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ON A 1-DIMENSIONAL PLANAR CONTINUUM WITHOUT THE FIXED POINT PROPERTY

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Introduction. In [**5**; **6**] the author considers the following two problems posed by Professor Lloyd Tucker.

Problem 1. Does there exist a 1-dimensional continuum X without the fixed point property such that every retract of X has the fixed point property with respect to one-to-one maps?

Problem 2. In Problem 1 replace "one-to-one" by "onto."

In [**5**] the author shows that an example of G. S. Young [**8**, p. 884] is an arcwise connected continuum which answers Problem 1 in the affirmative. In [**6**] the author gives an example of an arcwise connected continuum which answers both Problems 1 and 2 simultaneously. The purpose of this paper is to given an example of a 1-dimensional planar continuum X which answers both problems simultaneously and thus provide an affirmative answer to Problem 1 in [**6**].

We remark that the construction of X is a modification of a continuum in [**6**] which involves removing the interior of a triod and adding countably many "sin $1/x$ arcs."

The closure of a subset A of a topological space shall be denoted by Cl A.

1. Construction of the continuum X. Let C_1 denote the continuum in the right half xy-plane which is the union of two segments $[d_1, d_2]$, $[d_2, d_3]$, and the closure of the graph of $y = -5 + \sin(\pi/x)$, $0 < x \leqq 1$, where the points d_1, d_2, d_3 have coordinates $(0, 6)$, $(1, 6)$, $(1, -5)$ respectively. Let C_2 be the image of C_1 under the rotation of the origin O through an angle of π. Now let A be an infinite ray lying in the xy-plane with endpoint $a = (2, 6)$ such that A

(1) is disjoint from $C_1 \cup C_2$ and

(2) "converges" to $C_1 \cup C_2$ in such a way that

   (a) there is a sequence of arcs $S_1, S_2, S_3, \ldots$ filling up A such that $S_i \cap S_j = \emptyset$ for $j \neq i-1, i+1$, and is an endpoint of each for $j = i-1, i+1$, and

   (b) $C_1 = \lim S_{2j-1}, \quad C_2 = \lim S_{2j}$.

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It may be assumed that A has been contructed so that the intersection of each S_i with the set $\{(x, y) | -5 \leqq y \leqq 5\}$ is a vertical line segment with length 10 such that S_{2j-1} passes through $p_{2j-1} = (1 + 1/j, 0)$, and S_{2j} passes through $p_{2j} = (-1 - 1/j, 0)$.

Let A_{2j-1} denote the curve whose equation is

$$y = \frac{1}{2j-1} \sin \left(\frac{\pi}{j(j+1)(x-1) - j} \right) \quad \text{for } 1 + \frac{1}{j+1} < x \leqq 1 + \frac{1}{j}.$$

Thus A_{2j-1} is a $\sin(1/x)$ curve whose closure joins the point p_{2j-1} to the limiting interval in S_{2j+1} whose length is $2/(2j-1)$ and whose midpoint is p_{2j+1}. Let A_{2j} be the image of A_{2j-1} under the rotation of the xy-plane about the origin O through an angle of π.

Let g be a homeomorphism on $\mathrm{Cl}(\bigcup_{i=1}^{\infty} A_{2i})$ which preserves x-coordinates such that the image of $\mathrm{Cl}(\bigcup_{i=1}^{\infty} A_{2i})$ under g lies in the set $\{(x, y) | -2 \leqq x \leqq -1, -3x - 3 \leqq y \leqq -5x - 5\}$. We define $B_{2i}(q_{2i})$ to be the image of $A_{2i}(p_{2i})$ under the homeomorphism g. Then the intersection of $\mathrm{Cl}(\bigcup_{i=1}^{\infty} A_{2i})$ and $\mathrm{Cl}(\bigcup_{i=1}^{\infty} B_{2i})$ consists of the single point $q = (-1, 0)$.

Define $X = C_1 \cup C_2 \cup A \cup (\bigcup_{i=1}^{\infty} A_i) \cup (\bigcup_{i=1}^{\infty} B_{2i})$. Then X is a 1-dimensional (indeed, rational) planar continuum.

We define a fixed point free map $f: X \to X$ as follows. Restricted to $C_1 \cup C_2$, f is a rotation in the xy-plane about O through an angle of π. Also f is a continuous function mapping the path component $X - C_1 \cup C_2$ into itself such that for each i, f maps S_i onto S_{i+1}, and the restrictions $f|A_i$, $f|B_{2i}$ homeomorphically map the sets A_i, B_{2i} onto the sets A_{i+1}, A_{2i+1} respectively. The only other precaution that we must take to insure the continuity of f is to make certain that as a point moves far out on A, its image under f is very near its reflection through the origin O.

2. Proof that every retract of X has the fixed point property with respect to one-to-one maps and with respect to onto maps. (i) First we show that X itself has the fixed point property with respect to one-to-one maps and with respect to onto maps.

