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Cuntz Algebra States Defined by Implementers of Endomorphisms of the CAR Algebra

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Abstract. We investigate representations of the Cuntz algebra $\mathcal{O}_2$ on antisymmetric Fock space $F_a(\mathcal{K}_1)$ defined by isometric implementers of certain quasi-free endomorphisms of the CAR algebra in pure quasi-free states φ_{P_1}. We pay special attention to the vector states on $\mathcal{O}_2$ corresponding to these representations and the Fock vacuum, for which we obtain explicit formulae. Restricting these states to the gauge-invariant subalgebra $\mathcal{F}_2$, we find that for natural choices of implementers, they are again pure quasi-free and are, in fact, essentially the states φ_{P_1}. We proceed to consider the case for an arbitrary pair of implementers, and deduce that these Cuntz algebra representations are irreducible, as are their restrictions to $\mathcal{F}_2$.

The endomorphisms of $B\big(F_a(\mathcal{K}_1)\big)$ associated with these representations of $\mathcal{O}_2$ are also considered.

1 Introduction

In [2], generalizing the results of [15], the author studies the isometric implementation of non-surjective quasi-free endomorphisms of the self-dual CAR algebra in pure quasi-free states. The motivation for this was the author's speculation that implementable quasi-free endomorphisms might be useful in the construction of localized endomorphisms for free Fermi fields with non-abelian gauge groups. The explicit construction of sequences of implementers of such quasi-free endomorphisms is the main result of [2]. By definition, these satisfy the Cuntz relations and thus define representations of the Cuntz algebras on antisymmetric Fock space.

The Cuntz algebras are a class of C^*-algebras whose importance cannot be overstated. Since their introduction in [7], they and their representation theory have been the focus of a great deal of attention. In recent years, for example (in a setting very different from ours), there has been much interest in particular Cuntz algebra representations and their connection with multiresolution wavelet theory [4], [6], [12]. The study of Cuntz algebra representations on a Hilbert space $\mathcal{H}$ is not only interesting in its own right, but also because of its relation with the theory of endomorphisms of $B(\mathcal{H})$ [3], [13]. In this paper we set about investigating such representations which are defined by the implementers of quasi-free endomorphisms, in particular, addressing the question of their irreducibility.

In order to test for irreducibility, the approach we adopt is to consider the vector states on the Cuntz algebras associated with them and the antisymmetric Fock

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vacuum vector. Now the implementers constructed in [2] do not lend themselves in full generality to computations of the type to be found here, and so we carry out our investigations for the quasi-free endomorphism associated with the unilateral shift, with its implementation in specified quasi-free states. (Note that this means that we are dealing with representations of the Cuntz algebra $\mathcal{O}_2$.) We begin by obtaining explicit formulae for these states (for natural choices of implementers) and find that in fact, the vector states we obtain take a surprising and very interesting form. Consequently we focus further attention on them, particularly considering their restriction to the subalgebra fixed under the gauge action. It turns out that on identifying the gauge-invariant subalgebra $\mathcal{F}_2$ with the CAR algebra, these states when restricted to $\mathcal{F}_2$ and considered as states on the CAR algebra are again pure and quasi-free. Unexpectedly, they are in fact the quasi-free states in which the endomorphism is being implemented. (Though this finding arises as a by-product of our investigations into the Cuntz algebra representations, we feel it is our most interesting result.) Showing that the vacuum vector is cyclic for these Cuntz algebra representations, and also for their restrictions to the gauge-invariant subalgebra, we are able to deduce that both representations are irreducible. We also consider the case for an arbitrary pair of implementers.

Certainly, even for the specific quasi-free states chosen here, one could make similar computations to those found in this note for alternative quasi-free endomorphisms. For example one could simply consider the endomorphisms generated by higher powers of the unilateral shift, and still obtain simple formulae for the analogous vector states on the Cuntz algebras $\mathcal{O}_{2^n}$, for $n \geq 2$. However, it is the interesting formulae that we obtain in our case (closely related to the original quasi-free states in which the endomorphism is being implemented) that we feel justifies our concentration on these particular choices.

The paper is organized as follows. In Section 2 we set notation and state background definitions and results that will be needed in the sequel. In Section 3 the explicit computation of the Cuntz algebra state is given for a natural choice of implementers. We proceed in Section 4 to show that the vacuum vector is cyclic for these Cuntz algebra representations, and also for their restrictions to the gauge-invariant subalgebra. We then identify the gauge-invariant subalgebra with the CAR algebra, and show that these states, as states on the CAR algebra, are the pure quasi-free states in which the endomorphism is being implemented. From this we deduce irreducibility of the representations of $\mathcal{O}_2$ and $\mathcal{F}_2$, and discuss the case for an arbitrary pair of implementers. We conclude Section 4 with some further observations on these states and on induced endomorphisms of $B(\mathcal{H})$.

2 Preliminaries

The *self-dual* CAR algebra $A^{SDC}(\mathcal{K}, \Gamma)$ over a Hilbert space $\mathcal{K}$ with distinguished conjugation Γ, is the unital C^*-algebra generated by the range of a complex linear map B on $\mathcal{K}$, satisfying the *self-dual canonical anti-commutation relations*:

$$B(f)B(g)^* + B(g)^*B(f) = \langle g, f\rangle 1, \quad B(f)^* = B(\Gamma f)$$

for $f, g \in \mathcal{K}$. A projection P on $\mathcal{K}$ is said to be a *basis projection* if $\Gamma P \Gamma = 1 - P$.

If V is an isometry in $B(\mathcal{K})$, commuting with Γ, then it is said to be a *Bogoliubov operator* and it induces a unital isometric $*$-endomorphism ϱ_V of $A^{SDC}(\mathcal{K},\Gamma)$ via the map

$$B(k) \mapsto B(Vk), \quad k \in \mathcal{K}.$$

Such an endomorphism is called *quasi-free*, and is a $*$-automorphism if and only if V is unitary.

For each $S \in B(\mathcal{K})$ with $0 \leq S \leq 1$ and $\Gamma S \Gamma = 1 - S$, there exists a *quasi-free state*, φ_S, on $A^{SDC}(\mathcal{K},\Gamma)$ [1], [10]. This state is pure if and only if S is a (basis) projection. We omit its definition here, but it suffices to say that for P_1 a basis projection on $\mathcal{K}$, the GNS-decomposition of φ_{P_1} can be identified with the triple $\left(\pi_{P_1}, F_a(\mathcal{K}_1), \Omega_{P_1}\right)$, where Ω_{P_1} is the usual Fock vacuum vector, and π_{P_1} is the representation given by

$$\pi_{P_1}\left(B(k)\right) = a(P_1 k)^* + a(P_1 \Gamma k), \quad \text{for } k \in \mathcal{K},$$

with $a(\cdot)^*$ and $a(\cdot)$ the creation and annihilation operators on $F_a(\mathcal{K}_1)$ and $\mathcal{K}_1 := P_1\mathcal{K}$.

