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HOCHSCHILD (CO)HOMOLOGY OF $\mathbb{Z}_2 \times \mathbb{Z}_2$-GALOIS COVERINGS OF QUANTUM EXTERIOR ALGEBRAS

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Abstract

Let $A_q = k\langle x, y\rangle/(x^2, xy + qyx, y^2)$ be the quantum exterior algebra over a field k with char $k \neq 2$, and let Λ_q be the $\mathbb{Z}_2 \times \mathbb{Z}_2$-Galois covering of A_q. In this paper the minimal projective bimodule resolution of Λ_q is constructed explicitly, and from it we can calculate the k-dimensions of all Hochschild homology and cohomology groups of Λ_q. Moreover, the cyclic homology of Λ_q can be calculated in the case where the underlying field is of characteristic zero.

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1. Introduction

Let Λ be a finite-dimensional algebra (associative with identity) over a field k. We consider the enveloping algebra $\Lambda^e = \Lambda^{op} \otimes_k \Lambda$. For a finitely-generated bimodule ${}_\Lambda X_\Lambda$, the ith Hochschild homology and cohomology of Λ with coefficients in X, denoted by $HH_i(\Lambda, X)$ and $HH^i(\Lambda, X)$, are the groups $\mathrm{Tor}_i^{\Lambda^e}(\Lambda, X)$ and $\mathrm{Ext}^i_{\Lambda^e}(\Lambda, X)$ respectively, for each $i \geq 0$ [7]. Of particular interest is the case $X = \Lambda$, and in this case we shall write $HH_i(\Lambda) = HH_i(\Lambda, \Lambda)$ and $HH^i(\Lambda) = HH^i(\Lambda, \Lambda)$. It is well known that the Hochschild homology and cohomology of an algebra are subtle invariants of associative algebras under Morita equivalence, tilting equivalence, derived equivalence, and so on, and have played a fundamental role in the representation theory of Artin algebras: Hochschild cohomology is closely related to simple connectedness, separability and deformation theory [1, 8, 14, 18, 25]; Hochschild homology is closely related to the oriented cycle and the global dimension of algebras [3, 16, 19].

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In [18], Happel asked the following question. If the Hochschild cohomology groups $HH^n(\Lambda)$ of a finite-dimensional algebra Λ over a field k vanish for all sufficiently-large n, is the global dimension of Λ finite? In 2005, Buchweitz, Green, Madsen and Solberg gave a negative answer to this question by exhibiting the Hochschild cohomology behaviour of a family of pathological algebras $A_q = k\langle x, y\rangle/(x^2, xy + qyx, y^2)$ (the so-called quantum exterior algebras) [5]. Moreover, these algebras have been studied to exhibit some other pathologies and thus give negative answers to some open problems, such as Tachikawa's conjecture, Ringel's problem, and so on [20, 23, 26]. Recently, this class of algebras has been extensively studied. Their Hochschild homology and cohomology have been calculated explicitly, and the Hochschild cohomology rings of A_q have been determined via generators and relations [5, 27].

During the last few years, several results and tools from algebraic topology, such as covering theory, have been adapted to the representation theory of noncommutative finite-dimensional associative algebras over a field k [13]. A comparison of the Hochschild cohomology of the algebras involved in a Galois covering of the Kronecker algebra with a cyclic group of order 2 was initiated in [22]. A Cartan–Leray spectral sequence related to the Hochschild–Mitchell (co)homology of a Galois covering of linear categories was obtained in [9]. The skew category, Galois covering and smash product of a category over a ring are studied in [10, 11]. It is well known that there are strong connections between skew group algebras, Galois coverings and smash products of graded algebras [12]. Moreover, it is proved in [17] that a finite-dimensional quiver algebra is Koszul if and only if its finite Galois covering algebra with Galois group G satisfying char $k \nmid |G|$ is Koszul. As an example, the Galois covering algebra Λ_q of quantum exterior algebra A_q with Galois group G satisfying char $k \nmid |G|$ is Koszul again. In this note we consider the case where G is the noncyclic Abelian group $\mathbb{Z}_2 \times \mathbb{Z}_2$. We first provide a minimal projective resolution of Λ_q. Applying this minimal projective resolution, we calculate the Hochschild (co)homology of Λ_q, and the cyclic homology of Λ_q can be calculated in the case where the underlying field is of characteristic zero. These examples will be helpful to understand deeply the Hochschild homology and cohomology behaviour of Galois covering algebras of Koszul algebras with finite Galois groups.

2. Minimal projective bimodule resolutions

Throughout this paper, we fix a base field k with char $k \neq 2$ and let $\mathbb{Z}_2 = \{\bar{0}, \bar{1}\}$ be the residue group modulo 2. We always suppose that

$$A_q = k\langle x, y\rangle/(x^2, xy + qyx, y^2)$$

is the quantum exterior algebra, with $q \in k\backslash\{0\}$, unless otherwise specified.

Let Q be the quiver given by four points $\bar{0}, \bar{1}, \bar{0}', \bar{1}'$ and eight arrows $x_{\bar{0}}, y_{\bar{0}}, x_{\bar{1}}, y_{\bar{1}}$,

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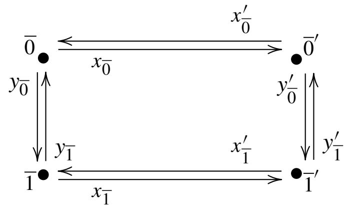
$x'_{\bar{0}}$, $y'_{\bar{0}}$, $x'_{\bar{1}}$, $y'_{\bar{1}}$ as follows:



We sometimes write arrows x_i, y_i, x'_i, y'_i instead of $x_{\bar{i}}, y_{\bar{i}}, x'_{\bar{i}}, y'_{\bar{i}}$ respectively, $i = 0, 1$, and assume that, for any nonnegative integers k, j, if $k \equiv j \pmod 2$, then $x_j = x_k$, $x'_j = x'_k$, $y_j = y_k$ and $y'_j = y'_k$. For an arrow $\rho \in Q$, we denote the length of ρ by $l(\rho)$, and let $o(\rho)$, $t(\rho)$ be the origin and terminus of ρ respectively. Denote by I the ideal of the path algebra kQ generated by

$$R := \{x_i x'_i,\ y_i y_{i+1},\ x'_i x_i,\ y'_i y'_{i+1},\ x_i y'_i + q y_i x_{i+1},\ x'_i y_i + q y'_i x'_{i+1} \mid i = 0, 1\}.$$

For information on quivers we refer to [2]. Set $\Lambda_q = kQ/I$. Then Λ_q is just the $\mathbb{Z}_2 \times \mathbb{Z}_2$-Galois covering of the quantum exterior algebra over k, which is a Koszul algebra (see [17]). Denote by e_0, e'_0, e_1, e'_1 the primitive orthogonal idempotents corresponding to the points $\bar{0}, \bar{0}', \bar{1}, \bar{1}'$ respectively. Then we can order the paths in Q in left length-lexicographic order by choosing

$$e_0 \prec e'_0 \prec e_1 \prec e'_1 \prec x_0 \prec x'_0 \prec x_1 \prec x'_1 \prec y_0 \prec y'_0 \prec y_1 \prec y'_1,$$

namely, $u_1 \ldots u_s \prec v_1 \ldots v_t$ with u_i and v_i being arrows if $s < t$, or $s = t$ but $u_i = v_i$ for $0 \le i < r$ and $u_r \prec v_r$ for some $1 \le r \le s$. Thus, Λ_q has an ordered k-basis

$$\mathcal{B} = \{e_i, e'_i, x_i, x'_i, y_i, y'_i, y_i x_{i+1}, y'_i x'_{i+1} \mid i = 0, 1\},$$

if we identify the elements of $\mathcal{B}$ with their images in Λ_q. We always write a composition of paths from left to right.