Suppose $h: X \to X$ is a one-to-one map. Then h is a homeomorphism of X into itself. Moreover, X is locally connected at $q = (-1, 0)$ and there is a connected neighborhood V of q such that $V - \{q\}$ consists of four components. Since no other point in X has this property, it follows that $h(q) = q$.

Now suppose that $h: X \to X$ is an onto map. We claim that $h(X - C_1 \cup C_2) = X - C_1 \cup C_2$. To see this, suppose $h(X - C_1 \cup C_2) \cap (C_1 \cup C_2) \neq \emptyset$. Since path components must be preserved, it follows that $h(X - C_1 \cup C_2) \subset C_1 \cup C_2$. Therefore $h(X) \subset C_1 \cup C_2$ which is a contradiction.

The remaining argument parallels a portion of the argument found in case (ii) of [**6**]. Let e be a homeomorphism from A onto the non-negative real numbers. Since $h(A)$ can contain no A_i or B_{2i}, there is a retraction $r: h(A) \cup$

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$A \to A$ defined by

$$r(x) = \begin{cases} x & \text{if } x \in A, \\ p_i & \text{if } x \in A_i, \\ q_{2i} & \text{if } x \in B_{2i}. \end{cases}$$

Define a map H from A into the real numbers $\mathbf{R}$ by $H(x) = erh(x) - e(x)$ for all x in A. If $h(A_1)$ contains an infinite subarc of A_1, it follows that h has a fixed point in Cl A_1. Hence, since h is onto, we may assume that an infinite subarc of A_1 must lie in $h(A_i)$ or $h(B_{2j})$ for some $i > 1$ or $2j > 1$. Assume that it is $h(A_i)$. Then $h(p_{i+2}) \in \text{Cl } A_1$ and hence $H(p_{i+2}) < 0$. If $h(a) \neq a$, then $H(a) > 0$. Since H is a continuous function from A into $\mathbf{R}$, it follows that there is a point c in A such that $H(c) = 0$. Thus $rh(c) = c$. If $h(c) \in A$, then $h(c) = c$. Hence suppose $h(c)$ lies in the interior of some A_i or B_{2j}. Assume that $h(c) \in \text{Int } A_i$ and thus $c = p_i$. If h maps Cl A_i into itself, then h has a fixed point. Otherwise, regard A_i as a directed arc with initial point p_i, and let a_i be the first point in A_i such that $h(a_i) = p_i$. It then follows that there is a point in A_i between p_i and a_i which is fixed under h.

(ii) We now consider the cases where Y is a proper retract of X. The arguments for these cases are similar to those used in case (iii) of [**6**], and we shall not repeat them in detail.

First we consider the case of a proper retract Y which contains an infinite subarc of A. Then, as in [**6**, p. 181], it is easy to see that Y must be a continuum which contains all but finitely many of the A_i and B_{2i}. Consequently, the argument for this case is completely analogous to that used for X itself.

It is easy to show that the only retracts of X lying in $C_1 \cup C_2$ are singleton points or compact arcs (see [**6**, p. 181]).

Finally, we consider the case of the retracts of X which lie in the path component $X - C_1 \cup C_2$. Any dendrite D in $X - C_1 \cup C_2$ has the fixed point property and is a retract of X [**2**, p. 138]. Also any arcwise connected continuum Y consisting of a compact subarc of A and finitely many A_i and B_{2i} is a retract of X which has the fixed point property [**6**, p. 182]. Consequently, any continuum of the form $D \cup Y$ is a retract of X which has the fixed point property, and this completes (ii).

Remarks. 1) In [**5**] the author asks if a planar and arcwise connected example can be found. Of course, such an example could not contain a simple closed curve. In fact, no such example can exist. This is a consequence of C. L. Hagopian's recent announcement that every arcwise connected planar continuum containing no simple curve has the fixed point property (see question 4 of [**1**]).

2) By a simply connected space we mean an arcwise connected space whose fundamental group is trivial. In [**6**] the author asks if there exists a simply connected 1-dimensional continuum X which does not have the fixed point property with respect to homeomorphisms.

We claim that such a space X could contain no simple closed curve. To see this, suppose C is a simple closed curve in X. Since X is 1-dimensional and C is homeomorphic to the unit circle S^1, it follows that C is a retract of X [**4**, p. 83]. If $p \in C$, then the inclusion map i: $C \to X$ induces a monomorphism i_*: $\pi_1(C, p) \to \pi_1(X, p)$ of the corresponding fundamental groups [**3**, p. 150]. Since $\pi_1(C, p)$ is infinite cyclic, it follows that X is not simply connected.

Lee Mohler has shown that no such example can exist by solving the following more general problem. In [**7**] he shows that every arcwise connected continuum containing no simple closed curve has the fixed point property with respect to homeomorphisms.

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