indeed, if $\binom{r}{s} \in V^-(m, h, \alpha)$, then $r \in R_h^-$ and $\partial_\alpha(r) + s \in P_{m-1}$. Therefore $Xr \in R_h$, and using (3) we get as well that

$$\partial_\alpha(Xr) + Xs = X\partial_\alpha(r) + \partial_{\alpha * r}(X) + Xs = X(\partial_\alpha(r) + s) + \langle \alpha, r\rangle \in P_m.$$

Definition The Kronecker module $\mathbb{V}(m, h, \alpha)$ is

$$V^-(m,h,\alpha) \overset{a}{\underset{b}{\rightrightarrows}} V(m,h,\alpha) \tag{7}$$

where

$$a\colon \begin{pmatrix} r \\ s \end{pmatrix} \mapsto \begin{pmatrix} r \\ s \end{pmatrix} \text{ and } b\colon \begin{pmatrix} r \\ s \end{pmatrix} \mapsto X\begin{pmatrix} r \\ s \end{pmatrix}, \text{ for each } \begin{pmatrix} r \\ s \end{pmatrix} \text{ in } V^-(m,h,\alpha).$$

The space R_h is infinite-dimensional over K exactly when h is positive on an infinite subset of $\mathrm{K} \cup \{\infty\}$ or h is infinite-valued at some θ of $\mathrm{K} \cup \{\infty\}$. When R_h is finite-dimensional, the modules $\mathbb{V}(m, h, \alpha)$ are also finite-dimensional and completely understood [1, p. 302]. So, we make the blanket assumption that in defining $\mathbb{V}(m, h, \alpha)$, the pole space R_h is infinite-dimensional. In [4] it is shown that every purely simple module of rank-2 is an extension of a finite-dimensional $\mathbb{F}_k$ by an infinite-dimensional $\mathbb{F}_h$. In [12, §2] every such extension is realized as a module $\mathbb{V}(m, h, \alpha)$. Thus $\mathbb{V}(m, h, \alpha)$'s include among them all possible purely simple rank-2 modules.

Our study of the endomorphism algebra $\operatorname{End}\mathbb{V}(m, h, \alpha)$ is based on [14, Theorem 2.2] which facilitates the use of linear algebra. It says that the endomorphisms of $\mathbb{V}(m, h, \alpha)$ are the $K(X)$-linear operators on $K(X)^2$ which leave the infinite-dimensional K-linear space $V(m, h, \alpha)$ invariant. Furthermore, such maps leave $V^-(m, h, \alpha)$ invariant. We represent the endomorphisms of $\mathbb{V}(m, h, \alpha)$ as 2×2 matrices of rational functions

$$\varphi = \begin{bmatrix} s & t \\ u & v \end{bmatrix} \tag{8}$$

acting on $K(X)^2$ in the usual way. If I is the identity operator and $\lambda \in K$, the scalar operator λI is obviously an endomorphism. If these are the only endomorphisms, we say that $\operatorname{End}\mathbb{V}(m, h, \alpha)$ is trivial.

In [15, Theorem 3.2] we showed that if a height function h admits an indecomposable $\mathbb{V}(m,h,\alpha)$ with non-trivial endomorphisms, then either $h(\theta)=\infty$ for some θ in $\mathrm{K}\cup\{\infty\}$ or $h(\theta)\geq 2$ for infinitely many θ in $\mathrm{K}\cup\{\infty\}$. In [16, Theorem 1.1] we showed further that, subject to char $K\neq 2$, if any height function h admits an indecomposable $\mathbb{V}(m,h,\alpha)$ with non-trivial endomorphisms, then $h(\theta)<\infty$ for some θ in $\mathrm{K}\cup\{\infty\}$. One of our objectives is the following theorem which gives the constraints on a height function h for it to admit purely-simple $\mathbb{V}(m,h,\alpha)$ with non-trivial endomorphisms.

Theorem 1.1 *Assuming K has characterictic not 2, a height function h will admit a purely simple Kronecker module $\mathbb{V}(m,h,\alpha)$ having non-trivial endomorphisms if and only if h attains the value ∞ at least once on $\mathrm{K}\cup\{\infty\}$ and h is finite-valued at least twice on* $\mathrm{K}\cup\{\infty\}$.

2 A Regulator for (h,α) and Pure Simplicity

Take a height function h with infinite-dimensional pole space R_h, and a functional $\alpha\colon K(X)\to K$. The deriver ∂_α is a K-linear operator on the space $K(X)$. Every rational function t acts on $K(X)$ as the *multiplier* $s\mapsto ts$. We identify t with its multiplier. The deriver ∂_α leaves R_h invariant, but a multiplier t need not. Nevertheless, the space tR_h lies inside the pole space R_k where

$$k(\theta)=h(\theta)+\max\{0,\operatorname{ord}_\theta(t)\}\ \text{ for every }\theta\text{ in }\mathrm{K}\cup\{\infty\},$$

and the pole space R_k is a finite-dimensional extension of R_h. Let $\mathcal{A}$ denote the K-subalgebra of $\operatorname{End}_K K(X)$ that is generated by ∂_α and by all multipliers. Put

$$\mathfrak{I}=\{\sigma\in\mathcal{A}:\sigma(R_h)\ \text{ is finite-dimensional over }K\}.$$

The operators in $\mathfrak{I}$ are said to have *finite rank on R_h*. Since the dimension of R_h is infinite, $\mathfrak{I}$ is a proper subspace of $\mathcal{A}$. Clearly $\mathfrak{I}$ is a left ideal, but one can check that $\mathfrak{I}$ is also a right ideal using the fact that, for every σ in $\mathcal{A}$, the image $\sigma(R_h)$ is inside a finite-dimensional extension of R_h, see also [12, Lemma 2.1].

While $\mathcal{A}$ is typically a non-commutative algebra containing $K(X)$, the quotient algebra $\mathcal{A}/\mathfrak{I}$ is a commuative $K(X)$-algebra. For the proof see [12, Lemma 2.2]. Briefly it suffices to check that a multiplier t commutes with ∂_α modulo the ideal $\mathfrak{I}$. From the deriver property (3) we have $\partial_\alpha(\mathrm{tr})t-t\partial_\alpha(r)=\partial_{\alpha*r}(t)$ for all r in R_h. The formula (4) shows that for all r, the functions $\partial_{\alpha*r}(t)$ lie in the smallest pole space containing t, a finite-dimensional space. Thus $\partial_\alpha\circ t-t\circ\partial_\alpha\in\mathfrak{I}$.

Clearly $\mathcal{A}/\mathfrak{I}$ is generated as a $K(X)$-algebra by the image $\partial_\alpha+\mathfrak{I}$ of ∂_α under the canonical projection $\mathcal{A}\to\mathcal{A}/\mathfrak{I}$. If $K(X)[Y]$ is the algebra of polynomials in Y over $K(X)$ we are entitled to the substitution map $\epsilon\colon K(X)[Y]\to\mathcal{A}/\mathfrak{I}$ where $Y\mapsto\partial_\alpha+\mathfrak{I}$. The unique monic generator $f(Y)$ of $\ker\epsilon$ is the polynomial in $K(X)[Y]$ that we call the *regulator* of the pair (h,α). Given a polynomial $g(Y)$ in $K(X)[Y]$ we shall let $g(\partial_\alpha)$ stand for any preimage in $\mathcal{A}$ of $g(\partial_\alpha+\mathfrak{I})$ under the projection $\mathcal{A}\to\mathcal{A}/\mathfrak{I}$. No matter what preimage $g(\partial_\alpha)$ is taken, we always have

$$f(Y)\ \text{ divides }\ g(Y)\quad\Leftrightarrow\quad g(\partial_\alpha+\mathfrak{I})=0\quad\Leftrightarrow\quad g(\partial_\alpha)\ \text{ has finite rank on }R_h.$$

To summarize, the regulator of (h, α) is the polynomial $f(Y)$ in $K(X)[Y]$ uniquely specified by the following properties:

- $f(Y)$ is monic or zero;
- $f(\partial_\alpha)$ has finite rank on R_h;
- $f(Y)$ divides $g(Y)$ in $K(X)[Y]$ if and only if $g(\partial_\alpha)$ has finite rank on R_h.

In order to establish some familiarity with the regulator, let us next compute some examples.

Example 1 (Zero regulator) Take the height function h that assumes the value ∞ on all of $\mathrm{K} \cup \{\infty\}$. For this h the pole space is the full pole algebra $R_h = K(X)$. To specify a functional α it suffices to give its values on the standard basis of $K(X)$. Put

$$\langle \alpha, X_\theta \rangle = \theta \text{ for every } \theta \text{ in } K$$

$$\alpha = 0 \text{ on the rest of the standard basis of } K(X).$$

We will show that for this (h, α) the regulator is 0. To get there, for each θ in K, define the height function $h_\theta \colon \mathrm{K} \cup \{\infty\} \to \{\infty, 0, 1, 2, \dots\}$ by $h_\theta(\theta) = h(\theta) = \infty$ and $h(\nu) = 0$ when $\nu \in \mathrm{K} \cup \{\infty\}, \nu \neq \theta$. The pole space for h_θ is $R_{h_\theta} = K[X_\theta]$, the space of polynomials in X_θ, with basis $1, X_\theta, X_\theta^2, X_\theta^3, \dots$.