Now we construct a minimal projective bimodule resolution $(P_\bullet, \delta_\bullet)$ of Λ_q. For each $n \ge 0$, we first construct certain elements

$$\{f_j^{(n,i)}, f_j^{(n,i')} \mid 0 \le j \le n, i = 0, 1\}.$$

Let

$$f_0^{(0,i)} = e_i, \qquad f_0^{(0,i')} = e'_i,$$
$$f_0^{(1,i)} = x_i, \qquad f_1^{(1,i)} = y_i, \qquad f_0^{(1,i')} = x'_i, \qquad f_1^{(1,i')} = y'_i.$$

For any $\rho \in \{x_0, y_0, x_1, y_1\}$, we define

$$\rho^{(l)} = \begin{cases} \rho' & \text{if } l \text{ is odd;} \\ \rho & \text{if } l \text{ is even.} \end{cases}$$

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Then we can define

$$\{f_j^{(n,i)},\ f_j^{(n,i')}\}_{j=0}^{n}, \quad i = 0, 1,$$

for all $n \geq 2$ inductively by setting

$$\begin{aligned} f_j^{(n,i)} &= q^j f_j^{(n-1,i)} x_{i+j}^{(n+j+1)} + f_{j-1}^{(n-1,i)} y_{i+j+1}^{(n+j)}, \\ f_j^{(n,i')} &= q^j f_j^{(n-1,i')} x_{i+j}^{(n+j)} + f_{j-1}^{(n-1,i')} y_{i+j+1}^{(n+j+1)}, \end{aligned} \tag{2.1}$$

and

$$f_{-1}^{(n,i)} = f_{-1}^{(n,i')} = f_{n+1}^{(n,i)} = f_{n+1}^{(n,i')} = 0, \quad i = 0, 1.$$

Let

$$\Gamma^{(n)} = \{f_j^{(n,i)},\ f_j^{(n,i')} \mid 0 \leq j \leq n,\ i = 0, 1\}.$$

Clearly, $|\Gamma^{(n)}| = 4(n+1)$.

Recall that a nonzero element

$$x = \sum_{i=1}^{s} \alpha_i p_i \in kQ$$

is said to be *uniform* if there exist vertices u and v in Q_0 such that $o(p_i) = u$ and $t(p_i) = v$ for all p_i, $i = 1, 2, \ldots, s$. Note that $f_j^{(n,i)}$ and $f_j^{(n,i')}$ are linear combinations of some paths in kQ for all $0 \leq j \leq n$, which are uniform. Thus for any $f \in \Gamma^{(n)}$, we usually denote by $o(f)$ and $t(f)$ the common origins and termini of all the paths occurring in f. Set

$$\Gamma_{ij}^{(n)} = \{f \in \Gamma^{(n)} \mid o(f) = i \text{ and } t(f) = j\}, \quad i, j = 0, 1, 0', 1'.$$

Let $\otimes := \otimes_k$. Let

$$P_n := \coprod_{f \in \Gamma^{(n)}} \Lambda_q o(f) \otimes t(f) \Lambda_q \quad (\forall\, n \geq 0).$$

Note that

$$\begin{aligned} f_j^{(n,i)} &= x_i f_j^{(n-1,i')} + q^{n-j} y_i f_{j-1}^{(n-1,i+1)}; \\ f_j^{(n,i')} &= x_i' f_j^{(n-1,i)} + q^{n-j} y_i' f_{j-1}^{(n-1,(i+1)')}. \end{aligned} \tag{2.2}$$

So we can define $\delta_n : P_n \to P_{n-1}$ by setting

$$\begin{aligned} \delta_n(o(f_j^{(n,i)}) \otimes t(f_j^{(n,i)})) = {} & x_i \otimes t(f_j^{(n-1,i')}) + (-1)^n q^j o(f_j^{(n-1,i)}) \otimes x_{i+j}^{(n+j+1)} \\ & + q^{n-j} y_i \otimes t(f_{j-1}^{(n-1,i+1)}) \\ & + (-1)^n o(f_{j-1}^{(n-1,i)}) \otimes y_{i+j+1}^{(n+j)}; \end{aligned}$$

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$$\delta_n(o(f_j^{(n,i')}) \otimes t(f_j^{(n,i')})) = x_i' \otimes t(f_j^{(n-1,i)}) + (-1)^n q^j o(f_j^{(n-1,i')}) \otimes x_{i+j}^{(n+j)} + q^{n-j} y_i' \otimes t(f_{j-1}^{(n-1,(i+1)')}) + (-1)^n o(f_{j-1}^{(n-1,i')}) \otimes y_{i+j+1}^{(n+j+1)}.$$

THEOREM 2.1. *The complex* $(P_\bullet, \delta_\bullet)$

$$\cdots \longrightarrow P_{n+1} \xrightarrow{\delta_{n+1}} P_n \xrightarrow{\delta_n} \cdots \xrightarrow{\delta_3} P_2 \xrightarrow{\delta_2} P_1 \xrightarrow{\delta_1} P_0 \longrightarrow 0$$

is a minimal projective bimodule resolution of the covering algebra $\Lambda_q = kQ/I$.

PROOF. We consider the minimal projective bimodule resolution of Λ_q constructed in [6, Section 9]. Let $X = \{x_0, y_0, x_1, y_1, x_0', y_0', x_1', y_1'\}$. Since Λ_q is a Koszul algebra (see [17]), we only need to prove that $\Gamma^{(n)}$ is a k-basis of the k-vector space

$$K_n := \bigcap_{p+q=n-2} X^p R X^q.$$

Note that $XK_{n-1} \cap K_{n-1}X \subset K_n$. By Formulae (2.1) and (2.2),

$$f_j^{(n,0)},\ f_j^{(n,0')},\ f_j^{(n,1)},\ f_j^{(n,1')} \in K_n,$$

by induction on n. Clearly, $\Gamma^{(n)}$ is k-linearly independent.

Denote by $(R^\perp)$ the ideal of kQ generated by

$$R^\perp := \{y_i x_{i+1} - q x_i y_i',\ y_i' x_{i+1} - q x_i' y_i' \mid i = 0, 1\}.$$

Then the quadratic dual of the Koszul algebra Λ_q is just the algebra $\Lambda_q^! = kQ/(R^\perp)$, which is isomorphic to the Yoneda algebra $E(\Lambda_q)$ of Λ_q (see [4, Theorem 2.10.1]). Thus the Betti numbers of a minimal projective resolution of Λ_q over Λ_q^{e} are $\dim_k K_n = 4(n+1)$. Hence $\Gamma^{(n)}$ is a k-basis of K_n.

Finally, the maps $\delta_\bullet$ are determined by [6, p. 354]; see also [15]. □

3. Hochschild homology and cyclic homology

In this section we calculate the k-dimensions of Hochschild homology groups and cyclic homology groups (in the case char $k = 0$) of the covering algebra Λ_q. Let X and Y be the sets of uniform elements in kQ; then one defines

$$X \odot Y = \{(p, q) \in X \times Y \mid t(p) = o(q) \text{ and } t(q) = o(p)\}.$$

We denote by $k(X \odot Y)$ the vector space that has as basis the set $X \odot Y$.