With $P_2 := 1 - P_1$, when $A \in B(\mathcal{K})$ we denote by $A_{\alpha,\beta}$ the operator $P_\alpha A P_\beta$ where $\alpha, \beta = 1, 2$.

Appearing in [2], the following definition relates to arbitrary C^*-algebras, $*$-endomorphisms and representations, though its motivation lies in the work of [8], [9] and [14]. The subsequent result is Theorem 3.3 of [2].

Definition 1 A $*$-endomorphism ϱ of a C^*-algebra A is *isometrically implementable* in a representation $(\pi, \mathcal{H})$ if there exists a (possibly finite) sequence $\{\Psi_n\}_{n\in I}$ in $B(\mathcal{H})$ with relations

$$\Psi_m^* \Psi_n = \delta_{mn} 1, \quad \sum_{n\in I} \Psi_n \Psi_n^* = 1$$

which implements ϱ by

$$\pi \circ \varrho = \sum_{n\in I} \Psi_n \pi(\cdot) \Psi_n^*,$$

with convergence of the sums with respect to the strong operator topology if I is infinite.

Theorem 2 *A quasi-free endomorphism ϱ_V of $A^{SDC}(\mathcal{K},\Gamma)$ is isometrically implementable in a Fock representation π_{P_1} if and only if V_{12} is Hilbert-Schmidt.*

In [2], for ϱ_V a quasi-free endomorphism of $A^{SDC}(\mathcal{K},\Gamma)$ and P_1 a basis projection on $\mathcal{K}$ such that $[V, P_1]$ is Hilbert-Schmidt, a sequence of implementers of ϱ_V in π_{P_1} is constructed. For this construction it is assumed at the outset that $\mathcal{K}_1 = L^2(\mathbf{R}^d)$, though as in the case of [15], the reason for this is a notational one and all of the results hold in the general case [2].

As a generalization of the operator $\Lambda(U)$ of [15], in [2] the author defines the associate $\Lambda(V)$ of the isometry V:

$$
\begin{aligned}
\Lambda(V)_{11} &= V_{11} - P_1 - V_{12}V_{22}^{-1}V_{21} + V_{11}^{-1\,*}V_{21}^*P_{\ker V_{22}^*}V_{21}\\
\Lambda(V)_{12} &= V_{12}V_{22}^{-1} - V_{11}^{-1\,*}V_{21}^*P_{\ker V_{22}^*}\\
\Lambda(V)_{21} &= (V_{22}^{-1} - V_{12}^*V_{11}^{-1\,*}V_{21}^*P_{\ker V_{22}^*})V_{21}\\
\Lambda(V)_{22} &= P_2 - V_{22}^{-1} + V_{12}^*V_{11}^{-1\,*}V_{21}^*P_{\ker V_{22}^*}.
\end{aligned}
\tag{1}
$$

For the meaning of the operators V_{ii}^{-1} see [2].

Let $\{e_1, \dots, e_{L_V}\}$ be an orthonormal basis for $\ker V_{11}$, where $L_V < \infty$ by the Hilbert-Schmidt assumption on $[V, P_1]$. Define $\mathcal{D}$ to be the dense subspace $\pi_{P_1}\big(A^{SDC}(\mathcal{K},\Gamma)\big)\,\Omega_{P_1}$ of $F_a(\mathcal{K}_1)$. Then on $\mathcal{D}$, an operator $\Psi_0(V)$ is defined by

$$
\begin{aligned}
\Psi_0(V) := \Big[\det\big(P_1 + \Lambda(V)_{12}\Lambda(V)_{12}{}^*\big)\Big]^{-1/4}&\\
\cdot \sum_{(\sigma,s)\in\mathcal{P}_{L_V}} (-1)^s \operatorname{sign}\sigma\, a(V_{12}\Gamma e_{\sigma(1)})^*\Psi\cdots a(V_{12}\Gamma e_{\sigma(s)})^*\Psi&\\
:\exp\Big(b\big(\Lambda(V)\big)/2\Big):a(e_{\sigma(s+1)})\Psi\cdots a(e_{\sigma(L_V)})\Psi&,
\end{aligned}
\tag{2}
$$

where $\mathcal{P}_{L_V}$ is the index set with elements, pairs (σ, s) with $s \in \{0, \dots, L_V\}$ and σ a permutation of order L_V with $\sigma(1) < \dots < \sigma(s)$ and $\sigma(s{+}1) < \dots < \sigma(L_V)$. Further Ψ is the unitary in $B\big(F_a(\mathcal{K}_1)\big)$ implementing α_{-1} in π_{P_1} and satisfying $\Psi\Omega = \Omega$.

It follows that $\Psi_0(V)$ has a continuous extension to an isometry, denoted by the same symbol, on $F_a(\mathcal{K}_1)$ [2, Lemma 4.3].

For $n \in \mathbf{N}$, define I_n to be the set of 2^n multiindices $\alpha = (\alpha_1, \dots, \alpha_l)$ with

$$0 \le l \le n, \quad 1 \le \alpha_1 < \dots < \alpha_l \le n \quad \text{and} \quad \alpha := 0 \quad \text{for } l = 0,$$

and for $k \in \ker V^*$, define $\psi(k) := \pi_{P_1}\big(B(k)\big)\,\Psi$. Then fixing an orthonormal basis $\{k_i\}_{i=1}^m$ for $\ker V^* \cap \operatorname{ran}\big(P_1 - \Lambda(V)_{12}{}^*\big)$, and defining for $\beta = (\beta_1, \dots, \beta_r) \in I_m$,

$$\psi_\beta := \psi(k_{\beta_1})\cdots\psi(k_{\beta_r}) \quad \text{and} \quad \Psi_\beta(V) := \psi_\beta\Psi_0(V),$$

the following is the main result of [2].

Theorem 3 *$m = \frac{1}{2}\operatorname{ind} V^*$, and the 2^m isometries $\{\Psi_\beta(V)\}_{\beta\in I_m}$ implement ϱ_V in π_{P_1}.*

Definition 4 For $2 \le n < \infty$, the *Cuntz algebra* $\mathcal{O}_n$ [7] is defined to be the universal C^*-algebra generated by n isometries $s_0, s_1, \dots, s_{n-1}$ satisfying

$$s_i^*s_j = \delta_{i,j}1, \quad \sum_{i=0}^{n-1} s_is_i^* = 1, \quad \text{for } i, j \in \{0, 1, \dots, n-1\}. \tag{3}$$

Then by definition, each sequence of implementers $\{\Psi_\beta(V)\}_{\beta\in I_m}$ of Theorem 3 defines a representation of the Cuntz algebra $\mathcal{O}_{2^m}$ on $F_a(\mathcal{K}_1)$.