We will first show that the regulator of (h_θ, α) is $Y + \theta$. Using (4) we can see that

$$\partial_\alpha(X_\theta^j) = -\theta X_\theta^j \text{ for any integer } j \geq 1.$$

Thus the operator $\partial_\alpha + \theta$ maps the space $X_\theta K[X_\theta]$, of polynomials in X_θ with zero constant term, to 0. Consequently the pole space $K[X_\theta]$ of h_θ goes to a finite-dimensional space under $\partial_\alpha + \theta$. In fact it goes to K. Thus we may conclude that $Y + \theta$ regulates (h_θ, α).

Now suppose that $f(Y)$ regulates (h, α). By the nature of regulators $f(\partial_\alpha)$ maps $K(X)$ to a finite-dimensional space. Consequently, $f(\partial_\alpha)(K[X_\theta])$ is finite-dimensional for every θ in K. Since $K[X_\theta]$ is the pole space for the height function h_θ, the polynomial $f(Y)$ must be divisible by the regulator $Y + \theta$ of (h_θ, α). It follows that $f(Y) = 0$ because there are infinitely many $Y + \theta$ that divide it.

Example 2 (Any irreducible as regulator) Let I be an infinite subset of K. Then take h to be the height function defined by

$$h(\theta) = \begin{cases} 1 & \text{if } \theta \in I, \\ 0 & \text{if } \theta \in \mathrm{K} \cup \{\infty\} \setminus I, \end{cases}$$

having pole space $R_h = K + \sum_{\theta \in I} K X_\theta$. As usual α is a functional. Take any monic polynomial

$$f(X, Y) = Y^n + r_{n-1}(X) Y^{n-1} + \cdots + r_1(X) Y + r_0(X) \in K(X)[Y].$$

Using the chosen h, let us develop a reasonably concrete condition for the operator

$$f(X, \partial_\alpha) = \partial_\alpha^n + r_{n-1}(X) \circ \partial_\alpha^{n-1} + \cdots + r_1(X) \circ \partial_\alpha + r_0(X)$$

to have finite rank on R_h. For each θ in I put $\eta_\theta = -\langle \alpha, X_\theta \rangle$. There are only finitely many θ in I at which the rational functions $r_j(X)$ could possibly have a pole. Let J be the cofinite set of θ in I for which every $r_j(\theta)$ is defined. When $\theta \in J$ we need to examine what $f(X, \partial_\alpha)$ does to X_θ. Using (4) we obtain that $\partial_\alpha(X_\theta) = \eta_\theta X_\theta$, and consequently $\partial_\alpha^j(X_\theta) = \eta_\theta^j X_\theta$ for all θ in J and $j \geq 0$. Then for θ in J we get

$$f(X, \partial_\alpha)(X_\theta) = \eta_\theta^n X_\theta + r_{n-1}(X)\eta_\theta^{n-1} X_\theta + \cdots + r_1(X)\eta_\theta X_\theta + r_0(X) X_\theta. \tag{9}$$

Furthermore, if $\theta \in J$, the scalars $f(\theta, \eta_\theta) = \eta_\theta^n + r_{n-1}(\theta)\eta_\theta^{n-1} + \cdots + r_1(\theta)\eta_\theta + r_0(\theta)$ are defined. As j runs down from $n-1$ to 0, each of the summands $r_j(X)\eta^j X_\theta$ in (9) can be rewritten as $r_j(X)\eta^j X_\theta = r_j(\theta)\eta^j X_\theta + \eta^j(r_j(X) - r_j(\theta))X_\theta$. After substituting these expressions into the summation (9) we get for each θ in J that

$$f(X, \partial_\alpha)(X_\theta) = f(\theta, \eta_\theta) X_\theta + \sum_{j=n-1}^{0} \eta^j (r_j(X) - r_j(\theta)) X_\theta. \tag{10}$$

Since each rational function $r_j(X) - r_j(\theta)$ vanishes at θ, the rational function $(r_j(X) - r_j(\theta))X_\theta$ has no pole at θ. In fact this function sits in the smallest pole space containing $r_j(X)$, a finite-dimensional space. Consequently the sum

$$\sum_{j=n-1}^{0} \eta^j (r_j(X) - r_j(\theta)) X_\theta$$

lies in the smallest pole space containing all $r_j(X)$, still a finite-dimensional space. Call it S.

We now use (10) to show that

$f(X, \partial_\alpha)$ has finite rank on R_h if and only if for all but finitely many θ in I the points (θ, η_θ) lie on the curve $f(X, Y) = 0$.

Suppose that $f(X, \partial_\alpha)$ has finite rank on R_h. It follows that $f(X, \partial_\alpha)(X_\theta)$ lie in a common finite-dimensional space, say T, for all θ in J. Looking at (10) we deduce that $f(\theta, \eta_\theta)X_\theta \in S + T$ for all θ in J. Since $S + T$ is finite-dimensional, the pole set for the functions $f(\theta, \eta_\theta)X_\theta$, as θ runs over J, has to be finite. This implies that $f(\theta, \eta_\theta) = 0$ for all but finitely many θ in J. Since J is cofinite in I we get that (θ, η_θ) are on the curve $f(X, Y) = 0$ for all but finitely many θ in I.

Conversely, suppose that (θ, η_θ) are on the curve $f(X, Y) = 0$ for all but finitely many θ in I. This cofinite set of θ lies in J. Using (10) we deduce that $f(X, \partial_\alpha)X_\theta \in S$ for all but finitely many θ in J. As θ runs over this cofinite subset of J, the functions X_θ span a subspace that is of finite codimension in the pole space R_h. Since the operator $f(X, \partial_\alpha)$ maps a space of finite codimension in R_h to a finite-dimensional space, it does the same to R_h. In other words $f(X, \partial_\alpha)$ has finite rank on R_h.

Now, to arrange for any monic, irreducible polynomial $f(X, Y)$ in $K(X)[Y]$ to regulate a pair (h, α) we simply take I and h as above, and take α to be any functional such that $(\theta, -\langle \alpha, X_\theta \rangle)$ is on the curve $f(X, Y) = 0$ for all but finitely many θ in I. As we saw $f(X, \partial_\alpha)$ has finite rank on R_h. Consequently the regulator of (h, α) divides $f(X, Y)$. Since $f(X, Y)$ is irreducible and monic in Y, it must be the regulator of (h, α).

Endomorphisms of $\mathbb{V}(m, h, \alpha)$ and Their Tag Polynomial

As noted, an endomorphism of $\mathbb{V}(m, h, \alpha)$ is a $K(X)$-linear operator on $K(X)^2$ that leaves the space $V(m, h, \alpha)$ invariant. Take such φ represented as the matrix in (8). The polynomial

$$tY^2 + (v - s)Y - u \tag{11}$$

associated to φ will have a role to play. We shall call it the *tag polynomial* of φ.

Proposition 2.1 *If φ as in* (8) *is an endomorphism of* $\mathbb{V}(m, h, \alpha)$*, then the regulator of* (h, α) *divides the tag polynomial of* φ*.*

Proof By the nature of regulators, one must check that the operator

$$t \circ \partial_\alpha^2 + (v - s) \circ \partial_\alpha - u$$

has finite rank on R_h. For every r in R_h the vector $\left(\begin{smallmatrix} r \\ -\partial_\alpha(r) \end{smallmatrix}\right)$ lies in $V(m, h, \alpha)$, by (5). Hence

$$\begin{bmatrix} s & t \\ u & v \end{bmatrix} \begin{pmatrix} r \\ -\partial_\alpha(r) \end{pmatrix} = \begin{pmatrix} sr - t\partial_\alpha(r) \\ ur - v\partial_\alpha(r) \end{pmatrix} \in V(m, h, \alpha).$$

Thus $\partial_\alpha(sr - t\partial_\alpha(r)) + ur - v\partial_\alpha(r) \in P_m$. Since P_m is finite-dimensional, the operator $\partial_\alpha \circ s - \partial_\alpha \circ t \circ \partial_\alpha + u - v \circ \partial_\alpha$ has finite rank on R_h. However, since $\mathcal{A}/\mathfrak{I}$ is commutative, this operator has the same projection in $\mathcal{A}/\mathfrak{I}$ as the operator $-(t \circ \partial_\alpha^2 + (v - s) \circ \partial_\alpha - u)$, and that projection is 0. By the definition of $\mathfrak{I}$, the operator $t \circ \partial_\alpha^2 + (v - s) \circ \partial_\alpha - u$ has finite rank on R_h. ∎

Next observe that the tag polynomial $tY^2 + (v - s)Y - u$ is zero if and only if the endomorphism φ is trivial. Indeed, if the tag polynomial is zero, then $\varphi = sI$. Since $\left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right) \in V(m, h, \alpha)$, so also $\left(\begin{smallmatrix} 0 \\ s \end{smallmatrix}\right) = \varphi\left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right) \in V(m, h, \alpha)$. This causes s to be in P_m. On the other hand the set of all s such that sI is an endomorphism forms a K-algebra, and the only K-algebra inside P_m is K. After making the simple observation that polynomials of degree 0 are never regulators, a useful fact emerges from Proposition 2.1.

Proposition 2.2 *If* $\operatorname{End} \mathbb{V}(m, h, \alpha)$ *is non-trivial, then the regulator of* (h, α) *must be linear or quadratic in* Y.

When the Regulator Is Linear

In searching for non-trivial endomorphisms we need only worry about those pairs (h, α) that have linear or quadratic regulators. We can quickly dispose of the linear case, whose proof already appears in [15, Proposition 2.5].