Applying the functor $\Lambda_q \otimes_{\Lambda_q^{\mathrm{e}}} (\cdot)$ to the minimal projective bimodule resolution $(P_\bullet, \delta_\bullet)$, we have the following result.

LEMMA 3.1. *We have* $\Lambda_q \otimes_{\Lambda_q^e} (P_\bullet, \delta_\bullet) = (N_\bullet, \tau_\bullet)$, *where* $N_n \cong k(\mathcal{B} \odot \Gamma^{(n)})$ *and* $\tau_n : N_n \to N_{n-1}$ *is given by*

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$$\begin{aligned}\tau_n(b, f_j^{(n,i)}) &= (bx_i, f_j^{(n-1,i')}) + (-1)^n q^j (x_{i+j}^{(n+j+1)} b, f_j^{(n-1,i)}) \\ &\quad + q^{n-j}(by_i, f_{j-1}^{(n-1,i+1)}) + (-1)^n (y_{i+j+1}^{(n+j)} b, f_{j-1}^{(n-1,i)}); \\ \tau_n(b, f_j^{(n,i')}) &= (bx_i', f_j^{(n-1,i)}) + (-1)^n q^j (x_{i+j}^{(n+j)} b, f_j^{(n-1,i')}) \\ &\quad + q^{n-j}(by_i', f_{j-1}^{(n-1,(i+1)')}) + (-1)^n (y_{i+j+1}^{(n+j+1)} b, f_{j-1}^{(n-1,i')}).\end{aligned}$$

PROOF. Clearly,

$$\begin{aligned}N_n = \Lambda_q \otimes_{\Lambda_q^e} P_n &= \Lambda_q \otimes_{E^e} \coprod_{f\in\Gamma^{(n)}} (o(f) \otimes_k t(f)) \\ &\cong \coprod_{\alpha,\beta\in\{e_0,e_0',e_1,e_1'\}} \alpha\Lambda_q\beta \otimes_k \beta\Gamma_{ji}^{(n)}\alpha,\end{aligned}$$

where E is the maximal semisimple subalgebra of Λ_q. Thus $\mathcal{B} \odot \Gamma^{(n)}$ forms a k-basis of N_n by definition.

From the isomorphism above, we have the commutative diagram

$$\begin{array}{ccccccc} \cdots \longrightarrow & \Lambda_q \otimes_{\Lambda_q^e} P_n & \xrightarrow{1\otimes\delta_n} & \Lambda_q \otimes_{\Lambda_q^e} P_{n-1} & \longrightarrow \cdots \\ & \wr\downarrow & & \wr\downarrow & \\ \cdots \longrightarrow & k(\mathcal{B}\odot\Gamma^{(n)}) & \xrightarrow{\tau_n} & k(\mathcal{B}\odot\Gamma^{(n-1)}) & \longrightarrow \cdots \end{array}$$

So the differentials τ_n can be induced by δ_n in the minimal projective resolution $(P_\bullet, \delta_\bullet)$. □

Note that $HH_n(\Lambda_q) = \mathrm{Ker}\,\tau_n/\mathrm{Im}\,\tau_{n+1}$ by definition:

$$\begin{aligned}\dim_k HH_n(\Lambda_q) &= \dim_k \mathrm{Ker}\,\tau_n - \dim_k \mathrm{Im}\,\tau_{n+1} \\ &= \dim_k N_n - \dim_k \mathrm{Im}\,\tau_n - \dim_k \mathrm{Im}\,\tau_{n+1}. \end{aligned} \tag{3.1}$$

Consequently, to calculate the dimensions of Hochschild homology groups of Λ_q, we only need to determine the $\dim_k N_n$ and $\dim_k \mathrm{Im}\,\tau_n$.

For any $(b, f) \in \mathcal{B} \odot \Gamma^{(n)}$, $l(b)$ and n must have the same parity. If n is odd, then

$$\begin{aligned}\mathcal{B}\odot\Gamma^{(n)} &= (\{x_0\}\odot\Gamma_{0'0}^{(n)}) \cup (\{x_0'\}\odot\Gamma_{00'}^{(n)}) \cup (\{x_1\}\odot\Gamma_{1'1}^{(n)}) \cup (\{x_1'\}\odot\Gamma_{11'}^{(n)}) \\ &\quad \cup (\{y_0\}\odot\Gamma_{10}^{(n)}) \cup (\{y_0'\}\odot\Gamma_{1'0'}^{(n)}) \cup (\{y_1\}\odot\Gamma_{01}^{(n)}) \cup (\{y_1'\}\odot\Gamma_{0'1'}^{(n)});\end{aligned}$$

if n is even, then

$$\begin{aligned}\mathcal{B}\odot\Gamma^{(n)} &= (\{e_0\}\odot\Gamma_{00}^{(n)}) \cup (\{e_0'\}\odot\Gamma_{0'0'}^{(n)}) \cup (\{e_1\}\odot\Gamma_{11}^{(n)}) \cup (\{e_1'\}\odot\Gamma_{1'1'}^{(n)}) \\ &\quad \cup (\{y_0x_1\}\odot\Gamma_{1'0}^{(n)}) \cup (\{y_0'x_1'\}\odot\Gamma_{10'}^{(n)}) \\ &\quad \cup (\{y_1x_0\}\odot\Gamma_{0'1}^{(n)}) \cup (\{y_1'x_0'\}\odot\Gamma_{01'}^{(n)}).\end{aligned}$$

Hence $\dim_k N_n = 4(n+1)$.

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We now define an order on $\mathcal{B} \odot \Gamma^{(n)}$ as follows:

$$(b_1, f_{j_1}^{(n,i_1)}) \prec (b_2, f_{j_2}^{(n,i_2)}) \quad \text{if } j_1 < j_2, \text{ or } j_1 = j_2 \text{ but } b_1 \prec b_2,$$

for any $(b_1, f_{j_1}^{(n,i_1)}), (b_2, f_{j_2}^{(n,i_2)}) \in \mathcal{B} \odot \Gamma^{(n)}$ and $i_1, i_2 \in \{0, 0', 1, 1'\}$.

We still denote by τ_n the matrix of the differentials τ_n under the ordered bases above, and write

$$A_i = \begin{pmatrix} 1 & q^i & 0 & 0 \\ q^i & 1 & 0 & 0 \\ 0 & 0 & 1 & q^i \\ 0 & 0 & q^i & 1 \end{pmatrix}_{4\times 4}, \quad B_i = \begin{pmatrix} q^{n-i} & 0 & 1 & 0 \\ 0 & q^{n-i} & 0 & 1 \\ 1 & 0 & q^{n-i} & 0 \\ 0 & 1 & 0 & q^{n-i} \end{pmatrix}_{4\times 4}.$$