Denote by $\mathcal{F}_n$, the gauge-invariant subalgebra of $\mathcal{O}_n$, which is the closure of the linear span of monomials of the form

$$s_{i_1}s_{i_2}\cdots s_{i_m}s_{j_m}^*\cdots s_{j_2}^*s_{j_1}^*$$

for $m = 0, 1, \ldots$. Then let θ be its canonical identification with $\bigotimes_1^\infty M_n$,

$$\theta\colon \bigotimes_1^\infty M_n \ni e_{i_1j_1}^{(1)}\otimes e_{i_2j_2}^{(2)}\otimes\cdots\otimes e_{i_mj_m}^{(m)} \mapsto s_{i_1}s_{i_2}\cdots s_{i_m}s_{j_m}^*\cdots s_{j_2}^*s_{j_1}^* \in \mathcal{F}_n, \tag{4}$$

where e_{ij}, for $i, j \in \{0, 1, \ldots, n-1\}$, are the usual matrix units in M_n.

Further, if $\mathcal{H}$ is a Hilbert space with orthonormal basis $\{f_i\}_{i=1}^\infty$, then $\bigotimes_1^\infty M_2$ can be identified with $A^{\mathrm{CAR}}(\mathcal{H})$, the CAR algebra over $\mathcal{H}$ in the complex formalism [10], via the map

$$\iota\colon A^{\mathrm{CAR}}(\mathcal{H}) \ni a(f_k) \mapsto \Big(\prod_{l=1}^{k-1}(e_{00}^{(l)} - e_{11}^{(l)})\Big)\, e_{01}^{(k)} \in \bigotimes_1^\infty M_2. \tag{5}$$

3 Computation of the State on $\mathcal{O}_2$

Let $\mathcal{H} = l^2$ and define a conjugation J on $\mathcal{H}$ by $J(\xi_1, \xi_2, \ldots) = (\overline{\xi_1}, \overline{\xi_2}, \ldots)$. With V the unilateral shift on $\mathcal{H}$, denote by T the isometry $V \oplus JVJ = V \oplus V$ on $\mathcal{K} = \mathcal{H} \oplus \mathcal{H}$.

For $n \geq 2$ and $m \in \mathbf{N}_0$ define a projection $P_{n,m}$ on $\mathcal{H}$ by

$$P_{n,m}(\xi_1, \xi_2, \xi_3, \ldots) = (0, \ldots, 0, \xi_n, \xi_{n+1}, \ldots, \xi_{n+m}, 0, \ldots)$$

and a basis projection $P_1 = P_{n,m} \oplus (1 - P_{n,m})$ on $\mathcal{K}$ with respect to the conjugation,

$$\Gamma = \begin{pmatrix} 0 & J \\ J & 0 \end{pmatrix}.$$

Remark 5 It will become clear that all of our computations could be carried out more generally, when in place of $P_{n,m}$, we have the projection onto $\overline{\mathrm{lin}}\{e_i\}_{i\in I}$, where $\{e_j\}_{j=1}^\infty$ is the canonical orthonormal basis for l^2 and I is any finite or infinite subset of $\mathbf{N}$, and furthermore that similar results occur. For these computations when for example $I = \varnothing, I = \mathbf{N}, I = \{1, \ldots, n\}, I = \{n, n+1, n+2, \ldots\}$, see [11].

Now with $\{f_i\}_{i=1}^\infty$ the canonical orthonormal basis for l^2, simplifying notation let Ψ_0 and Ψ_1 be the isometric implementers of ϱ_T in π_{P_1} of Theorem 3, where we choose

$$e_1 = (f_{n+m} \oplus 0), \quad e_2 = (0 \oplus f_{n-1}) \tag{6}$$

to be the orthonormal basis for $\ker T_{11}$. The operator Λ defined in (1) here has components,

$$\Lambda(T)_{11} = T_{11} - P_1, \quad \Lambda(T)_{12} = 0, \quad \Lambda(T)_{21} = 0, \quad \Lambda(T)_{22} = P_2 - T_{22}^{-1}, \tag{7}$$

and we note that in fact

$$T_{ii}^{-1} = T_{ii}^{*} \quad \text{for } i = 1, 2. \tag{8}$$

Thus take $k_1 = (0 \oplus f_1)$ to be the orthonormal basis of $\ker T^* \cap \operatorname{ran}\big(P_1 - \Lambda(T)_{12}^*\big) = \ker T^* \cap \mathcal{K}_1$.

Then for s_0, s_1 the generators of $\mathcal{O}_2$, define the representation π of $\mathcal{O}_2$ on $F_a(\mathcal{K}_1)$ by $\pi(s_i) = \Psi_i$, for $i = 0, 1$.

Definition 6 Let ω_π be the state on $\mathcal{O}_2$ defined by

$$\omega_\pi(s_\mu s_\nu^*) := \langle \Psi_\mu \Psi_\nu^* \Omega, \Omega \rangle,$$

for all multiindices μ and ν.

Of course there is freedom in the choice of implementers, and we discuss this in Subsection 4.2. In fact any other pair of implementers is of the form

$$\Psi_0' := u_{00}\Psi_0 + u_{01}\Psi_1, \quad \Psi_1' := u_{10}\Psi_0 + u_{11}\Psi_1, \tag{9}$$

for $U = [u_{ij}]_{i,j=0,1} \in U(2)$ [2]. However, if we are to compute the state on $\mathcal{O}_2$ defined by the implementers of this quasi-free endomorphism, certainly Ψ_0 and Ψ_1 above are the most natural choices to begin with.

3.1 The Adjoints of the Implementers

In order to compute the state ω_π, we first need to understand how products of the adjoints, Ψ_0^* and Ψ_1^*, act on the vacuum Ω. From the definitions one can verify that

$$\Psi_0^* = -\, a(e_2)^* a(e_1)^* : \exp\Big(b\big(\Lambda(T)^*\big)\Big) : \tag{10}$$

$$-\, \Psi a(e_2)^* : \exp\Big(b\big(\Lambda(T)^*\big)\Big) : \Psi a(T_{12}\Gamma e_1) \tag{11}$$

$$+\, \Psi a(e_1)^* : \exp\Big(b\big(\Lambda(T)^*\big)\Big) : \Psi a(T_{12}\Gamma e_2) \tag{12}$$

$$-\, : \exp\Big(b\big(\Lambda(T)^*\big)\Big) : a(T_{12}\Gamma e_2) a(T_{12}\Gamma e_1) \tag{13}$$

and $\Psi_1^* = \Psi_0^* \Psi a(k_1)$. We will require the following, which are easily checked.