Proposition 2.3 *If the regulator of* (h, α) *is linear, then* $\mathbb{V}(m, h, \alpha)$ *has a non-zero, finite-dimensional summand.*

When $\mathbb{V}(m, h, \alpha)$ have a finite-dimensional direct summand, their endomorphism algebras are non-commutative, but well understood, [14, §3]. Thereby the case of a linear regulator requires no further attention.

When the Regulator Is Quadratic

If the regulator is quadratic, things get more complicated, but we can quickly squeeze out some constraints on $\operatorname{End} \mathbb{V}(m, h, \alpha)$. Supposing that the pair (h, α) is regulated by the quadratic polynomial

$$Y^2 + pY + q, \quad \text{where } p, q \in K(X), \tag{12}$$

the matrix

$$D = \begin{bmatrix} p & -1 \\ q & 0 \end{bmatrix} \tag{13}$$

will play a significant role. We call D the *generic matrix* for the quadratic regulator $Y^2 + pY + q$.

Proposition 2.4 *If* (h, α) *has quadratic regulator and generic matrix as in* (12) *and* (13)*, respectively, then every endomorphism of* $\mathbb{V}(m, h, \alpha)$ *takes the form*

$$tD + (\partial_\alpha(t) + \ell)I \quad \textit{where } t \in R_h \textit{ and } \ell \in P_m. \tag{14}$$

Proof Let φ as in (8) be an endomorphism. By Proposition 2.1, $Y^2 + pY + q$ divides the tag polynomial $tY^2 + (v - s)Y - u$. Hence,

$$v - s = tp, \quad -u = tq, \quad \text{and thus } s = v - tp, \ u = -tq.$$

This means that

$$\varphi = \begin{bmatrix} v - tp & t \\ -tq & v \end{bmatrix} = -tD + vI.$$

Since φ is an endomorphism and $\binom{0}{1} \in V(m, h, \alpha)$, its image $\binom{t}{v}$ under φ is also in $V(m, h, \alpha)$. By the definition (5) we can write $v = -\partial_\alpha(t) + \ell$ where $t \in R_h$ and $\ell \in P_m$. Consequently, every endomorphism takes the form $\varphi = -tD + (-\partial_\alpha(t) + \ell)I$, where $t \in R_h$ and $\ell \in P_m$. After replacing t by $-t$, the desired parametrization of φ emerges. ∎

An interesting result, already implicit in [12], now follows.

Corollary 2.5 *If the regulator of* (h, α) *is quadratic with generic matrix* D*, then* $\operatorname{End} \mathbb{V}(m, h, \alpha) \subseteq K(X)[D]$*. In particular* $\operatorname{End} \mathbb{V}(m, h, \alpha)$ *is commutative.*

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Proof The algebra $K(X)[D]$ is the algebra of all matrices of the form $tD + uI$ where $t, u \in K(X)$. By Proposition 2.4 all endomorphisms are special matrices of this sort. ■

The characteristic polynomial of D is $Y^2 - pY + q$, a close relative of the regulator. It is plain to see that the regulator $Y^2 + pY + q$ has no roots, a repeated root, or distinct roots in $K(X)$ if and only if the characteristic poynomial of D has the same respective properties. The algebra $K(X)[D]$ is isomorphic to $K(X)[Y]/(Y^2 - pY + q)$, and this is

- a field extension of $K(X)$ when $Y^2 - pY + q$ has no roots in $K(X)$,
- isomorphic to the simple extension $K(X) \ltimes K(X)$ when $Y^2 - pY + q$ has only one root in $K(X)$,
- isomorphic to the product algebra $K(X) \times K(X)$ when $Y^2 - pY + q$ has two roots in $K(X)$.

Thus the regulator anticipates what End $\mathbb{V}$ will look like, according to the next result whose proof is immediate from Corollary 2.5.

Proposition 2.6 *Let* (h, α) *have quadratic regulator as in* (12) *with generic matrix as in* (13).

- *If the regulator has no root in* $K(X)$, *then* $\operatorname{End}\mathbb{V}(m, h, \alpha)$ *sits inside the quadratic field extension* $K(X)[D]$.
- *If the regulator has one repeated root in* $K(X)$, *then* $\operatorname{End}\mathbb{V}(m, h, \alpha)$ *sits inside the algebra* $K(X) \ltimes K(X)$.
- *If the regulator has non-repeated roots in* $K(X)$, *then* $\operatorname{End}\mathbb{V}(m, h, \alpha)$ *sits inside the algebra* $K(X) \times K(X)$.

Two small items come out of Proposition 2.6. First, if the quadratic regulator has no repeated root, then $\operatorname{End}\mathbb{V}(m, h, \alpha)$ has no nilpotents. Second, if the regulator has only one or no root, then $\operatorname{End}\mathbb{V}(m, h, \alpha)$ has no proper idempotents, and $\mathbb{V}(m, h, \alpha)$ is indecomposable.

Quadratic Regulator, Only Trivial Endomorphisms

Proposition 2.6 also lets us show that the converse of Proposition 2.2 fails. As noted earlier, a linear regulator for (h, α) does imply that $\mathbb{V}(m, h, \alpha)$ has non-trivial endomorphisms. As well, for certain height functions h a quadratic regulator for (h, α) implies that $\operatorname{End}\mathbb{V}(m, h, \alpha)$ is non-trivial [13, 16]. However, in general a quadratic regulator for (h, α) need not force $\mathbb{V}(m, h, \alpha)$ to have non-trivial endomorphisms. To see this fact, take any irreducible polynomial $f(Y)$ in $K(X)[Y]$ of degree 2 in Y. With this polynomial $f(Y)$ take the pair (h, α) of Example 2 that is regulated by $f(Y)$. Since $f(Y)$ has no root in $K(X)$, Proposition 2.6 causes $\operatorname{End}\mathbb{V}(m, h, \alpha)$ to be an integral domain. The height function h of Example 2 satisfies the singularity condition of [14, Theorem 3.4]. Namely, h never assumes the value ∞. That result tells us that $\operatorname{End}\mathbb{V}(m, h, \alpha)$ is either the product $K \times K$ or the trivial extension $K \ltimes S$ for some K-linear space S. The only domain that is compatible with these two possibilities is $K = K \ltimes (0)$. Thus we have that $\operatorname{End}\mathbb{V}(m, h, \alpha)$ is trivial.

Pure Simplicity and Non-Zero Regulator

Next we show how the regulator of (h, α) impinges on the pure simplicity of a module $\mathbb{V}(m, h, \alpha)$. A submodule of $\mathbb{V}(m, h, \alpha)$ is called pure provided it splits inside any of its finite-dimensional extensions that are in $\mathbb{V}(m, h, \alpha)$. The module $\mathbb{V}(m, h, \alpha)$ is called purely simple provided it has no proper pure submodules. Purely simple modules are thereby indecomposable in a strong sense. We shall adopt the criterion for pure simplicity that is in [2, Proposition 2.1]. This says that $\mathbb{V}(m, h, \alpha)$ is purely simple if and only if every non-zero module homomorphism $\mathbb{V}(m, h, \alpha) \to \mathcal{R}$ has finite-dimensional kernel. In order to avoid undue digression we can view a module homomorphism $\mathbb{V}(m, h, \alpha) \to \mathcal{R}$ simply as the restriction to $V(m, h, \alpha)$ of a $K(X)$-linear map $K(X)^2 \to K(X)$. From this point of view it is easy to see from the definition (5) of $V(m, h, \alpha)$ that the integer m is irrelevant to the pure-simplicity of $\mathbb{V}(m, h, \alpha)$. This is because for any positive integer m the space $V(1, h, \alpha)$ has finite codimension in $V(m, h, \alpha)$. Only the pair (h, α) matters for pure simplicity.

Proposition 2.7 *Suppose that the regulator $f(Y)$ of (h, α) is non-zero. Then the module $\mathbb{V}(m, h, \alpha)$ is purely simple if and only if $f(Y)$ has no root in $K(X)$.*

Proof Suppose that u in $K(X)$ is a root of $f(Y)$ and write $f(Y) = (Y - u)g(Y)$ where $g(Y) \in K(X)[Y]$, and $g(Y) \neq 0$. The operators $g(\partial_\alpha)$ and $\partial_\alpha - u$ lie in the algebra $\mathcal{A}$ used to define the regulator. By the nature of regulators, the operator $(\partial_\alpha - u) \circ g(\partial_\alpha)$ has finite rank on R_h, while the operator $g(\partial_\alpha)$ has infinite rank on R_h. Thus the image $g(\partial_\alpha)(R_h)$ is an infinite-dimensional space that lies in a finite-dimensional extension of R_h. Consequently, $R_h \cap g(\partial_\alpha)(R_h)$ is an infinite-dimensional space. This intersection goes to a finite-dimensional space under $\partial_\alpha - u$, because $(\partial_\alpha - u) \circ g(\partial_\alpha)$ has finite rank on R_h. It follows that $\ker(\partial_\alpha - u) \cap R_h \cap g(\partial_\alpha)(R_h)$ is infinite-dimensional. Thus the kernel of $\partial_\alpha - u$ restricted to R_h is infinite-dimensional.