It follows from the descriptions of the differentials τ_n in Lemma 3.1 that if n is odd, then

$$\tau_n = \begin{pmatrix} 0 & & & & & & \\ 0 & A_2 & & & & & \\ & -B_1 & 0 & & & & \\ & & 0 & A_4 & & & \\ & & & -B_3 & \ddots & & \\ & & & & \ddots & 0 & & \\ & & & & & 0 & A_{n-1} & \\ & & & & & & -B_{n-2} & 0 \\ & & & & & & & 0 \end{pmatrix}_{4(n+1)\times 4n};$$

and if n is even, then

$$\tau_n = \begin{pmatrix} A_0 & & & & & & \\ 0 & 0 & & & & & \\ & B_2 & A_2 & & & & \\ & & 0 & \ddots & & & \\ & & & \ddots & 0 & & \\ & & & & B_{n-2} & A_{n-2} & \\ & & & & & 0 & 0 \\ & & & & & & B_n \end{pmatrix}_{4(n+1)\times 4n}.$$

Clearly,

$$\operatorname{rank}(\tau_n) = \sum_{i\in\{1,3,5,\ldots,n-2\}} \operatorname{rank}\begin{pmatrix} A_{i+1} \\ -B_i \end{pmatrix},$$

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if n is odd; and

$$\begin{aligned}\operatorname{rank}(\tau_n) &= \operatorname{rank}(A_0) + \operatorname{rank}(B_n) + \sum_{i\in\{2,4,6,\ldots,n-2\}} \operatorname{rank}\begin{pmatrix} B_i & A_i \end{pmatrix}\\ &= 4 + \sum_{i\in\{2,4,6,\ldots,n-2\}} \operatorname{rank}\begin{pmatrix} B_i & A_i \end{pmatrix},\end{aligned}$$

if n is even.

LEMMA 3.2. *Let Λ_q be the $\mathbb{Z}_2 \times \mathbb{Z}_2$-Galois covering of A_q, with q an rth ($r > 2$) primitive root of unity. If n (>2) is odd,*

$$\operatorname{rank}(\tau_n) = \begin{cases} 2n-k-1 & \text{if } r \text{ is odd and } n = 2kr-1, \text{ for some } k \geq 1,\\ & \text{or } r \text{ is even and } n = kr-1, \text{ for some } k \geq 1;\\ 2n-2 & \text{otherwise.}\end{cases}$$

and if $n (>2)$ is even, then

$$\operatorname{rank}(\tau_n) = \begin{cases} 2n-k+1 & \text{if } r \text{ is odd and } n = 2kr, \text{ for some } k \geq 1,\\ & \text{or } r \text{ is even and } n = kr, \text{ for some } k \geq 1;\\ 2n & \text{otherwise.}\end{cases}$$

PROOF.

CASE I. Suppose n is odd.

For $i = 1, 3, 5, \ldots, n-2$, by elementary operations, each

$$\begin{pmatrix} A_{i+1} \\ -B_i \end{pmatrix}$$

can be changed into

$$\begin{pmatrix} 0 & 0 & -q^{n-i} & -q^{n+1}\\ 0 & 0 & -q^{n+1} & -q^{n-i}\\ 0 & 0 & 1 & q^{i+1}\\ 0 & 0 & q^{i+1} & 1\\ 0 & 0 & q^{2(n-i)}-1 & 0\\ 0 & 0 & 0 & q^{2(n-i)}-1\\ -1 & 0 & 0 & 0\\ 0 & -1 & 0 & 0 \end{pmatrix}_{8\times 4}.$$

Note that $\operatorname{rank}\begin{pmatrix} A_{i+1} \\ -B_i \end{pmatrix} = 3$ or 4, and $\operatorname{rank}\begin{pmatrix} A_{i+1} \\ -B_i \end{pmatrix} = 3$ if and only if

$$\begin{cases} q^{2(n-i)} = 1;\\ q^{2(i+1)} = 1 \end{cases} \iff \begin{cases} q^{2(n+1)} = 1;\\ q^{2(i+1)} = 1. \end{cases}$$

Moreover, $q^{2(n+1)} = 1$ if and only if either of the following ((1) or (2)) is satisfied:

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(1) r is odd and $2(n+1)=4kr$, for some $k\geq 1$;
(2) r is even and $2(n+1)=2kr$, for some $k\geq 1$.
Since i is odd, we have $q^{2(i+1)}=1$ if and only if either of the following ((3) or (4)) is satisfied:
(3) r is odd and $2(i+1)=4k_1r$, for some $k_1\geq 1$;
(4) r is even and $2(i+1)=2k_1r$, for some $k_1\geq 1$.

If (1) and (3) are satisfied, then r is odd, $n=2kr-1$ and $k_1=1,2,\ldots,k-1$. So the number of i satisfying

$$\operatorname{rank}\begin{pmatrix}A_{i+1}\\ -B_i\end{pmatrix}=3$$

is $k-1$, and $\operatorname{rank}(\tau_n)=2n-k-1$.

If (2) and (4) are satisfied, then r is even, $n=kr-1$ and $k_1=1,2,\ldots,k-1$. So the number of i satisfying

$$\operatorname{rank}\begin{pmatrix}A_{i+1}\\ -B_i\end{pmatrix}=3$$

is $k-1$, and $\operatorname{rank}(\tau_n)=2n-k-1$.

Otherwise, for each i,

$$\operatorname{rank}\begin{pmatrix}A_{i+1}\\ -B_i\end{pmatrix}=4.$$

So $\operatorname{rank}(\tau_n)=4\times((n-1)/2)=2n-2$.

Case II. Suppose n is even.

For $i=2,4,6,\ldots,n-2$, by elementary operations, each $(B_i\quad A_i)$ can be changed into

$$\begin{pmatrix}0&0&1-q^{2(n-i)}&0&1&q^i&-q^{n-i}&-q^n\\ 0&0&0&1-q^{2(n-i)}&q^i&1&-q^n&-q^{n-i}\\ 1&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0\end{pmatrix}_{4\times 8}.$$

Note that $\operatorname{rank}(B_i\quad A_i)=3$ or 4, and $\operatorname{rank}(B_i\quad A_i)=3$ if and only if

$$\begin{cases}q^{2(n-i)}=1;\\ q^{2i}=1.\end{cases}\iff\begin{cases}q^{2n}=1;\\ q^{2i}=1.\end{cases}$$

Moreover, we have $q^{2n}=1$ if and only if either of the following ((5) or (6)) is satisfied:
(5) r is odd and $2n=4kr$, for some $k\geq 1$;
(6) r is even and $2n=2kr$, for some $k\geq 1$.

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Since i is even, we have $q^{2i} = 1$ if and only if either of the following ((7) or (8)) is satisfied:

(7) r is odd and $2i = 4k_1 r$, for some $k_1 \geq 1$;

(8) r is even and $2i = 2k_1 r$, for some $k_1 \geq 1$.

If (5) and (7) are satisfied, then r is odd, $n = 2kr$ and $k_1 = 1, 2, \ldots, k-1$. So the number of i satisfying $\operatorname{rank}(B_i \quad A_i) = 3$ is $k-1$, and $\operatorname{rank}(\tau_n) = 2n - k + 1$.

If (6) and (8) are satisfied, then r is even, $n = kr$ and $k_1 = 1, 2, \ldots, k-1$. So the number of i satisfying $\operatorname{rank}(B_i \quad A_i) = 3$ is $k-1$, and $\operatorname{rank}(\tau_n) = 2n - k + 1$.

