CORRESPONDENCE.

"EXPECTATION OF LIFE."

To the Editor.

SIR,—Is there anywhere to be found an exact and accurate definition of this function, called by some writers "expectation of life," by others "mean duration of life"? Mr. Peter Gray, in his valuable work entitled *Tables and Formulæ for the Computation of Life Contingencies*, has taken great pains to point out the errors of some preceding writers (see chapter v., pp. 59 to 72); and while treating the subject himself in a clear and satisfactory way, has been rather severe upon the inaccuracies of others. He therefore cannot, I think, reasonably object, if I direct attention to what seems to me a serious inaccuracy in his definition. He says, "By the mean duration of life at a specified age and according to a given table of mortality, is implied, the *average* number of years that, in the case of a single life, will be enjoyed by each individual of the specified age." WILL BE *enjoyed by* EACH *individual!!!* It is of course obvious that *each* individual cannot enjoy the *average* number of years, and that the preceding definition must be amended, by substituting the "average number of years which persons of the specified age, taken one with another, enjoy according to the given table of mortality."

Several writers object wholly to the use of the phrase "expectation of life"; and I notice in particular that Professor De Morgan in this *Journal*, vol. xii. p. 33, speaks of "the average life, or expectation, as it is *wrongly* called." Now, I can understand the objection made to the phrase, that it is somewhat misleading to persons who have not studied the subject accurately; though I do not admit that to be a sufficient reason for abandoning the use of an old phrase, which is convenient and well understood, as a technical term in the science of life contingencies. But what I do not understand is, why it should be said that this function is *wrongly called* "the expectation." It appears to me that the phrase is used in a sense strictly analogous to the sense it bears in other cases. Take the case of a number of persons drawing lots for a prize, what is meant by the "expectation of gain" possessed by any one of the persons? Or again, take the case of an Insurance Company with a given number of lives insured for specified sums, what is the "expectation of loss" in a particular year? It appears to me that the phrase "expectation of life" is strictly analogous to these other two phrases.

It is well known that the value of an annuity on a given life is less than that of an annuity certain for a term equal to the expectation of life; but I am only aware of one attempt at a strict proof of this proposition; that, namely, given by myself in the tenth volume of this *Journal*, p. 52. In that proof, however, I only established that the annuity is less than the annuity certain for the term of years indicated by the *nearest integer* to the number expressing the expectation of life. I now propose to complete the demonstration.

The expectation of life at a given age expresses the average number of payments of an annuity that would be made to persons of that age. Thus, then, in comparing the value of a life annuity with that of the annuity certain for a term equal to the expectation of life, we must take account, not only of the entire years included in that expectation, but also of the additional fraction of a year included therein. For example, if the expectation be equal to $n+\delta$, where n is an integer, and δ a fraction less than unity, we must compare with the value of the life annuity of £1 that of an annuity certain for n years with a further payment of δ at the end of the $(n+1)$th year. It will be noticed that the annuity being supposed, as usual, to cease with the payment at the end of the year preceding that in which death takes place, and not being continued up to the day of death, we therefore take for comparison the curtate and not the complete expectation of life.

Now, the age being x, the value of the life annuity is

$$\frac{l_{x+1}}{l_x}v+\frac{l_{x+2}}{l_x}v^2+\ \dots\dots\ +\frac{l_{x+z}}{l_x}v^z$$

$$=p_1v+p_2v^2+\ \dots\dots\ +p_zv^z, \text{ suppose;}$$

and the value of the annuity certain is

$$v+v^2+v^3+\ \dots\dots\ +v^n+\delta v^{n+1}.$$

Also, $$n+\delta=\frac{l_{x+1}+l_{x+2}+\ \dots\dots\ +l_{x+z}}{l_x}$$

$$=p_1+p_2+\ \dots\dots\ +p_z \quad \dots\dots \quad (1);$$

this equation expressing that if money be supposed to bear no interest, *i.e.*, $v=1$, then the value of the life annuity is equal to that of the annuity

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certain. Then the value of the life annuity will be greater or less than than that of the annuity certain, according as

$$p_1v+p_2v^2+\ .\ .\ .\ +p_zv^z > \text{ or } < v+v^2+\ .\ .\ .\ +v^n+\delta v^{n+1}\ .\ .\ (2)$$

and therefore according as

$$p_{n+2}v^{n+2}+p_{n+3}v^{n+3}+\ .\ .\ .\ +p_zv^z > \text{ or } <$$
$$(1-p_1)v+(1-p_2)v^2+\ .\ .\ .\ +(1-p_n)v^n+(\delta-p_{n+1})v^{n+1}.$$

Now, since $v<1$, the first member of this inequality is less than

$$v^{n+2}(p_{n+2}+p_{n+3}+\ .\ .\ .\ +p_z)\ .\ .\ .\ .\ (3)$$

and the second member is greater than

$$(1-p_1+1-p_2+\ .\ .\ .\ +1-p_n+\delta-p_{n+1})v^{n+1}$$
$$>(n+\delta-p_1-p_2-\ .\ .\ .\ -p_{n+1})v^{n+1}$$

or, from (1),

$$>(p_{n+2}+p_{n+3}+\ .\ .\ .\ +p_z)v^{n+1}.$$

But this is greater than (3), and therefore returning to (2), we see that in all cases

$$p_1v+p_2v^2+\ .\ .\ .\ +p_zv^z < v+v^2+\ .\ .\ .\ +v^n+\delta v^{n+1},$$

or the life annuity is less than the annuity certain for the term equal to the expectation.

The reason for this result may be briefly explained as follows:—Comparing the life annuity and the annuity certain, and observing that the life annuity is on the average equivalent to an annuity certain of which the payment at the end of the first year is p_1, at the end of the second p_2, &c., we see that the total payments made in the two cases are equal, but in the life annuity, the payments are longer postponed. Thus taking the mth year, we have in the one case £1 payable at the end of the mth year, and in the other p_m payable at the end of the mth year, and $1-p_m$ payable at a time later than n years; and the value of the former is clearly the larger.

The reasoning of Jones on this point (Art. 127) appears to be not quite conclusive.

I am, Sir,

Your obedient servant,

T. B. SPRAGUE.

Equity and Law Life Assurance Society,
18, *Lincoln's Inn Fields,*
1st *March,* 1867.