The mapping

$$\tau\colon V(m, h, \alpha) \to K(X) \text{ where } \tau\colon \begin{pmatrix} r \\ s \end{pmatrix} \mapsto ur + s,$$

defines a non-zero module homomorphism $\mathbb{V}(m, h, \alpha) \to \mathcal{R}$. For each r in R_h, the element $\left(\begin{smallmatrix} r \\ -\partial_\alpha(r) \end{smallmatrix}\right) \in V(m, h, \alpha)$. We have

$$\tau \begin{pmatrix} r \\ -\partial_\alpha(r) \end{pmatrix} = ur - \partial_\alpha(r) = -(\partial_\alpha - u)(r).$$

Since $\ker(\partial_\alpha - u)$ is infinite-dimensional, so is $\ker \tau$. If $\tau \neq 0$, we have found a non-zero homomorphism $\mathbb{V}(m, h, \alpha) \to \mathcal{R}$ with infinite-dimensional kernel, so that using [2, Proposition 2.1], $\mathbb{V}(m, h, \alpha)$ is not purely simple. If $\tau = 0$, then $\partial_\alpha - u$ vanishes on R_h. In that case the regulator is linear, namely $Y - u$. From this it follows by Proposition 2.3 that $\mathbb{V}(m, h, \alpha)$ has a finite-dimensional summand, and thus $\mathbb{V}(m, h, \alpha)$ is still not purely simple.

Conversely suppose $\mathbb{V}(m,h,\alpha)$ is not purely simple. In that case there is a non-zero homomorphism $\mathbb{V}(m,h,\alpha) \to \mathcal{R}$ with infinite-dimensional kernel. This means there are rational functions u, t not both zero such that the map

$$\tau\colon V(m,h,\alpha) \to K(X) \text{ given by } \tau\colon \begin{pmatrix} r \\ s \end{pmatrix} \mapsto ur + ts$$

has infinite-dimensional kernel. If $t = 0$, then $u \neq 0$ and $\ker \tau$ is the finite-dimensional space $(0) \times P_m$ that, as revealed by (5), sits inside $V(m,h,\alpha)$. Hence $t \neq 0$, and multiplying through by t^{-1} we can suppose $t = 1$. The space R_h is embedded with finite codimension inside V according to $r \mapsto \left(\begin{smallmatrix} r \\ -\partial_\alpha(r) \end{smallmatrix}\right)$. Hence the composite $R_h \to K(X)$ of this embedding followed by τ, namely the map $r \mapsto (u - \partial_\alpha)(r)$, has infinite-dimensional kernel. Divide $f(Y)$ by $Y - u$ to get

$$f(Y) = g(Y)(Y-u) + r \text{ where } g(Y) \in K(X)[Y],\ r \in K(X).$$

Since $f(Y)$ is the regulator, the K-linear operator $g(\partial_\alpha) \circ (\partial_\alpha - u) + r$ has finite rank on R_h. Because $\partial_\alpha - u$ has infinite-dimensional kernel inside R_h, the same holds for $g(\partial_\alpha) \circ (\partial_\alpha - u)$. Consequently the multiplier r, being the difference of a finite rank operator and an operator with an infinite-dimensional kernel, must have non-zero kernel. This forces $r = 0$. Hence u is root of $f(Y)$. ■

Corollary 2.8 *If the module $\mathbb{V}(m,h,\alpha)$ is purely simple and has non-trivial endomorphisms, then the regulator of (h,α) is a quadratic irreducible over $K(X)$. Furthermore,* $\operatorname{End} \mathbb{V}(m,h,\alpha)$ *embeds in a quadratic field extension of $K(X)$.*

Proof From Proposition 2.2 the regulator is linear or quadratic. By Proposition 2.7 it has no root in $K(X)$. In that case it has to be quadratic and irreducible. In addition Proposition 2.6 ensures that $\operatorname{End} \mathbb{V}(m,h,\alpha)$ sits inside a quadratic field extension of $K(X)$. ■

Pure Simplicity and Zero Regulator

When the regulator of (h,α) is 0, pure simplicity of $\mathbb{V}(m,h,\alpha)$ can go either way.

If (h,α) is taken as in Example 1, the regulator is 0. We check now that for such (h,α) the modules $\mathbb{V}(m,h,\alpha)$ are not purely simple. Take the mapping

$$\tau\colon V(m,h,\alpha) \to K(X)$$

defined by $\tau\colon \left(\begin{smallmatrix} r \\ s \end{smallmatrix}\right) \mapsto s$ for every $\left(\begin{smallmatrix} r \\ s \end{smallmatrix}\right)$ in $V(m,h,\alpha)$. As noted prior to Proposition 2.7 such a map determines a module homomorphism $\mathbb{V}(m,h,\alpha) \to \mathcal{R}$. Since $\left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right) \in V(m,h,\alpha)$, we see that τ is non-zero. The elements $\left(\begin{smallmatrix} r \\ -\partial_\alpha(r) \end{smallmatrix}\right)$ where $r \in K(X)$ belong to $V(m,h,\alpha)$. When $r \in K[X]$ we see from (4) and the definition of α that $\partial_\alpha(r) = 0$. Thus $\left(\begin{smallmatrix} r \\ 0 \end{smallmatrix}\right) \in V(m,h,\alpha)$ for every polynomial r. Since these elements are in the kernel of τ, we have produced a non-zero homomorphism $\mathbb{V}(m,h,\alpha) \to \mathcal{R}$ with infinite-dimensional kernel. According to [2, Proposition 2.1], the module $\mathbb{V}(m,h,\alpha)$ is not purely simple.

On the other hand, here is a scheme for building a pair (h,α) with regulator 0 such that $\mathbb{V}(m,h,\alpha)$ is purely simple. Let I be an infinite subset of K and let h be the height function of Example 2 with pole space $R_h = K + \sum_{\theta\in I} KX_\theta$. Suppose that a function $\eta\colon I \to K$ denoted by $\theta \to \eta_\theta$ can be chosen so that for any polynomial $f(X,Y)$ in $K(X)[Y]$, monic in Y, the set

$$\{\theta \in I : f(\theta,\eta_\theta) \text{ is defined and } f(\theta,\eta_\theta) \neq 0\}$$

is infinite. Let α be any functional that satisfies $\langle \alpha, X_\theta\rangle = -\eta_\theta$ for all θ in I. According to the criterion of Example 2, no monic polynomial $f(X,Y)$ can cause the operator $f(X,\partial_\alpha)$ to have finite-rank on R_h. Consequently the regulator of such (h,α) will have to be 0. Suppose in addition that the function $\eta\colon I \to K$ can be chosen so that for any rational function r in $K(X)$, the set of agreement

$$\{\theta \in I : r(\theta) \text{ is defined and } r(\theta) = \eta_\theta\}$$

is finite. In that case [7, Theorem D] shows that the module $\mathbb{V}(m,h,\alpha)$ will be purely simple.

In light of the above remarks, a pair (h,α) having 0 regulator and such that $\mathbb{V}(m,h,\alpha)$ is purely simple will be found provided there is an infinite subset I of K and a function $\eta\colon I \to K$ such that

- the graph of η misses every curve $f(X,Y) = 0$ for infinitely many θ in I;
- no rational function r agrees with η on an infinite subset of I.

The procurement of an explicit formula for such a function for every field K and every infinite subset I remains a small mystery. Nevertheless, if K is the complex numbers $\mathbb{C}$ and I is the set of positive integers, the exponential function $\eta_\theta = e^\theta$ for every positive integer θ, does even more. In fact, given a monic polynomial

$$f(X,Y) = Y^n + r_{n-1}(X)Y^{n-1} + \cdots + r_1(X)Y + r_0(X) \text{ in } \mathbb{C}(X)[Y],$$

the equation $f(\theta, e^\theta) = 0$ can only have finitely many solutions in positive integers θ. This is because as θ tends to ∞ through positive integers, the term $e^{n\theta}$ grows faster than the lower terms $r_j(\theta)e^{j\theta}$. Thus any curve $f(X,Y) = 0$ meets the graph of η on only a finite set, which certainly fulfills the first requirement above. In addition taking $f(X,Y)$ of degree $n = 1$, we see that the second requirement is met as well. Thus we have at least one pair (h,α) regulated by 0 and such that $\mathbb{V}(m,h,\alpha)$ is purely simple.

3 Analyzing and Building Endomorphisms

We represent a $K(X)$-linear operator φ on $K(X)^2$ as usual by (8), and look for computational devices for deciding when φ is an endomorphism of $\mathbb{V}(m,h,\alpha)$.

The Maximal Subspace $\mathfrak{J}$ of $\operatorname{End}\mathbb{V}(m,h,\alpha)$

Let $\mathfrak{J}$ denote the space of those ψ that are endomorphisms of $\mathbb{V}(m,h,\alpha)$ and for which $\nu = -\partial_\alpha(t)$. In other words $\mathfrak{J}$ is the space of endomorphisms that take the

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form

$$\begin{bmatrix} s & t \\ u & -\partial_\alpha(t) \end{bmatrix} \text{ where } s, t, u \in K(X). \tag{15}$$

The set of scalar endomorphisms is denoted naturally as KI.

Proposition 3.1 $\operatorname{End} \mathbb{V}(m, h, \alpha) = KI \oplus \mathcal{J}$.