$$T_{11}^{*\,k} e_1 = f_{n+m-k} \oplus 0, \quad k = 1, \dots, m \quad \text{and} \quad T_{11}^{*\,m+1} e_1 = 0, \tag{14}$$

$$T_{11}^{*\,k} e_2 = 0 \oplus f_{n-1+k}, \quad k = 1, \dots, n-2 \quad \text{and} \quad T_{11}^{*\,n-1} e_2 = 0, \tag{15}$$

and

$$\langle e_i, T_{12}\Gamma e_2 \rangle = 0, \quad i, j = 1, 2, \tag{16}$$

$$\langle T_{11}^{*\,k} e_1, T_{12}\Gamma e_1 \rangle = 0, \quad k = 1, \dots, m+1, \tag{17}$$

$$\langle T_{11}^{*\,k} e_1, T_{12}\Gamma e_2 \rangle = \delta_{k,m}, \tag{18}$$

$$\langle T_{11}^{*\,k} e_2, T_{12}\Gamma e_i \rangle = 0, \quad i = 1, 2 \quad \text{and} \quad k = 1, \dots, n-1. \tag{19}$$

Note that from (15), ${T_{11}^*}^{n-2}e_2 = k_1$.

Furthermore, the first of the commutation relations of of [2, Lemma 4.1] will also be needed. In our case it states that

$$:\exp\Big(b\big(\Lambda(T)^*\big)\Big):a(f)^* = a(T_{11}^* f)^* :\exp\Big(b\big(\Lambda(T)^*\big)\Big): \quad \text{for all } f \in \mathcal{K}_1. \tag{20}$$

To write down expressions for $\Psi_\nu^*\Omega$ for multiindices ν, it is necessary to consider the cases

(i) $n < m+1$,
(ii) $n = m+1$ and
(iii) $n > m+1$ separately.

We shall assume henceforth that (i) holds, although all of the results certainly hold in an identical fashion for all $n \geq 2$ and $m \in \mathbf{N}_0$.

In the first place, from (14)–(20) we find that for $k = 1, \dots, n-1$,

$${\Psi_0^*}^k\Omega = (-1)^k a(e_2)^* a(e_1)^* a(T_{11}^* e_2)^* a(T_{11}^* e_1)^* \cdots a({T_{11}^*}^{k-1} e_2)^* a({T_{11}^*}^{k-1} e_1)^*\Omega, \tag{21}$$

obtained through k applications of (10) to Ω, and that

$${\Psi_0^*}^n\Omega = 0. \tag{22}$$

Then noting that

$$\langle {T_{11}^*}^r e_1, k_1\rangle = 0, \quad r = 0,1,2,\dots \quad \text{and} \tag{23}$$

$$\langle {T_{11}^*}^r e_2, k_1\rangle = \delta_{r,n-2}, \tag{24}$$

it is clear that for $k = 0, 1, \dots, n-2$,

$$\Psi_1^* {\Psi_0^*}^k \Omega = 0. \tag{25}$$

Moreover, from (14)–(20), (23)–(24) and since ${T_{11}^*}^{n-2}e_2 = k_1$, it follows that for $k = 0, 1, \dots, m-n+2$,

$$\begin{aligned} {\Psi_1^*}^k {\Psi_0^*}^{n-1}\Omega = (-1)^n t(k) a(e_2)^* a(e_1)^* a(T_{11}^* e_2)^* a(T_{11}^* e_1)^* \\ \cdots a({T_{11}^*}^{n-2} e_2)^* a({T_{11}^*}^{n-2} e_1)^* \cdot a({T_{11}^*}^{n-1} e_1)^* \\ \cdots a({T_{11}^*}^{n-2+k} e_1)^*\Omega \end{aligned} \tag{26}$$

where

$$t(k) := \begin{cases} +1 & \text{if } k \equiv 2,3 \bmod 4 \\ -1 & \text{if } k \equiv 0,1 \bmod 4. \end{cases}$$

The expression in (26) is the result of k applications of (10) $(\times \Psi a(k_1))$ to ${\Psi_0^*}^{n-1}\Omega$.

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Observe that when $k = m - n + 2$, the term ${T_{11}^*}^m e_1$ $(= T_{12}\Gamma e_2)$ appears immediately before Ω. Thus (14)–(20), (23)–(24) and the equality, ${T_{11}^*}^{n-2} e_2 = k_1$ imply that for $r = 1, \dots, n-1$, r applications of (12) $(\times \Psi a(k_1))$ to ${\Psi_1^*}^{m-n+2}\Psi_0^{n-1}\Omega$ yields the equality,

$$\begin{aligned}(27)\quad &{\Psi_1^*}^r {\Psi_1^*}^{m-n+2} {\Psi_0^*}^{n-1}\Omega \\ &= (-1)^n t(m-n)t(r)\cdot \hat{a}(e_2)^* a(e_1)^* \hat{a}(T_{11}^* e_2)^* a(T_{11}^* e_1)^* \\ &\quad \cdots \hat{a}({T_{11}^*}^{r-2} e_2)^* a({T_{11}^*}^{r-2} e_1)^* \hat{a}({T_{11}^*}^{r-1} e_2)^* a({T_{11}^*}^{r-1} e_1)^* \\ &\quad \cdot a({T_{11}^*}^{r} e_2)^* a({T_{11}^*}^{r} e_1)^* \\ &\quad \cdots a({T_{11}^*}^{n-2} e_2)^* a({T_{11}^*}^{n-2} e_1)^* \cdot a({T_{11}^*}^{n-1} e_1)^* \cdots a({T_{11}^*}^{m} e_1)^*\Omega,\end{aligned}$$

where by $\hat{a}(\cdot)$ we mean omit $a(\cdot)$.

Then from (27), with $r = n-1$, and by (23), we clearly have that

$$(28)\quad {\Psi_1^*}^{m+2}{\Psi_0^*}^{n-1}\Omega = \Psi_1^* {\Psi_1^*}^{n-1} {\Psi_1^*}^{m-n+2} {\Psi_0^*}^{n-1}\Omega = 0.$$

Moreover from (15)–(20), we deduce that for $k = 0, 1, \dots, m$,

$$(29)\quad \Psi_0^* {\Psi_1^*}^k {\Psi_0^*}^{n-1}\Omega = 0.$$

Then finally, from (14) and (16)–(20), for $l \in \mathbf{N}_0$ we have

$$\begin{aligned}(30)\quad {\Psi_0^*}^l {\Psi_1^*}^{m+1} {\Psi_0^*}^{n-1}\Omega &= (-1)^{l(m+1)} {\Psi_1^*}^{m+1} {\Psi_0^*}^{n-1}\Omega \\ &= (-1)^{l(m+1)+n} t(m-n)t(n-1)\cdot a(e_1)^* a(T_{11}^* e_1)^* \\ &\quad \cdots a({T_{11}^*}^m e_1)^*\Omega,\end{aligned}$$

this expression being the result of l applications of (12) to ${\Psi_1^*}^{m+1}{\Psi_0^*}^{n-1}\Omega$.