Otherwise, for each i, $\operatorname{rank}(B_i \quad A_i) = 4$. So

$$\operatorname{rank}(\tau_n) = 4 + 4 \times \frac{n-2}{2} = 2n.$$

The proof is complete. □

LEMMA 3.3. *Let Λ_q be the $\mathbb{Z}_2 \times \mathbb{Z}_2$-Galois covering of A_q. If q ($\neq 0$) is not an rth ($r > 2$) primitive root of unity, then for $n > 2$*

$$\operatorname{rank}(\tau_n) = \begin{cases} 2n - 2 & \text{if } n \text{ is odd and } q\ (\neq 0) \text{ is not a root of unity;} \\ 2n & \text{if } n \text{ is even and } q\ (\neq 0) \text{ is not a root of unity;} \\ \frac{3}{2}(n-1) & \text{if } n \text{ is odd and } q = \pm 1; \\ \frac{3}{2}n + 1 & \text{if } n \text{ is even and } q = \pm 1. \end{cases}$$

PROOF.

CASE I. Suppose n is odd. If q ($\neq 0$) is not a root of unity, then $q^{2(n+1)} \neq 1$ for $n > 2$, and $\operatorname{rank}(\tau_n) = 2n - 2$; if $q = \pm 1$, then $q^{2(n+1)} = q^{2(i+1)} = 1$, and $\operatorname{rank}(\tau_n) = (3/2)\,(n-1)$.

CASE II. Suppose n is even. If q ($\neq 0$) is not a root of unity, then $q^{2n} \neq 1$ for $n > 2$, and $\operatorname{rank}(\tau_n) = 2n$; if $q = \pm 1$, then $q^{2n} = q^{2i} = 1$, and $\operatorname{rank}(\tau_n) = (3/2)n + 1$. This completes the proof. □

For $n = 0, 1, 2$, direct computations show that

$$\dim_k HH_0(\Lambda_q) = 4;$$
$$\dim_k HH_1(\Lambda_q) = 4;$$
$$\dim_k HH_2(\Lambda_q) = \begin{cases} 4 & \text{if } q \neq \pm 1, \pm\sqrt{-1}; \\ 5 & \text{if } q = \pm 1, \pm\sqrt{-1}. \end{cases}$$

THEOREM 3.4. *Let Λ_q be the $\mathbb{Z}_2 \times \mathbb{Z}_2$-Galois covering of the quantum exterior algebra A_q. If q is an rth ($r > 2$) primitive root of unity, then for $n > 2$*

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$$\dim_k HH_n(\Lambda_q) = \begin{cases} k+3 & \textit{if } r \textit{ is odd and } n = 2kr-2 \textit{ or } n = 2kr, \textit{ for some } k \geq 1, \\ & \textit{or } r \textit{ is even and } n = kr-2 \textit{ or } n = kr, \textit{ for some } k \geq 1; \\ 2k+2 & \textit{if } r \textit{ is odd and } n = 2kr-1, \textit{ for some } k \geq 1, \\ & \textit{or } r \textit{ is even and } n = kr-1, \textit{ for some } k \geq 1; \\ 4 & \textit{otherwise.} \end{cases}$$

PROOF. By Lemma 3.2 and the formula

$$\dim_k HH_n(\Lambda_q) = \dim_k N_n - \dim_k \mathrm{Im}\ \tau_n - \dim_k \mathrm{Im}\ \tau_{n+1},$$

we can get the result directly. □

THEOREM 3.5. *Let Λ_q be the $\mathbb{Z}_2 \times \mathbb{Z}_2$-Galois covering of the quantum exterior algebra A_q. If q ($\neq 0$) is not an rth ($r > 2$) primitive root of unity, then for $n > 2$,*

$$\dim_k HH_n(\Lambda_q) = \begin{cases} 4 & \textit{if } q\ (\neq 0) \textit{ is not a root of unity;} \\ n+3 & \textit{if } q = \pm 1. \end{cases}$$

PROOF. By Lemma 3.3, we have $\dim_k(\tau_n) + \dim_k(\tau_{n+1}) = 4n$ if q ($\neq 0$) is not a root of unity; and $\dim_k(\tau_n) + \dim_k(\tau_{n+1}) = 3n+1$ if $q = \pm 1$. The theorem follows from the formula (3.1) as desired. □

Denote by $HC_n(\Lambda_q)$ the nth cyclic homology group of Λ_q (see [21]).

COROLLARY 3.6. *Let Λ_q be the $\mathbb{Z}_2 \times \mathbb{Z}_2$-Galois covering of the quantum exterior algebra A_q. If q is an rth ($r > 2$) primitive root of unity and* char $k = 0$, *then*

$$\dim_k HC_n(\Lambda_q) = \begin{cases} k+3 & \textit{if } r \textit{ is odd and } n = 2kr-1 \textit{ or } n = 2kr-2, \textit{ for some } k \geq 1, \\ & \textit{or } r \textit{ is even and } n = kr-1 \textit{ or } n = kr-2, \textit{ for some } k \geq 1; \\ 4 & \textit{otherwise.} \end{cases}$$

PROOF. By [21, Theorem 4.1.13],

$$\begin{aligned} \dim_k HC_n(\Lambda_q) - \dim_k HC_n(k^4) = {} & -(\dim_k HC_{n-1}(\Lambda_q) - \dim_k HC_{n-1}(k^4)) \\ & + (\dim_k HH_n(\Lambda_q) - \dim_k HH_n(k^4)). \end{aligned}$$

Thus

$$\dim_k HC_n(\Lambda_q) - \dim_k HC_n(k^4) = \sum_{i=0}^{n} (-1)^{n-i} (\dim_k HH_i(\Lambda_q) - \dim_k HH_i(k^4)).$$

It is well known that

$$\dim_k HH_i(k^4) = \begin{cases} 4 & \text{if } i = 0; \\ 0 & \text{if } i \geq 1 \end{cases} \quad \text{and} \quad \dim_k HC_i(k^4) = \begin{cases} 4 & \text{if } i \text{ is even;} \\ 0 & \text{if } i \text{ is odd.} \end{cases}$$

By Theorem 3.4, we can obtain the result. □

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COROLLARY 3.7. *Let Λ_q be the $\mathbb{Z}_2 \times \mathbb{Z}_2$-Galois covering of the quantum exterior algebra A_q. If q ($\neq 0$) is not an rth ($r > 2$) primitive root of unity and* $\operatorname{char} k = 0$, *then*

$$\dim_k HC_n(\Lambda_q) = \begin{cases} \dfrac{n-1}{2} + 4 & \text{if } q = \pm 1 \text{ and } n \text{ is odd;} \\ \dfrac{n}{2} + 4 & \text{if } q = \pm 1 \text{ and } n \text{ is even;} \\ 4 & \text{if } q\ (\neq 0) \text{ is not a root of unity.} \end{cases}$$

PROOF. Similarly to the proof of Corollary 3.6, we can get this corollary by Theorem 3.5. □

For completeness, we also consider the degenerate case when $q = 0$. Then

$$A_0 = k\langle x, y\rangle/(x^2, xy, y^2),$$

and its $\mathbb{Z}_2 \times \mathbb{Z}_2$-Galois covering algebra $\Lambda_0 = kQ/(R_0)$, where the quiver Q is as in Section 2 and

$$R_0 := \{x_i x_i', y_i y_{i+1}, x_i' x_i, y_i' y_{i+1}', x_i y_i', x_i' y_i \mid i = 0, 1\}.$$