Proof Let φ as in (8) be an endomorphism of $\mathbb{V}(m, h, \alpha)$. If $p \in P_m$, the element $\binom{0}{p} \in V(m, h, \alpha)$. Hence its φ-image $\binom{tp}{vp}$ is in $V(m, h, \alpha)$, so that by (5)

$$\partial_\alpha(tp) + vp \in P_m \text{ for all } p \text{ in } P_m.$$

By (3) applied to $\partial_\alpha(tp)$ the above becomes

$$(\partial_\alpha(t) + v)p + \partial_{\alpha * t}(p) \in P_m \text{ for all } p \text{ in } P_m.$$

Put $p = 1$ and use (4) to obtain $\partial_\alpha(t) + v \in P_m$. Again by (4), $\partial_{\alpha * t}(p)$ is a polynomial of degree at most $m - 2$. A consideration of degrees above implies that $\partial_\alpha(t) + v \in K$. Putting $\lambda = \partial_\alpha(t) + v$ we have

$$\varphi = \lambda I + \begin{bmatrix} s - \lambda & t \\ u & -\partial_\alpha(t) \end{bmatrix}.$$

We have shown that $\operatorname{End} V(m, h, \alpha) = KI + \mathcal{J}$. To see that the sum is direct, just observe that the only scalar matrix of the type (15) is the zero matrix. ∎

Testing Membership in $\mathcal{J}$

In light of Proposition 3.1, $\mathbb{V}(m, h, \alpha)$ will possess a non-trivial endomorphism if and only if $\mathcal{J}$ has a non-zero endomorphism. In the next result we interpret in more detail what it means for a matrix of the form (15) to be in $\mathcal{J}$. The proof comes from a straightforward application of (5) and of (3).

Proposition 3.2 *A matrix φ as in* (15) *belongs to $\mathcal{J}$ if and only if*

$$tP_m \subseteq R_h, \tag{16}$$

$$(s - t \circ \partial_\alpha)R_h \subseteq R_h, \tag{17}$$

$$(\partial_\alpha(s) + u + \partial_{\alpha * s} - \partial_{\alpha * t} \circ \partial_\alpha)\, R_h \subseteq P_m. \tag{18}$$

Proof The elements of $V(m, h, \alpha)$ are sums of elements $\binom{0}{p}$ and $\binom{r}{-\partial_\alpha(r)}$ where $p \in P_m$ and $r \in R_h$. Our φ is an endomorphism if and only if φ maps all such elements into $V(m, h, \alpha)$.

For p in P_m we have $\varphi\left(\begin{smallmatrix}0\\p\end{smallmatrix}\right)=\left(\begin{smallmatrix}tp\\-\partial_\alpha(t)p\end{smallmatrix}\right)$. This element is in $V(m,h,\alpha)$ if and only if $tp\in R_h$ and $\partial_\alpha(tp)-\partial_\alpha(t)p\in P_m$. Using (3) for $\partial_\alpha(tp)$ we see that

$$\partial_\alpha(tp)-\partial_\alpha(t)p=\partial_{\alpha*t}(p),$$

which from (4) certainly lies in P_m. Thus we see that φ maps each $\left(\begin{smallmatrix}0\\p\end{smallmatrix}\right)$ into $V(m,h,\alpha)$ if and only if (16) holds.

If $r\in R_h$ the φ-image of $\left(\begin{smallmatrix}r\\-\partial_\alpha(r)\end{smallmatrix}\right)$ is $\left(\begin{smallmatrix}sr-t\partial_\alpha(r)\\ur+\partial_\alpha(t)\partial_\alpha(r)\end{smallmatrix}\right)$. This element lies in $V(m,h,\alpha)$ if and only if $sr-t\partial_\alpha(r)\in R_h$ and $\partial_\alpha(sr-t\partial_\alpha(r))+ur+\partial_\alpha(t)\partial_\alpha(r)\in P_m$. The first of these conditions is just (17). By applying (3) to $\partial_\alpha(sr)$ and to $\partial_\alpha(t\partial_\alpha(r))$ the second condition simplifies down to $\partial_\alpha(s)r+ur+\partial_{\alpha*s}(r)-\partial_{\alpha*t}(\partial_\alpha(r))\in P_m$ which is what (18) says. ■

Since $1\in P_m$, condition (16) implies $t\in R_h$. Also since $1\in R_h$ and since derivers kill scalars as per (4), condition (17) implies $s\in R_h$.

Controlling the Height Function

If the module $\mathbb{V}(m,h,\alpha)$ has $h(\infty)<\infty$, then an argument based on equivalence of height functions (see *e.g.*, [3, §3] or [17, §6]) shows that $\mathbb{V}(m,h,\alpha)$ is isomorphic to a module $\mathbb{V}(m,k,\beta)$ in which $k(\infty)=m$. This observation makes it acceptable to operate under the assumption that $\mathbb{V}(m,h,\alpha)$ is a module in which $h(\infty)\geq m$.

Proposition 3.3 *If $\mathbb{V}(m,h,\alpha)$ is such that $h(\infty)\geq m$ and $\left[\begin{smallmatrix}s&t\\u&-\partial_\alpha(t)\end{smallmatrix}\right]$ as in* (15) *belongs to $\mathfrak{J}$, then $\partial_\alpha(s)+u=0$.*

Proof Since $1\in R_h$ and derivers kill scalars, we get $\partial_\alpha(s)+u\in P_m$ from condition (18). The assumption on h gives $X^m\in R_h$. From (18) again we get

$$(\partial_\alpha(s)+u)X^m+(\partial_{\alpha*s}-\partial_{\alpha*t}\circ\partial_\alpha)(X^m)\in P_m.$$

Since derivers reduce degree as in (4), we see that $(\partial_{\alpha*s}-\partial_{\alpha*t}\circ\partial_\alpha)(X^m)\in P_m$. Consequently $(\partial_\alpha(s)+u)X^m\in P_m$. This forces $\partial_\alpha(s)+u=0$. ■

In light of Proposition 3.3 we see that when $h(\infty)\geq m$, condition (18) simplifies to the statement

$$(\partial_{\alpha*s}-\partial_{\alpha*t}\circ\partial_\alpha)R_h\subseteq P_m. \tag{19}$$

Let us gather the bits we have obtained so far in this section.

Proposition 3.4 *If $\mathbb{V}(m,h,\alpha)$ is such that $h(\infty)\geq m$, then a matrix φ as in* (8) *belongs to $\mathfrak{J}$ if and only if φ takes the form*

$$\varphi=\begin{bmatrix}s&t\\-\partial_\alpha(s)&-\partial_\alpha(t)\end{bmatrix}\quad \textit{where } s,t\in R_h, \tag{20}$$

and (16), (17), *and* (19) *hold.*

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Of the conditions (16), (17), and (19), the most important is the last one. If $\theta \in K$, let $K[X_\theta]$ be the pole space of polynomials in X_θ. The statement $K[X_\theta] \subseteq R_h$ is equivalent to having $h(\theta) = \infty$. Also the statement $K[X] \subseteq R_h$ is equivalent to having $h(\infty) = \infty$.

Proposition 3.5 *Suppose s, t in $K(X)$ satisfy* (19) *for a given height function h and a positive integer m. If $\theta \in K$ and $h(\theta) = \infty$, then $\partial_{\alpha * s} = \partial_{\alpha * t} \circ \partial_\alpha$ on $K[X_\theta]$. Furthermore, if $h(\infty) = \infty$, then $\partial_{\alpha * s} = \partial_{\alpha * t} \circ \partial_\alpha$ on $K[X]$.*

Proof Since $K[X_\theta] \subseteq R_h$, (19) gives $(\partial_{\alpha * s} - \partial_{\alpha * t} \circ \partial_\alpha)K[X_\theta] \subseteq P_m$. The operator $\partial_{\alpha * s} - \partial_{\alpha * t} \circ \partial_\alpha$ is a deriver because [16, Proposition 2.3] shows that the set of derivers is an algebra. Due to (4) this deriver kills scalars and leaves the space $X_\theta K[X_\theta]$ invariant. Since the intersection $P_m \cap X_\theta K[X_\theta] = (0)$ it follows that $\partial_{\alpha * s} - \partial_{\alpha * t} \circ \partial_\alpha$ vanishes on $K[X_\theta]$.

If $h(\infty) = \infty$, then $K[X] \subseteq R_h$ and (19) gives $(\partial_{\alpha * s} - \partial_{\alpha * t} \circ \partial_\alpha)K[X] \subseteq P_m$. Hence the deriver $\partial_{\alpha * s} - \partial_{\alpha * t} \circ \partial_\alpha$ has finite rank on $K[X]$. An examination of (4) shows that only the zero deriver on $K[X]$ can have finite rank on $K[X]$. Thus $\partial_{\alpha * s} - \partial_{\alpha * t} \circ \partial_\alpha$ vanishes on $K[X]$. ∎

From Propositions 3.4 and 3.5 we extract the following.

Corollary 3.6 *If $\mathbb{V}(m, h, \alpha)$ is such that $h(\infty) \geq m$ and φ as in* (8) *belongs to $\mathfrak{J}$, then φ takes the form* (20) *and for any θ in K where $h(\theta) = \infty$, $\partial_{\alpha * s} = \partial_{\alpha * t} \circ \partial_\alpha$ on $K[X_\theta]$. Furthermore, if $h(\infty) = \infty$, then $\partial_{\alpha * s} = \partial_{\alpha * t} \circ \partial_\alpha$ on $K[X]$.*

Laurent Coefficients of Functions and Functionals

For each θ in K, the field $K((X - \theta))$ of Laurent series in $X - \theta$ is the completion of $K(X)$ with respect to the valuation ord_θ. There is a unique field embedding $K(X) \to K((X-\theta))$ that fixes the common subalgebra $K[X-\theta, X_\theta]$ of Laurent polynomials in $X - \theta$. The field $K((X^{-1}))$ of Laurent series in X^{-1} is the completion of $K(X)$ using the valuation ord_∞. Again there is a unique field embedding $K(X) \to K((X^{-1}))$ that fixes the common subalgebra $K[X^{-1}, X]$ of Laurent polynomials in X^{-1}. Such matters are discussed in [11, Ch. 2]. If r is a non-zero rational function and $\theta \in K$, we shall denote the image of r in $K((X-\theta))$ simply as r again, and write its expansion in $K((X - \theta))$ as follows:

$$(21) \qquad r = \sum_{n=m}^{\infty} r_n^\theta (X - \theta)^n \ \text{ where } m = -\mathrm{ord}_\theta(r),\ r_n^\theta \in K \text{ and } r_m^\theta \neq 0.$$

Likewise the image of r in $K((X^{-1}))$ will be denoted simply as r again, and as an expansion in $K((X^{-1}))$ we write

$$(22) \qquad r = \sum_{n=m}^{\infty} r_n^\infty X^{-n} \ \text{ where } m = -\mathrm{ord}_\infty(r),\ r_n^\infty \in K \text{ and } r_m^\infty \neq 0.$$

The valuations ord_θ and ord_∞ extend to valuations on $K((X-\theta))$ and $K((X^{-1}))$, respectively. The extended valuations give the negative of the least power of $X-\theta$ or of X^{-1} that appears in a given expansion. These completions were exploited in [16] to study the endomorphism algebras of the modules $\mathbb{V}(m,h,\alpha)$ in the important case where $R_h = K(X)$, but we need them here as well.