We now gather together the above results, (21), (22) and (25)–(30), in the following proposition.

Proposition 7 *For arbitrary multiindex $\nu = (\nu_1, \dots, \nu_s)$ with $s \in \mathbf{N}$, the following hold.*
For $s \in [1, n-1]$,

$$\begin{aligned}(31)\quad \Psi_\nu^*\Omega = \delta_{\nu_1,0}\cdots\delta_{\nu_s,0}\cdot(-1)^s a(e_2)^* a(e_1)^* \cdot a(T_{11}^* e_2)^* a(T_{11}^* e_1)^* \\ \cdots a({T_{11}^*}^{s-1} e_2)^* a({T_{11}^*}^{s-1} e_1)^*\Omega.\end{aligned}$$

For $s \in [n, m+1]$,

$$\begin{aligned}(32)\quad \Psi_\nu^*\Omega &= \delta_{\nu_1,0}\cdots\delta_{\nu_{n-1},0}\delta_{\nu_n,1} \\ &\quad \cdots\delta_{\nu_s,1}\cdot(-1)^n t(s-n+1) a(e_2)^* a(e_1)^* a(T_{11}^* e_2)^* a(T_{11}^* e_1)^* \\ &\quad \cdots a({T_{11}^*}^{n-2} e_2)^* a({T_{11}^*}^{n-2} e_1)^* \cdot a({T_{11}^*}^{n-1} e_1)^* \cdots a({T_{11}^*}^{s-1} e_1)^*\Omega.\end{aligned}$$

For $s \in [m+2, n+m-1]$,

(33)
$$\begin{aligned}\Psi_\nu^*\Omega = \delta_{\nu_1,0}\cdots\delta_{\nu_{n-1},0}\delta_{\nu_n,1}\\ \cdots\delta_{\nu_s,1}\cdot(-1)^n t(m-n)t(s-m-1)\hat{a}(e_2)^*a(e_1)^*\hat{a}(T_{11}^*e_2)^*a(T_{11}^*e_1)^*\\ \cdots\hat{a}({T_{11}^*}^{s-m-3}e_2)^*a({T_{11}^*}^{s-m-3}e_1)^*\hat{a}({T_{11}^*}^{s-m-2}e_2)^*a({T_{11}^*}^{s-m-2}e_1)^*\\ \cdot a({T_{11}^*}^{s-m-1}e_2)^*a({T_{11}^*}^{s-m-1}e_1)^*\\ \cdots a({T_{11}^*}^{n-2}e_2)^*a({T_{11}^*}^{n-2}e_1)^*\cdot a({T_{11}^*}^{n-1}e_1)^*\cdots a({T_{11}^*}^{m}e_1)^*\Omega.\end{aligned}$$

For $s \geq n+m$,

$$\begin{aligned}\Psi_\nu^*\Omega = \delta_{\nu_1,0}\cdots\delta_{\nu_{n-1},0}\delta_{\nu_n,1}\cdots\delta_{\nu_{n+m},1}\delta_{\nu_{n+m+1},0}\\ \cdots\delta_{\nu_s,0}\cdot(-1)^{(s-n-m)(m+1)}(-1)^n t(m-n)t(n-1)\cdot a(e_1)^*a(T_{11}^*e_1)^*\\ \cdots a({T_{11}^*}^{m}e_1)^*\Omega.\end{aligned} \tag{34}$$

3.2 The State

We are now in a position to write down a formula for the state.

Proposition 8 *Let* $s, t \in \mathbf{N}$ *and* $\mu = (\mu_1, \ldots, \mu_t)$, $\nu = (\nu_1, \ldots, \nu_s)$ *be multiindices. For* $s \in [1, n]$,

$$\langle \Psi_\mu \Psi_\nu^* \Omega, \Omega\rangle = \delta_{t,s}\cdot\delta_{\mu,\nu}\cdot\delta_{\mu_1,0}\cdots\delta_{\mu_t,0}. \tag{35}$$

For $s \in [n, m+n-1]$,

$$\langle \Psi_\mu \Psi_\nu^* \Omega, \Omega\rangle = \delta_{t,s}\cdot\delta_{\mu,\nu}\cdot\delta_{\mu_1,0}\cdots\delta_{\mu_{n-1},0}\cdot\delta_{\mu_n,1}\cdots\delta_{\mu_t,1}. \tag{36}$$

For $s \geq n+m$,

$$\begin{aligned}\langle \Psi_\mu \Psi_\nu^* \Omega, \Omega\rangle = (-1)^{(s+t)(m+1)}\chi_{\{m+n,m+n+1,\ldots\}}(t)\cdot\delta_{\mu_1,0}\\ \cdots\delta_{\mu_{n-1},0}\cdot\delta_{\mu_n,1}\cdots\delta_{\mu_{n+m},1}\cdot\delta_{\mu_{n+m+1},0}\cdots\delta_{\mu_t,0}\cdot\delta_{\nu_1,0}\\ \cdots\delta_{\nu_{n-1},0}\cdot\delta_{\nu_n,1}\cdots\delta_{\nu_{n+m},1}\cdot\delta_{\nu_{n+m+1},0}\cdots\delta_{\nu_s,0},\end{aligned} \tag{37}$$

where

$$\chi_{\{m+n,m+n+1,\ldots\}}(t) := \begin{cases}1 & \text{if } t \geq m+n\\ 0 & \text{otherwise.}\end{cases}$$

Proof Follows from Proposition 7.

Remark 9 The state is therefore obviously *not* gauge-invariant.

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4 Irreducible Representations of $\mathcal{O}_2$ and $\mathcal{F}_2$

4.1 Cyclicity of Ω for π

We begin with a notational definition.

Definition 10 For $k \in \mathcal{K}_1$ and $i \in \mathbf{N}$,

$$a_{T^i}(k) := a\big((T^i)_{11}k\big) + a\big((T^i)_{12}\Gamma k\big)^* = \pi_{P_1}\big(B(T^i k)\big)^*.$$

Then with $t \in \mathbf{N}$ and $\mu_i \in \{0,1\}$, $1 \le i \le t$, neglecting signs we have,

$$\begin{aligned}\Psi_{\mu_1}\cdots\Psi_{\mu_t}\Omega = (\pm)a(k_{\mu_1})^* a_T(k_{\mu_2})^* \\ \cdots a_{T^{t-1}}(k_{\mu_t})^* \cdot a_{T^{t-1}}(T_{12}\Gamma e_1)^* a_{T^{t-1}}(T_{12}\Gamma e_2)^* \\ \cdots a_T(T_{12}\Gamma e_1)^* a_T(T_{12}\Gamma e_2)^* a(T_{12}\Gamma e_1)^* a(T_{12}\Gamma e_2)^*\Omega,\end{aligned} \tag{38}$$

where if $\mu_i = 0$ delete the term $a_{T^{i-1}}(k_{\mu_i})^*$ from the expression.