THEOREM 3.8. *Let Λ_0 be the $\mathbb{Z}_2 \times \mathbb{Z}_2$-Galois covering of A_0 as the above; then*

$$\dim_k HH_n(\Lambda_0) = 4.$$

PROOF. Clearly, Λ_0 is a quadratic monomial algebra. Denote by $\Lambda_0^!$ the quadratic dual of Λ_0; then

$$\Lambda_0^! = kQ/(y_0 x_0', y_1 x_1', y_0' x_0, y_1' x_1),$$

and

$$\begin{aligned}\Gamma^n = \{f_j^{(n,i)} &= x_i x_i' x_i \ldots x_i^{(n-j+1)} y_i^{(n-j)} y_{i+1}^{(n-j)} \ldots y_{i+j-1}^{(n-j)}, f_j^{(n,i')} \\ &= x_i' x_i x_i' \ldots x_i^{(n-j)} y_i^{(n-j+1)} y_{i+1}^{(n-j+1)} \ldots y_{i+j-1}^{(n-j+1)} \mid 0 \le j \le n, i = 0, 1\}\end{aligned}$$

is a k-basis of $\Lambda_0^!$. Thus, by [24], we can get a minimal projection resolution of Λ_0,

$$(P_\bullet^0, \delta_\bullet^0) : \cdots \to P_n^0 \xrightarrow{\delta_n^0} P_{n-1}^0 \longrightarrow \cdots \longrightarrow P_2^0 \xrightarrow{\delta_2^0} P_1^0 \xrightarrow{\delta_1^0} P_0^0 \longrightarrow 0,$$

where

$$P_n^0 = \coprod_{f \in \Gamma^{(n)}} \Lambda_0 o(f) \otimes t(f) \Lambda_0,$$

and the differentials are given by

$$\delta_n^0(o(f) \otimes t(f)) = L^{n-1} \otimes t(f) + (-1)^n o(f) \otimes R^{n-1},$$

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where L^{n-1} and R^{n-1} are the subpaths of f satisfying $f = L^{n-1}g = hR^{n-1}$ for some $g, h \in \Gamma^{(n-1)}$.

Applying the functor $\Lambda_0 \otimes_{\Lambda_0^e} (\cdot)$ to $(P_\bullet^0, \delta_\bullet^0)$, we get the homology complex $(N_\bullet^0, \tau_\bullet^0)$ of Λ_0, and $\dim_k N_n^0 = 4(n+1)$,

$$\dim_k \mathrm{Im}\, \tau_n^0 = \begin{cases} 2n-2 & \text{if } n \text{ is odd;} \\ 2n & \text{if } n \text{ is even.} \end{cases}$$

So $\dim_k HH_n(\Lambda_0) = 4$. □

Similar to Corollary 3.6, we have the following result.

COROLLARY 3.9. *Let Λ_0 be the $\mathbb{Z}_2 \times \mathbb{Z}_2$-Galois covering of A_0 and* char $k = 0$*; then*

$$\dim_k HC_n(\Lambda_0) = 4.$$

4. Hochschild cohomology

In this section we calculate the k-dimensions of Hochschild cohomology groups of the covering algebra Λ_q. Let X and Y be the sets of uniform elements in kQ; then one defines

$$X/\!/Y = \{(p, q) \in X \times Y \mid o(p) = o(q) \text{ and } t(p) = t(q)\}.$$

We denote by $k(X/\!/Y)$ the vector space that has as basis the set $X/\!/Y$.

Applying the functor $\mathrm{Hom}_{\Lambda_q^e}(\cdot, \Lambda_q)$ to the minimal projective bimodule resolution $(P_\bullet, \delta_\bullet)$, we have the following result.

LEMMA 4.1. *We have $\mathrm{Hom}_{\Lambda_q^e}((P_\bullet, \delta_\bullet), \Lambda_q) = (M^\bullet, \varphi^\bullet)$, where $M^n \cong k(\mathcal{B}/\!/\Gamma^{(n)})$ and $\varphi^{n+1} : M^n \to M^{n+1}$ is given by*

$$\begin{aligned}\varphi^{n+1}(b, f_j^{(n,i)}) &= (x_i' b, f_j^{(n+1,i')}) + (-1)^{n+1} q^j (b x_{i+j}^{(n+j)}, f_j^{(n+1,i)}) \\ &\quad + q^{n-j}(y_{i+1} b, f_{j+1}^{(n+1,i+1)}) + (-1)^{n+1}(b y_{i+j}^{(n+j)}, f_{j+1}^{(n+1,i)}); \\ \varphi^{n+1}(b, f_j^{(n,i')}) &= (x_i b, f_j^{(n+1,i)}) + (-1)^{n+1} q^j (b x_{i+j}^{(n+j+1)}, f_j^{(n+1,i')}) \\ &\quad + q^{n-j}(y_{i+1}' b, f_{j+1}^{(n+1,(i+1)')}) + (-1)^{n+1}(b y_{i+j}^{(n+j+1)}, f_{j+1}^{(n+1,i')}).\end{aligned}$$

PROOF. Clearly,

$$\begin{aligned}M^n &= \mathrm{Hom}_{\Lambda_q^e}(P_n, \Lambda_q) = \mathrm{Hom}_{\Lambda_q^e}\Big(\coprod_{f \in \Gamma^{(n)}} \Lambda_q o(f) \otimes t(f)\Lambda_q, \Lambda_q\Big) \\ &\cong \coprod_{f \in \Gamma^{(n)}} (o(f) \otimes t(f)\Lambda_q) \cong \coprod_{f \in \Gamma^{(n)}} (o(f)\Lambda_q t(f)).\end{aligned}$$

Thus $\mathcal{B}/\!/\Gamma^{(n)}$ forms a k-basis of M^n by definition.

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Applying the commutative diagram

$$\begin{array}{ccccccc} \cdots \longrightarrow & \mathrm{Hom}_{\Lambda_q^e}(P_n, \Lambda_q) & \xrightarrow{\delta_{n+1}^*} & \mathrm{Hom}_{\Lambda_q^e}(P_{n+1}, \Lambda_q) & \longrightarrow \cdots \\ & \wr\downarrow & & \wr\downarrow & \\ \cdots \longrightarrow & k(\mathcal{B}/\!/\,\Gamma^{(n)}) & \xrightarrow{\varphi^{n+1}} & k(\mathcal{B}/\!/\,\Gamma^{(n+1)}) & \longrightarrow \cdots \end{array}$$

we can induce the differentials φ^n by δ_n in the minimal projective resolution $(P_\bullet, \delta_\bullet)$. □

Clearly, for any $(b, f) \in \mathcal{B}/\!/\,\Gamma^{(n)}$, $l(b)$ and n must have the same parity. Thus $\dim_k M^n = 4(n+1)$.