Each functional α can be assigned an expansion in the subalgebra $K[[X-\theta]]$ of $K((X-\theta))$ by putting

$$\alpha^\theta = \sum_{n=0}^{\infty} \langle \alpha, X_\theta^{n+1} \rangle (X-\theta)^n. \tag{23}$$

Also we can put

$$\alpha^\infty = \sum_{n=0}^{\infty} \langle \alpha, X^n \rangle X^{-n}, \tag{24}$$

in the subalgebra $K[[X^{-1}]]$ of $K((X^{-1}))$. Through its coefficients the expansion α^θ determines the action of α on the space $X_\theta K[X_\theta]$, and likewise the expansion α^∞ determines the action of α on the space of polynomials $K[X]$. Taken together these expansions determine the functional on $K(X)$.

With the notation (21), (22), (23), and (24) adopted above, products such as $r\alpha^\theta$ and $r\alpha^\infty$ taken in $K((X-\theta))$ and $K((X^{-1}))$, respectively, are clearly defined. In [16, Proposition 2.6] it is shown that for a functional α and a rational function r, the coefficients of the functional $\alpha * r$ are given by

$$(\alpha * r)^\theta = r\alpha^\theta + \partial_\alpha(r) \ \text{ for every } \theta \text{ in } K, \tag{25}$$

$$(\alpha * r)^\infty = r\alpha^\infty - X\partial_\alpha(r). \tag{26}$$

Given functionals α and β, it follows from [16, Proposition 2.3] that the composite $\partial_\alpha \circ \partial_\beta$ is again a deriver. If $\theta \in K$ and γ is another functional, the proposition also says that

$$\partial_\gamma = \partial_\alpha \circ \partial_\beta \ \text{ on the pole space } K[X_\theta] \ \Leftrightarrow \ \gamma^\theta = -\alpha^\theta \beta^\theta. \tag{27}$$

Furthermore, it says that

$$\partial_\gamma = \partial_\alpha \circ \partial_\beta \ \text{ on the pole space } K[X] \ \Leftrightarrow \ \gamma^\infty = X^{-1}\alpha^\infty \beta^\infty. \tag{28}$$

For each θ in K the field $K(X)$ sits inside its θ-completion $K((X-\theta))$. Thus any polynomial $g(Y)$ in $K(X)[Y]$ is also a polynomial in $K((X-\theta))[Y]$. Likewise $g(Y)$ is also in $K((X^{-1}))[Y]$. Now we are enabled to make the following interpretation of the identities in Proposition 3.5.

Proposition 3.7 *Let α be a functional, let $s, t \in K(X)$ and $\theta \in K$. Then*

$$\partial_{\alpha * s} = \partial_{\alpha * t} \circ \partial_\alpha \ \textit{ on } \ K[X_\theta] \tag{29}$$

if and only if

$$\alpha^\theta \text{ is a root in } K((X-\theta)) \text{ of } tY^2 + (s+\partial_\alpha(t))Y + \partial_\alpha(s). \tag{30}$$

Furthermore,

$$\partial_{\alpha * s} = \partial_{\alpha * t} \circ \partial_\alpha \text{ on } K[X] \tag{31}$$

if and only if

$$\alpha^\infty \text{ is a root in } K((X^{-1})) \text{ of } tY^2 - X(s+\partial_\alpha(t))Y + X^2\partial_\alpha(s). \tag{32}$$

Proof Let γ be the functional that gives the deriver $\partial_{\alpha * t} \circ \partial_\alpha$.

Using (4), the condition (29) is equivalent to having $\alpha * s$ agree with γ on $X_\theta K[X_\theta]$. In turn, using (23), this is the same as having $(\alpha * s)^\theta = \gamma^\theta$. From (27) and (25), this becomes $s\alpha^\theta + \partial_\alpha(s) = -(\alpha * t)^\theta \alpha^\theta = -(t\alpha^\theta + \partial_\alpha(t))\alpha^\theta$. After expanding out and rearranging, this becomes (30).

Again from (4), condition (31) is the same as having $\alpha * s$ agree with γ on $K[X]$. In turn using (24) this is the same as $(\alpha * s)^\infty = \gamma^\infty$. From (28) and (26) this becomes

$$s\alpha^\infty - X\partial_\alpha(s) = X^{-1}(\alpha * t)^\infty \alpha^\infty = X^{-1}(t\alpha^\infty - X\partial_\alpha(t))\alpha^\infty.$$

By expanding and rearranging, this comes down to

$$X^{-1}t(\alpha^\infty)^2 - (s+\partial_\alpha(t))\alpha^\infty + X\partial_\alpha(s) = 0,$$

which is the same as (32) after multiplication by$X$. ∎

Purely Simple Modules with Non-Trivial Endomorphisms

Regrettably, it seems necessary to avoid characteristic 2 in the following result.

Theorem 3.8 *Suppose* char $K \neq 2$. *If* $\mathbb{V}(m,h,\alpha)$ *is purely simple with non-trivial endomorphisms, then* h *must assume the value* ∞ *somewhere on* $\mathrm{K} \cup \{\infty\}$ *and be finite valued at least twice on* $\mathrm{K} \cup \{\infty\}$.

Proof As explained prior to Proposition 3.3 we can assume that $h(\infty) \geq m$. This enables us to apply Corollary 3.6.

Since $\mathbb{V}(m,h,\alpha)$ has non-trivial endomorphisms, there must be a non-zero endomorphism φ in $\mathcal{J}$. Corollary 3.6 applies for this non-zero φ, which takes the form of (20). By Proposition 3.7 it follows that (30) holds for all θ where $h(\theta) = \infty$. For such θ the discriminant $w = (s+\partial_\alpha(t))^2 - 4t\partial_\alpha(s)$ of the polynomial in (30) must be a perfect square in $K((X-\theta))$. Consequently the integers $\mathrm{ord}_\theta(w)$ are even for all θ in K where $h(\theta) = \infty$. In case $h(\infty) = \infty$, Corollary 3.6 followed by Proposition 3.7 yield (32). Now the discriminant $X^2((s+\partial_\alpha(t))^2 - 4t\partial_\alpha(s))$ of the polynomial in (32) is a perfect square in $K((X^{-1}))$. This is the same as saying that when $h(\infty) = \infty$, the function w above is a perfect square in $K((X^{-1}))$. In turn this causes $\operatorname{ord}_\infty(w)$ to be an even integer. Hence $\operatorname{ord}_\theta(w)$ is even for all θ in $\mathrm{K} \cup \{\infty\}$ at which $h(\theta) = \infty$.

By Corollary 2.8 the regulator of (h, α) is irreducible in $K(X)[Y]$ and quadratic. Since φ is given as in (20), its tag polynomial in accordance with (11) is

$$tY^2 - (s + \partial_\alpha(t))Y + \partial_\alpha(s).$$

By Proposition 2.1 the regulator of (h, α) divides the tag polynomial of φ. Since $\varphi \neq 0$, the tag polynomial of φ is non-zero, and thereby is an irreducible quadratic in Y. The discriminant of the tag polynomial is the very same function w encountered above.

Because $\operatorname{char} K \neq 2$, irreducibility of the tag polynomial implies that w is not a perfect square in $K(X)$. From the unique factorization as in (1) of w, $\operatorname{ord}_\theta(w)$ must be odd for some θ in K. The well-known identity

$$\operatorname{ord}_\infty(w) + \sum_{\theta \in K} \operatorname{ord}_\theta(w) = 0$$

shows that the number of θ in $\mathrm{K} \cup \{\infty\}$ for which $\operatorname{ord}_\theta(w)$ is odd must be even. In particular $\operatorname{ord}_\theta(w)$ is odd at least twice on $\mathrm{K} \cup \{\infty\}$. We saw at the beginning of the proof that $\operatorname{ord}_\theta(w)$ is even whenever $h(\theta) = \infty$. Consequently $h(\theta) < \infty$ at least twice on $\mathrm{K} \cup \{\infty\}$.

To get that $h(\theta) = \infty$ at least once on $\mathrm{K} \cup \{\infty\}$, we note from Corollary 2.8 that $\operatorname{End} \mathbb{V}(m, h, \alpha)$ has no zero divisors. Consequently, if h never assumed the value ∞, we could invoke [14, Theorem 3.4] to contradict the assumption that $\operatorname{End} \mathbb{V}(m, h, \alpha)$ is non-trivial. ∎

Example 3 (A general construction using $h(\infty) = \infty$) We turn to the construction of a broad family of purely simple $\mathbb{V}(m, h, \alpha)$ that possess non-trivial endomorphisms. Asssuming $\operatorname{char} K \neq 2$, this construction is both a simplification and a generalization of [13, Lemma 4.8].