Now from (34) and (37), for $s, t \ge n+m$,

$$\begin{aligned}&\langle \Psi_0^{n-1}\Psi_1^{m+1}\Psi_0^{t-n-m}\Psi_0^{*s-n-m}\Psi_1^{*m+1}\Psi_0^{*n-1}\Omega, \Omega\rangle \\ &= (\pm)\langle a_{T^t}(e_1)^* a_{T^t}(T_{11}^* e_1)^* \cdots a_{T^t}({T_{11}^*}^m e_1)^* \Psi_0^{n-1}\Psi_1^{m+1}\Psi_0^{t-n-m}\Omega, \Omega\rangle \\ &= (\pm)1.\end{aligned} \tag{39}$$

Lemma 11 *For all $s, t \ge n+m$,*

$$\Psi_0^{n-1}\Psi_1^{m+1}\Psi_0^{t-n-m}\Psi_0^{*s-n-m}\Psi_1^{*m+1}\Psi_0^{*n-1}\Omega = (\pm)\Omega.$$

Proof This follows immediately from (38), (39) and the computations,

$$a_{T^t}({T_{11}^*}^i e_1)^* = \begin{cases} a\big((T^t)_{11}{T_{11}^*}^i e_1\big)^* = a(f_{n+m+t-1} \oplus 0)^* & t \le i \\ a\big(\Gamma(T^t)_{21}{T_{11}^*}^i e_1\big) = a(0 \oplus f_{n+m+t-1}) & t \ge i+1, \end{cases} \tag{40}$$

$$a_{T^k}(T_{12}\Gamma e_1)^* = a\big((T^k)_{11}T_{12}\Gamma e_1\big)^* = a(0 \oplus f_{n+m+1+k})^*, \quad \text{for all } k \in \mathbf{N}, \tag{41}$$

$$a_{T^k}(T_{12}\Gamma e_2)^* = \begin{cases} a\big((T^k)_{11}T_{12}\Gamma e_2\big)^* = a(f_{n+k} \oplus 0)^* & \text{if } k \le m \\ a\big((T^k)_{12}T_{21}e_2\big) = a(0 \oplus f_{n+k}) & \text{if } k \ge m+1, \end{cases} \tag{42}$$

$$a_{T^k}(k_1)^* = \begin{cases} a\big((T^k)_{11}k_1\big)^* = a(0 \oplus f_{k+1})^* & \text{if } k \le n-2 \text{ or } k \ge n+m \\ a\big((T^k)_{12}\Gamma k_1\big) = a(f_{k+1} \oplus 0) & \text{if } n-1 \le k \le n+m-1. \end{cases} \tag{43}$$

■

Proposition 12 *Ω is cyclic for π, and so π is the GNS-representation of ω_π.*

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Proof As a result of (38) and (40)–(43), for $s, t \geq n+m$ we have

$$
\begin{aligned}
&\Psi_{\mu_1} \cdots \Psi_{\mu_t} \Psi_0^{*s-n-m} \Psi_1^{*m+1} \Psi_0^{*n-1} \Omega \\
&\quad = (\pm) a_{T^t}(e_1)^* a_{T^t}(T_{11}^* e_1)^* \cdots a_{T^t}(T_{11}^{*\,m} e_1)^* \Psi_{\mu_1} \cdots \Psi_{\mu_t} \Omega \\
&\quad = (\pm) a\Big(\Gamma(T^t)_{21} e_1\Big)\, a\Big(\Gamma(T^t)_{21} T_{11}^* e_1\Big) \cdots a\Big(\Gamma(T^t)_{21} T_{11}^{*\,m} e_1\Big) \cdot a(k_{\mu_1})^* \\
&\qquad \cdots a\Big((T^{n-2})_{11} k_{\mu_{n-1}}\Big)^* \cdot a\Big((T^{n-1})_{12} \Gamma k_{\mu_n}\Big) \\
&\qquad \cdots a\Big((T^{n+m-1})_{12} \Gamma k_{\mu_{n+m}}\Big) \cdot a\Big((T^{n+m})_{11} k_{\mu_{n+m+1}}\Big)^* \\
&\qquad \cdots a\Big((T^{t-1})_{11} k_{\mu_t}\Big)^* \cdot a\Big((T^{t-1})_{11} T_{12} \Gamma e_1\Big)^* a\Big((T^{t-1})_{12} T_{21} e_2\Big) \\
&\qquad \cdots a\Big((T^{m+1})_{11} T_{12} \Gamma e_1\Big)^* a\Big((T^{m+1})_{12} T_{21} e_2\Big) \\
&\qquad\quad \cdot a\Big((T^{m})_{11} T_{12} \Gamma e_1\Big)^* a\Big((T^{m})_{11} T_{12} \Gamma e_2\Big)^* \\
&\qquad \cdots a(T_{11} T_{12} \Gamma e_1)^* a(T_{11} T_{12} \Gamma e_2)^* a(T_{12} \Gamma e_1)^* a(T_{12} \Gamma e_2)^* \Omega,
\end{aligned} \tag{44}
$$

where again, as in (38), if $\mu_i = 0$ omit the term $a_{T^{i-1}}(k_{\mu_i})^*$.

Note from (43) that

$$\{(T^k)_{11} k_1 : 0 \leq k \leq n-2, k \geq n+m\} \cup \{(T^k)_{12} \Gamma k_1 : n-1 \leq k \leq n+m-1\}$$

forms an orthonormal basis for $\mathcal{K}_1$.

Then from Lemma 11, it clearly follows that with appropriate choice of μ_i, for $i = 1, \dots, t$, the orthonormal basis $\{a(f_{i_1})^* \cdots a(f_{i_r})^* \Omega : i_1 < \cdots < i_r, r \in \mathbf{N}_0\}$ for $F_a(\mathcal{K}_1)$ can be obtained through (44).

Thus the proof is complete. ■

4.2 Restriction to $\mathcal{F}_2$

We shall now consider the state ω_π restricted to $\mathcal{F}_2$.