Now, we can define an order on $\mathcal{B}/\!/\,\Gamma^{(n)}$ as follows:

$$(b_1, f_{j_1}^{(n,i_1)}) \prec (b_2, f_{j_2}^{(n,i_2)}) \quad \text{if } j_1 < j_2 \text{ or } j_1 = j_2 \text{ but } b_1 \prec b_2,$$

for any

$$(b_1, f_{j_1}^{(n,i_1)}), (b_2, f_{j_2}^{(n,i_2)}) \in \mathcal{B}/\!/\,\Gamma^{(n)} \quad \text{and} \quad i_1, i_2 \in \{0, 0', 1, 1'\}.$$

We still denote by φ^n the matrix of the differentials φ^n under the ordered bases above, and write

$$C_i = \begin{pmatrix} -q^i & 1 & 0 & 0 \\ 1 & -q^i & 0 & 0 \\ 0 & 0 & -q^i & 1 \\ 0 & 0 & 1 & -q^i \end{pmatrix}_{4\times 4}, \quad D_i = \begin{pmatrix} -1 & 0 & q^{n-i} & 0 \\ 0 & -1 & 0 & q^{n-i} \\ q^{n-i} & 0 & -1 & 0 \\ 0 & q^{n-i} & 0 & -1 \end{pmatrix}_{4\times 4}.$$

It follows from the description of the differentials φ^n in Lemma 4.1 that if n is odd, then

$$\varphi^n = \begin{pmatrix} C_0 & D_1 & & & & & & \\ & 0 & 0 & & & & & \\ & & C_2 & D_3 & & & & \\ & & & 0 & 0 & & & \\ & & & & \ddots & \ddots & & \\ & & & & & C_{n-3} & D_{n-2} & & \\ & & & & & & 0 & 0 & \\ & & & & & & & C_{n-1} & D_n \end{pmatrix}_{4n\times 4(n+1)};$$

and if n is even, then

$$\varphi^n = \begin{pmatrix} 0 & qD_2 & & & & & & \\ 0 & -qC_0 & 0 & & & & & \\ & & 0 & qD_4 & & & & \\ & & & -qC_2 & \ddots & & & \\ & & & & \ddots & 0 & & \\ & & & & & 0 & qD_n & 0 \\ & & & & & & -qC_{n-2} & 0 \end{pmatrix}_{4n\times 4(n+1)}.$$

Clearly,

$$\operatorname{rank}(\varphi^n) = \begin{cases} \sum_{i\in\{0,2,4,\ldots,n-1\}} \operatorname{rank}(C_i \quad D_{i+1}) & \text{if } n \text{ is odd;} \\ \sum_{i\in\{0,2,4,\ldots,n-2\}} \operatorname{rank}\begin{pmatrix} D_{i+2} \\ -C_i \end{pmatrix} & \text{if } n \text{ is even.} \end{cases}$$

LEMMA 4.2. *Let Λ_q be the $\mathbb{Z}_2 \times \mathbb{Z}_2$-Galois covering of A_q, with q an rth $(r > 2)$ primitive root of unity. If n (>2) is odd, then*

$$\operatorname{rank}(\varphi^n) = \begin{cases} 2n-k+1 & \textit{if } r \textit{ is odd and } n = 2kr+1, \textit{ for some } k \geq 1, \\ & \textit{or } r \textit{ is even and } n = kr+1, \textit{ for some } k \geq 1; \\ 2n+2 & \textit{otherwise}; \end{cases}$$

and if n (>2) is even, then

$$\operatorname{rank}(\varphi^n) = \begin{cases} 2n-k-1 & \textit{if } r \textit{ is odd and } n = 2kr+2, \textit{ for some } k \geq 1, \\ & \textit{or } r \textit{ is even and } n = kr+2, \textit{ for some } k \geq 1; \\ 2n & \textit{otherwise}. \end{cases}$$

PROOF.

CASE I. Suppose n is odd.

For $i = 0, 2, 4, \ldots, n-1$, by elementary operations, each $(C_i \quad D_{i+1})$ can be changed into

$$\begin{pmatrix} -q^i & 1 & -q^{n-1} & q^{n-i-1} & q^{2(n-i-1)}-1 & 0 & 0 & 0 \\ 1 & -q^i & q^{n-i-1} & -q^{n-1} & 0 & q^{2(n-i-1)}-1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}_{4\times 8}.$$

Note that $\operatorname{rank}(C_i \quad D_{i+1}) = 3$ or 4, and $\operatorname{rank}(C_i \quad D_{i+1}) = 3$ if and only if

$$\begin{cases} q^{2(n-i-1)} = 1; \\ q^{2i} = 1 \end{cases} \iff \begin{cases} q^{2(n-1)} = 1; \\ q^{2i} = 1. \end{cases}$$

Moreover, we have $q^{2(n-1)} = 1$ if and only if either of the following ((1′) or (2′)) is satisfied:

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(1′) r is odd and $2(n-1) = 4kr$, for some $k \geq 1$;
(2′) r is even and $2(n-1) = 2kr$, for some $k \geq 1$.
Since i is even, we have $q^{2i} = 1$ if and only if either of the following ((3′) or (4′)) is satisfied:
(3′) r is odd and $2i = 4k_1 r$, for some $k_1 \geq 0$;
(4′) r is even and $2i = 2k_1 r$, for some $k_1 \geq 0$.

If (1′) and (3′) are satisfied, then r is odd, $n = 2kr + 1$ and $k_1 = 0, 1, \ldots, k$. So the number of i satisfying $\operatorname{rank}(C_i \quad D_{i+1}) = 3$ is $k+1$, and $\operatorname{rank}(\varphi^n) = 2n - k + 1$.

If (2′) and (4′) are satisfied, then r is even, $n = kr + 1$ and $k_1 = 0, 1, \ldots, k$. So the number of i satisfying $\operatorname{rank}(C_i \quad D_{i+1}) = 3$ is $k+1$, and $\operatorname{rank}(\varphi^n) = 2n - k + 1$.

Otherwise, $\operatorname{rank}(\varphi^n) = 4 \times ((n+1)/2) = 2n + 2$.

CASE II. Suppose n is even.

For $i = 0, 2, 4, \ldots, n-2$, by elementary operations, each

$$\begin{pmatrix} D_{i+2} \\ -C_i \end{pmatrix}$$

can be changed into

$$\begin{pmatrix}
-1 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 \\
0 & 0 & q^{2(n-i-2)} - 1 & 0 \\
0 & 0 & 0 & q^{2(n-i-2)} - 1 \\
0 & 0 & q^{n-2} & -q^{n-i-2} \\
0 & 0 & -q^{n-i-2} & q^{n-2} \\
0 & 0 & q^i & -1 \\
0 & 0 & -1 & q^i
\end{pmatrix}_{8\times 4}.$$

Note that $\operatorname{rank}\begin{pmatrix} D_{i+2} \\ -C_i \end{pmatrix} = 3$ or 4, and $\operatorname{rank}\begin{pmatrix} D_{i+2} \\ -C_i \end{pmatrix} = 3$ if and only if

$$\begin{cases} q^{2(n-i-2)} = 1; \\ q^{2i} = 1 \end{cases} \iff \begin{cases} q^{2(n-2)} = 1; \\ q^{2i} = 1. \end{cases}$$

Moreover, we have $q^{2(n-2)} = 1$ if and only if either of the following ((5′) or (6′)) is satisfied:
(5′) r is odd and $2(n-2) = 4kr$, for some $k \geq 1$;
(6′) r is even and $2(n-2) = 2kr$, for some $k \geq 1$.
Since i is even, we have $q^{2i} = 1$ if and only if either of the following ((7′) or (8′)) is satisfied:
(7′) r is odd and $2i = 4k_1 r$, for some $k_1 \geq 1$;
(8′) r is even and $2i = 2k_1 r$, for some $k_1 \geq 1$.

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If (5′) and (7′) are satisfied, then r is odd, $n = 2kr + 2$ and $k_1 = 0, 1, \ldots, k$. So the number of i such that

$$\operatorname{rank}\begin{pmatrix} D_{i+2} \\ -C_i \end{pmatrix} = 3$$

is $k+1$, and $\operatorname{rank}(\varphi^n) = 2n - k - 1$.