Start with any quadratic polynomial

$$f(Y) = Y^2 + pY + q \tag{33}$$

where $p, q \in K[X], p \neq 0, q \neq 0$ and $\deg q < \deg p$. A simple check confirms that $f(Y)$ is irreducible over $K(X)$. Let $w = p^2 - 4q$ be the discriminant of $f(Y)$. Since $\deg p > \deg q$, we see that $\deg w = 2 \deg p \geq 2$, and w has at least one root in K. If w had only one root in K, then w, being of even degree, would be a perfect square. Hence $f(Y)$ would have roots in $K(X)$ contrary to its irreducibility. Thus w has at least two roots. Let k be the height function defined by the formula:

$$k(\theta) = \begin{cases} 0 & \text{when } \theta \text{ is a root of the discriminant } w, \\ \infty & \text{when } \theta \in K \text{ but not a root of } w, \\ \infty & \text{when } \theta = \infty. \end{cases} \tag{34}$$

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We will now build a functional α such that (k, α) is regulated by the chosen polynomial $f(Y)$. By Proposition 2.7 this will ensure that the module $\mathbb{V}(m, k, \alpha)$ is purely simple. In addition, we will see that the generic matrix D as in (13) of $f(Y)$ is an endomorphism of $\mathbb{V}(m, k, \alpha)$. The choice of m will not matter.

A good way to specify a functional is to select a series σ_θ in $K[[X-\theta]]$ for each θ in K and series σ_∞ in $K[[X^{-1}]]$, and then to take α to be the unique functional for which α^θ as in (23) and α^∞ as in (24) equals σ_θ and σ_∞, respectively. Thus we speak of selecting the α^θ and α^∞ that determine α.

For θ that is a root of w, it will not matter how α^θ is selected, but just to fix it let α^θ be zero. To select α^θ when θ is a non-root of w, notice that $\operatorname{ord}_\theta(w) = 0$. Since this is an even number, w is a perfect square in $K((X-\theta))$, and also $\operatorname{ord}_\theta(\sqrt{w}) = 0$. Consequently the polynomial $Y^2 - pY + q$, having discriminant w, has roots $(p \pm \sqrt{w})/2$ in $K((X-\theta))$ These roots are in the subalgebra $K[[X-\theta]]$ since p and $\sqrt{w}$ are in $K[[X-\theta]]$. Now select α^θ to be any one of the two roots of $Y^2 - pY + q$ that lie in $K[[X-\theta]]$.

To select α^∞, notice that the polynomial $Y^2 + XpY + X^2q$ has roots in the field $K((X^{-1}))$ because its discriminant X^2w is such that $\operatorname{ord}_\infty(X^2w) = 2 + 2\deg p$, an even number. That makes the discriminant a perfect square in $K((X^{-1}))$ and assures the desired roots in $K((X^{-1}))$. One of those two roots must lie in the subalgebra $K[[X^{-1}]]$. To see this let σ, τ be the roots in $K((X^{-1}))$ of $Y^2 + XpY + X^2q$. Then $\sigma + \tau = -Xp$ and $\sigma\tau = X^2q$. Taking the valuation $\operatorname{ord}_\infty$ and noting that $\deg p > \deg q$ we get

$$\begin{aligned}\max\{\operatorname{ord}_\infty(\sigma), \operatorname{ord}_\infty(\tau)\} &\geq \operatorname{ord}_\infty(\sigma+\tau) = 1 + \deg p \geq 2 + \deg q\\ &= \operatorname{ord}_\infty(\sigma\tau) = \operatorname{ord}_\infty(\sigma) + \operatorname{ord}_\infty(\tau).\end{aligned}$$

When the sum of two integers is no more than their maximum, one of the integers is at most 0. Thus either $\operatorname{ord}_\infty(\sigma) \leq 0$ or $\operatorname{ord}_\infty(\tau) \leq 0$. This says exactly that one of σ or τ is in $K[[X^{-1}]]$. Incidentally, the other root is not in $K[[X^{-1}]]$ since the roots add up to $-Xp$ which is not in $K[[X^{-1}]]$. Now select α^∞ to be that root of $Y^2 + XpY + X^2q$ which lies in $K[[X^{-1}]]$.

Having completely specified the functional α, we begin to show that the generic matrix $D = \left[\begin{smallmatrix} p & -1 \\ q & 0 \end{smallmatrix}\right]$ as in (13), for the chosen polynomial $f(Y)$, is an endomorphism of $\mathbb{V}(m, k, \alpha)$. For that we need to show that $q = -\partial_\alpha(p)$. We can decompose a Laurent series σ in $K((X^{-1}))$ as $\sigma = u + \tau$ where u is a polynomial in X and $\tau \in K[[X^{-1}]]$. We may call u the polynomial part of σ, and denote it by $P_\infty(\sigma)$. We have selected α^∞ so that $(\alpha^\infty)^2 + Xp\alpha^\infty + X^2q = 0$ and thus $X^{-2}(\alpha^\infty)^2 + X^{-1}p\alpha^\infty + q = 0$. Taking the polynomial part of this and noting that $P_\infty(X^{-2}(\alpha^\infty)^2) = 0$, we obtain $P_\infty(q) = -P_\infty(X^{-1}p\alpha^\infty)$. Using [16, Proposition 2.2] we have $P_\infty(\partial_\alpha(p)) = P_\infty(X^{-1}p\alpha^\infty)$. (In these formulas from [16], p, q were denoted by p^∞, q^∞, respectively.) Hence $P_\infty(q) = -P_\infty(\partial_\alpha(p))$. Since q and $\partial_\alpha(p)$ are polynomials in X, each equals its own polynomial part. Therefore $q = -\partial_\alpha(p)$.

Now to check that the generic matrix D is an endomorphism of $\mathbb{V}(m, k, \alpha)$, it will do to verify that D meets the criteria of Proposition 3.4. Because $k(\infty) = \infty$ we have

$$p \in K[X] \subseteq R_k,$$

and since $q = -\partial_\alpha(p)$, the matrix D takes the form (20) and satisfies (16) using $t = -1$. Condition (17) becomes $(p + \partial_\alpha)R_k \subseteq R_k$. This is met because derivers leave pole spaces invariant and because p as seen above lies in the pole algebra of R_k. To get the important condition (19) notice k was defined so that

$$R_k = K[X] + \sum_{w(\theta)\neq 0} K[X_\theta].$$

Thus (19) will be met if we can show that $\partial_{\alpha * p} = -\partial_\alpha^2$ on every $K[X_\theta]$ for which $w(\theta) \neq 0$, and that $\partial_{\alpha * p} = -\partial_\alpha^2$ on $K[X]$. These latter conditions are nothing but (29) when $w(\theta) \neq 0$ and (31), where we specialize $s = p$ and $t = -1$. We now see this by calling upon Proposition 3.7. Indeed if $w(\theta) \neq 0$, then α^θ, as a root of $Y^2 - pY + q$ in $K[[X - \theta]]$, has been selected precisely to satisfy (30) with $s = p, t = -1$. Also α^∞, as a root of $Y^2 + XpY + q$ in $K[[X^{-1}]]$, has been selected precisely to satisfy (32) with $s = p, t = -1$.

Our module $\mathbb{V}(m, k, \alpha)$ now has a non-trivial endomorphism. It remains to check that (k, α) is regulated by $Y^2 + pY + q$. Proposition 2.7 will ensure the desired pure simplicity of $\mathbb{V}(m, k, \alpha)$. It will also afford us a broad family of constructed regulators. Since D is an endomorphism, Proposition 2.1 says that the regulator of (k, α) divides the tag polynomial (11) of D. The tag polynomial of D is $-(Y^2 + pY + q)$, an irreducible quadratic. The regulator must be the monic polynomial $Y^2 + pY + q$ that we started with. Our construction is complete.

Theorem 3.9 *Suppose* char $K \neq 2$. *If h is a height function such that $h(\infty) = \infty$, and $h(\theta) < \infty$ for at least two θ in K, then there is a purely simple module $\mathbb{V}(m, h, \alpha)$ which admits non-trivial endomorphisms.*

Proof Let η, ζ be two elements of K for which $h(\eta) < \infty$ and $h(\zeta) < \infty$. Using equivalence of height functions, as in [3, §3] or [17, §6], there is no loss in generality by presuming $h(\eta) = h(\zeta) = 0$. Take the polynomial

$$f(Y) = Y^2 + \left(X - \frac{\eta + \zeta}{2}\right)Y + \left(\frac{\eta - \zeta}{4}\right)^2.$$

Since $\eta \neq \zeta$, this polynomial lies in the family given by (33). Its discriminant is

$$w = \left(X - \frac{\eta + \zeta}{2}\right)^2 - 4\left(\frac{\eta - \zeta}{4}\right)^2 = X^2 - (\eta + \zeta)X + \eta\zeta,$$

having exactly the roots η and ζ. For this $f(Y)$ take the height function k precisely as in (34) and select α as in the construction following (34). The construction was designed so that the resulting module $\mathbb{V}(m, k, \alpha)$ is purely simple and has the endomorphism

$$D = \begin{bmatrix} p & -1 \\ q & 0 \end{bmatrix} \text{ where } p = X - \frac{\eta + \zeta}{2} \text{ and } q = \left(\frac{\eta - \zeta}{4}\right)^2.$$

Notice that $h \leq k$ and since $h(\infty) = \infty$ we have

$$p \in K[X] \subseteq R_h \subseteq R_k. \tag{35}$$

Since $f(Y)$ regulates (k, α), the operator $f(\partial_\alpha)$ has finite rank on R_k. Hence $f(\partial_\alpha)$ has finite rank on the lesser space R_h. Thus the regulator of (h, α) must be a monic factor of $f(Y)$. Because $f(Y)$ is already monic and irreducible, $f(Y)$ must be the regulator of (h, α). By Proposition 2.7, $\mathbb{V}(m, h, \alpha)$ is purely simple.