Proposition 13 *Ω is cyclic for the representation π restricted to $\mathcal{F}_2$, and so $\pi|_{\mathcal{F}_2}$ is the GNS-representation of $\omega_\pi|_{\mathcal{F}_2}$.*

Proof Exactly as for Proposition 12, taking $s = t$. ■

Now making the identifications, θ of $\mathcal{F}_2$ with $\bigotimes_1^\infty M_2$ given in (4), and ι of $\bigotimes_1^\infty M_2$ with $A^{\mathrm{CAR}}(\mathcal{H})$ in (5), where $\{f_i\}_{i=1}^\infty$ is a fixed orthonormal basis for a Hilbert space $\mathcal{H}$, we consider the state $\omega_\pi|_{\mathcal{F}_2}$ as a state on $A^{\mathrm{CAR}}(\mathcal{H})$.

First note that on $\bigotimes_1^\infty M_2$, for $r \in \mathbf{N}$,

$$\omega_\pi|_{\mathcal{F}_2}(e_{\mu_1 \cdots \mu_r, \nu_1 \cdots \nu_r}) = \delta_{\mu,\nu} \cdot \delta_{\mu_1,0} \cdots \delta_{\mu_{n-1},0} \cdot \delta_{\mu_n,1} \cdots \delta_{\mu_{n+m},1} \cdot \delta_{\mu_{n+m+1},0} \cdots \delta_{\mu_r,0}. \tag{45}$$

When Q is a projection on $\mathcal{H}$, let ω_Q denote the associated quasi-free state on $A^{\mathrm{CAR}}(\mathcal{H})$ [10].

Proposition 14 *Under the identifications above, $\omega_\pi = \omega_{Q_{n,m}}$, where $Q_{n,m}$ is defined as the projection of $\mathcal{H}$ onto $\operatorname{lin}\{f_i\}_{i=n}^{n+m}$.*

Proof Assume that N, M and $i_1, \ldots, i_N, j_1, \ldots, j_M \in \mathbf{N}$, and that $i_k \neq i_l$ if $k \neq l$ and $j_r \neq j_s$ if $r \neq s$. Then consider the monomial,

$$a(f_{i_N})^* \cdots a(f_{i_1})^* a(f_{j_1}) \cdots a(f_{j_M}). \tag{46}$$

First let $1 \leq N = M \leq m+1$ and $i_1 = j_1, \ldots, i_N = j_N$, with $i_1, \ldots, i_N \in \{n, n+1, \ldots, n+m\}$. Then

$$\begin{aligned} &a(f_{i_N})^* \cdots a(f_{i_1})^* a(f_{i_1}) \cdots a(f_{i_N}) \qquad (47)\\ &\quad = a(f_{i_1})^* a(f_{i_1}) \cdots a(f_{i_N})^* a(f_{i_N})\\ &\quad = (e_{00}^{(1)} + e_{11}^{(1)}) \cdots (e_{00}^{(i_1-1)} + e_{11}^{(i_1-1)}) e_{11}^{(i_1)} \cdot \cdots \cdot (e_{00}^{(1)} + e_{11}^{(1)})\\ &\qquad \cdots (e_{00}^{(i_N-1)} + e_{11}^{(i_N-1)}) e_{11}^{(i_N)}, \qquad (48) \end{aligned}$$

and of course applying $\omega_{Q_{n,m}}$ to (47) gives 1, as does $\omega_\pi|_{\mathcal{F}_2}$ to (48).

Then suppose that $N = M$ and $i_1, \ldots, i_N, j_1, \ldots, j_N \in \{n, n+1, \ldots, n+m\}$, but that there exists k, $1 \leq k \leq N$ with $j_k \notin \{i_1, \ldots, i_N\}$. Then (46) is taken to 0 by $\omega_{Q_{n,m}}$, as is its image in $\bigotimes_1^\infty M_2$, by $\omega_\pi|_{\mathcal{F}_2}$, since $e_{01}^{(j_k)}$ will appear in each term in the latter.

Now suppose that $N = M$, (not necessarily in $[1, m+1]$), and that there exists k, with $1 \leq k \leq N$, such that $i_k \notin \{n, n+1, \ldots, n+m\}$. Then again, certainly (46) and its image in $\bigotimes_1^\infty M_2$ are taken to 0 by $\omega_{Q_{n,m}}$ and $\omega_\pi|_{\mathcal{F}_2}$ respectively.

Similarly if $i_k \notin \{n, n+1, \ldots, n+m\}$.

Finally, if $N \neq M$, and $i_1, \ldots, i_N, j_1, \ldots, j_M \in \{n, n+1, \ldots, n+m\}$ are any indices, then (46), taken to 0 by $\omega_{Q_{n,m}}$, also has its image taken to 0 by $\omega_\pi|_{\mathcal{F}_2}$, as $e_{01}^{(k)}$ or $e_{10}^{(k)}$ (or both) appear for some k (or k's) in each term in the latter.

Thus $\omega_{Q_{n,m}}$ and $\omega_\pi|_{\mathcal{F}_2}$ agree on (46) for all N, M and all indices, and so we are done. ∎

Corollary 15 *With Ω as cyclic vector, the representation,*

$$\pi \circ \theta \circ \iota \colon A^{\mathrm{CAR}}(\mathcal{H}) \to B\big(F_a(\mathcal{K}_1)\big),$$

is a GNS-representation for the pure quasi-free state $\omega_{Q_{n,m}}$.

Proof Immediate from Propositions 13 and 14. ∎

Of course, the choice of $\mathcal{H}$ and of orthonormal basis has been totally arbitrary. Taking $\mathcal{H} = l^2$ and $\{f_i\}_{i=1}^\infty$ to be the canonical orthonormal basis, for example, would obviously mean that $Q_{n,m} = P_{n,m}$.

Let $U = [u_{i,j}]_{i,j=0,1} \in U(2)$ and

$$\Psi_0' := u_{00}\Psi_0 + u_{01}\Psi_1, \quad \Psi_1' := u_{10}\Psi_0 + u_{11}\Psi_1 \tag{49}$$

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be another pair of implementers. Define π' to be the representation of $\mathcal{O}_2$ on $F_a(\mathcal{K}_1)$ given by $s_i \mapsto \Psi_i'$, for $i = 0, 1$, and let $\omega_{\pi'}$ denote the state, $\omega_{\pi'}(s_\mu s_\nu^*) := \langle \Psi_\mu' \Psi_\nu'^* \Omega, \Omega\rangle$ on $\mathcal{O}_2$. Then under the identifications above, with $\{f_i\}_{i=1}^\infty$ any fixed orthonormal basis of a Hilbert space $\mathcal{H}$, we have the following.