If (6′) and (8′) are satisfied, then r is even, $n = kr + 2$ and $k_1 = 0, 1, \ldots, k$. So the number of i such that

$$\operatorname{rank}\begin{pmatrix} D_{i+2} \\ -C_i \end{pmatrix} = 3$$

is $k+1$, and $\operatorname{rank}(\varphi^n) = 2n - k - 1$.

Otherwise, $\operatorname{rank}(\varphi^n) = 4 \times (n/2) = 2n$. The proof is complete. □

LEMMA 4.3. *Let Λ_q be the $\mathbb{Z}_2 \times \mathbb{Z}_2$-Galois covering of A_q. If q ($\neq 0$) is not an rth ($r > 2$) primitive root of unity, then for $n > 2$,*

$$\operatorname{rank}(\varphi^n) = \begin{cases} 2n+2 & \text{if } n \text{ is odd and } q\ (\neq 0) \text{ is not a root of unity;} \\ 2n & \text{if } n \text{ is even and } q\ (\neq 0) \text{ is not a root of unity;} \\ \frac{3}{2}(n+1) & \text{if } n \text{ is odd and } q = \pm 1; \\ \frac{3}{2}n & \text{if } n \text{ is even and } q = \pm 1. \end{cases}$$

PROOF.

CASE I. Suppose n is odd. If q ($\neq 0$) is not a root of unity, then $q^{2(n-1)} \neq 1$ for $n > 2$, and $\operatorname{rank}(\varphi^n) = 2n + 2$. If $q = \pm 1$, then $q^{2(n-1)} = q^{2i} = 1$, and $\operatorname{rank}(\varphi^n) = (3/2)\,(n+1)$.

CASE II. Suppose n is even. If q ($\neq 0$) is not a root of unity, then $q^{2(n-2)} \neq 1$ for $n > 2$, and $\operatorname{rank}(\varphi^n) = 2n$. If $q = \pm 1$, then $q^{2(n-2)} = q^{2i} = 1$, and $\operatorname{rank}(\varphi^n) = (3/2)n$. This completes the proof. □

For $n = 0, 1, 2$, direct computations show that

$$\dim_k HH^0(\Lambda_q) = 1;$$
$$\dim_k HH^1(\Lambda_q) = 2;$$
$$\dim_k HH^2(\Lambda_q) = \begin{cases} 3 & \text{if } q = \pm 1, \pm\sqrt{-1}; \\ 1 & \text{if } q \neq \pm 1, \pm\sqrt{-1}. \end{cases}$$

THEOREM 4.4. *Let Λ_q be the $\mathbb{Z}_2 \times \mathbb{Z}_2$-Galois covering of the quantum exterior algebra A_q. If q is an rth ($r > 2$) primitive root of unity, then for $n > 2$,*

$$\dim_k HH^n(\Lambda_q) = \begin{cases} k+1 & \textit{if } r \textit{ is odd and } n = 2kr \textit{ or } n = 2kr+2, \textit{ for some } k \geq 1, \\ & \textit{or } r \textit{ is even and } n = kr \textit{ or } n = kr+2, \textit{ for some } k \geq 1; \\ 2k+2 & \textit{if } r \textit{ is odd and } n = 2kr+1, \textit{ for some } k \geq 1, \\ & \textit{or } r \textit{ is even and } n = kr+1, \textit{ for some } k \geq 1; \\ 0 & \textit{otherwise}. \end{cases}$$

PROOF. Note that $HH^n(\Lambda_q) = \operatorname{Ker} \varphi^{n+1}/\operatorname{Im} \varphi^n$ by definition, and

$$\begin{aligned} \dim_k HH^n(\Lambda_q) &= \dim_k \operatorname{Ker} \varphi^{n+1} - \dim_k \operatorname{Im} \varphi^n \\ &= \dim_k M^n - \dim_k \operatorname{Im} \varphi^{n+1} - \dim_k \operatorname{Im} \varphi^n \\ &= \dim_k M^n - \operatorname{rank} \varphi^{n+1} - \operatorname{rank} \varphi^n. \end{aligned} \tag{4.1}$$

The theorem follows from Lemma 4.2. □

THEOREM 4.5. *Let Λ_q be the $\mathbb{Z}_2 \times \mathbb{Z}_2$-Galois covering of the quantum exterior algebra A_q. If q ($\neq 0$) is not an rth ($r > 2$) primitive root of unity, then for $n > 2$,*

$$\dim_k HH^n(\Lambda_q) = \begin{cases} 0 & \textit{if } q\ (\neq 0) \textit{ is not a root of unity}; \\ n+1 & \textit{if } q = \pm 1. \end{cases}$$

PROOF. By Lemma 4.3,

$$\dim_k(\varphi^n) + \dim_k(\varphi^{n+1}) = 4n + 4$$

if q ($\neq 0$) is not a root of unity; and

$$\dim_k(\varphi^n) + \dim_k(\varphi^{n+1}) = 3n + 3$$

if $q = \pm 1$. The theorem follows from the formula (4.1) as desired. □

COROLLARY 4.6. *If q ($\neq 0$) is not an rth ($r > 2$) primitive root of unity, then the Hilbert series of the $\mathbb{Z}_2 \times \mathbb{Z}_2$-Galois covering of the algebra Λ_q is*

$$\sum_{n=0}^{\infty} \dim_k HH^n(\Lambda_q) t^n = \begin{cases} \dfrac{1}{(1-t)^2} & \textit{if } q = \pm 1; \\ 1 + 2t + t^2 & \textit{if } q\ (\neq 0) \textit{ is not a root of unity}. \end{cases}$$

PROOF. This follows from Theorem 4.5 and the fact that

$$\sum_{n=0}^{\infty} (n+1) t^n = \frac{1}{(1-t)^2}.$$ □

Theorem 4.5 shows that the $\mathbb{Z}_2 \times \mathbb{Z}_2$-Galois covering of the algebra Λ_q also give a family of counterexamples to Happel's question in the case where q ($\neq 0$) is not a root of unity. For completeness, we also consider the degenerate case when $q = 0$.

THEOREM 4.7. *Let Λ_0 be the $\mathbb{Z}_2 \times \mathbb{Z}_2$-Galois covering of A_0; then*

$$\dim_k HH^n(\Lambda_0) = \begin{cases} 1 & \text{if } n = 0; \\ 5 & \text{if } n = 1; \\ 2n & \text{if } n \geq 2 \text{ is even}; \\ 2n+2 & \text{if } n \geq 2 \text{ is odd}. \end{cases}$$

PROOF. Denote the cohomology complex of Λ_0 by $(M_0^\bullet, \varphi_0^\bullet)$. We can get

$$\dim_k M_0^n = 4(n+1);$$
$$\dim_k \mathrm{Im}\, \varphi_0^1 = 3;$$
$$\dim_k \mathrm{Im}\, \varphi_0^n = \begin{cases} 2n+2 & \text{if } n \geq 2 \text{ is odd}; \\ 0 & \text{if } n \geq 2 \text{ is even}. \end{cases}$$

Thus, we can get this theorem by the formula (4.1). □

Similar to Corollary 4.6, we have the following result.

COROLLARY 4.8. *The Hilbert series of the $\mathbb{Z}_2 \times \mathbb{Z}_2$-Galois covering algebra Λ_0 is*

$$\sum_{n=0}^{\infty} \dim_k HH^n(\Lambda_0) t^n = 1 + t + \frac{4t(1+t)}{(1-t^2)^2}.$$

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