To see that D remains an endomorphism of $\mathbb{V}(m, h, \alpha)$ it suffices to invoke Proposition 3.4 for the matrix D and for the original height function h. Because of (35) the matrix D is of the type (20), and we can easily see that (16) and (17) hold. Finally (19) is true because it already holds using the larger space R_k. ∎

Completing the Main Theorem

Theorems 3.8 and 3.9 in conjunction with our final result furnish the proof of Theorem 1.1. Our final result also offers a slightly different construction of a purely simple module having non-trivial endomorphisms.

Theorem 3.10 *Suppose* $\operatorname{char} K \neq 2$. *If h is a height function such that*

$$h(\infty) < \infty,\ h(\eta) < \infty,\ h(\zeta) = \infty \ \textit{for some}\ \eta \ \textit{and some}\ \zeta \ \textit{in}\ K,$$

then there is a purely simple $\mathbb{V}(m, h, \alpha)$ *which admits non-trivial endomorphisms.*

Proof Take any positive integer m. Using equivalence of height functions we can use [3, §3] or [17, §6] and safely suppose that $h(\infty) = m$ and $h(\eta) = 0$. Let $\nu = \sqrt{\zeta - \eta}$ be one of the square roots of $\zeta - \eta$. Take the polynomial $f(Y) = Y^2 + 2\nu X_\zeta Y - X_\zeta$. This polynomial is irreducible over $K[X_\zeta]$ by Eisenstein's criterion, and being monic it is irreducible over $K(X)$, the fraction field of $K[X_\zeta]$. The discriminant of $f(Y)$ is

$$w = (2\nu X_\zeta)^2 + 4X_\zeta = 4\frac{X + \nu^2 - \zeta}{(X - \zeta)^2} = 4\frac{X - \eta}{(X - \zeta)^2}.$$

Since $\eta \neq \zeta$, we see that $\operatorname{ord}_\infty(w) = \operatorname{ord}_\eta(w) = -1$, $\operatorname{ord}_\zeta(w) = 2$ and $\operatorname{ord}_\theta(w) = 0$ for all other θ in K.

Now we specify a functional α so that $f(Y)$ is the regulator of (h, α) and so that its generic matrix

$$D = \begin{bmatrix} 2\nu X_\zeta & -1 \\ -X_\zeta & 0 \end{bmatrix}$$

is an endomorphism of $\mathbb{V}(m, h, \alpha)$. To select α it suffices to specify the expansions α^θ for each θ in $\mathrm{K} \cup \{\infty\}$. The choices of α^η and α^∞ do not matter, but to fix things put them equal to 0.

To specify α^θ when $\theta \neq \eta$ consider the polynomial $g(Y) = Y^2 - 2\nu X_\zeta Y - X_\zeta$ whose discriminant is w, the same as for $f(Y)$. For each θ in K, other than η and ζ, we have $\operatorname{ord}_\theta(w) = 0$. Hence w has a square root, say σ, in $K((X - \theta))$. In fact

$\sigma \in K[[X-\theta]]$ because $\mathrm{ord}_\theta(\sigma)$ must be 0. Hence the roots $(\nu X_\zeta \pm \sigma)/2$ of $g(Y)$ are in $K[[X-\theta]]$. Select α^θ to be any one of these roots of $g(Y)$.

It remains to specify α^ζ. Since $\mathrm{ord}_\zeta(w)$ is the even number 2, the discriminant w of the polynomial $g(Y)$ is a perfect square in $K((X-\zeta))$. Hence $g(Y)$ has a root in $K((X-\zeta))$. One of these two roots lies in the subalgebra $K[[X-\zeta]]$. Indeed, if ρ, τ are the roots of $g(Y)$ in $K((X-\zeta))$, we have $\rho\tau = -X_\zeta$ so that $\mathrm{ord}_\zeta(\rho)+\mathrm{ord}_\zeta(\tau) = 1$. Consequently $\mathrm{ord}_\zeta(\rho) \le 0$ or $\mathrm{ord}_\zeta(\tau) \le 0$, meaning that one of these roots lies in the subalgebra $K[[X-\zeta]]$. Select α^ζ to be that root of $g(Y)$ which lies $K[[X-\zeta]]$. Now α is completely specified.

In preparation for D to satisfy Proposition 3.4, we check that

$$\partial_\alpha(2\nu X_\zeta) = X_\zeta. \tag{36}$$

If $\sigma \in K((X-\zeta))$, let $P_\zeta(\sigma)$ denote the finite part of the expansion of σ that involves only the positive powers of X_ζ. In particular $P_\zeta(\sigma) = 0$ if and only if $\sigma \in K[[X-\zeta]]$. In [16] we refer to $P_\zeta(\sigma)$ as the principal ζ-part of σ. We have chosen α so that α^ζ is a root of $g(Y)$, *i.e.*, $(\alpha^\zeta)^2 - 2\nu X_\zeta \alpha^\zeta - X_\zeta = 0$. Take the principal ζ-part and notice that $P_\zeta((\alpha^\zeta)^2) = 0$ to obtain $-2\nu P_\zeta(X_\zeta\alpha^\zeta) = P_\zeta(X_\zeta)$. From [16, Proposition 2.1] we have that $P_\zeta(X_\zeta\alpha^\zeta) = -P_\zeta(\partial_\alpha(X_\zeta))$. These last two equalities give $2\nu P_\zeta(\partial_\alpha(X_\zeta)) = P_\zeta(X_\zeta)$. After observing that $P_\zeta(X_\zeta) = X_\zeta$ and from (4) that $P_\zeta(\partial_\alpha(X_\zeta)) = \partial_\alpha(X_\zeta)$, we obtain the desired $\partial_\alpha(2\nu X_\zeta) = X_\zeta$.

Now we check that D as above is an endomorphism of $\mathbb{V}(m, h, \alpha)$ by testing D against the criteria of Proposition 3.4, with $s = 2\nu X_\zeta, t = -1$. Since $h(\zeta) = \infty$ we have $X_\zeta \in K[X_\zeta] \subseteq R_h$. We checked that $-X_\zeta = -\partial_\alpha(2\nu X_\zeta)$ and clearly $0 = -\partial_\alpha(-1)$. Hence D takes the form (20). To get (16) observe that $(-1)P_m = P_m \subseteq R_h$ simply because $h(\infty) = m$. The condition (17) becomes $(2\nu X_\zeta + \partial_\alpha)R_h \subseteq R_h$. This holds because derivers leave pole spaces invariant and because, as seen above, X_ζ is in the pole algebra of R_h.

Most importantly we have to verify that (19) holds for D. Since $h(\infty) = m$, the height function h is such that

$$R_h \subset P_{m+1} + \sum_{\substack{\theta\in K\\ \theta\neq\eta}} K[X_\theta].$$

To obtain (19) it therefore suffices to show that using $s = 2\nu X_\zeta, t = -1$ we have

$$(\partial_{\alpha * s} - \partial_{\alpha * t} \circ \partial_\alpha) P_{m+1} \subseteq P_m \qquad \text{and} \tag{37}$$

$$\partial_{\alpha * s} = \partial_{\alpha * t} \circ \partial_\alpha \ \text{ on every } K[X_\theta] \text{ other than } K[X_\eta]. \tag{38}$$

According to (4), derivers lower the degree of any polynomial. Hence (37) holds no matter what s and t are. To get (38) we apply Proposition 3.7. The functional α was deliberately selected so that α^θ is a root of $g(Y)$ for every θ in K other than η. With $s = 2\nu X_\zeta$ and $t = -1$, we have $\partial_\alpha(t) = 0$ and $\partial_\alpha(s) = X_\zeta$, using (36). Then $g(Y)$ becomes $-tY^2 - (s + \partial_\alpha(t))Y - \partial_\alpha(s)$. As this is the negative of the polynomial in (30), Proposition 3.7 gives (38). Thus D is an endomorphism of $\mathbb{V}(m, h, \alpha)$.

Finally the regulator of (h, α) divides the tag polynomial of D. According to (11) that tag polynomial is $-Y^2 - 2\nu X_\zeta Y + X_\zeta$, the negative of our monic irreducible $f(Y)$. Thus the regulator of (h, α) is $f(Y)$ and Proposition 2.7 shows that $\mathbb{V}(m, h, \alpha)$ is purely simple. ■

Open Problem

We have seen that if $\mathbb{V}(m, h, \alpha)$ is purely simple with non-trivial endomorphisms, then $\operatorname{End}\mathbb{V}(m, h, \alpha)$ embeds in the quadratic function field $K(X)[Y]/(f)$, where f is the regulator of (h, α). It is not known exactly which K-subalgebras of this field arise as $\operatorname{End}\mathbb{V}(m, h, \alpha)$. For instance, subfields other than K do not arise, see [12]. Preliminary investigations indicate that the geometry of the curve given by f will be relevant.

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