Proposition 16 *$\omega_{\pi'}|_{\mathcal{F}_2}$ considered as a state on $A^{\mathrm{CAR}}(\mathcal{H})$ is quasi-free if and only if either*

(i) $\Psi_0' = u_{00}\Psi_0$ *and* $\Psi_1' = u_{11}\Psi_1$, *or*
(ii) $\Psi_0' = u_{01}\Psi_1$ *and* $\Psi_1' = u_{10}\Psi_0$,

where in case (i), $\omega_{\pi'}|_{\mathcal{F}_2} = \omega_{Q_{n,m}}$, and in case (ii), $\omega_{\pi'}|_{\mathcal{F}_2} = \omega_{1-Q_{n,m}}$.

Proof Since under the identifications above,

$$\omega_{\pi'}|_{\mathcal{F}_2}(e_{01}^{(1)}) = u_{00}\overline{u_{10}},$$

it follows that if $\omega_{\pi'}|_{\mathcal{F}_2}$ is to be quasi-free, either u_{00} or u_{10} has to be zero. That is, either (i) or (ii) holds, where in (i), $|u_{00}| = |u_{11}| = 1$, and in (ii), $|u_{01}| = |u_{10}| = 1$. ■

Of course, different choices of u in (i) and (ii) respectively, give different extensions of $\omega_{Q_{n,m}}$ and $\omega_{1-Q_{n,m}}$ to $\mathcal{O}_2$.

Proposition 17 *If $\{\Psi_0', \Psi_1'\}$ is any pair of implementers of ϱ_T in π_{P_1}, with π' and $\omega_{\pi'}$ as defined above we have that π' is irreducible, as is its restriction to $\mathcal{F}_2$.*

Proof Clearly $\omega_\pi|_{\mathcal{F}_2}$ is a pure state on $\mathcal{F}_2$, so that by Propositions 12 and 13, ω_π is pure and $\pi|_{\mathcal{F}_2}$ and π are irreducible.

Then the claim follows since any other pair of implementers Ψ_0' and Ψ_1' are given by (49) for some $U \in U(2)$. ■

We end this section with some observations.

Definition 18 A pure state ω on $\mathcal{O}_n$ is said to be *strongly asymptotically shift invariant* [5] of order k if it satisfies

$$\omega \circ \sigma^{k+1} = \omega \circ \sigma^k.$$

If $(\pi, \mathcal{H}, \Omega)$ is its GNS-decomposition, then it is defined to be *finitely correlated* [5] if the subspace $\mathcal{S} \subset \mathcal{H}$ generated linearly by Ω and vectors of the form

$$\pi(s_{i_1}^* s_{i_2}^* \cdots s_{i_m}^*)\Omega,$$

where $i_1, \ldots, i_m \in \mathbf{N}$ and $m = 1, 2, \ldots$, is finite dimensional.

Proposition 19 *ω_π is:*

(1) *finitely correlated, and*
(2) *strongly asymptotically shift invariant of order k, if and only if $k \geq n+m$.*

Proof (1) is an immediate consequence of Propositions 7 and 12, while a simple calculation is enough to verify (2). ■

Denote by $\rho_{\Psi'}$, the endomorphism of $B\big(F_a(\mathcal{K}_1)\big)$,

$$\rho_{\Psi'} := \mathrm{Ad}_{\pi'} = \Psi_0'(\cdot)\Psi_0'^* + \Psi_1'(\cdot)\Psi_1'^*,$$

where $\{\Psi_0', \Psi_1'\}$ is any pair of implementers of ϱ_T in π_{P_1}.

Proposition 20 *The endomorphism $\rho_{\Psi'}$ is a shift on $B\big(F_a(\mathcal{K}_1)\big)$ admitting a pure, normal invariant state.*

Proof From Proposition 17 we have that $\pi|_{\mathcal{F}_2}$ is irreducible, thus ρ_Ψ is a shift [3], [13]. Then we have that for all $l \in \mathbf{N}_0$, the vector state corresponding to $\Psi_0^{*l}\Psi_1^{*m+1}\Psi_0^{*n-1}\Omega$ is pure, normal and ρ_Ψ-invariant.

The result then follows since for each pair $\{\Psi_0', \Psi_1'\}$, we have by (9) and [13] that $\rho_{\Psi'}$ and ρ_Ψ are conjugate. ■

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References

[1] H. Araki, *On quasifree states of* CAR *and Bogoliubov automorphisms.* Publ. Res. Inst. Math. Sci. **6**(1970/71), 385–442.

[2] C. Binnenhei, *Implementation of endomorphisms of the* CAR *algebra.* Rev. Math. Phys. **7**(1995), 833–869.

[3] O. Bratteli, P. E. T. Jorgensen and G. L. Price, *Endomorphisms of $B(\mathcal{H})$.* In: Quantization of Nonlinear Partial Differential Equations (eds. W. B. Arveson *et al.*), Proc. Sympos. Pure Math. **59**(1996), 93–138.

[4] O. Bratteli and P. E. T. Jorgensen, *A connection between multiresolution theory of scale N and representations of the Cuntz algebra $\mathcal{O}_N$.* In: Operator Algebras and Quantum Field Theory (Rome 1996), Internat. Press, Cambridge, MA, 1997, 151–163.

[5] ———, *Endomorphisms of $B(\mathcal{H})$ II. Finitely Correlated States on $\mathcal{O}_n$.* J. Funct. Anal. (2) **145**(1997), 323–373.

[6] ———, *Isometries, Shifts, Cuntz algebras and multiresolution wavelet analysis of scale N*. Integral Equations Operator Theory (4) **28**(1997), 382–443.

[7] J. Cuntz, *Simple C^*-algebras generated by isometries.* Comm. Math. Phys. **57**(1977), 173–185.

[8] S. Doplicher and J. E. Roberts, *Fields, statistics and non-abelian gauge groups*. Comm. Math. Phys. **28**(1972), 331–348.

[9] ———, *Why there is a field algebra with a compact gauge group describing the superselection structure in particle physics*. Comm. Math. Phys. (1) **131**(1990), 51–107.

[10] D. E. Evans and Y. Kawahigashi, *Quantum Symmetries on Operator Algebras.* Oxford University Press, 1998.

[11] M. J. Gabriel, *Quasi-free Maps of the* CAR*, Cuntz and Pimsner Algebras*. PhD Thesis, University of Wales, Swansea, July 2000.
[12] E. Jeong, *Irreducible representations of the Cuntz algebra $\mathcal{O}_n$*. Proc. Amer. Math. Soc. (12) **127**(1999), 3583–3590.
[13] M. Laca, *Endomorphisms of $B(\mathcal{H})$ and Cuntz algebras*. J. Operator Theory (1) **30**(1993), 85–108.
[14] J. E. Roberts, *Cross products of von Neumann algebras by group duals*. Symposia Mathematica Vol. **XX**, Academic Press, London, 1976, 335–363.
[15] S. N. M. Ruijsenaars, *On Bogoliubov Transformations. II. The General Case*. Ann. Physics **116**(1978), 105–134.

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